

GROUP ACTIONS ON BOOLEAN ALGEBRAS

LEO LONG

ABSTRACT. This paper provides an introduction to Boolean algebra and group actions on it. We first define the Boolean algebra B_n and some key concepts regarding partially ordered sets. We show that the Boolean algebra B_n is graded of rank n , rank-symmetric, rank-unimodal, and Sperner. We then introduce the quotient poset B_n/G and show that it also has the four properties of B_n . Finally, we demonstrate two applications of B_n/G in graph theory and combinatorics, respectively.

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1. INTRODUCTION

Boolean algebra, the power set of a finite set equipped with inclusion partial order, is a fundamental algebraic structure with a wide range of applications in logic, set theory, graph theory, and combinatorics. One may notice that Boolean algebras have some beautiful properties: since $\binom{n}{k} = \binom{n}{n-k}$ for all natural numbers $n \geq k$, if we divide a Boolean algebra into “levels” based on the number of elements in each subset, we will obtain a “symmetric” diagram such that the number of subsets at level k is the same as that of level $n - k$; since $\binom{n}{k}$, as a function of k , first increases, attains its maximum at $k = \lfloor \frac{n}{2} \rfloor$, and then decreases, the diagram obtained will have most elements in the middle level and least elements at top and bottom. These two properties correspond to rank-symmetry and rank-unimodality, which we will define rigorously in the next section. Along with the Sperner property, which we will also define later, these three beautiful properties enable us to solve some problems in an elegant way using Boolean algebras.

However, in many cases, we are interested in the quotients of Boolean algebras. That is, we may group subsets that are equivalent under a specific group action, which leads us to consider the structure of the resulting quotient poset. Surprisingly, three beautiful properties we mentioned previously: rank-symmetry,

rank-unimodality, and the Sperner property, still hold for the quotient poset. This enables some further applications of Boolean algebras, some of which will be discussed later in the paper.

We will demonstrate two applications in graph theory and combinatorics, respectively. First, we will show that the set of isomorphism classes of simple graphs on a vertex set is isomorphic to some quotient of Boolean algebra, which leads to applications in the enumeration of nonisomorphic graphs and solves a problem in extremal graph theory. Second, we connect quotients of Boolean algebras to integer partitions and rectangular Young lattices, which results in an application in combinatorics.

2. BOOLEAN ALGEBRAS

We first review some key definitions about partially ordered sets.

Definition 2.1. A **partially ordered set (poset)** P is a finite set equipped with a partial order $\leq$, which is a binary relation such that for all $x, y, z \in P$, we have (1) $x \leq x$, (2) $x \leq y$ and $y \leq x$ implies $x = y$, and (3) $x \leq y$ and $y \leq z$ implies $x \leq z$.

One can see that for all finite collections of sets P , we can turn P into a poset by defining, for all $x, y \in P$, $x \leq y$ if $x \subseteq y$. In this paper, we are interested in a particular type of collection of sets: Boolean algebra, which is defined as follows.

Definition 2.2. P is called a **Boolean algebra** of rank n if it consists of all subsets of an n -element set S and $x \leq y$ in P if and only if $x \subseteq y$.

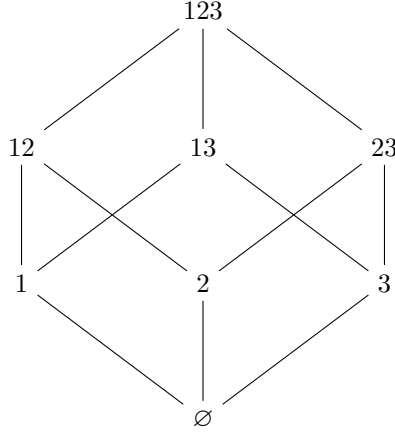
In this case, we denote P by B_S . In particular, if $S = \{1, \dots, n\}$, we denote B_S by B_n . By the previous argument, all Boolean algebras are posets. In this paper, we will denote sets using the abbreviated notation when it does not lead to confusion. For example, we may denote $\{1, 2, 3\}$ simply by 123.

Definition 2.3. Let P be a poset. For all $x, y \in P$, we say y **covers** x if $x < y$ and there does not exist $z \in P$ such that $x < z < y$. We write $x < y$.

Note that $x < y$ means $x \leq y$ and $x \neq y$. The notion of cover provides us a way to represent posets graphically. Since the relation $\leq$ is transitive, given all cover relations, we can uniquely determine all relations of a poset. The **Hasse diagram** is a diagram that depicts covering relations of a poset. In the Hasse diagram of a poset P , if $x < y$ in P , then an edge is drawn between x and y with y drawn above x . By the previous argument, we can recover all relations of a poset from its Hasse diagram. [Figure 1](#) shows the Hasse diagram of the Boolean algebra B_3 .

Definition 2.4. We introduce the following definitions regarding posets.

- (1) Two posets P and Q are **isomorphic** if there exists a bijection $\phi : P \rightarrow Q$ such that $x \leq y$ in P if and only if $\phi(x) \leq \phi(y)$ in Q . The bijection is called an **isomorphism** between P and Q .
- (2) An **automorphism** of a poset P is an isomorphism $\phi : P \rightarrow P$.
- (3) Let P be a poset. A **chain** C in P is a totally ordered subset of P . That is, for all $x, y \in C$, we have either $x \leq y$ or $y \leq x$.
- (4) We say that a finite chain has **length** n if it has $n + 1$ elements.
- (5) A chain is **maximal** if it is not contained in a larger chain.
- (6) A poset P is **graded of rank** n if every maximal chain in P has length n .

FIGURE 1. Hasse diagram of the Boolean algebra B_3 .

- (7) A chain $x_0 < x_1 < \cdots < x_n$ is **saturated** if $x_i \lessdot x_{i+1}$ for each i .
- (8) Let P be a graded poset. We say $x \in P$ has **rank** i if the largest saturated chain in P with top element x has length i . We denote this by $\rho(x) = i$.
- (9) Let P be a graded poset. Then $P_i = \{x \in P : \rho(x) = i\}$ is the i th **level** of P .

Example 2.5. We now present examples corresponding to the definitions introduced in Definition 2.4.

- (1) For each n -element set S , B_S is isomorphic to B_n . Therefore, in this paper, we will often only show that certain results hold for the Boolean algebra B_n , since then the results can be easily generalized to all Boolean algebras.
- (2) An automorphism ϕ of B_2 is defined by $\phi(\emptyset) = \emptyset$, $\phi(1) = 2$, $\phi(2) = 1$, and $\phi(12) = 12$.
- (3) $\{\emptyset, 1, 12, 123\}$ is a chain in B_3 while $\{\emptyset, 1, 23, 123\}$ is not.
- (4) The chain in (3) has length 3. Note that each chain of length n can be written in the form $x_0 < x_1 < \cdots < x_n$.
- (5) The chain in (3) is a maximal chain in B_3 while $\emptyset < 1 < 12$ is not.
- (6) B_n is graded of rank n .
- (7) $2 < 23$ is a saturated chain in B_3 while $\emptyset < 1 < 123$ is not.
- (8) $\rho(12) = 2$ in B_3 since the largest saturated chains in B_3 with top element 12 are $\emptyset < 1 < 12$ and $\emptyset < 2 < 12$.
- (9) $(B_n)_0 = \{\emptyset\}$ and $(B_n)_n = \{\{1, \dots, n\}\}$ for all n .

Note that each maximal chain in P has the form $x_0 < x_1 < \cdots < x_n$, where $x_i \in P_i$ for each i . After defining these basic concepts concerning posets, we are now ready to deduce some properties of Boolean algebras. The first two properties are rank-symmetry and rank-unimodality.

Definition 2.6. A graded poset P of rank n is **rank-symmetric** if $|P_i| = |P_{n-i}|$ for all $0 \leq i \leq n$ and **rank-unimodal** if there exists $0 \leq j \leq n$ such that

$$|P_0| \leq |P_1| \leq \cdots \leq |P_j| \geq |P_{j+1}| \geq \cdots \geq |P_n|.$$

To show that B_n is rank-symmetric and rank-unimodal, we can calculate the size of each level of B_n explicitly, which requires the following proposition.

Proposition 2.7. $\rho(x) = |x|$ for all $x \in B_n$.

Proof. Let $x = \{x_1, \dots, x_k\} \in B_n$. Then $|x| = k$. Since

$$\emptyset < \{x_1\} < \{x_1, x_2\} < \dots < \{x_1, \dots, x_k\} = x$$

is a saturated chain with length k and top element x , we have $\rho(x) \geq k$. To show $\rho(x) \leq k$, suppose there exists some chain C in B_n with length greater than k and top element x . Then C has at least $k+2$ elements. Since $|x| = k$, all elements in C must have sizes less than or equal to k , and there exist two distinct elements in C of the same size. However, a finite set cannot be a subset of a different set with the same size. Thus, C is not a chain, a contradiction, and we must have $\rho(x) \leq k$. $\square$

Hence, we have $|(B_n)_i| = |\{x \in B_n : |x| = i\}| = \binom{n}{i}$. Since $\binom{n}{i} = \binom{n}{n-i}$ for all $0 \leq i \leq n$ and $\binom{n}{0} \leq \binom{n}{1} \leq \dots \leq \binom{n}{\lfloor n/2 \rfloor} \geq \binom{n}{\lfloor n/2 \rfloor + 1} \geq \dots \geq \binom{n}{n}$, we obtain the following theorem.

Theorem 2.8. B_n is rank-symmetric and rank-unimodal.

In many combinatorial problems, we are interested in an extremal question: how large can a collection of subsets be if none of them is contained in another? This question leads us to the concept of an antichain.

Definition 2.9. Let P be a poset. An **antichain** A in P is a subset of P such that for all $x, y \in A$, x and y are not comparable. That is, we do not have $x < y$ for all $x, y \in A$.

We first show a basic yet important property of graded posets regarding antichains.

Proposition 2.10. If P is a graded poset, then each level of P is an antichain.

Proof. Suppose that for some level P_i , there exist $x, y \in P_i$ such that $x < y$. By definition, there exists a saturated chain C with length i and top element x . Thus, $C \cup \{y\}$ is a chain with length $i+1$ and top element y , contradicting $\rho(y) = i$. $\square$

Let us return to our previous question. Notice that the question is equivalent to asking: What is the size of the largest antichain in B_n ? By [Proposition 2.10](#), we know that each level of B_n is an antichain; by [Theorem 2.8](#), we know that the largest level of B_n is its $\lfloor \frac{n}{2} \rfloor$ th level. Therefore, one guess is to choose all subsets of size $\lfloor \frac{n}{2} \rfloor$, which gives us an antichain of size $\binom{n}{\lfloor n/2 \rfloor}$. However, is this the largest antichain that we can choose? Motivated by the conjecture, we are now ready to define the Sperner property.

Definition 2.11. We say that a graded poset P of rank n has the **Sperner property** or is **Sperner** if

$$\max\{|A| : A \text{ is an antichain in } P\} = \max\{|P_i| : 0 \leq i \leq n\}.$$

Therefore, our conjecture is equivalent to stating that B_n is Sperner. We will spend the rest of this section showing this result. While there are direct proofs of the Sperner property of B_n , we will use the order-matching method, since it is much easier to generalize to the quotient posets that will be introduced later.

Definition 2.12. Let P be a graded poset. An **order-matching** from P_i to P_{i+1} is an injective function $\mu : P_i \rightarrow P_{i+1}$ such that $x < \mu(x)$ for all $x \in P_i$. Similarly, an order-matching from P_i to P_{i-1} is an injective function $\mu : P_i \rightarrow P_{i-1}$ such that $\mu(x) < x$ for all $x \in P_i$.

Example 2.13. An order-matching μ from $(B_4)_1$ to $(B_4)_2$ is defined by $\mu(1) = 12$, $\mu(2) = 23$, $\mu(3) = 13$, and $\mu(4) = 14$.

In the rest of the paper, for all finite sets S , we will use $\mathbb{R}S$ to denote the real vector space with basis S .

Definition 2.14. Let P be a graded poset. A linear transformation $U : \mathbb{R}P_i \rightarrow \mathbb{R}P_{i+1}$ is an **order-raising operator** if $Ux \in \text{span}\{y \in P_{i+1} : x < y\}$ for all $x \in P_i$.

We can think of an injective order-raising operator as a “weighted” order-matching. Instead of choosing one $y \in P_{i+1}$ that satisfies $x < y$ to match x , we assign weights to all $y \in P_{i+1}$ such that $x < y$. One may ask a question: Does the existence of an injective order-raising operator imply the existence of an order-matching? The question leads to the following proposition.

Proposition 2.15. *Let P be a graded poset and $U : \mathbb{R}P_i \rightarrow \mathbb{R}P_{i+1}$ an order-raising operator. If U is injective, then there exists an order-matching $\mu : P_i \rightarrow P_{i+1}$; if U is surjective, then there exists an order-matching $\mu : P_{i+1} \rightarrow P_i$.*

Proof. Assume U is an injective order-raising operator. Consider the matrix of U with respect to bases P_i and P_{i+1} . Since U is injective, $\text{rank}(U) = |P_i|$. Thus, $[U]$ has $|P_i|$ linearly independent rows. Let A be the $|P_i| \times |P_i|$ matrix obtained from choosing the $|P_i|$ linearly independent rows of $[U]$. Let $x_1, \dots, x_{|P_i|}$ be elements of P_i by the order they appear in columns of A , and let $y_1, \dots, y_{|P_i|}$ be elements of P_{i+1} by the order they appear in rows of A . Since the rows of A are linearly independent, $\det(A) \neq 0$. By the definition of determinant, there exists $\sigma \in S_{|P_i|}$ such that $a_{1,\sigma(1)} \cdots a_{|P_i|,\sigma(|P_i|)} \neq 0$. Hence, for all $1 \leq k \leq |P_i|$, $a_{k,\sigma(k)} \neq 0$. Since U is an order-raising operator, $y_k > x_{\sigma(k)}$ for all k . Therefore, there exists an order-matching $\mu : P_i \rightarrow P_{i+1}$ defined by $\mu(x_k) = y_{\sigma^{-1}(k)}$ for all $1 \leq k \leq |P_i|$. The case when U is surjective is proved similarly. $\square$

After showing this key proposition, one question remains: Why do we care about the existence of an order-matching at all? How is it related to the Sperner property? Consider a graded poset P with only two levels, P_0 and P_1 . Suppose there exists an order-matching $\mu : P_0 \rightarrow P_1$, and we wish to build an antichain larger than the size of P_1 across two levels. Thus, we must choose some elements in P_0 . Say we include k elements in P_0 , namely $\{x_1, \dots, x_k\} \subseteq P_0$, in our antichain. Then by the definition of order-matching, k elements in P_1 , namely $\mu(\{x_1, \dots, x_k\})$, cannot be included in the antichain. Therefore, including elements in P_0 does not increase the size of our antichain since it results in elements in P_1 being excluded. Hence, the largest antichain we can choose is P_1 . Extending this notion gives us the following proposition.

Proposition 2.16. *Let P be a graded poset of rank n . If there exist order-matchings $P_0 \rightarrow P_1 \rightarrow \cdots \rightarrow P_k \leftarrow P_{k+1} \leftarrow \cdots \leftarrow P_n$ for some $0 \leq k \leq n$, then P is rank-unimodal and Sperner.*

Proof. Since order-matchings are injective, we have

$$|P_0| \leq |P_1| \leq \cdots \leq |P_k| \geq |P_{k+1}| \geq \cdots \geq |P_n|,$$

and P is rank-unimodal.

To show that P is Sperner, we will decompose P into disjoint chains. Let μ_i denote the order-matching from P_i to P_{i+1} for all $0 \leq i \leq k-1$, and let μ_j denote the order-matching from P_j to P_{j-1} for all $k+1 \leq j \leq n$. We will construct chains as follows. For each $x \in P_0$, construct the chain

$$x < \mu_0(x) < (\mu_1 \circ \mu_0)(x) < \cdots < (\mu_{k-1} \circ \cdots \circ \mu_0)(x) \in P_k;$$

for each $x \in P_n$, construct the chain

$$x > \mu_n(x) > (\mu_n \circ \mu_{n-1})(x) > \cdots > (\mu_n \circ \cdots \circ \mu_{k+1})(x) \in P_k;$$

for each $x \in P_1 \setminus \mu_0(P_0)$, construct the chain

$$x < \mu_1(x) < \cdots < (\mu_{k-1} \circ \cdots \circ \mu_0)(x) \in P_k;$$

similarly, for each $x \in P_{n-1} \setminus \mu_n(P_n)$, construct the chain

$$x > \mu_{n-1}(x) > \cdots > (\mu_{k+1} \circ \cdots \circ \mu_{n-1})(x) \in P_k.$$

Continue this process until we reach P_k , where we combine chains that intersect in P_k and construct a singleton chain for each element in P_k not included in any previous chains. Hence, each element of P is in at least one of the chains constructed. Since order-matchings are injective, the chains are disjoint. Therefore, we have partitioned elements of P into disjoint chains that start or end in P_k . Hence, the number of chains is at most $|P_k|$. Let A be an antichain. Since A can intersect each chain at most once, we have $|A| \leq |P_k|$. Thus, P_k is the largest antichain in P , and P is Sperner. $\square$

We now give an explicit example of how we construct disjoint chains in the proof of [Proposition 2.16](#).

Example 2.17. Consider the Boolean algebra B_4 . One can verify that one set of order-matchings is given by

$$\begin{aligned} \mu_0(\emptyset) &= 1, \\ \mu_1(1) &= 12, \mu_1(2) = 23, \mu_1(3) = 13, \mu_1(4) = 14, \\ \mu_3(123) &= 12, \mu_3(124) = 14, \mu_3(134) = 13, \mu_4(234) = 23, \\ \mu_4(1234) &= 123. \end{aligned}$$

Using the algorithm in the proof of [Proposition 2.16](#), we decompose B_4 into six disjoint chains, namely $\emptyset < 1 < 12 < 123 < 1234$, $2 < 23 < 234$, $3 < 13 < 134$, $4 < 14 < 124$, and singleton chains 24 and 34. [Figure 2](#) shows the disjoint chains graphically.

Combining [Proposition 2.15](#) and [Proposition 2.16](#) yields the following corollary.

Corollary 2.18. *Let P be a graded poset of rank n . If, for some $0 \leq k \leq n$, there exist order-raising operators*

$$\mathbb{R}P_0 \xrightarrow{U_0} \mathbb{R}P_1 \xrightarrow{U_1} \cdots \xrightarrow{U_{k-1}} \mathbb{R}P_k \xrightarrow{U_k} \mathbb{R}P_{k+1} \xrightarrow{U_{k+1}} \cdots \xrightarrow{U_{n-1}} \mathbb{R}P_n,$$

where U_i is injective when $i < k$ and surjective when $i \geq k$, then P is rank-unimodal and Sperner.

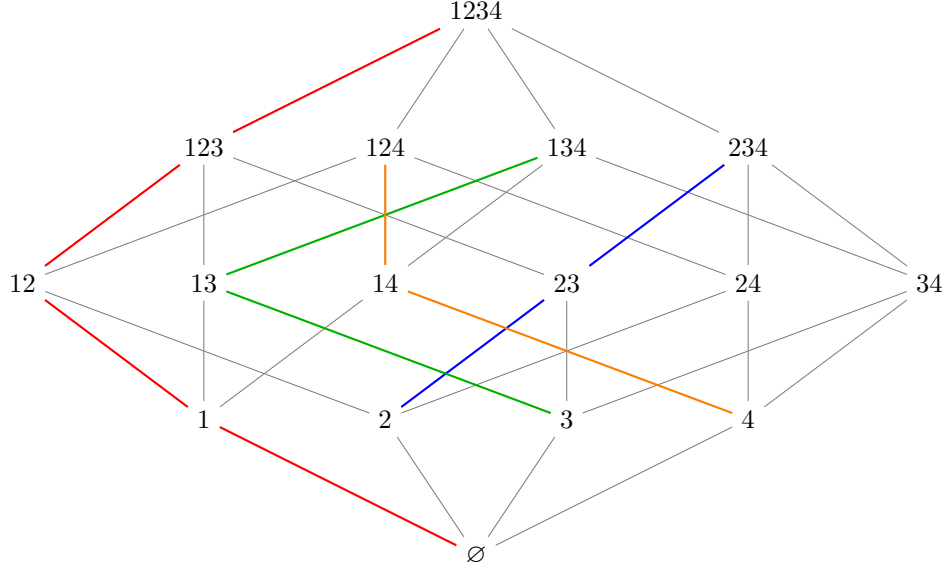


FIGURE 2. Hasse diagram of the Boolean algebra B_4 . Colored paths indicate disjoint chains. Note that 24 and 34 are singleton chains.

Therefore, to show that B_n is Sperner, we need to find appropriate order-raising operators $U_i : \mathbb{R}(B_n)_i \rightarrow \mathbb{R}(B_n)_{i+1}$ for all $0 \leq i < n$. Fortunately, a simple linear transformation will do the job. For each $0 \leq i < n$, define the linear transformation $U_i : \mathbb{R}(B_n)_i \rightarrow \mathbb{R}(B_n)_{i+1}$ by

$$U_i x = \sum_{\substack{y \in (B_n)_{i+1} \\ x < y}} y$$

for $x \in (B_n)_i$, which is by definition an order-raising operator. Since the largest level of B_n is $(B_n)_{\lfloor n/2 \rfloor}$, we wish to show that U_i is injective when $i < \frac{n}{2}$ and surjective when $i \geq \frac{n}{2}$, which is the following proposition.

Proposition 2.19. *U_i is injective when $i < \frac{n}{2}$ and surjective when $i \geq \frac{n}{2}$.*

Proof. For each i , consider the linear transformation $D_i : \mathbb{R}(B_n)_{i+1} \rightarrow \mathbb{R}(B_n)_i$ of U_i defined by

$$D_i y = \sum_{\substack{x \in (B_n)_i \\ x < y}} x.$$

Note that

$$\langle U_i x, y \rangle = \langle x, D_i y \rangle = \begin{cases} 1 & \text{if } x < y \\ 0 & \text{otherwise} \end{cases}$$

for all $0 \leq i < n$, $x \in (B_n)_i$, and $y \in (B_n)_{i+1}$ under the standard inner product. Hence, D_i is the adjoint of U_i , and we have $[D_i] = [U_i]^T$ for all i . We also define $U_{-1} = D_{-1} = U_n = D_n = 0$.

Now consider the operators $D_i U_i$ and $U_{i-1} D_{i-1}$, which are positive semidefinite and thus have nonnegative eigenvalues by the previous statement. Let $x \in (B_n)_i$.

Then we have

$$\begin{aligned}
D_i U_i x &= D_i \left(\sum_{\substack{y \in (B_n)_{i+1} \\ x < y}} y \right) \\
&= \sum_{\substack{y \in (B_n)_{i+1} \\ x < y}} D_i y \\
&= \sum_{\substack{y \in (B_n)_{i+1} \\ x < y}} \sum_{\substack{z \in (B_n)_i \\ z < y}} z.
\end{aligned}$$

If $z = x$, there are $n - i$ choices of $y \in (B_n)_{i+1}$ such that $x = z < y$; if $|z \cap x| = i - 1$, there is one choice of y , namely $y = x \cap z$, such that $x < y$ and $z < y$; otherwise, there does not exist $y \in (B_n)_{i+1}$ such that $x < y$ and $z < y$. Therefore,

$$D_i U_i x = (n - i)x + \sum_{\substack{z \in (B_n)_i \\ |x \cap z| = 1}} z.$$

Similarly, we have

$$U_{i-1} D_{i-1} x = ix + \sum_{\substack{z \in (B_n)_i \\ |x \cap z| = 1}} z.$$

Hence,

$$D_i U_i = U_{i-1} D_{i-1} + (n - 2i)I$$

for all i . Since $U_{i-1} D_{i-1}$ has nonnegative eigenvalues, $D_i U_i$ has positive eigenvalues when $i < \frac{n}{2}$. Therefore, $D_i U_i$ is invertible and U_i is injective if $i < \frac{n}{2}$.

We also have

$$U_i D_i = D_{i+1} U_{i+1} + (2i + 2 - n)I$$

for all i . Similarly, since $D_{i+1} U_{i+1}$ has nonnegative eigenvalues, $U_i D_i$ has positive eigenvalues when $i \geq \frac{n}{2}$. Thus, $U_i D_i$ is invertible and U_i is surjective if $i \geq \frac{n}{2}$. $\square$

By [Corollary 2.18](#) and [Proposition 2.19](#), we have the following theorem and corollary.

Theorem 2.20. B_n is Sperner.

Corollary 2.21. Let S be an n -element set. Let A be a collection of subsets of S such that none of the elements in A is contained in another. Then $|A| \leq \binom{n}{\lfloor n/2 \rfloor}$.

3. THE QUOTIENT POSET B_n/G

In many situations, we are interested not only in the behavior of the Boolean algebra B_n itself, but also in its structure under symmetry. For instance, we can uniquely represent a simple graph with n vertices as a subset of $(B_n)_2$, and we consider two simple graphs to be isomorphic if their edge sets are “equivalent” under the permutation of vertices. Such examples motivate the study of B_n/G , where G is a subgroup of S_n that acts on B_n . We first review some key definitions concerning group actions. In the following definitions, let G be a group acting on a set X .

Definition 3.1. We say that $x, y \in X$ are G -**equivalent** if there exists $g \in G$ such that $gx = y$.

Note that G -equivalence is an equivalence relation. Hence, it makes sense to define the orbit under a group action.

Definition 3.2. The **orbits** of X under the action of G are the equivalence classes resulting from G -equivalence. The set of orbits of X under G is denoted X/G .

Example 3.3. Consider $\theta \in \mathbb{R}$ acting on $\mathbb{R}^2$ by a counterclockwise rotation of θ radians. Then we have $\mathbb{R}^2/\mathbb{R} = \{C_r : r \geq 0\}$, where C_r is the circle centered at $(0, 0)$ with radius r .

We will denote the orbit of X under G containing $x \in X$ by $Gx = \{gx : g \in G\}$. Notice that for $x, y \in X$, $Gx = Gy$ if and only if x and y are G -equivalent.

Now we want to define the quotient poset B_n/G . As we mentioned at the beginning of the section, our interest in the behavior of B_n under certain permutations of its elements motivates the study of B_n/G . Thus, G should be a subgroup of S_n . But how should we define the action of G on B_n ? Since B_n is a poset, a natural requirement of the action is that it must preserve the partial order. In other words, for all $\sigma \in S_n$, the action of σ on B_n should be an automorphism. Consider the action defined by

$$\sigma\{x_1, \dots, x_k\} = \{\sigma(x_1), \dots, \sigma(x_k)\}$$

for all $\sigma \in S_n$ and $\{x_1, \dots, x_k\} \in B_n$. Then for $x, y \in B_n$ such that $x \subseteq y$, we have $\sigma x \subseteq \sigma y$ for all $\sigma \in S_n$. Thus, the action of σ is indeed an automorphism. Now we are ready to define B_n/G .

Definition 3.4. Let G be a subgroup of S_n . The **quotient poset** B_n/G is the set of orbits of B_n under the action of G defined above, equipped with the partial order defined as follows. For all $\mathcal{O}, \mathcal{O}' \in B_n/G$, define $\mathcal{O} \leq \mathcal{O}'$ if there exist $x \in \mathcal{O}$ and $y \in \mathcal{O}'$ such that $x \subseteq y$ in B_n .

In the remainder of this section, unless otherwise specified, assume G is a subgroup of S_n . We now show that $\leq$ is indeed a partial order.

Proposition 3.5. *The relation $\leq$ defined above is a partial order.*

Proof. Let $\mathcal{O}, \mathcal{O}', \mathcal{O}'' \in B_n/G$.

(1) Take any $x \in \mathcal{O}$. Then $\mathcal{O} \leq \mathcal{O}$ follows from $x \subseteq x$.

(2) Assume $\mathcal{O} \leq \mathcal{O}'$ and $\mathcal{O}' \leq \mathcal{O}$. Then there exist $x_1, x_2 \in \mathcal{O}$ and $y_1, y_2 \in \mathcal{O}'$ such that $x_1 \subseteq y_1$ and $y_2 \subseteq x_2$. Then we have $|x_1| \leq |y_1| = |y_2| \leq |x_2|$. Since $|x_1| = |x_2|$, we must have $|x_1| = |y_1|$ and thus $x_1 = y_1$. Hence, $\mathcal{O} = \mathcal{O}'$.

(3) Assume $\mathcal{O} \leq \mathcal{O}'$ and $\mathcal{O}' \leq \mathcal{O}''$. Then there exist $x \in \mathcal{O}$, $y_1, y_2 \in \mathcal{O}'$, and $z \in \mathcal{O}''$ such that $x \subseteq y_1$ and $y_2 \subseteq z$. Take $\sigma \in G$ such that $\sigma y_1 = y_2$, and we have $\sigma x \subseteq \sigma y_1 = y_2 \subseteq z$. Therefore, $\mathcal{O} \leq \mathcal{O}''$. $\square$

Example 3.6. Let $G = \langle (1\ 2\ 3\ 4) \rangle \leq S_4$. Then $B_4/G = \{\{\emptyset\}, \{1, 2, 3, 4\}, \{12, 23, 34, 14\}, \{13, 24\}, \{123, 234, 134, 124\}, \{1234\}\}$. Figure 3 shows the Hasse diagram of B_4/G .

Recall that the Boolean algebra B_n is graded of rank n , rank-symmetric, rank-unimodal, and Sperner. One may wonder: Does the quotient poset B_n/G have similar properties? The answer is yes. In the remainder of this section, we show that these properties also hold for B_n/G . We first show that B_n/G is graded of rank n and rank-symmetric.

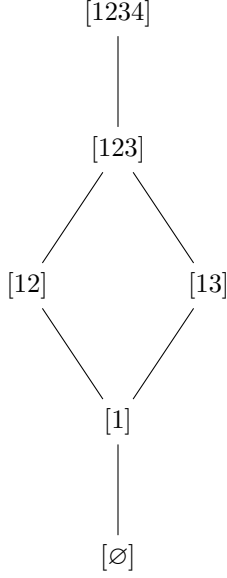


FIGURE 3. Hasse diagram of the quotient poset $B_4 / \langle (1\ 2\ 3\ 4) \rangle$. Note that each orbit is represented by an element of the orbit. For example, $\{13, 24\}$ is represented by $[13]$.

Proposition 3.7. B_n/G is graded of rank n , and $\rho([x]) = |x|$ for all $x \in B_n$.

Proof. By the definition of the action of G on B_n , we have $|\sigma x| = |x|$ for all $\sigma \in G$ and $x \in B_n$. Hence, elements of the same orbit in B_n/G must have the same size, and it makes sense to define $|[x]| = |x|$ for all $x \in B_n$.

Let $\mathcal{O}_0 < \mathcal{O}_1 < \dots < \mathcal{O}_k$ be a maximal chain in B_n/G . Since $[\emptyset] \leq \mathcal{O} \leq [\{1, \dots, n\}]$ for all $\mathcal{O} \in B_n/G$, we have $\mathcal{O}_0 = [\emptyset]$ and $\mathcal{O}_k = [\{1, \dots, n\}]$. We now claim that $|\mathcal{O}_{i+1}| = |\mathcal{O}_i| + 1$ for all i . We have $|\mathcal{O}_{i+1}| \geq |\mathcal{O}_i| + 1$ by the definition of B_n/G . By means of contradiction, assume there exists j some that $|\mathcal{O}_{j+1}| \geq |\mathcal{O}_j| + 2$. Take $x \in \mathcal{O}_j$ and $y \in \mathcal{O}_{j+1}$. Since $|y| - |x| \geq 2$, there exists $z \in B_n$ such that $x \subset z \subset y$. Therefore, $\mathcal{O}_j < [z] < \mathcal{O}_{j+1}$, contradicting that the chain is maximal. Thus, the size of the elements of the orbits increases from 0 to n with an increase of 1 for each orbit. Hence, the chain must have length n , and B_n/G is graded of rank n .

The proof of $\rho(x) = |[x]|$ is similar to that in [Proposition 2.7](#), except we consider $[x]$ instead of x . $\square$

The previous proposition implies that $(B_n/G)_i = (B_n)_i/G$ for all $0 \leq i \leq n$.

Proposition 3.8. B_n/G is rank-symmetric.

Proof. Consider the complement function

$$c : B_n \rightarrow B_n \quad x \mapsto \{1, \dots, n\} \setminus x.$$

For all $\sigma \in G$, since σ is a permutation of B_n , we have $\sigma c(x) = c(\sigma x)$. For each $0 \leq i \leq n$, define

$$f_i : (B_n)_i/G \rightarrow (B_n)_{n-i}/G \quad [x] \mapsto [c(x)].$$

Note that if $\sigma x = y$ for some $\sigma \in G$ and $x, y \in (B_n)_i$, then

$$f_i([y]) = [c(y)] = [c(\sigma x)] = [\sigma c(x)] = [c(x)] = f_i(x).$$

Therefore, f_i is well-defined. Since f_i has an inverse, namely f_{n-i} , f_i is a bijection. Hence, $|(B_n)_i/G| = |(B_n)_{n-i}/G|$, and B_n/G is rank-symmetric. $\square$

Now we wish to show that B_n/G is rank-unimodal and Sperner. Intuitively, we want to mimic the approach we used to show that B_n is Sperner. That is, define appropriate order-raising operators from $\mathbb{R}(B_n/G)_i$ to $\mathbb{R}(B_n/G)_{i+1}$ for each i . However, in $(B_n/G)_i$, we wish to take a vector for each orbit that treats each element in the orbit equally. A natural choice is the orbit sum

$$v_{\mathcal{O}} = \sum_{x \in \mathcal{O}} x,$$

where $\mathcal{O} \in (B_n/G)_i$. Note that $v_{\mathcal{O}}$ is invariant under G since G only permutes its summands.

For all $\sigma \in S_n$ and $0 \leq i \leq n$, we extend σ to an action on $\mathbb{R}(B_n)_i$ by defining

$$\sigma \left(\sum_{x \in (B_n)_i} c_x x \right) = \sum_{x \in (B_n)_i} c_x \sigma x$$

for real numbers c_x . Then each element in $\mathbb{R}\{v_{\mathcal{O}} : \mathcal{O} \in (B_n/G)_i\}$ is invariant under G . Consider the subspace

$$\mathbb{R}(B_n)_i^G = \{v \in \mathbb{R}(B_n)_i : \sigma v = v \text{ for all } \sigma \in G\}$$

of $\mathbb{R}(B_n)_i$ in which each element, by definition, is also invariant under G . We claim that $\mathbb{R}\{v_{\mathcal{O}} : \mathcal{O} \in (B_n/G)_i\} = \mathbb{R}(B_n)_i^G$.

Proposition 3.9. $\{v_{\mathcal{O}} : \mathcal{O} \in (B_n/G)_i\}$ is a basis for $\mathbb{R}(B_n)_i^G$.

Proof. Assume there exists $c_{\mathcal{O}} \in \mathbb{R}$ for all $\mathcal{O} \in (B_n/G)_i$ such that

$$\sum_{\mathcal{O} \in (B_n/G)_i} c_{\mathcal{O}} v_{\mathcal{O}} = 0.$$

Expanding the LHS gives

$$\sum_{\mathcal{O} \in (B_n/G)_i} c_{\mathcal{O}} \sum_{x \in \mathcal{O}} x = \sum_{x \in (B_n)_i} c_{[x]} x = 0.$$

Since $(B_n)_i$ is linearly independent, $c_{\mathcal{O}} = 0$ for all $\mathcal{O}$, and $v_{\mathcal{O}}$'s are linearly independent.

We have $\sigma x \in \mathcal{O}$ for all $\sigma \in G$, $\mathcal{O} \in (B_n/G)_i$, and $x \in \mathcal{O}$. Since σ is a permutation of $(B_n)_i$, it also permutes the elements of $\mathcal{O}$. Hence, $\sigma v_{\mathcal{O}} = v_{\mathcal{O}}$, and $v_{\mathcal{O}} \in \mathbb{R}(B_n)_i^G$ for all $\mathcal{O}$. It suffices to show that $\{v_{\mathcal{O}} : \mathcal{O} \in (B_n/G)_i\}$ spans $\mathbb{R}(B_n)_i^G$. Let $v = \sum_{x \in (B_n)_i} c_x x \in \mathbb{R}(B_n)_i^G$. Since $\sigma v = v$ for all $\sigma \in G$,

$$\sum_{x \in (B_n)_i} c_x \sigma x = \sum_{x \in (B_n)_i} c_x x = \sum_{x \in (B_n)_i} c_{\sigma x} \sigma x,$$

and we have $c_x = c_{\sigma x}$ for all $x \in (B_n)_i$. Thus, we can write

$$v = \sum_{\mathcal{O} \in (B_n/G)_i} c_{\mathcal{O}} \sum_{x \in \mathcal{O}} x = \sum_{\mathcal{O} \in (B_n/G)_i} c_{\mathcal{O}} v_{\mathcal{O}}.$$

Therefore, $\text{span}\{v_{\mathcal{O}} : \mathcal{O} \in (B_n/G)_i\} = \mathbb{R}(B_n)_i^G$. $\square$

The above proposition gives us good candidates for the order-raising operators from $\mathbb{R}(B_n/G)_i$ to $\mathbb{R}(B_n/G)_{i+1}$. Since $v_{\mathcal{O}}$'s form a basis for $\mathbb{R}(B_n)_i^G$ and $\mathcal{O}$'s form a basis for $\mathbb{R}(B_n/G)_i$, $\mathbb{R}(B_n)_i^G$ and $\mathbb{R}(B_n/G)_i$ are isomorphic vector spaces. Recall the order-raising operator $U_i : \mathbb{R}(B_n)_i \rightarrow \mathbb{R}(B_n)_{i+1}$ defined in the previous section. Consider its restriction to $\mathbb{R}(B_n)_i^G$. If we can show that U_i maps $\mathbb{R}(B_n)_i^G$ to a subset of $\mathbb{R}(B_n)_{i+1}^G$, we can naturally define the operator $\hat{U}_i$ according to the following commutative diagram.

$$\begin{array}{ccc} (\mathbb{R}B_n)_i^G & \xrightarrow{U_i} & (\mathbb{R}B_n)_{i+1}^G \\ \cong \downarrow & & \downarrow \cong \\ \mathbb{R}(B_n/G)_i & \xrightarrow{\hat{U}_i} & \mathbb{R}(B_n/G)_{i+1} \end{array}$$

Hence, we need the following proposition.

Proposition 3.10. $U_i v \in \mathbb{R}(B_n)_{i+1}^G$ for all $v \in \mathbb{R}(B_n)_i^G$.

Proof. Let $\sigma \in G$. Since σ is an automorphism of B_n , for all $x, y \in B_n$, $x < y$ in B_n if and only if $\sigma x < \sigma y$. Then for all $x \in B_n$,

$$\begin{aligned} \sigma U_i x &= \sigma \left(\sum_{\substack{y \in (B_n)_{i+1} \\ x < y}} y \right) \\ &= \sum_{\substack{y \in (B_n)_{i+1} \\ \sigma x < \sigma y}} \sigma y \\ &= U_i \sigma x. \end{aligned}$$

Therefore, $\sigma U_i v = U_i \sigma v = U_i v$ for all $v \in \mathbb{R}(B_n)_i^G$. $\square$

Thus, for all $\mathcal{O} \in (B_n/G)_i$, by [Proposition 3.9](#) and [Proposition 3.10](#), we may uniquely write

$$U_i v_{\mathcal{O}} = \sum_{\mathcal{O}' \in (B_n/G)_{i+1}} c_{\mathcal{O}\mathcal{O}'} v_{\mathcal{O}'},$$

where $c_{\mathcal{O}\mathcal{O}'}$'s are real. Then we define the linear transformation $\hat{U}_i : \mathbb{R}(B_n/G)_i \rightarrow \mathbb{R}(B_n/G)_{i+1}$ by

$$\hat{U}_i \mathcal{O} = \sum_{\mathcal{O}' \in (B_n/G)_{i+1}} c_{\mathcal{O}\mathcal{O}'} \mathcal{O}'$$

for all $\mathcal{O} \in (B_n/G)_i$. We show that $\hat{U}_i$'s are the appropriate order-raising operators.

Proposition 3.11. $\hat{U}_i$ is an injective order-raising operator if $i < \frac{n}{2}$ and a surjective order-raising operator if $i \geq \frac{n}{2}$.

Proof. To show that $\hat{U}_i$ is order-raising, we need to show that $c_{\mathcal{O}\mathcal{O}'} \neq 0$ implies $\mathcal{O} < \mathcal{O}'$. Since U_i is order-raising, if $c_{\mathcal{O}\mathcal{O}'} \neq 0$, then there exist $x \in \mathcal{O}$ and $x' \in \mathcal{O}'$ such that $x < x'$, which is exactly the definition of $\mathcal{O} < \mathcal{O}'$. Hence, $\hat{U}_i$ is order-raising for all i .

By [Proposition 2.19](#), U_i is injective when $i < \frac{n}{2}$. Thus, its restriction to $\mathbb{R}(B_n)_i^G$ is also injective when $i < \frac{n}{2}$. Note that U_i restricted to $\mathbb{R}(B_n)_i^G$ and $\hat{U}_i$ are the same

transformation under the natural isomorphism between $\mathbb{R}(B_n)_i^G$ and $\mathbb{R}(B_n/G)_i$. Therefore, $\hat{U}_i$ is injective if $i < \frac{n}{2}$.

Similar to U_i , since for all $\mathcal{O} \in (B_n/G)_{i+1}$, we can write

$$D_i v_{\mathcal{O}} = \sum_{\mathcal{O}' \in (B_n/G)_i} d_{\mathcal{O}\mathcal{O}'} \mathcal{O}',$$

we may define $\hat{D}_i : \mathbb{R}(B_n/G)_{i+1} \rightarrow \mathbb{R}(B_n/G)_i$ by

$$\hat{D}_i \mathcal{O} = \sum_{\mathcal{O}' \in (B_n/G)_i} d_{\mathcal{O}\mathcal{O}'} \mathcal{O}'$$

for all $\mathcal{O} \in (B_n/G)_{i+1}$. Since U_i is surjective when $i \geq \frac{n}{2}$ and $D_i = U_i^*$, D_i is injective when $i \geq \frac{n}{2}$ and so is its restriction on $\mathbb{R}(B_n/G)_{i+1}$. Therefore, $\hat{D}_i$ is injective when $i \geq \frac{n}{2}$. Since $\hat{U}_i = \hat{D}_i^*$, $\hat{U}_i$ is surjective when $i \geq \frac{n}{2}$. $\square$

Combining [Corollary 2.18](#) and [Proposition 3.11](#) gives the following theorem.

Theorem 3.12. *B_n/G is rank-unimodal and Sperner.*

4. APPLICATIONS

In the preceding section, we have shown that for each subgroup G of S_n , the quotient poset B_n/G is graded of rank n , rank-symmetric, rank-unimodal, and Sperner. The last section of the paper gives two applications of these properties of B_n/G .

4.1. Graph Theory. The first application concerns graph theory. Before we dive into the application, we first recall some important definitions. For a set S and a nonnegative integer n , we denote by $\binom{S}{n}$ the set of n -element subsets of S .

Definition 4.1. A **simple graph** is an ordered pair $G = (V, E)$, where V is a finite set, whose elements are called vertices, and $E \subseteq \binom{V}{2}$ is a set of 2-element subsets of V , whose elements are called edges.

We can think of simple graphs as graphs with no loops and no multiple edges. Note that in this subsection, G , by default, denotes a simple graph rather than a subgroup of S_n . Given a simple graph G , we may denote its vertex set and edge set by $V(G)$ and $E(G)$ respectively.

Definition 4.2. Let $G = (V, E)$ and $G' = (V', E')$ be two simple graphs. We say G and G' are **isomorphic graphs** if there exists a bijection $\phi : V \rightarrow V'$ such that for all $u, v \in V$, $\{u, v\} \in E$ if and only if $\{\phi(u), \phi(v)\} \in E'$.

Intuitively, two graphs are isomorphic if they are the same graph up to a permutation of vertices.

Definition 4.3. Let $G = (V, E)$ be a simple graph. A simple graph $G' = (V', E')$ is a **spanning subgraph** of G if $V' = V$ and $E' \subseteq E$.

That is, a spanning subgraph of G is a graph that maintains all the vertices of G while possibly deleting some edges. [Figure 4](#) gives an example of a spanning subgraph. Note that for any collection C of graphs, we can turn C into a poset by defining, for all $G, G' \in C$, $G \leq G'$ if G is a spanning subgraph of G' .

We are interested in two questions regarding simple graphs. The first is about the number of edges: Given the number of vertices n , how many nonisomorphic

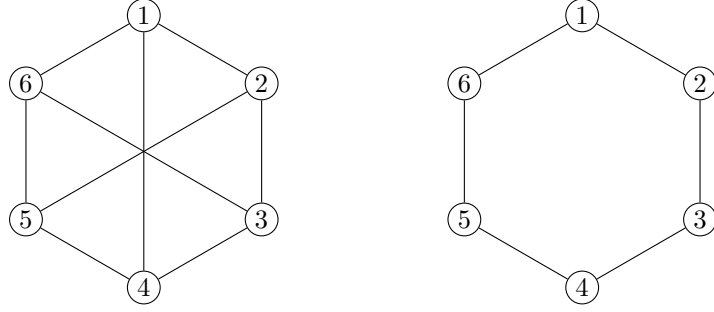


FIGURE 4. The right graph is a spanning subgraph of the left.

simple graphs with i edges exist? For all $0 \leq i \leq \binom{n}{2}$, denote this number by $f_{n,i}$. Then we obtain, for each n , a sequence $(f_{n,0}, \dots, f_{n,\binom{n}{2}})$. From Figure 5, we see that for $n = 4$, the sequence is $(1, 1, 2, 3, 2, 1, 1)$.

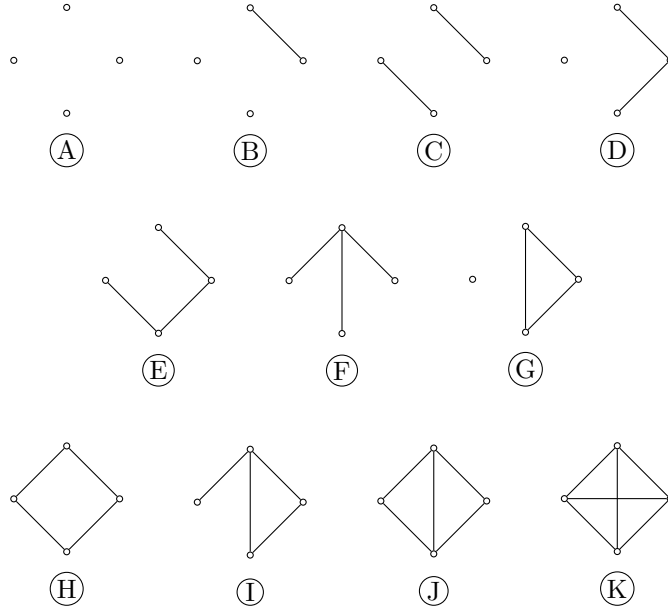


FIGURE 5. All 11 nonisomorphic simple graphs with 4 vertices.

Calculating the exact value of $f_{n,i}$ for large n is a notoriously difficult question in graph theory. In fact, there is no known closed-form formula for $f_{n,i}$. However, from the result we calculated for $n = 4$, one may guess whether the sequence $(f_{n,i})_{0 \leq i \leq \binom{n}{2}}$ is symmetric and unimodal for all n . The answer is yes, and this will be the first result we prove in this subsection.

Theorem 4.4. *The sequence $(f_{n,i})_{0 \leq i \leq \binom{n}{2}}$ is symmetric and unimodal for all n .*

The second question is a famous problem in extremal graph theory: Given the number of vertices n , what is the largest collection of simple graphs we can build

such that none of them is isomorphic to a spanning subgraph of another? From Figure 5, we see that for $n = 4$, the three graphs with $3 = \frac{1}{2}\binom{4}{2}$ edges form a collection that satisfies the requirement. In fact, for $n = 4$, this collection is the largest we can take. But is it true that, for all n , the largest collection of graphs that satisfies the requirement is the set of nonisomorphic graphs with $\lfloor \frac{1}{2}\binom{n}{2} \rfloor$ edges? This will be the second result we prove in this subsection.

Theorem 4.5. *Fix $n \in \mathbb{N}$. Let A be a collection of simple graphs on n vertices such that none of the graphs in A is isomorphic to a spanning subgraph of another. Then $|A|$ is maximized by choosing the family of nonisomorphic simple graphs with $\lfloor \frac{1}{2}\binom{n}{2} \rfloor$ edges.*

We first formally define the poset of nonisomorphic simple graphs on a vertex set. Let $V = \{1, \dots, n\}$ and $\mathcal{G}_n = \{\text{simple graphs on the vertex set } V\} / \cong$, the set of isomorphism classes of simple graphs on the vertex set V . We can naturally turn $\mathcal{G}_n$ into a poset by defining, for all $[G], [H] \in \mathcal{G}_n$, $[G] \leq [H]$ if there exist $G' \in [G]$ and $H' \in [H]$ such that G' is a spanning subgraph of H' . One can verify that $\leq$ here is indeed a partial order. Figure 6 demonstrates the Hasse diagram of the poset $\mathcal{G}_4$.

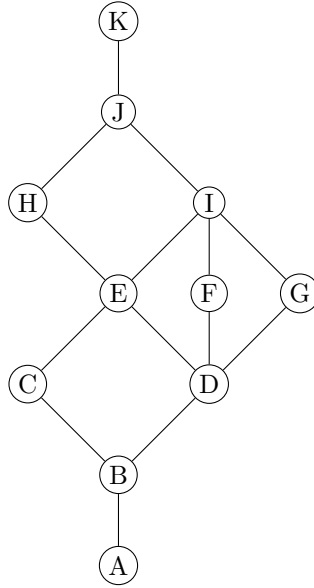


FIGURE 6. Hasse diagram of $\mathcal{G}_4$ with graphs labeled as in Figure 5.

We now show that $\mathcal{G}_n$ is a graded poset.

Proposition 4.6. *$\mathcal{G}_n$ is graded of rank $\binom{n}{2}$, and $\rho([G]) = |E(G)|$, the number of edges of G , for all $[G] \in \mathcal{G}_n$.*

Proof. Since $|E(G)| = |E(G')|$ for all $G \cong G'$, we can define $|[G]| = |E(G)|$ for all $[G] \in \mathcal{G}_n$. Let $[G_0] < [G_1] < \dots < [G_k]$ be a maximal chain in $\mathcal{G}_n$. Note that for all graphs G on the vertex set $\{1, \dots, n\}$, E_n is a spanning subgraph of G and G is a spanning subgraph of K_n , where E_n is the edgeless graph on n vertices and K_n is

the complete graph on n vertices. Thus, we must have $[[G_0]] = 0$ and $[[G_k]] = \binom{n}{2}$. The rest of the proof of the proposition is analogous to that of [Proposition 3.7](#), except that we are working on $\mathcal{G}_n$ instead of quotients of Boolean algebra, and the partial order is defined by spanning subgraphs rather than inclusion of sets. $\square$

Let us return to our two main questions in this subsection. By [Proposition 4.6](#), we see that showing [Theorem 4.4](#) and [Theorem 4.5](#) is equivalent to showing that $\mathcal{G}_n$ is rank-symmetric, rank-unimodal, and Sperner. Therefore, if we can show that $\mathcal{G}_n$ is isomorphic to B_m/Γ for some m and $\Gamma \leq S_m$, then [Theorem 4.4](#) and [Theorem 4.5](#) follow immediately from [Proposition 3.8](#) and [Theorem 3.12](#).

Proposition 4.7. *Set $m = \binom{n}{2}$. Then $\mathcal{G}_n$ is isomorphic to B_m/Γ for some $\Gamma \leq S_m$.*

Proof. Recall that $\mathcal{G}_n$ is the set of isomorphism classes of simple graphs on the vertex set $V = \{1, \dots, n\}$. Let G_n denote the set of all simple graphs on the vertex set V . Consider $E(K_n) = \binom{V}{2}$, the edge set of the complete graph K_n . Note that each simple graph in G_n is uniquely determined by its edge set $E \subseteq E(K_n)$, and each subset E of $E(K_n)$ uniquely defines a simple graph $G = (V, E)$ in G_n . By the definition of the poset of simple graphs, we also have, for all $G, G' \in G_n$, $G \leq G'$ in G_n if and only if $E(G) \subseteq E(G')$ in $B_{E(K_n)}$. Thus, $G_n \cong B_{E(K_n)}$.

Consider the map

$$\phi : S_n \rightarrow S_{E(K_n)} \quad \sigma \mapsto \hat{\sigma},$$

where $\hat{\sigma} : E(K_n) \rightarrow E(K_n)$ is defined by

$$\hat{\sigma}\{i, j\} = \{\sigma(i), \sigma(j)\}.$$

Then $\phi(S_n)$ is a subgroup of $S_{E(K_n)}$ isomorphic to S_n . Intuitively, $\phi(S_n)$ is the group of permutations of edges in $E(K_n)$ induced by permutations of the vertex set V . Let $\phi(S_n)$ act on $B_{E(K_n)}$ by defining, for all $\hat{\sigma} \in \phi(S_n)$ and $E \subseteq E(K_n)$, $\hat{\sigma}E = \{\hat{\sigma}(e) : e \in E\}$. By the definition of isomorphic graphs, for all $G, G' \in G_n$, we have $G \cong G'$ if and only if $E(G') = \hat{\sigma}E(G)$ for some $\hat{\sigma} \in \phi(S_n)$. Hence, $\mathcal{G}_n \cong B_{E(K_n)}/\phi(S_n)$.

Since $|E(K_n)| = \binom{n}{2} = m$, we have $B_{E(K_n)} \cong B_m$. Let $E(K_n) = \{e_1, \dots, e_m\}$. Note that each $\hat{\sigma} \in \phi(S_n)$ corresponds to some $\tau \in S_m$ defined by $\tau(i) = j$ such that $\hat{\sigma}(e_i) = e_j$. This defines an injective homomorphism $\pi : \phi(S_n) \rightarrow S_m$. Set $\Gamma = \pi(\phi(S_n)) \leq S_m$. Then we have $\Gamma \cong \phi(S_n)$, and thus $\mathcal{G}_n$ is isomorphic to B_m/Γ . $\square$

Combining [Proposition 4.7](#), [Proposition 3.8](#), and [Theorem 3.12](#) gives [Theorem 4.4](#) and [Theorem 4.5](#) as desired.

4.2. Rectangular Young's Lattice and Combinatorics. In this subsection, we introduce the rectangular Young's lattice and illustrate its relationship with quotients of Boolean algebras. We then apply properties of rectangular Young's lattices to solve a problem in combinatorics.

Definition 4.8. Let n be a nonnegative integer. A partition of n is a nonincreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of nonnegative integers such that $\sum_{i \geq 1} \lambda_i = n$.

We write $|\lambda| = n$ or $\lambda \vdash n$ if λ is a partition of n . Note that we usually omit 0's in a partition. For instance, we may write $(3, 3, 2, 2, 1)$ in place of $(3, 3, 2, 2, 1, 0, \dots)$. In this paper, we use an even more abbreviated notation when it does not lead to

confusion. For example, we denote $(3, 3, 2, 2, 1)$ by 33221. The partition $(0, 0, \dots)$ is denoted by 0.

Example 4.9. There are five partitions of 4: 4, 31, 22, 211, and 1111.

Just as we use Hasse diagrams to visualize posets, we can use Young diagrams to represent partitions geometrically.

Definition 4.10. The **Young diagram** of a partition λ , denoted $D(\lambda)$, is a left-justified array of squares with λ_i squares in the i th row.

The term “left-justified” in the definition above indicates that all rows of a Young diagram are aligned along the left edge, so that each row begins in the same column. Hence, given a partition λ , we can uniquely determine the shape of $D(\lambda)$. Figure 7 shows the Young diagram of the partition $5331 \vdash 12$.

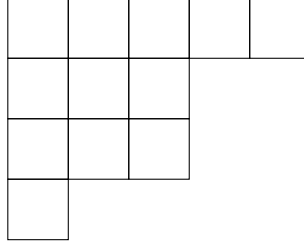


FIGURE 7. Young diagram of the partition $(5, 3, 3, 1)$.

We are now ready to define the rectangular Young’s lattice.

Definition 4.11. Let $m, n \in \mathbb{Z}^+$. The **rectangular Young’s lattice** $L(m, n)$ is the set of partitions with at most m parts and no part greater than n equipped with the partial order $\leq$ defined as follows: for all $\lambda, \mu \in L(m, n)$, $\lambda \leq \mu$ if $\lambda_i \leq \mu_i$ for all i .

One can verify that $\leq$ here is indeed a partial order and thus makes $L(m, n)$ a poset.

Example 4.12. $L(2, 3) = \{0, 1, 2, 3, 11, 21, 22, 31, 32, 33\}$. Figure 8 shows the Hasse diagram of the poset $L(2, 3)$.

Young diagrams provide us with a good way to interpret the poset $L(m, n)$: Note that $L(m, n)$ is the set of partitions with Young diagrams contained in an $m \times n$ rectangular grid, and for all $\lambda, \mu \in L(m, n)$, $\lambda \leq \mu$ is equivalent to $D(\lambda) \subseteq D(\mu)$.

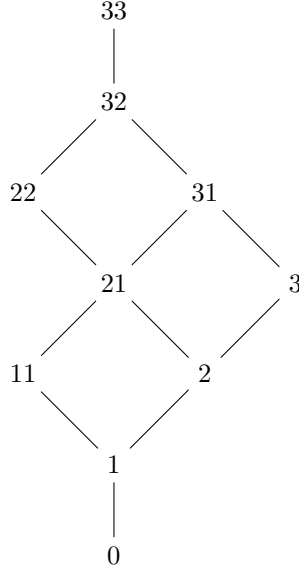
Using the approach in Proposition 3.7, one can show that $L(m, n)$ is graded of rank mn , and $\rho(\lambda) = |\lambda|$ for all $\lambda \in L(m, n)$. We also show that $L(m, n)$ is rank-symmetric.

Proposition 4.13. $L(m, n)$ is rank-symmetric.

Proof. Consider the complement map $c : L(m, n) \rightarrow L(m, n)$ defined by

$$c(\lambda)_i = \begin{cases} n - \lambda_{m+1-i} & \text{if } i \leq m \\ 0 & \text{if } i > m. \end{cases}$$

Since $c(c(\lambda)) = \lambda$ for all $\lambda \in L(m, n)$, c is a bijection. Note that geometrically, mapping λ to $c(\lambda)$ is equivalent to mapping its Young diagram to its complement

FIGURE 8. Hasse diagram of $L(2,3)$.

(rotated 180 degrees) in an $m \times n$ rectangular grid. Figure 9 depicts how c maps 421 to 32 in $L(3,4)$.

Therefore, for all $\lambda \in L(m,n)$, we have

$$\rho(c(\lambda)) = |c(\lambda)| = mn - |\lambda| = mn - \rho(\lambda).$$

Since c is a bijection, $L(m,n)$ is rank-symmetric. □

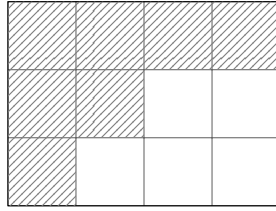


FIGURE 9. Depiction of c mapping 421 to 32 in $L(3,4)$. The shaded region represents the partition 421, and the unshaded region represents the partition 32.

Given our previous discussion of quotients of Boolean algebras and the fact that $L(m,n)$ is a rank-symmetric graded poset, one may naturally wonder: Is $L(m,n)$ isomorphic to some B_k/G , where k is a positive integer and $G \leq S_k$? This is indeed the case, and since $L(m,n)$ is graded of rank mn , the only possible choice for k here is $k = mn$.

Let $R(m,n)$ denote the set of mn squares in the $m \times n$ rectangular grid. We can define the partial order on $B_{R(m,n)}$ by inclusion of diagrams, which turns $B_{R(m,n)}$ into a poset isomorphic to B_{mn} . Figure 10 draws $R(3,4)$ with each of its squares labeled. Note that each partition in $L(m,n)$ corresponds uniquely to a Young

diagram contained in the $m \times n$ rectangular grid, and each diagram contained in the $m \times n$ grid corresponds to a subset of $R(m, n)$. Therefore, instead of B_{mn} , we may consider $B_{R(m, n)}$, which is much easier to visualize.

1	2	3	4
5	6	7	8
9	10	11	12

FIGURE 10. $R(3, 4)$ with each of its squares labeled.

Hence, the problem now is to find a suitable $G \leq S_{R(m, n)}$ and show that $B_{R(m, n)}/G$ is isomorphic to $L(m, n)$. Since the number of squares in each row of a Young diagram is nonincreasing, we can uniquely determine a Young diagram given the set of its rows. Thus, the permutations in G must permute the rows of $R(m, n)$. Since we only care about the number of squares, rather than the order of squares, in each row, the permutations in G must permute the squares within each row of $R(m, n)$. Taking into account these two properties of Young diagrams, we now define our choice of G .

Definition 4.14. The **wreath product** of S_n and S_m , denoted $S_n \wr S_m$, is a subgroup of $S_{R(m, n)}$ consisting of elements in $S_{R(m, n)}$ that permute rows of $R(m, n)$ and then permute squares within each row of $R(m, n)$.

Example 4.15. Assume σ and τ permute Figure 10 to the left and right diagrams in Figure 11, respectively. Then $\sigma \in S_4 \wr S_3$ while $\tau \notin S_4 \wr S_3$, since some elements in the same row initially (1 and 3, for example) are in different rows after applying τ .

5	8	6	7
12	11	9	10
3	2	4	1

5	8	11	7
12	3	9	10
6	2	4	1

FIGURE 11. Figure 10 after applying σ and τ .

Just as we did for Boolean algebras, we define the action of $S_n \wr S_m$ on $B_{R(m, n)}$ by

$$\sigma S = \{\sigma(s) : s \in S\}$$

for all $\sigma \in S_n \wr S_m$ and $S \subseteq R(m, n)$. We now show that $S_n \wr S_m$ is indeed the correct choice.

Proposition 4.16. $L(m, n)$ is isomorphic to $B_{R(m, n)}/(S_n \wr S_m)$.

Proof. We first show that each orbit in $B_{R(m,n)}/(S_n \wr S_m)$ contains exactly one Young diagram. Let $A \subseteq R(m,n)$, and suppose that A has A_i squares in the i th row. By definition, elements in $S_n \wr S_m$ can only permute the number of squares in rows. That is, for all $\sigma \in S_n \wr S_m$ and $1 \leq i \leq m$, we have $(\sigma A)_i = A_{\pi(i)}$ for some $\pi \in S_m$. Thus, the only possible Young diagram in $(S_n \wr S_m)A$ is the one representing the partition $\lambda = (\lambda_1, \dots, \lambda_m)$, where $\lambda_1 \geq \dots \geq \lambda_m$ is the unique nonincreasing ordering of $A_1, \dots, A_m$. We also have $\sigma S = D(\lambda)$, where $\sigma \in S_n \wr S_m$ is the permutation of $R(m,n)$ that left-justifies the rows of S by permuting elements within each row and makes the number of squares in each row nondecreasing by permuting rows. Hence, $D(\lambda)$ is in the orbit.

We can thus define a bijection

$$\phi : L(m,n) \rightarrow B_{R(m,n)}/(S_n \wr S_m) \quad \lambda \mapsto [D(\lambda)].$$

It suffices to show that ϕ is a poset isomorphism. If $\lambda \leq \mu$ in $L(m,n)$, then $D(\lambda) \subseteq D(\mu)$ by definition and therefore $[D(\lambda)] \leq [D(\mu)]$. Now assume for some $\mathcal{O}, \mathcal{O}' \in B_{R(m,n)}/(S_n \wr S_m)$, there exist $D \in \mathcal{O}$ and $D' \in \mathcal{O}'$ such that $D \subseteq D'$. Then for each i , the i th largest row of D' is at least as large as the i th largest row of D . Hence, the Young diagram in the orbit of D is contained in that of D' , and we have $\phi^{-1}(\mathcal{O}) \leq \phi^{-1}(\mathcal{O}')$. $\square$

By [Proposition 4.16](#) and [Theorem 3.12](#), we have the following theorem.

Theorem 4.17. *$L(m,n)$ is rank-unimodal and Sperner.*

Finally, we present an application of [Theorem 4.17](#) to combinatorics. Let k be a positive integer, S a finite subset of $\mathbb{R}^+$, and $x > 0$. Define

$$f_k(S, x) = \left| \left\{ S' \in \binom{S}{k} : \sum_{s \in S'} s = x \right\} \right|.$$

For example, $f_2(\{1, 2, 3, 4\}, 5) = 2$, since $1 + 4 = 2 + 3 = 5$.

We are interested in maximizing $f_k(S, x)$ by varying x and the elements of S with k and the size of S fixed. Let

$$M_k(n) = \max \left\{ f_k(S, x) : S \in \binom{\mathbb{R}^+}{n}, x > 0 \right\}.$$

Let us consider the $n = 4$ and $k = 2$ case. Note that when $S = \{s_1 < s_2 < s_3 < s_4\}$ is a set with equal gaps, that is, $s_{i+1} - s_i$ is fixed for all i , we have $f_2(S, s_1 + s_4) = 2$ since $s_1 + s_4 = s_2 + s_3$. Since $M_2(4) \leq 2$ by definition, the optimal choice here is indeed making S an arithmetic progression. For the sake of convenience, we may take $S = \{1, 2, 3, 4\}$.

We claim that it is optimal to choose $S = \{1, \dots, n\} = [n]$ for all n . That is, there exists $y \in \mathbb{R}^+$ such that for all $S \in \binom{\mathbb{R}^+}{n}$ and $x > 0$, we have $f_k(S, x) \leq f_k([n], y)$. We first determine our choice of y .

Proposition 4.18. *$f_k([n], x) \leq f_k([n], \lfloor \frac{k(n+1)}{2} \rfloor)$ for all k and n .*

Proof. Let $x > 0$. Let $S = \{s_1 < \dots < s_k\} \subseteq [n]$ such that $s_1 + \dots + s_k = x$. Since $1 \leq s_1 < \dots < s_k \leq n$, if we define $\lambda_i = s_i - i$ for each i , we have

$$0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq n - k$$

and

$$\lambda_1 + \cdots + \lambda_k = x - \binom{k+1}{2}.$$

One can also show that we can recover some $S \in \binom{[n]}{k}$ in which elements add up to x given a sequence $(\lambda_1, \dots, \lambda_k)$ satisfying the two properties above. Thus, $f_k([n], x)$ is equal to the number of sequences $(\lambda_1, \dots, \lambda_k)$ with the two properties. For all $S \in \binom{[n]}{k}$ in which elements add up to x , we denote the corresponding sequence by $\lambda(S)$.

Note that sequences with the two properties are exactly the partitions of $x - \binom{n+1}{2}$ with at most k parts and no part greater than $n-k$. Hence, the number of sequences is the number of elements in the $(x - \binom{k+1}{2})$ th level of $L(k, n-k)$. Since $L(k, n-k)$ is rank-symmetric and rank-unimodal by [Proposition 4.13](#) and [Theorem 4.17](#), the largest level of $L(k, n-k)$ is the $\lfloor \frac{k(n-k)}{2} \rfloor$ th level. Therefore, $f_k([n], x)$ is maximized by taking

$$x = \lfloor \frac{k(n-k)}{2} \rfloor + \binom{k+1}{2} = \lfloor \frac{k(n+1)}{2} \rfloor,$$

and the result follows. $\square$

Now we are ready to prove our final theorem.

Theorem 4.19. $f_k(S, x) \leq f_k([n], \lfloor \frac{k(n+1)}{2} \rfloor)$ for all $k, n \in \mathbb{Z}^+$, $S \in \binom{\mathbb{R}^+}{n}$, and $x > 0$.

Proof. Suppose $S = \{s_1 < \cdots < s_n\}$. Let $A_1, \dots, A_m$ be distinct k -element subsets of S whose elements sum to the same value. For each i , if $A_i = \{s_{j_1} < \cdots < s_{j_k}\}$, then define $\hat{A}_i = \{j_1, \dots, j_k\}$. Then we have $\lambda(\hat{A}_i) \in L(k, n-k)$ for all i . We claim that for all $i \neq j$, $\lambda(\hat{A}_i)$ and $\lambda(\hat{A}_j)$ are incomparable in $L(k, n-k)$.

By means of contradiction, assume $\lambda(\hat{A}_i) \leq \lambda(\hat{A}_j)$ for some i and j . Suppose $\hat{A}_i = \{i_1 < \cdots < i_k\}$ and $\hat{A}_j = \{j_1 < \cdots < j_k\}$. Since $i_r - r \leq j_r - r$ for all r , we have $i_r \leq j_r$ and thus $s_{i_r} \leq s_{j_r}$ for all r . But since $s_{i_1} + \cdots + s_{i_k} = s_{j_1} + \cdots + s_{j_k}$, we must have $s_{i_r} = s_{j_r}$ for all r , contradicting that A_i and A_j are distinct.

Therefore, $\{\lambda(\hat{A}_1), \dots, \lambda(\hat{A}_m)\}$ is an antichain in $L(k, n-k)$. By [Proposition 4.13](#) and [Theorem 4.17](#), the largest antichain in $L(k, n-k)$ is its $\lfloor \frac{k(n-k)}{2} \rfloor$ th level, which has the size $f_k([n], \lfloor \frac{k(n+1)}{2} \rfloor)$. Thus, we have $m \leq f_k([n], \lfloor \frac{k(n+1)}{2} \rfloor)$, and the theorem follows. $\square$

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