

MEASURE THEORETICAL NULLSETS AND DIOPHANTINE EQUATIONS

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ABSTRACT. This paper explains and builds off of foundational tools of measure theory. A main goal of the paper is to learn about the Lebesgue measure and examine an example of nullsets seen when determining the amount of rational approximations to a specific diophantine equation. The paper overviews and defines algebras, premeasures, outermeasures, measures and the Lebesgue measure, then explores diophantine equations of degree greater than or equal to two. Finally, we discuss how a particular diophantine inequality researched by Dirichlet has a set of infinite solutions with measure zero.

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INTRODUCTION

How far away is the grocery store on foot? To answer this, one would likely calculate the length of road walked from the person's current location to the front entrance of a grocery store. For a person living in a city whose neighborhood has a lot of construction or unkempt sidewalks, there could be severe gaps in the road. What if for every step that person took on the sidewalk, the next step would be on a missing stretch of concrete? If we measure the length of that road by summing up the length of the existing sidewalk, we get a value close to zero due to the amount of missing concrete. Even if two people live the same "distance" away from the grocery store, this example shows that by measuring by road length, a person could be measure zero away from the store while living far away. Intuitively, that is a contradiction.

For notions of length, area, and volume in pathological spaces, the ability to convey a concrete value that feels accurate can be difficult. Measure theory provides a way to broaden our idea of what can be measured by generalizing pre-existing ideas

about measurement. Measures can be found useful in the many areas of math where set theory is present – even polynomials. Rather than proving a set of solutions to a problem doesn't exist, one can show that the set of satisfactory solutions has measure zero. The two statements are equivalent, but at times, one can be easier to prove. In relation to homogenous diophantine equations, many mathematicians have explored what the size of certain solution sets are. These questions often resemble the form “How many integer solutions are there to an equation of this type? Finitely many? Infinitely many?” For such questions, measures and nullsets will be invaluable. Before getting into such examples of nullsets, we will first have to discuss to how many sets we can extend this abstraction of measurement to, what properties these measures have, and what rules they obey. Before that, we must ask, “What is a measure?”

1. ALGEBRAS

To begin the study of measures, we must first discuss σ -algebras. Ideally, a measure would have an input for the object being measured, and an output value revealing the issued measurement. For this to happen, the measure has to be a function, and every function needs a domain. A σ -algebra is defined as the type of sets that can be measured. It therefore makes sense for a σ -algebra to be the domain of a measure. In order to measure a set, we must be able to measure its entirety, and also have an understanding of what its lack is: what a “zero” measurement is. If we want to measure some subset of the larger set, then we must also be able to measure that sets complement. Finally, if we can measure multiple disjoint subsets individually, we must be able to know what the measure of their sum is. Since these sets are different and not overlapping, the measure of the individual subsets added together should be equal to the subsets added first and then measured. We make these intuitive notions of measurement clear in the following definitions.

Definition 1.1. A σ -algebra $\mathcal{A}$ on a nonempty set X is a subset $\mathcal{A} \subseteq \mathcal{P}(X)$ such that

- $\emptyset$ and X are elements of $\mathcal{A}$,
- If $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$, and
- If $E_1, E_2, E_3, \dots$ is a countable sequence of disjoint $E_j \in \mathcal{A}$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$.

Definition 1.2. The σ -algebra $\mathcal{B}_X$ that is the collection of open sets in a topological space X is called the Borel σ -algebra on X .

Remark 1.3. Since $\mathcal{B}_X$ is a σ -algebra and is therefore closed under complements, $\mathcal{B}_X$ also contains the complements of open sets, which are closed sets. Note that $\mathcal{B}_X$ contains open and closed sets, countable intersections of open sets, countable unions of closed sets, and so on.

2. MEASURES

Now that we have defined σ -algebras, we can define measures. Much like algebras, the definition of a measure is rooted in notions of measurement that we already know. To preserve this definition, we want to make sure that the measurement of nothing is zero, and that the measurement of a union of disjoint sets is the same as the sum of measurements of those sets separately. We use these ideas to define measures in the following way.

Definition 2.1. Let X be a set with a σ -algebra $\mathcal{A}$. Then a measure μ on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \rightarrow [0, \infty]$ such that

- $\mu(\emptyset) = 0$, and
- If $E_1, E_2, E_3, \dots$ is a countable sequence of disjoint $E_j \in \mathcal{A}$, then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

Definition 2.2. If X is a set with a σ -algebra $\mathcal{A}$, then $(X, \mathcal{A})$ is a measurable space and the sets $\mathcal{A}$ are measurable sets. If μ is a measure on $(X, \mathcal{A})$, then $(X, \mathcal{A}, \mu)$ is a measure space.

With this definition of a measure we can derive the following properties.

Theorem 2.3. Let $(X, \mathcal{A}, \mu)$ be a measure space. Then the following are true:

- If $E, F \in \mathcal{A}$ and $E \subseteq F$, then $\mu(E) \leq \mu(F)$.
- If $\{E_j\}_1^{\infty}$ is a sequence in $\mathcal{A}$, then $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$.
- If $\{E_j\}_1^{\infty}$ is a sequence in $\mathcal{A}$ and $E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$, then $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{i \rightarrow \infty} \mu(E_i)$.
- If $\{E_i\}_1^{\infty}$ is a sequence in $\mathcal{A}$ and $E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$ with $\mu(E_1) < \infty$, then $\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{i \rightarrow \infty} \mu(E_i)$.

Definition 2.4. If $(X, \mathcal{A}, \mu)$ is a measure space, a set $E \in \mathcal{A}$ with $\mu(E) = 0$ is called a nullset.

Remark 2.5. By the properties of a measure, a countable union of nullsets is a nullset. If a statement about points is true except for the points that lie in nullsets, we say that it is true almost everywhere, denoted **a.e.** If $E \subseteq \mathcal{A}$ is a nullset and all subsets of E are contained in $\mathcal{A}$, then μ is a complete measure.

While measures are well defined, they may not be easy to find. The next section outlines a way to build a measure.

3. OUTER MEASURES

In order to find an exact measure for a space, we must first approximate it. After continually improving our approximations, we will arrive at the intended measure. We initiate the approximation process by defining outer measures. The intuition for an outer measure is to approximate the measure of the set from the outside. For example, in calculus we approximate the area below the curve of an increasing function $f(x)$ by splitting the domain into subintervals of a given length n and covering the area under the curve in rectangles of width n and height $f(a)$ where a is the leftmost point in a given interval of length n . Much like earlier concepts were abstracted, we abstract this notion of approximating from outside the desired value of a function by extending our intuition to sets.

Definition 3.1. If $A \subseteq \mathcal{P}(X)$ is an algebra, then a function $\mu_o : A \rightarrow [0, \infty]$ is called a premeasure if

- $\mu_o(\emptyset) = 0$,

- If $E_1, E_2, E_3, \dots$ is a countable sequence of disjoint $E_j \in A$, then

$$\mu_o(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu_o(E_i).$$

Definition 3.2. An outer measure on a non empty set X is a function $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ such that

- $\mu^*(\emptyset) = 0$,
- If $A \subseteq B$ for $B \subseteq X$, then $\mu^*(A) \leq \mu^*(B)$,
- If $\{E_j\}_1^{\infty}$ is a sequence in $\mathcal{P}(X)$, then $\mu^*(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$.

Note that the third axiom for outer measures does not require a sequence to be comprised of disjoint sets in order to understand its outer measure. In contrast to the axioms for a measure, the definition for an outer measure is looser and defines an inequality between the measure of an infinite union of sets in a sequence and the countably infinite sum of measures of sets in a sequence. As such, it is clear that a measure is always an outer measure, while the opposite is not always true. It would be useful to examine this difference in definition more closely.

Definition 3.3. If μ^* is an outer measure on X , a set $B \subseteq X$ is called μ^* -measurable if and only if for all $E \subseteq X$ with $\mu^*(E) < \infty$, $\mu^*(E) \geq \mu^*(E \cap B) + \mu^*(E \cap B^c)$.

Remark 3.4. Let μ_o be a premeasure on an algebra $A \subseteq \mathcal{P}(X)$. For any $E \subseteq X$, let

$$\mu^* = \inf \left\{ \sum_1^{\infty} \mu_o(A_j) \mid A_j \in A, E \subseteq \bigcup_1^{\infty} A_j \right\}.$$

Then μ^* is an outer measure. In this way, a premeasure on X can induce an outer measure on X .

Theorem 3.5. Carathéodory's Theorem. Let μ^* be an outer measure on X . Then the collection $\mathcal{M}$ of μ^* -measurable sets forms a σ -algebra, and the restriction of μ^* to $\mathcal{M}$ is a complete measure.

We have established a relationship between pre-measures, outer measures and measures. As outlined above, in order to build a measure on X , we begin by defining an algebra on X . We use that algebra as the domain for a premeasure μ_o , such that μ_o is a function from the algebra to $[0, \infty]$ satisfying certain properties. We then take the infimum of the set of all premeasures of subsets of X such that each subset is covered by a countable collection of open sets. That infimum is an outer measure μ^* . From there we consider the restriction of μ^* only to μ^* -measurable sets. That restricted outer measure is a complete measure whose domain is the σ -algebra comprised of μ^* -measurable sets.

Using this pathway, we can build a specific measure called the Lebesgue measure. The domain of this measure is a σ -algebra that is a subset of $\mathbb{R}$. The Lebesgue measure defines measure as the length of an interval in $\mathbb{R}$.

4. THE LEBESGUE MEASURE

The Lebesgue measure is arguably the most important measure in measure theory because it can measure things in our physical world. While other measures can

be useful, understanding the Lebesgue Measure gives us insight into traditional definitions of length, area, and volume, as well as the inner workings of calculus. The Lebesgue measure can be used in higher dimensional spaces, using the Lebesgue integral. For the purpose of this paper, we will focus on the Lebesgue measure over $\mathbb{R}^1$.

Definition 4.1. A Borel Measure is a measure μ with a domain of the Borel σ -algebra.

Remark 4.2. Consider the set of intervals of the forms $(a, b]$, (a, ∞) where $a, b \in \mathbb{R}$. Then such a set with the inclusion of $\mathbb{R}$ and $\emptyset$ form an algebra, specifically $\mathcal{B}_{\mathbb{R}}$.

Definition 4.3. Let μ be the Borel σ -algebra over $\mathbb{R}$. A function F such that $F(x) = \mu((-\infty, x])$ is called the distribution function of μ , and F is increasing and continuous from the right.

Definition 4.4. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ defined by $F(x) = x$ be increasing and continuous from the right. If $(a_j, b_j]$ with $j = 1, \dots, n$ are disjoint intervals, let

$$m_o \left(\bigcup_{j=1}^{\infty} (a_j, b_j] \right) = \sum_{j=1}^{\infty} [F(b_j) - F(a_j)]$$

with $m_o(\emptyset) = 0$. Then m_o is a premeasure on the algebra $\mathcal{B}_{\mathbb{R}}$. Note that this suggests that $m_o : \mathcal{B}_{\mathbb{R}} \rightarrow [0, \infty]$ is defined by $m_o((a, b]) = F(b) - F(a)$.

Definition 4.5. Let m_o be defined above, and for any $E \subseteq \mathbb{R}$ let

$$m^* = \inf \left\{ \sum_1^{\infty} m_o(A_j) \mid A_j \in \mathcal{B}_{\mathbb{R}}, E \subseteq \bigcup_1^{\infty} A_j \right\}.$$

Then, m^* is an outer measure on $\mathcal{B}_{\mathbb{R}}$.

Proposition 4.6. *The restriction of m^* to m^* -measurable sets is a measure. The completion of that measure is called a Lebesgue Measure. The domain of the Lebesgue measure, the collection of Lebesgue measurable sets, is denoted $\mathcal{L}$.*

Here is another way to build the set, by defining a larger set known as the set of Lebesgue-Stieltjes measures. This method is based closely on material from source [2].

Theorem 4.7. *If $F : \mathbb{R} \rightarrow \mathbb{R}$ is any increasing, right continuous function, then there exists a unique Borel measure μ_F such that $\mu_F((a, b]) = F(b) - F(a)$ for all a, b . If μ is some Borel measure on $\mathbb{R}$ and we define*

$$F(x) = \begin{cases} \mu((0, x]), & x > 0 \\ 0, & x = 0 \\ -\mu((0, x]), & x < 0 \end{cases}$$

then F is increasing and right continuous, and we know that this μ must be our unique measure μ_F .

Definition 4.8. The completed measure $\overline{\mu_F}$ is a Lebesgue-Stieltjes measure associated to F .

Remark 4.9. Pick any Lebesgue-Stieltjes measure $\overline{\mu}_F$ on $\mathbb{R}$ associated to the function F . Let the domain of $\overline{\mu}_F$ be denoted $\mathcal{M}$. Then, for any $E \subseteq \mathcal{M}$ we have

$$\begin{aligned}\overline{\mu}_F(E) &= \inf \left\{ \sum_{j=1}^{\infty} [F(b_j) - F(a_j)] \mid E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j] \right\} \\ &= \inf \left\{ \sum_{j=1}^{\infty} \overline{\mu}_F((a_j, b_j]) \mid E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j] \right\} \\ &= \inf \left\{ \sum_{j=1}^{\infty} \overline{\mu}_F((a_j, b_j)) \mid E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j) \right\}.\end{aligned}$$

Definition 4.10. The Lebesgue measure on $\mathbb{R}$ is the complete measure μ_F associated to the function $F(x) = x$. The Lebesgue measure is denoted m . The domain of m is called the class of Lebesgue measurable sets, denoted $\mathcal{L}$. The restriction $m|_{\mathbb{B}_{\mathbb{R}}}$ is the Lebesgue measure.

The Lebesgue measure has the following properties:

Definition 4.11. If $E \subseteq \mathbb{R}$ and $s, r \in \mathbb{R}$ then $E + s = \{x + s \mid x \in E\}$, $rE = \{rx \mid x \in E\}$.

Theorem 4.12. If $E \in \mathcal{L}$, then $E + s \in \mathcal{L}$ and $rE \in \mathcal{L}$ for all $s, r \in \mathbb{R}$. Additionally, $m(E + s) = m(E)$ and $m(rE) = |r|m(E)$.

Now that we have surveyed the fundamentals of measure theory and have an understanding of the Lebesgue measure, we can see an example of how measures appear in other areas of math.

5. DIRICHLET THEOREM ON DIOPHANTINE EQUATIONS

We will shortly apply our knowledge of nullsets and the Lebesgue measure to a well discussed problem in number theory regarding solutions to diophantine equations. Before we see this example, we will explore similar diophantine equations and discuss some results that will be helpful in understanding our final example. The following definitions will be essential to our discussion.

Definition 5.1. A diophantine equation is a polynomial equation with two or more unknowns and integer coefficients, focusing only on integer solutions.

Definition 5.2. A homogenous polynomial is a polynomial with all its terms to the same degree.

The first example we will dissect is the simple diophantine equation $y^d - 2x^d = 1$ and the conditions needed for integer solution pairs (x, y) to exist. We begin by hypothesizing that for solutions (x, y) to $y^2 - 2x^2 = 1$, the ratio $\frac{y}{x}$ must be a “fairly good” rational approximation to $\sqrt{2}$. That is to say, $\frac{y}{x}$ and $\sqrt{2}$ are within a small given distance from each other.

Proposition 5.3. Consider the diophantine equation $y^2 - 2x^2 = 1$. If $x, y \in \mathbb{Z}$ are solutions to the equation, then

$$\left| \sqrt{2} - \left| \frac{y}{x} \right| \right| \leq \frac{1}{|x|^2}.$$

Proof. Let $y^2 - 2x^2 = 1$. Divide both sides by x^2 to have

$$\begin{aligned} \left| \left(\frac{y}{x} \right)^2 - 2 \right| &= \frac{1}{|x|^2} \\ \left| \left(\frac{y}{x} \right)^2 - (\sqrt{2})^2 \right| &= \frac{1}{|x|^2} \\ \left| \left| \frac{y}{x} \right| - \sqrt{2} \right| \left| \left| \frac{y}{x} \right| + \sqrt{2} \right| &= \frac{1}{|x|^2} \\ \left| \left| \frac{y}{x} \right| - \sqrt{2} \right| &= \frac{1}{|x|^2} \cdot \frac{1}{\left| \frac{y}{x} \right| + \sqrt{2}} \\ &\leq \frac{1}{|x|^2}. \end{aligned}$$

□

We expand our intuition from the results of $d = 2$ to find a similar result for $d \geq 3$.

Proposition 5.4. *Consider the diophantine equation $y^d - 2x^d = 1$. If $x, y \in \mathbb{Z}_{>0}$ are solutions and $d \geq 3$, then*

$$\left| 2^{1/d} - \frac{y}{x} \right| \leq \frac{1}{|x|^d}.$$

Proof. Let $y^d - 2x^d = 1$. We divide both sides by x^d and get

$$\begin{aligned} \left| \left(\frac{y}{x} \right)^d - 2 \right| &= \frac{1}{|x|^d} \\ \left| \left(\frac{y}{x} \right)^d - 2^{(1/d)d} \right| &= \frac{1}{|x|^d}. \end{aligned}$$

Next, we apply the difference of powers formula. Recall that $x, y > 0$ to see

$$\begin{aligned} \left| \left(\frac{y}{x} \right)^d - 2^{(1/d)d} \right| &= \left| \left(\frac{y}{x} \right)^d - (2^{1/d})^d \right| = \left| \left(\frac{y}{x} \right)^d - (2^{1/d})^{d-1} \left(\frac{y}{x} \right) + (2^{1/d})^{d-1} \left(\frac{y}{x} \right) - (2^{1/d})^{d-2} 2^{(1/d)} \left(\frac{y}{x} \right) + \dots \right. \\ &\quad \left. + (2^{1/d})^{d-2} 2^{(1/d)} \left(\frac{y}{x} \right) - (2^{1/d})^{d-1} 2^{(1/d)} + (2^{1/d})^{d-1} 2^{(1/d)} \right| = \frac{1}{|x|^d} \\ &= \frac{1}{|x|^d} \cdot \frac{1}{\left| \left(\frac{y}{x} \right)^{d-1} + \left(\frac{y}{x} \right)^{d-2} 2^{(1/d)} + \dots + \left(\frac{y}{x} \right) 2^{(1/d)(d-1)} + 2^{(1/d)d} \right|} \\ &\leq \frac{1}{|x|^d}. \end{aligned}$$

□

Abstracting the result from Proposition 5.4, we will explore rational approximations for more than just the irrational number $2^{1/d}$. We consider rational approximations for any general irrational number β . In discovering the number of integer solutions to diophantine equations of this type, we first want to determine when there will be infinitely many solutions. Similarly to before, we will examine equations with $d = 2$, and then equations with $d > 2$.

Proposition 5.5. *For any irrational $\beta \in \mathbb{R}$, there are infinitely many solutions to*

$$\left| \beta - \frac{y}{x} \right| \leq \frac{1}{|x|^2}.$$

Proof. For any $a \in \mathbb{R}$, let $\langle a \rangle \in [0, 1)$ be the fractional part of a such that $\langle a \rangle + I = a$ for $I \in \mathbb{Z}$. Let $Q \in \mathbb{N}$ and consider the sequence $\langle \beta \rangle, \langle 2\beta \rangle, \dots, \langle Q\beta \rangle \in [0, 1)$. If we split $[0, 1)$ into subintervals of length $\frac{1}{Q}$, we know that with Q many fractional parts of β , at least two of those numbers must be in one subinterval of length $\frac{1}{Q}$. This can be seen easily through pigeonhole principle. If at least two $q_i\beta$ do not lie in the same interval, we would have to assume that two of the $q_i\beta$ s have the same fractional part. If the two $q_i\beta$ s have the same fractional part, we know that the numbers differ only by an integer $I \in \mathbb{Z}$. In this case we would have

$$\begin{aligned}\langle q_i\beta \rangle &= \langle q_j\beta \rangle \\ q_i\beta &= q_j\beta + I \\ (q_i - q_j)\beta &= I \\ \beta &= \frac{I}{q_i - q_j} \in \mathbb{Z}\end{aligned}$$

which is a contradiction since β is irrational. Therefore, we know that no fractional parts of different $q_i\beta$ s can be the same, and at least two must lie in the same interval of length $\frac{1}{Q}$. We choose two of these numbers $q_1\beta$ and $q_2\beta$ with $1 \leq q_1 \leq q_2 \leq Q$ such that

$$\begin{aligned}|\langle q_2\beta \rangle - \langle q_1\beta \rangle| &\leq \frac{1}{Q} \\ |\langle (q_2 - q_1)\beta \rangle| &\leq \frac{1}{Q}.\end{aligned}$$

With

$$q_1 \leq q_2 \leq Q,$$

we also have

$$0 \leq q_2 - q_1 \leq Q - q_1 \leq Q.$$

Let $x = q_2 - q_1$. Then $0 \leq x \leq Q$ and there exists some $x\beta \in [0, 1)$ such that

$$|\langle x\beta \rangle| \leq \frac{1}{Q}.$$

We pick a strategic integer y such that $\langle x\beta \rangle + y = x\beta$ and therefore

$$\begin{aligned}|x\beta - y| &\leq \frac{1}{Q} \\ \left|\beta - \frac{y}{x}\right| &\leq \frac{1}{|x|Q} \\ &\leq \frac{1}{|x|^2}.\end{aligned}$$

□

Proposition 5.6. *If $d > 2$, then almost everywhere $\beta \in \mathbb{R}$ allows for a finite number of solutions to the diophantine inequality*

$$\left|\beta - \frac{y}{x}\right| \leq \frac{1}{|x|^d}.$$

To prove our proposition we will show that if D_d is the set of $\beta \in \mathbb{R}$ such that

$$\left| \beta - \frac{y}{x} \right| \leq \frac{1}{|x|^d}$$

has infinitely many solutions, then the measure of D_d must be zero.

Proof. For a fixed $x = x_o$, we consider β where

$$\left| \beta - \frac{y}{x_o} \right| \leq \frac{1}{|x_o|^d} \leq 1$$

$$\left| \beta - \frac{y}{x_o} \right| \leq 1$$

$$-1 \leq \beta - \frac{y}{x_o} \leq 1$$

$$\beta - 1 \leq \frac{y}{x_o} \leq \beta + 1.$$

Now consider $D_d \cap [0, 1)$. We know that if $\beta \in [0, 1)$ is a solution, then

$$-1 \leq \frac{y}{x_o} \leq 2,$$

$$-x_o \leq y \leq 2x_o$$

is true for our fixed x . On the other hand, we see from the given equation that

$$\beta \in \left(\frac{y}{x_o} - \frac{1}{|x_o|^d}, \frac{y}{x_o} + \frac{1}{|x_o|^d} \right),$$

$$\left\{ \beta \in [0, 1) \mid \left| \beta - \frac{y}{x_o} \right| \leq \frac{1}{|x_o|^d} \right\} \subseteq \left\{ \left(\frac{y}{x_o} - \frac{1}{|x_o|^d}, \frac{y}{x_o} + \frac{1}{|x_o|^d} \right) \mid -x_o \leq y \leq 2x_o \right\}.$$

It is clear that for fixed $x = x_o$, there are $3x_o$ choices for y . Using the Lebesgue measure m where the measure of a set is its length, we compute

$$\begin{aligned} m \left(\left\{ \beta \in [0, 1) \mid \left| \beta - \frac{y}{x_o} \right| \leq \frac{1}{|x_o|^d} \right\} \right) &\leq m \left(\left\{ \left(\frac{y}{x_o} - \frac{1}{|x_o|^d}, \frac{y}{x_o} + \frac{1}{|x_o|^d} \right) \mid -x_o \leq y \leq 2x_o \right\} \right) \\ &= 3x_o \cdot m \left(\left\{ \left(\frac{y}{x_o} - \frac{1}{|x_o|^d}, \frac{y}{x_o} + \frac{1}{|x_o|^d} \right) \right\} \right) \\ &= \frac{6}{|x_o|^{d-1}} \\ &\sim \frac{1}{|x_o|^{d-1}}. \end{aligned}$$

The above equation must only be considered when x_o is arbitrarily large. Otherwise, we would have a finite number of y such that $-x_o \leq y \leq 2x_o$, and therefore there would be a finite number of solutions (x, y) to our problem. Recall that we are currently examining when our diophantine inequality has infinitely many solutions. If we now consider arbitrary large values of x , we get that

$$m(D_d \cap [0, 1)) \leq \sum_{x > x_o}^{\infty} \frac{1}{|x|^{d-1}} \rightarrow 0$$

for all $d > 2$. Examining the sequence, it is clear that the tail end goes to zero. This happens for arbitrarily large $x > x_o$, which are the only values of x we can

consider for infinitely many solutions. Thus, the measure of $D_d \cap [0, 1)$ is zero and $D_d \cap [0, 1)$ is a nullset. This process can repeat for the intersection of D_d with any interval $[a, a + 1)$ for $a \in \mathbb{R}$. The result shows that each intersection has Lebesgue measure zero and is therefore a nullset. Since D_d is the union of these intervals, and the union of nullsets is a nullset, we must conclude that $m(D_d) = 0$. $\square$

CONCLUSION

As explored in this paper, abstracting the concept of measurement can lead to useful problem solving methods. These ideas can be extended well past the realm of geometry. Connecting the idea of nullsets to “falseness” is a particularly clever technique used in Proposition 5.6. It would be interesting to see how nullsets appear in other number theory problems and elsewhere that this technique is useful. Much of the analysis of diophantine equations presented in this paper are ideas used to build intuition for Thue’s Theorem. Thue’s theorem is a staple concept in number theory and questions the amount of solutions to any homogenous polynomial of degree greater than or equal to three. If said polynomial is irreducible over the integers, it is proven to have only finitely many integer solutions. For readers interested in diophantine equations, that is an accessible next step. Further research building off the contents of this paper can be done on the Lebesgue Integral as well as the study of measures in n -dimensional spaces.

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