TWISTING MAPS AND ADAMS SPECTRAL SEQUENCE

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Abstract. In this paper, we develop a new construction that enables discrete computation of n-ary Steenrod algebras and establish an explicit relationship between this and the computation of the E_{n+1}-page of the $\mathbb{F}_p$-Adams spectral sequence. As an application, we demonstrate how hidden extensions in the Adams spectral sequence detect extensions of homotopy groups.

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1. Introduction

1.1. **Motivations.** In [Bau06], Baues introduced and computed the secondary Steenrod algebra as a differential graded algebra. Nassau simplified this construction in [Nas12], and his model was used to develop an algorithm for computing the E_3 page of the Adams spectral sequence in [CPH22] by Chua. The basic ideas are as follows:

The secondary Steenrod algebra $\mathcal{A}_p^{(2)}$ is defined as a differential graded $\mathbb{Z}/p^2$-algebra, namely

$$\begin{tikzcd} \mathcal{B}_1 \arrow[d, "d^{\mathcal{B}}"] \\ \mathcal{B}_0. \end{tikzcd}$$

Furthermore, there is an exact sequence

$$0 \to \mathcal{A}_p[+1] \to \mathcal{B}_1 \xrightarrow{d^{\mathcal{B}}} \mathcal{B}_0 \xrightarrow{\pi_{\mathcal{B}}} \mathcal{A}_p \to 0,$$

where $\mathcal{A}_p$ denotes the Steenrod algebra and $(\mathcal{A}_p[+1])_k = (\mathcal{A}_p)_{k-1}$. Specifically, there is a map of right $\mathcal{A}_p^{(2)}$-modules

$$\tau : \Sigma\mathcal{A}_p^{(2)}[+1] \to \mathcal{A}_p^{(2)}$$

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that induces a splitting

$$\mathcal{B}_1 \simeq \ker \pi_{\mathcal{B}} \oplus \mathcal{A}_p\{\tau\}$$

as a right $\mathcal{B}_0$-module, where τ has degree $|\tau| = 1$.

Given a spectrum X, to compute its d_2 differential, we first compute its E_2-page whose (s,t)-term is $E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_p}^{s,s+t}(H^*(X;\mathbb{F}_p), \mathbb{F}_p)$. This can be computed from the projective resolution of $H^*(X)$ as an $\mathcal{A}_p$-module, given by

$$\cdots \longrightarrow \overline{X^{(2)}} \xrightarrow{\overline{\partial^{(2)}}} \overline{X^{(1)}} \xrightarrow{\overline{\partial^{(1)}}} \overline{X^{(0)}} \longrightarrow H^*(X;\mathbb{F}_p).$$

By [CPH22, Lemma 8.23], this chain complex can be lifted to a chain of free $\mathcal{A}_p^{(2)}$-modules

$$\begin{tikzcd}
\cdots \arrow[r] & X_1^{(3)} \arrow[r, "\partial_1^{(3)}"] \arrow[d, "d^{(3)}"] & X_1^{(2)} \arrow[r, "\partial_1^{(2)}"] \arrow[d, "d^{(2)}"] & X_1^{(1)} \arrow[r, "\partial_1^{(1)}"] \arrow[d, "d^{(1)}"] & X_1^{(0)} \arrow[d, "d^{(0)}"] \\
\cdots \arrow[r] & X_0^{(3)} \arrow[r, "\partial_0^{(3)}"'] & X_0^{(2)} \arrow[r, "\partial_0^{(2)}"'] & X_0^{(1)} \arrow[r, "\partial_0^{(1)}"'] & X_0^{(0)}
\end{tikzcd}$$

The d_2 differentials are captured by a specified null-homotopy of the composition $\partial_0^{(n)} \circ \partial_0^{(n+1)}$, namely by the maps h that make the following diagram commute:

$$\begin{tikzcd}
\cdots \arrow[r] & X_1^{(3)} \arrow[r, "\partial_1^{(3)}"] \arrow[d, "d^{(3)}"] & X_1^{(2)} \arrow[r, "\partial_1^{(2)}"] \arrow[d, "d^{(2)}"] & X_1^{(1)} \arrow[r, "\partial_1^{(1)}"] \arrow[d, "d^{(1)}"] & X_1^{(0)} \arrow[d, "d^{(0)}"] \\
\cdots \arrow[r] \arrow[urr, "h^{(4)}"] & X_0^{(3)} \arrow[r, "\partial_0^{(3)}"'] \arrow[urr, "h^{(3)}"] & X_0^{(2)} \arrow[r, "\partial_0^{(2)}"'] \arrow[urr, "h^{(2)}"] & X_0^{(1)} \arrow[r, "\partial_0^{(1)}"'] & X_0^{(0)}
\end{tikzcd}$$

There are exact sequences

$$0 \to \overline{X^{(n)}}\{\tau\} \to X_1^{(n)} \to X_0^{(n)} \to \overline{X^{(n)}} \xrightarrow{\pi_{(n)}} 0,$$

so we can decompose the target of $h^{(n+2)}$ as $\ker \pi_{(n)} \oplus \overline{X^{(n)}}\{\tau\}$. We can identify the projection of $h^{(n+2)}$ onto $\ker \pi_{(n)}$ with $\partial_0^{(n+2)} \circ \partial_0^{(n+1)}$ by commutativity. The essential information is captured by the map into the τ-component, which we denote by $h_\tau^{(n+2)}$. Since $h_\tau^{(n+2)}$ sends $\ker \pi_{(n+2)}$ to zero, the map $h_\tau^{(n+2)}$ can be regarded as an $\mathcal{A}_p$-module map from $\overline{X}^{(n+2)}$ to $\overline{X}^{(n)}\{\tau\}$. As stated in [CPH22, Lemma 9.2], these maps determine the d_2-differentials.

Therefore, one has an algorithm for all d_2-differentials by inductively determining $h_\tau^{(n)}$. The heart of Secondary Steenrod algebra is giving an explicit formula for the difference

$$t_{n+2} := \overline{\partial^{(n)}} \circ h_\tau^{(n+2)} - h_\tau^{(n+1)} \circ \overline{\partial^{(n+1)}}$$

to simplify the computation of all d_2-differential into a procedural computation.

However, all the procedures shown above rely heavily on the fact that $\mathcal{A}_p^{(2)}$ is an algebra over $\mathbb{Z}/p^2$, which allows it to have a model as a differential graded algebra. It seems unclear why the maps h_τ should be defined so.

In this paper, we design a concrete quasi-category for the modules over the n-ary Steenrod algebra[1] as in Theorem 2.16. Then, we present a canonical way to

[1]Actually, we do not deal with the category of modules over the n-ary Steenrod algebra directly, but rather with its dual in some sense, namely $\mathcal{M}od_{C\tau^n}$ in the category of synthetic spectra.

define h_τ and t, and generalize them to higher Steenrod algebras. See Definition 3.5 for the precise definition of the *twisting maps* t of modules over higher Steenrod algebras. Finally, in Theorem 6.11, we justify our construction as a generalization of the secondary Steenrod algebra by relating the first twisting map we define to the key formula

$$\overline{\partial^{(n)}} \circ h_\tau^{(n+2)} - h_\tau^{(n+1)} \circ \overline{\partial^{(n+1)}}$$

that appeared above.

1.2. **Conventions.**

Notations 1.1. Throughout this paper, we put $\mathcal{C}$ to be a stable presentable ∞-category equipped with a symmetric monoidal structure and a separated t-structure. Assume that the unit object $\mathbb{1} \in \mathcal{C}$ belongs to $\mathcal{C}_{\geq 0}$, that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimit separately in each variable, and that $\otimes$ carries $\mathcal{C}_{\geq 0} \times \mathcal{C}_{\geq 0}$ into $\mathcal{C}_{\geq 0}$.

Notations 1.2. We use $\tau_{\leq n}$ to denote the standard t-structure truncation of $\mathcal{C}$ and $\tau_{[a,b]}$ to denote $\tau_{\leq b} \circ \tau_{\geq a}$. Unless otherwise specified, we use $\pi_n(X)$ to denote the n-th homotopy group with respect to the t-structure and $X_{\leq n}$ to denote $\tau_{\leq n} X$.

Notations 1.3. Let A be a connective algebra object in $\mathcal{C}$. We denote by $\mathcal{M}od_A$ the category of bounded below left A-modules in $\mathcal{C}$.

Throughout this paper, ∞-categories should be regarded as quasi-categories and all limits and colimits are taken as homotopy limits and colimits in the ∞-category sense.

1.3. **Outline of the paper.**

Given a symmetric monoidal ∞-category $\mathcal{C}$ and a connective algebra object A, one can always define a quasi-category as the ∞-category of left modules over A. However, it is typically difficult to understand the information encoded in higher simplexes, unless the algebra A is discrete. In [LH17], Lurie and Hopkins introduced the functor Θ and a natural transformation π from Θ to the identity functor. In [PV22], Pstrągowski and Vankoughnett applied these functors to construct an equivalence of ∞-categories ([PV22, Theorem 3.8]):

$$\mathcal{M}od_{A_{\leq n}} \simeq \Theta\text{-Sect}_{A_{\leq n-1}}.$$

The right-hand side is a subcategory of $\mathrm{Fun}(\Delta^2, \mathcal{M}od_{A_{\leq n-1}})$, which means that we can use a diagram of $A_{\leq n-1}$-modules to represent a $A_{\leq n}$-module.

This motivated the author to repeat this construction to identify $\mathcal{M}od_{A_{\leq n}}$ as a subcategory of $\mathrm{Fun}((\Delta^2)^n, \mathcal{M}od_{A_{\leq 0}})$. In many favorable cases, the category $\mathcal{M}od_{A_{\leq 0}}$ is purely algebraic and equivalent to some ∞-derived category, so that one can understand each simplex explicitly. This idea is fully discussed and generalized to a symmetric monoidal version in Section 2.

Concretely, for any m, n such that $n \leq 2m+1$, we have a fiber-cofiber sequence

$$F \to A_{\leq n} \to A_{\leq m}$$

and a pullback square of associated algebras

$$\begin{tikzcd} A_{\leq n} \arrow[r] \arrow[d] & A_{\leq m} \arrow[d, "d"] \\ A_{\leq m} \arrow[r, "d_0"'] & A_{\leq m} \oplus \Sigma F, \end{tikzcd} \qquad (\star)$$

where d and d_0 are all sections of the natural projection $p : A_{\leq m} \oplus \Sigma F \to A_{\leq m}$. This induces a pullback square of categories of modules over the corresponding rings:

$$\begin{tikzcd} \mathcal{M}od_{A_{\leq n}} \arrow[r] \arrow[d] & \mathcal{M}od_{A_{\leq m}} \arrow[d, "d^*"] \\ \mathcal{M}od_{A_{\leq m}} \arrow[r, "d_0^*"] & \mathcal{M}od_{A_{\leq m} \oplus \Sigma F}. \end{tikzcd}$$

Definition 1.4 (Definition 2.2)**.** We define the functor $\Theta_m^{n-m} : \mathcal{M}od_{A_{\leq m}} \to \mathcal{M}od_{A_{\leq m}}$ as $\Theta_m^{n-m} := d_* d_0^*$.

Definition 1.5 (Proposition 2.4)**.** There is a natural transformation $\pi : \Theta \to \mathrm{Id}$ induced by the natural map $d_* d_0^* M \to d_* p_* p^* d_0^* M$.

We then construct an ∞-operad as the homotopy pullback in the following square:

Construction 1.6 (Construction 2.12)**.**

$$\begin{tikzcd} \Theta_m^{n-m}\text{-Sect}^{\otimes} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^2, \mathcal{M}od_{A_{\leq m}})^{\otimes} \arrow[d] \\ \mathcal{M}od^{\otimes}_{A_{\leq m}} \arrow[r] & \mathrm{Fun}(\Lambda^2_2, \mathcal{M}od_{A_{\leq m}})^{\otimes}. \end{tikzcd}$$

Here the horizontal map sends M to the diagram

$$\begin{tikzcd} M \arrow[dr, "id"] & & \Theta M \arrow[dl, "\pi"] \\ & M & \end{tikzcd} .$$

We then have the following symmetric monoidal equivalence:

Theorem 1.7 (Theorem 2.16)**.**

$$\Theta_m^{n-m}\text{-Sect}^{\otimes} \simeq \mathcal{M}od^{\otimes}_{A_{\leq n}}.$$

Hence, one can iterate this identification to obtain a simplicial set model of $\mathcal{M}od_{A_{\leq n}}$ for each n.

With this construction, it is natural to ask what these higher simplexes represent. For technical reasons, we restrict our attention to shift algebras and periodic objects. With a grading on the entire category, we can define:

Definition 1.8 (Definition 3.2)**.** A *shift algebra* is an associative algebra $A \in \mathrm{Alg}(\mathcal{C})$ equipped with a map $\tau : \Sigma A[-1] \to A$ of right A-modules which induces an isomorphism $\pi_*(A) \simeq \pi_0(A)[\tau]$, where the latter is the graded algebra in $\mathcal{C}^{\heartsuit}$ given by $(\pi_0(A)[\tau])_k \simeq \pi_0(A)[-k]$.

Definition 1.9 (Definition 3.3)**.** For a fixed shift algebra A and $1 \leq n \leq \infty$, we say X is a *periodic n-module* if $M \in \mathcal{M}od_{A_{\leq n}}$ satisfies that $\pi_0(A) \otimes_{A_{\leq n}} X$ is discrete. When $n = \infty$, we also refer to it as a *periodic module*.

The main feature of periodic modules is that their t-structure truncation coincides with their left tensoring with $A_{\leq n}$ over A. This simplifies the computation of $A_{\leq n} \otimes_A X$.

One finds that the essential homotopical information is always captured by the sections:

$$s : X \to \Theta X.$$

The intuition is as follows. Given a t-structure and a periodic object X over a periodic algebra $\mathbb{1}$, there are fiber-cofiber sequences:

$$\Sigma\pi_1(X) \to \tau_{[0,1]}X \to \pi_0(X),$$

and we are always concerned with the connecting map:

$$\pi_0(X) \to \Sigma^2\pi_1(X).$$

However, this map typically lives in $\mathcal{M}od_{\mathbb{1}_{\leq 1}}$ and we cannot describe it algebraically. The functor Θ can help. Indeed, $\Theta\pi_0(X)$ is the fiber of a $\mathbb{1}_{\leq 0}$-map $\pi_0(X) \to \Sigma^3\pi_1(X)$, so the connecting map can be factored as:

$$\pi_0(X) \to \Theta\pi_0(X) \to \Sigma^2\pi_1(X),$$

where the first map can be described completely algebraically. Therefore, we should think of Θ as a way to record the effect of $\pi_0(X)$ on $\pi_1(X)$ so that one can be specific about the connecting map.

As an application, we work in the category of synthetic spectra $\widehat{\mathcal{S}yn}_{\mathbb{F}_p}$ and develop a method to record the information of the n-ary Steenrod algebra algebraically. For any spectrum X, the section

$$s : C\tau \otimes \nu(X) \to \Theta^1_0(C\tau \otimes \nu(X))$$

can be regarded as a map between chain complexes of injective $\mathcal{A}^\vee_p$-comodules:

$$\begin{tikzcd}
X_0 \arrow[r, "{(id,d_0^{(2)})}"] \arrow[d] & X_0 \oplus X_2[-1] \arrow[d] \\
X_1 \arrow[r, "{(id,d_1^{(2)})}"] \arrow[d] & X_1 \oplus X_3[-1] \arrow[d] \\
X_2 \arrow[r, "{(id,d_2^{(2)})}"] \arrow[d] & X_2 \oplus X_4[-1] \arrow[d] \\
X_3 \arrow[r, "{(id,d_3^{(2)})}"] & X_3 \oplus X_5[-1].
\end{tikzcd}$$

Then the maps $d^{(2)}$ are exactly the duals of h_τ introduced in Section 1.1. Moreover, the map t is recorded by Θ^1_0 and can be seen from the fiber-cofiber sequence:

$$\Theta^1_0(C\tau \otimes \nu(X)) \xrightarrow{\pi} C\tau \otimes \nu(X) \xrightarrow{t} \Sigma^{2,-1}C\tau \otimes \nu(X),$$

where the second map expands into a map between chain complexes:

$$\begin{tikzcd}
X_0 \arrow[r, "t_0^{(2)}"] \arrow[d] & X_3[-1] \arrow[d] \\
X_1 \arrow[r, "t_1^{(2)}"] \arrow[d] & X_4[-1] \arrow[d] \\
X_2 \arrow[r, "t_2^{(2)}"] \arrow[d] & X_5[-1] \arrow[d] \\
X_3 \arrow[r, "t_3^{(2)}"] & X_6[-1].
\end{tikzcd}$$

This gives the dual of $t := \overline{\partial^{(n)}} \circ h_\tau^{(n+2)} - h_\tau^{(n+1)} \circ \overline{\partial^{(n+1)}}$. We generalize the map t for higher Steenrod algebras in Definition 3.5 and call it the *twisting map*. Hence the formula designed by Baues is actually one choice of the twisting map for Θ_0^1:

Theorem 1.10 (Theorem 6.11)**.** *Given a $\mathcal{A}_p^\vee$-comodule $\overline{M}^\vee$ and its dual $\mathcal{A}_p$-module $\overline{M}$, the free resolution of $\overline{M}$ is given by:*

$$\cdots \longrightarrow \overline{M^{(2)}} \xrightarrow{\overline{\partial^{(2)}}} \overline{M^{(1)}} \xrightarrow{\overline{\partial^{(1)}}} \overline{M^{(0)}} \longrightarrow \overline{M}$$

with its dual being an injective resolution in $\mathcal{C}omod_{\mathcal{A}_p^\vee}$:

$$0 \longrightarrow \overline{M}^\vee \longrightarrow \overline{M^{(0)}}^\vee \longrightarrow \overline{M^{(1)}}^\vee \longrightarrow \overline{M^{(2)}}^\vee \longrightarrow \cdots$$

Suppose that $\overline{M}^\vee$ can be lifted to a periodic $C\tau^2$-module. Then the first twisting map

$$t^{(2)} : \overline{M^{(n)}}^\vee \to \overline{M^{(n+3)}}^\vee[-1]$$

has a dual

$$(t^{(2)})^\vee : \overline{M^{(n+3)}} \to \overline{M^{(n)}}[+1]$$

sending each generator g to

$$\sum A(\alpha^i, \partial_0^{(n+2)}\partial_0^{(n+1)}(g_i)),$$

where $\overline{\partial^{(n+3)}}(g) = \sum \alpha^i g_i$ if we write $\{g_i\}$ for the generators of $\overline{M^{(n+2)}}$.

Example 1.11 (Example 6.14)**.** We recover the differential $d_2(h_4) = h_0h_3^2$ in the $\mathbb{F}_2$-Adams spectral sequence of the sphere. By computing the twisting map

$$(t^{(2)})^\vee(h_0h_3^2) = Sq^{16} + Sq^{(6,1,1)} + Sq^{(10,2)},$$

the commutative square

$$\begin{tikzcd}
\langle 1 \rangle & \langle 1 \rangle \oplus (\langle h_0^2 \rangle \oplus \langle h_1^2 \rangle \oplus \langle h_0h_2 \rangle \oplus \cdots)[-1] \arrow[l, "(id,0)"'] \\
\langle h_0 \rangle \oplus \langle h_1 \rangle \oplus \langle h_2 \rangle \oplus \cdots \arrow[u] & (\langle h_0 \rangle \oplus \langle h_1 \rangle \oplus \cdots) \oplus (\langle h_0^3 \rangle \oplus \langle h_0^2h_2 \rangle \oplus \cdots)[-1] \arrow[l, "(id,d^{(2)})"] \arrow[u]
\end{tikzcd}$$

implies the first differential $d_2(h_4) = h_0h_3^2$.

To discuss the explicit relationship between the sections and Adams differentials, we define a variation of the Adams spectral sequence as follows:

Construction 1.12 (Construction 5.5)**.** For a spectrum X, we define

$$\mathfrak{E}_k^{s,t}(X) := \pi_{t,t+s}(C\tau^{k-1} \otimes \nu X)$$

and we focus on the (2^k+1)-th pages. By considering the connecting map in the fiber-cofiber sequence:

$$\Sigma^{0,-2^k} C\tau^{2^k} \otimes \nu X \to C\tau^{2^k+1} \otimes \nu X \to C\tau^{2^k} \otimes \nu X \to \Sigma^{1,-2^k} C\tau^{2^k} \otimes \nu X,$$

we can define the differential $d_{2^{k+1}}^{\mathcal{S}yn}$ as

$$d_{2^{k+1}}^{\mathcal{S}yn} : \mathfrak{E}_{2^k+1}^{s,t}(X) = \pi_{t,t+s}(C\tau^{2^k} \otimes \nu X) \to \pi_{t-1,t+s+2^k}(C\tau^{2^k} \otimes \nu X) = \mathfrak{E}_{2^k+1}^{s+2^k+1,t-1}(X).$$

In this case, the underlying object of $\Theta_{2^k}^{2^k} C\tau^{2^k} \otimes \nu X$ is $C\tau^{2^k} \otimes \nu X \oplus \Sigma^{1,-2^k} C\tau^{2^k} \otimes \nu X$, so the connecting map is recorded as part of the data:

$$s : \pi_{t,t+s}(C\tau^{2^k} \otimes \nu X) \to \pi_{t,t+s}(\Theta_{2^k-1}^{2^k} C\tau^{2^k} \otimes \nu X).$$

This allows us to work on the connecting map in the category $\mathcal{M}od_{C\tau^{2^k}}$ rather than $\mathcal{M}od_{C\tau^{2^{k+1}}}$.

This also allows us to easily determine extension problems of the Adams spectral sequence using the product structure. Suppose we choose $h_0^{(2^k)}$ to be any lifting of $(h_0)^{2^k}$ along

$$\mathfrak{E}_{2^k+1}^{2^k,0}(\mathbb{S}) = \pi_{0,2^k}(C\tau^{2^k}) \to \pi_{0,2^k}(C\tau) = \mathfrak{E}_2^{2^k,0}(\mathbb{S}).$$

Then we have:

Theorem 1.13 (Theorem 5.11)**.** *There is a long exact sequence:*

$$\pi_{t,t+s+2^k}(X_{2^k}) \to \pi_{t,t+s}(X_{2^{k+1}}) \to \pi_{t,t+s}(X_{2^k}) \xrightarrow{d_{2^{k+1}}^{\mathcal{S}yn}} \pi_{t-1,t+s+2^k}(X_{2^k}).$$

This reduces to a short exact sequence:

$$0 \to \operatorname{coker}(d_{2^{k+1}}^{\mathcal{S}yn})_{s+2^k,t} \to \mathfrak{E}_{2^{k+1}}^{s,t} \to \ker(d_{2^{k+1}}^{\mathcal{S}yn})_{s,t} \to 0.$$

The extension is detected by $h_0^{(2^k)} \cdot (-) : \ker(d_{2^{k+1}}^{\mathcal{S}yn})_{s,t} \to \operatorname{coker}(d_{2^{k+1}}^{\mathcal{S}yn})_{s+2^k,t}$.

Example 1.14 (Example 5.13)**.** By the hidden extension

$$h_0^{(2)} \widetilde{h_3^2} = \tau h_0 d_0$$

and Theorem 5.11, we can deduce the first d_3-differential:

$$d_3(h_0 h_4) = h_0 d_0.$$

The relationship between this variation and the classical Adams spectral sequence is already known and discussed in [BHS19, Theorem 9.19], and we provide a refinement in Theorem 5.9.

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2. A simplicial model for the category of modules over truncated algebra

In this section, we will construct a quasi-category as a concrete model of $\mathcal{M}od_{A_{\leq n}}$ so that one can understand each simplex algebraically. We also discuss the symmetric monoidal structure and t-structure of the ∞-category so that we can completely identify this model with $\mathcal{M}od_{A_{\leq n}}$.

We follow Lurie's treatment in [Lur17, 7.4.1] of square-zero extensions and treat the Postnikov tower of an algebra as a sequence of square-zero extensions. We list the basic definitions and properties of square-zero extensions in Appendix A. Hereafter, we fix an $\mathbb{E}_k$-algebra $A \in \mathrm{Alg}^{(k)}(\mathcal{C}_{\geq 0})$ such that $\pi_0(A) \neq 0$ and denote the category of bounded below left modules over $A_{\leq n}$ by $\mathcal{M}od_{A_{\leq n}}$.

As shown in [Lur17, Corollary 7.4.1.27], $A_{\leq n} \to A_{\leq m}$ is a square-zero extension if $n \leq 2m+1$. Suppose its fiber is F. Then there is a pullback square of $\mathbb{E}_k$-algebras as in [Lur17, Remark 7.4.1.7]:

$$\begin{tikzcd} A_{\leq n} \arrow[r] \arrow[d] & A_{\leq m} \arrow[d, "d"] \\ A_{\leq m} \arrow[r, "d_0"'] & A_{\leq m} \oplus \Sigma F. \end{tikzcd} \qquad (\star)$$

Here $A_{\leq m} \oplus \Sigma F$ is the trivial square-zero extension of $A_{\leq m}$ by ΣF, and d and d_0 are both sections of the natural projection $p : A_{\leq m} \oplus \Sigma F \to A_{\leq m}$ so $d_0 = (id, 0)$ and $d = (id, \mathrm{der})$. der is a derivation determined by the structure of $A_{\leq n}$.

This induces a commutative square of ∞-categories of left modules:

Proposition 2.1. *The commutative square of ∞-categories of left modules induced by the diagram $(\star)$ is a pullback square of ∞-categories.*

$$\begin{tikzcd} \mathcal{M}od_{A_{\leq n}} \arrow[r] \arrow[d] & \mathcal{M}od_{A_{\leq m}} \arrow[d, "d^*"] \\ \mathcal{M}od_{A_{\leq m}} \arrow[r, "d_0^*"] & \mathcal{M}od_{A_{\leq m} \oplus \Sigma F} \end{tikzcd}$$

That is, there is a canonical equivalence between $\mathcal{M}od_{A_{\leq n}}$ and the ∞-category of triples (M, N, α), where $M, N \in \mathcal{M}od_{A_{\leq m}}$ and $\alpha : d_0^* M \simeq d^* N$. Furthermore, since d^* and d_0^* are symmetric monoidal functors, this pullback can be viewed as a pullback square of symmetric monoidal categories.

Proof. This first appeared in [LH17] and was restated in [PV22, Proposition 3.5]. Here we slightly generalize the statement to fit our needs and mimic the original proof. There is an adjunction

$$F : \mathcal{M}od_{A_{\leq n}} \rightleftarrows \mathcal{M}od_{A_{\leq m}} \times_{\mathcal{M}od_{A_{\leq m} \oplus \Sigma F}} \mathcal{M}od_{A_{\leq m}} : G,$$

where F is induced by base change along the diagram $(\star)$ and its right adjoint G is given informally by the formula

$$(M, N, \alpha) \mapsto M \times_{d_0^* M} N.$$

Here the right-hand side is taken as a limit of modules over $A_{\leq n}$. To show that F and G are categorical equivalences, we first show the unit of the adjunction is an equivalence:

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & A_{\leq m} \otimes_{A_{\leq n}} M \arrow[d] \\
A_{\leq m} \otimes_{A_{\leq n}} M \arrow[r] & (A_{\leq m} \oplus \Sigma F) \otimes_{A_{\leq n}} M
\end{tikzcd}$$

is a pullback square for every $M \in \mathcal{M}od_{A_{\leq n}}$. In fact, $(\star)$ can be viewed as a pullback square of $A_{\leq n}$-bimodules or a pushout square as bimodules since $\mathrm{BMod}_{A_{\leq n}}$ is stable. By tensoring with M on the right, we get a pushout-pullback square of left $A_{\leq n}$-modules.

Then we show that G is conservative. Similar to the argument above, G preserves cofibers, so it suffices to check that if (M, N, α) is a triple such that $M \times_{d_0^* M} N$ vanishes, then $M = 0$ and $N = 0$. Otherwise, we can assume M is nontrivial and take its lowest nontrivial homotopy group. Notice that d_0 and d induce identity maps on π_0, so M and $d_0^* M$ share the same lowest nontrivial homotopy group. Since $d^* N \simeq d_0^* M$, we have that $d^* N$ and thus N also have the same lowest nontrivial homotopy group. This implies that $M \times_{d_0^* M} N$ is nontrivial, which is a contradiction.

For the symmetric monoidal structure, we can view this diagram as a square over $\mathrm{N}(\mathrm{Fin}_*)$, which gives the equivalence as symmetric monoidal categories. □

Now we introduce the key functor of this paper, which was first introduced in [LH17] and used to develop obstruction theory in [PV22].

Definition 2.2. We define the functor $\Theta : \mathcal{M}od_{A_{\leq m}} \to \mathcal{M}od_{A_{\leq m}}$ as $\Theta := d_* d_0^*$.

Warning 2.3. Here we slightly abuse notation because Θ actually depends on n and m. We shall use Θ_m^{n-m} to specify when needed.

Proposition 2.4. *There is a natural transformation from* Θ *to* Id*, denoted by* π*.*

Proof. Let $p : A_{\leq m} \oplus \Sigma F \to A_{\leq m}$ be the projection map. Then d_0 and d are sections of p. There is a natural transformation given by:

$$\Theta M = d_* d_0^* M \to d_* p_* p^* d_0^* M \xrightarrow{\simeq} M.$$

Here the first map is the unit of the adjunction and the second equivalence follows from the fact that d_0 and d are sections of p. □

Clearly, the underlying object of ΘM is $\Theta M = (A_{\leq m} \oplus \Sigma F) \otimes_{A_{\leq m}} M$ since d_* does not change the underlying object.

Definition 2.5. We define the ∞-category Θ_m^{n-m}-Sect as the ∞-category of triples (M, s, h), where $M \in \mathcal{M}od_{A_{\leq m}}$, $s : M \to \Theta M$ is a section, and h is a homotopy from $\pi \circ s$ to Id_M.

Construction 2.6. We regard all ∞-categories mentioned below as quasi-categories. Then the ∞-category Θ_m^{n-m}-Sect is defined rigorously as the homotopy pullback of the diagram:

$$\begin{tikzcd}
\Theta_m^{n-m}\text{-Sect} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^2, \mathcal{M}od_{A_{\leq m}}) \arrow[d] \\
\mathcal{M}od_{A_{\leq m}} \arrow[r] & \mathrm{Fun}(\Lambda_2^2, \mathcal{M}od_{A_{\leq m}}).
\end{tikzcd}$$

Here the bottom horizontal map sends M to the diagram:

$$\begin{tikzcd} M \arrow[dr, "id"] & & \Theta M \arrow[dl, "\pi"] \\ & M & . \end{tikzcd}$$

Since $\Lambda^2_2 \to \Delta^2$ is a cofibration in the Joyal model of simplicial sets, $\mathrm{Fun}(\Delta^2, \mathcal{M}od_{A_{\leq m}}) \to \mathrm{Fun}(\Lambda^2_2, \mathcal{M}od_{A_{\leq m}})$ is an inner fibration. Then the pullback of this map, namely Θ^{n-m}_m-Sect $\to \mathcal{M}od_{A_{\leq m}}$, is still an inner fibration, so Θ^{n-m}_m-Sect is a quasi-category. Specifically, Θ^{n-m}_m-Sect is the strict pullback of this diagram.

Theorem 2.7. *There is an equivalence of ∞-categories*

$$\mathcal{M}od_{A_{\leq n}} \simeq \Theta^{n-m}_m\text{-Sect}.$$

Proof. This is a formal variation of Proposition 2.1 and the proof is similar to [PV22, Theorem 3.8]. As in Proposition 2.1, we can identify $\mathcal{M}od_{A_{\leq n}}$ as the category of triples (M, N, α), where $\alpha : d^*N \to d_0^*M$ is an equivalence. By adjunction, this corresponds to a map $\alpha' : N \to \Theta M$ as $A_{\leq m}$-modules.

Actually, α is an equivalence if and only if the composite $\pi \circ \alpha' : N \to \Theta M \to M$ is. Under the identification $p^*d^*N \simeq N$ and $p^*d_0^*M \simeq M$, the map $\pi \circ \alpha'$ can be written as $p^*\alpha : p^*d^*N \to p^*d_0^*M$. Since $p : A_{\leq m} \oplus \Sigma F \to A_{\leq m}$ induces an isomorphism on π_0, p^* is conservative. Therefore, $p^*\alpha$ is an equivalence if and only if α is. It follows that $\mathcal{M}od_{A_{\leq n}}$ is equivalent to the ∞-category of triples

$$\mathcal{D} \simeq \{(M, N, \alpha') | M, N \in \mathcal{M}od_{A_{\leq m}}, \alpha' : N \to \Theta M \text{ such that } \pi \circ \alpha' \text{ is an equivalence}\}.$$

We can also define the ∞-category of quintuples $\mathcal{E} := \{(M, N, \alpha', \beta, h)\}$ as in Construction 2.6, where M, N, α' are as above, β is an $A_{\leq m}$-module morphism from M to N, and h is a homotopy witnessing $\pi \circ \alpha' \circ \beta \simeq id_M$. The functor from $\mathcal{E}$ to $\mathcal{D}$:

$$f : \mathcal{E} \to \mathcal{D} : (M, N, \alpha', \beta, h) \mapsto (M, N, \alpha')$$

is a Cartesian fibration whose fibers are ∞-groupoids. We claim the fibers are all contractible so that f is an equivalence of ∞-categories. Indeed, the fiber over (M, N, α') is the space of pairs

$$(\beta : M \to N, h : \pi \circ \alpha' \circ \beta \simeq id_M),$$

which is the homotopy fiber of the composition map

$$(\pi \circ \alpha')_* : \mathcal{M}od_{A_{\leq m}}(M, N) \to \mathcal{M}od_{A_{\leq m}}(M, M)$$

over the identity. Since $\pi \circ \alpha'$ is an equivalence, this fiber is contractible.

Similarly, the map

$$g : \mathcal{E} \to \Theta^{n-m}_m\text{-Sect} : (M, N, \alpha', \beta, h) \mapsto (M, \alpha' \circ \beta, h)$$

is also fibered in spaces. The fiber over (M, s, h) can be identified with the space of objects N equipped with an equivalence $\beta : M \simeq N$, which is also contractible. Therefore, g is an equivalence of ∞-categories, implying the theorem. $\square$

Now we can inductively define the simplicial set model of $\mathcal{M}od_{A_{\leq n}}$ we want.

Construction 2.8. Let $\mathbb{A}$ be the abelian category of $\pi_0(A)$-modules in $\mathcal{C}^{\heartsuit}$. If $\mathbb{A}$ admits enough projective objects and $\mathcal{M}od_{A_{\leq 0}} \simeq \mathcal{D}^-(\mathbb{A})$, then we can start from a concrete quasi-category $\mathcal{D}^-(\mathbb{A}) := \mathrm{N}_{dg}(\mathrm{Ch}^-(\mathbb{A}^{\mathrm{proj}}))$. Here $\mathrm{Ch}^-(\mathbb{A}^{\mathrm{proj}})$ is the category of bounded below chain complexes of projective $\pi_0(A)$-modules in $\mathcal{C}^{\heartsuit}$. The advantage of this description of ∞-derived categories is that we can concretely describe all simplices and maps between them.

Suppose the model of $\mathcal{M}od_{A_{\leq m}}$ is defined. Then for $n \leq 2m+1$, we can apply Theorem 2.7 and define $\mathcal{M}od_{A_{\leq n}}$ as a simplicial set whose k-simplexes are given by:

$$\mathrm{Fun}(\Delta^2 \times \Delta^k, \mathcal{M}od_{A_{\leq m}}) \times_{\mathrm{Fun}(\Lambda^2_2 \times \Delta^k, \mathcal{M}od_{A_{\leq m}})} \mathrm{Fun}(\Delta^k, \mathcal{M}od_{A_{\leq m}}),$$

where the maps defining the pullback are the same as those in Construction 2.6 and the face maps and degeneracy maps are induced by the face maps and degeneracy maps of Δ^k.

Therefore, we have $\mathcal{M}od_{A_{\leq 1}} \simeq \Theta^1_0$-Sect and we treat Θ^1_0-Sect as the model of $\mathcal{M}od_{A_{\leq 1}}$. Then we apply:

$$\begin{aligned}\Theta^1_1\text{-Sect} &\simeq \mathcal{M}od_{A_{\leq 2}},\\ \Theta^2_1\text{-Sect} &\simeq \mathcal{M}od_{A_{\leq 3}},\\ \Theta^3_1\text{-Sect} &\simeq \mathcal{M}od_{A_{\leq 4}}\end{aligned}$$

to obtain the models for $\mathcal{M}od_{A_{\leq 2}}$, $\mathcal{M}od_{A_{\leq 3}}$, and $\mathcal{M}od_{A_{\leq 4}}$. Inductively, we obtain the model of $\mathcal{M}od_{A_{\leq n}}$ for all $n \geq 0$.

Remark 2.9. In the construction above, we define several different models for $\mathcal{M}od_{A_{\leq n}}$. We shall use $\Theta^{(b_1,b_2,\ldots)}_{(a_1,a_2,\ldots)}$-Sect to denote the category we obtain after inductively applying $\Theta^{b_1}_{a_1}, \Theta^{b_2}_{a_2}, \ldots$ to $\mathcal{M}od_{A_{\leq a_1}}$. Clearly, we should require $a_{i+1} = a_i + b_i$ and we have $\Theta^{(b_1,b_2,\ldots,b_k)}_{(a_1,a_2,\ldots,a_k)}$-Sect $\simeq \mathcal{M}od_{A_{\leq a_k+b_k}}$. In applications, we will use the models $\Theta^{(1,1,1,\ldots)}_{(0,1,2,\ldots)}$ associated with the tower:

$$\cdots \to A_{\leq 3} \to A_{\leq 2} \to A_{\leq 1} \to A_{\leq 0}$$

to get the full information and $\Theta^{(1,2,4,8,\ldots)}_{(0,1,3,7,\ldots)}$ associated with the tower:

$$\cdots \to A_{\leq 7} \to A_{\leq 3} \to A_{\leq 1} \to A_{\leq 0}$$

to simplify the computation.

To strengthen the above construction, we need to give the description of some additional structures in view of what should be enjoyed by $\mathcal{M}od_{A_{\leq n}}$. Now we further assume A to be $\mathbb{E}_\infty$ to ensure d_* is lax symmetric monoidal and we will define the symmetric monoidal structure of Θ-Sect and promote Theorem 2.7 to be a symmetric monoidal equivalence.

Before giving the construction, we need to refine the properties of Θ.

Proposition 2.10. $\Theta^\otimes := d_* d_0^*$ *is a lax symmetric monoidal functor from* $\mathcal{M}od^\otimes_{A_{\leq m}}$ *to* $\mathcal{M}od^\otimes_{A_{\leq m}}$. *There is a natural transformation* $\pi^\otimes$ *over* $\mathrm{N}(\mathrm{Fin}_*)$ *from* $\Theta^\otimes$ *to the identity functor.*

Proof. The first property is clear. For the second one, we observe that π is given by composing with the unit of a pair of (lax) symmetric monoidal functors, so π can be refined to be a natural transformation over $\mathrm{N}(\mathrm{Fin}_*)$. $\square$

Lemma 2.11. *Let $\mathcal{O}$ be a base ∞-category, and let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors between coCartesian fibrations over $\mathcal{O}$. Suppose G is an inner fibration preserving coCartesian arrows.*

For any object $C \in \mathcal{C}$ lying over $X \in \mathcal{O}$, and any morphism $u : X \to X'$, let $C \to u_!(C)$ be a coCartesian arrow lifting u. The image of $u_!$ under F admits a factorization $F(C) \to u_!F(C) \to F(u_!(C))$, where $F(C) \to u_!F(C)$ is a coCartesian arrow in $\mathcal{E}$ lifting u. (The morphism $u_!F(C) \to F(u_!(C))$ is well-defined up to homotopy and is called the Beck-Chevalley morphism*.)*

$(*)$ *Suppose that there are enough G-coCartesian arrows in $\mathcal{D}$ lifting $u_!F(C) \to F(u_!(C))$. Then the pullback $\mathcal{P} := \mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ is coCartesian over $\mathcal{O}$.*

Proof. For $(C, E, D) \in \mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ lying over $X \in \mathcal{O}$ and any morphism $u : X \to X'$, we define $C' := u_!(C)$, $E' := F(u_!(C))$, and D' such that $u_!(D) \to D'$ is coCartesian over the Beck-Chevalley morphism $u_!F(C) \to F(u_!(C))$. This gives an element of $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ lying over X' since $u_!F(C) \to F(u_!(C))$ is exactly $u_!(E) \to E'$. Now we obtain a morphism $(C, E, D) \to (C', E', D')$ lying over $X \to X'$. We check this morphism is coCartesian over u: Notice that for any $(U, W, V) \in \mathcal{P}$ lying over Y, we have pullback squares:

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(C', U) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(C, U) \arrow[d] \\
\mathrm{Map}_{\mathcal{O}}(X', Y) \arrow[r] & \mathrm{Map}_{\mathcal{O}}(X, Y),
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{D}}(D', V) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{D}}(u_!(D), V) \arrow[d] \\
\mathrm{Map}_{\mathcal{E}}(F(u_!(C)), W) \arrow[r] & \mathrm{Map}_{\mathcal{E}}(u_!F(C), W),
\end{tikzcd}$$

and similarly

$$\mathrm{Map}_{\mathcal{D}}(D, V) \simeq \mathrm{Map}_{\mathcal{D}}(u_!(D), V) \times_{\mathrm{Map}_{\mathcal{O}}(X,Y)} \mathrm{Map}_{\mathcal{O}}(X', Y),$$
$$\mathrm{Map}_{\mathcal{E}}(F(C), W) \simeq \mathrm{Map}_{\mathcal{E}}(u_!(F(C)), W) \times_{\mathrm{Map}_{\mathcal{O}}(X,Y)} \mathrm{Map}_{\mathcal{O}}(X', Y).$$

Then we have

$$\begin{aligned}
&\mathrm{Map}_{\mathcal{P}}((C', E', D'), (U, W, V)) \simeq \mathrm{Map}_{\mathcal{C}}(C', U) \times_{\mathrm{Map}_{\mathcal{E}}(E',W)} \mathrm{Map}_{\mathcal{D}}(D', V) \\
&\simeq \mathrm{Map}_{\mathcal{O}}(X', Y) \times_{\mathrm{Map}_{\mathcal{O}}(X,Y)} \mathrm{Map}_{\mathcal{C}}(C, U) \times_{\mathrm{Map}_{\mathcal{E}}(u_!(F(C)),W)} \mathrm{Map}_{\mathcal{D}}(u_!(D), V) \\
&\simeq \mathrm{Map}_{\mathcal{O}}(X', Y) \times_{\mathrm{Map}_{\mathcal{O}}(X,Y)} \mathrm{Map}_{\mathcal{P}}((C, E, D), (U, W, V)),
\end{aligned}$$

which completes the proof of the lemma. □

Construction 2.12. We promote Construction 2.6 to be diagrams of symmetric monoidal ∞-categories. Suppose we have a coCartesian fibration $\mathcal{M}od^{\otimes}_{A_{\le m}} \to \mathrm{N}(\mathrm{Fin}_*)$. Then we define $\Theta^{n-m}_m\text{-}\mathrm{Sect}^{\otimes}$ to be the homotopy pullback of the diagram:

$$\begin{tikzcd}
\Theta^{n-m}_m\text{-}\mathrm{Sect}^{\otimes} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^2, \mathcal{M}od_{A_{\le m}})^{\otimes} \arrow[d] \\
\mathcal{M}od^{\otimes}_{A_{\le m}} \arrow[r] & \mathrm{Fun}(\Lambda^2_2, \mathcal{M}od_{A_{\le m}})^{\otimes}.
\end{tikzcd}$$

Here the horizontal map sends M to

$$\begin{tikzcd} M \arrow[dr, "id"] & & \Theta M \arrow[dl, "\pi"'] \\ & M & \end{tikzcd}.$$

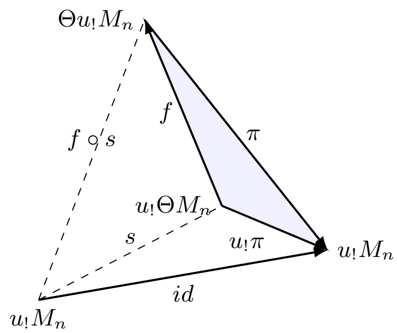
As shown in Proposition 2.10, $\pi : \Theta \to \mathrm{Id}$ is a natural transformation over $\mathrm{N}(\mathrm{Fin}_*)$, so the lower horizontal map is lax symmetric monoidal. The symmetric monoidal structures of $\mathrm{Fun}(\Delta^2, \mathcal{M}od_{A_{\leq m}})^{\otimes}$ and $\mathrm{Fun}(\Lambda^2_2, \mathcal{M}od_{A_{\leq m}})^{\otimes}$ are the standard ones by [Lur17, Remark 2.1.3.4], so the right vertical map is symmetric monoidal and is still an inner fibration.

Lemma 2.13. *The vertical functor*

$$p : \mathrm{Fun}(\Delta^2, \mathcal{M}od_{A_{\leq m}})^{\otimes} \to \mathrm{Fun}(\Lambda^2_2, \mathcal{M}od_{A_{\leq m}})^{\otimes}$$

satisfies the condition $(*)$ *in Lemma 2.11.*

Proof. Given a morphism $u : \langle n\rangle \to \langle m\rangle$ in $\mathrm{N}(\mathrm{Fin}_*)$, suppose that $u' : M_n \to u_!(M_n) = M_m$ is the coCartesian map in $\mathcal{M}od_{A_{\leq m}}$ lifting u. Then the corresponding Beck-Chevalley morphism is given as part of a tetrahedron:



where f is the Beck-Chevalley morphism of functor Θ, since Θ is lax symmetric monoidal and $u_!\pi = \pi \circ f$ since π is a natural transformation over $\mathrm{N}(\mathrm{Fin}_*)$. After filling the front triangle with $(u_!M_n, s, h)$, where $s : u_!M_n \to u_!\Theta M_n$, we define the left triangle to be the composition of f and s. Now we can fill the whole tetrahedron by using the lifting property of quasi-categories. This is exactly the p-coCartesian lifting in $\mathrm{Fun}(\Delta^2, \mathcal{M}od_{A_{\leq m}})^{\otimes}$ of the Beck-Chevalley morphism in $\mathrm{Fun}(\Lambda^2_2, \mathcal{M}od_{A_{\leq m}})^{\otimes}$. □

Proposition 2.14. Θ^{n-m}_m*-*$\mathrm{Sect}^{\otimes}$ *is coCartesian over* $\mathrm{N}(\mathrm{Fin}_*)$*, making* Θ^{n-m}_m*-*$\mathrm{Sect}^{\otimes}$ *a symmetric monoidal stable* ∞*-category. Furthermore,*

$$F : \Theta^{n-m}_m\text{-}\mathrm{Sect}^{\otimes} \to \mathcal{M}od^{\otimes}_{A_{\leq m}}$$

is symmetric monoidal.

Proof. This follows from combining Lemma 2.11 and Lemma 2.13. □

Remark 2.15. Informally speaking, the tensor product of

$$\begin{tikzcd} M \arrow[rr, "s"] \arrow[dr, "id"'] & & \Theta M \arrow[dl, "\pi"] \\ & M & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} N \arrow[rr, "t"] \arrow[dr, "id"'] & & \Theta N \arrow[dl, "\pi"] \\ & N & \end{tikzcd}$$

is

$$
\begin{tikzcd}
M \otimes_{A_{\le m}} N \arrow[r, "s\otimes t"] \arrow[dr, "id"'] & \Theta M \otimes_{A_{\le m}} \Theta N \arrow[r, "BC\text{-map}"] & \Theta(M \otimes_{A_{\le m}} N) \arrow[dl, "\pi"] \\
& M \otimes_{A_{\le m}} N &
\end{tikzcd}
$$

Now we can promote the equivalence in Theorem 2.7 to a symmetric monoidal equivalence.

Theorem 2.16. *There is a symmetric monoidal equivalence*

$$\Theta_m^{n-m}\text{-Sect}^{\otimes} \simeq \mathcal{M}od^{\otimes}_{A_{\le n}}.$$

Moreover, under this equivalence, the extension of scalars $\mathcal{M}od_{A_{\le n}} \to \mathcal{M}od_{A_{\le m}}$ corresponds to the forgetful functor $F : \Theta_m^{n-m}\text{-Sect}^{\otimes} \to \mathcal{M}od^{\otimes}_{A_{\le m}}$ sending (M, s, h) to M.

Proof. We only need to promote the equivalence in Theorem 2.7 to be symmetric monoidal. First, as in Proposition 2.1, we have a symmetric monoidal equivalence $\mathcal{M}od^{\otimes}_{A_{\le n}} \simeq \mathcal{M}od^{\otimes}_{A_{\le m}} \times_{\mathcal{M}od^{\otimes}_{A_{\le m}\oplus\Sigma F}} \mathcal{M}od^{\otimes}_{A_{\le m}}$, so we can still identify $\mathcal{M}od_{A_{\le n}}$ with the ∞-category of triples (M, N, α), where $\alpha : d^*N \to d_0^*M$ is an equivalence.

Similar to Construction 2.12 and following the notations in Theorem 2.7, we can define $\mathcal{D}^{\otimes}$ and $\mathcal{E}^{\otimes}$ whose product structure can be informally given by

$$
\begin{aligned}
&(M_1, N_1, \alpha_1') \otimes (M_2, N_2, \alpha_2') = (M_1 \otimes_{A_{\le m}} M_2, N_1 \otimes_{A_{\le m}} N_2, BC_\Theta \circ \alpha_1' \otimes \alpha_2'),\\
&(M_1, N_1, \alpha_1', \beta_1, h_1) \otimes (M_2, N_2, \alpha_2', \beta_2, h_2)\\
=&(M_1 \otimes_{A_{\le m}} M_2, N_1 \otimes_{A_{\le m}} N_2, BC_\Theta \circ \alpha_1' \otimes \alpha_2', \beta_1 \otimes \beta_2, BC_\Pi \circ h_1 \otimes h_2).
\end{aligned}
$$

Here BC_Θ is the Beck-Chevalley morphism of the lax symmetric monoidal functor Θ and BC_Π is the Beck-Chevalley morphism of the functor $\Pi : \mathcal{M}od_{A_{\le m}} \to \mathrm{Fun}(\Delta^1, \mathcal{M}od_{A_{\le m}})$ sending M to $\pi_M : \Theta M \to M$.

Now, the functors $f : \mathcal{E} \to \mathcal{D}$ and $g : \mathcal{E} \to \Theta_m^{n-m}\text{-Sect}$ can be modified to $f^{\otimes} : \mathcal{E}^{\otimes} \to \mathcal{D}^{\otimes}$ and $g^{\otimes} : \mathcal{E}^{\otimes} \to \Theta_m^{n-m}\text{-Sect}^{\otimes}$. One can check that both of them preserve the coCartesian morphisms, so the equivalence of ∞-categories is now an equivalence of coCartesian fibrations over $\mathrm{N}(\mathrm{Fin}_*)$. □

To complete the induction, the only remaining part is the symmetric monoidal ∞-category $\mathcal{M}od_{A_{\le 0}}$. There is a condition to ensure that $\mathcal{M}od_{A_{\le 0}}$ is the derived category:

Proposition 2.17. *When $\mathcal{M}od_{A_{\le 0}}$ is generated by discrete objects, the left modules over the zero-truncated ring $A_{\le 0}$ are given by the derived ∞-category over $\pi_0(A)$ in the 1-category $\mathbb{M}od_{\pi_0(A)}(\mathcal{C}^{\heartsuit})$, which is*

$$\mathcal{M}od_{A_{\le 0}} \simeq \mathcal{D}^-(\mathbb{M}od_{\pi_0(A)}(\mathcal{C}^{\heartsuit})).$$

The underlying category of the following has a concrete model:

$$\mathcal{D}^-(\mathbb{M}od_{\pi_0(A)}(\mathcal{C}^{\heartsuit})) \simeq \mathrm{N}_{dg}(\mathrm{Ch}^-(\mathbb{M}od_{\pi_0(A)}(\mathcal{C}^{\heartsuit})^{\mathrm{proj}})).$$

Proof. This is a corollary of [Lur18, C5.4.11]. □

To compute the homotopy groups, we need to further define the t-structure of Θ-Sect.

Construction 2.18. We begin with the t-structure of $\mathcal{D}^-(\mathbb{M}od_{\pi_0(A)}(\mathcal{C}^\heartsuit))$, which is clearly given by the homology of chain complexes.

Suppose that the t-structure of $\mathcal{M}od_{A_{\leq m}}$ is given by

$$((\mathcal{M}od_{A_{\leq m}})_{\geq 0}, (\mathcal{M}od_{A_{\leq m}})_{\leq 0}).$$

Then we define

$$(\Theta_m^{n-m}\text{-Sect})_{\geq 0} := F^{-1}((\mathcal{M}od_{A_{\leq m}})_{\geq 0}),$$

where $F : \Theta_m^{n-m}\text{-Sect} \to \mathcal{M}od_{A_{\leq m}}$ is the functor defined in Proposition 2.14.

Proposition 2.19. *When $\pi_0(A) \neq 0$, under the equivalence $\mathcal{M}od_{A_{\leq n}} \simeq \Theta\text{-Sect}_{A_{\leq m}}$, there is an equivalence of subcategories*

$$(\mathcal{M}od_{A_{\leq n}})_{\geq 0} \simeq (\Theta_m^{n-m}\text{-Sect})_{\geq 0}.$$

Proof. For simplicity of notation, we write Ind for the extension of scalars. It suffices to show that $\mathrm{Ind}^{-1}((\mathcal{M}od_{A_{\leq m}})_{\geq 0}) = (\mathcal{M}od_{A_{\leq n}})_{\geq 0}$. Given $M \in \mathcal{M}od_{A_{\leq n}}$, we have a spectral sequence

$$E_2^{p,q} = \mathrm{Tor}_p^{\pi_*(A_{\leq n})}(\pi_*(A_{\leq n-1}), \pi_*(M))_q \Longrightarrow \pi_{p+q}(\mathrm{Ind}(M)).$$

Then clearly the connective modules are sent to connective modules. If the lowest degree of the nontrivial homotopy groups of M is negative, denoted by k, then $E_2^{0,k} = (\pi_*(A_{\leq m}) \otimes_{\pi_*(A_{\leq n})} \pi_*(M))_k$ is nontrivial since $\pi_0(A)$ is assumed to be nontrivial. Therefore, $\mathrm{Ind}(M)$ is not connective and we complete the proof. $\square$

Now we can identify the ∞-category $\Theta_{(a_1,a_2,\dots,a_k)}^{(b_1,b_2,\dots,b_k)}$-Sect defined by induction with the original ∞-category $\mathcal{M}od_{A_{\leq n}}$. This gives us a concrete model to understand each simplex in $\mathcal{M}od_{A_{\leq n}}$ discretely, allowing us to be precise about the elementwise computation, which will be shown in the following sections.

3. Computational methods for periodic objects

This section will be devoted to the explicit computational method. Though the construction in Section 2 works for general stable ∞-categories, it is not often the case that the t-structure truncation and the extension of scalars along the Postnikov tower coincide, leading to computational difficulty. Therefore, we further restrict A to be a shift algebra. We still take $\mathcal{C}$ as in Notations 1.1.

Definition 3.1. A *grading* on $\mathcal{C}$ is a choice of distinguished autoequivalence which we denote by $c \to c[1]$, together with a natural equivalence $c[1] \otimes d \simeq (c \otimes d)[1]$.

Definition 3.2. A *shift algebra* is an associative algebra $A \in \mathrm{Alg}(\mathcal{C})$ equipped with a map $\tau : \Sigma A[-1] \to A$ of right A-modules which induces an isomorphism $\pi_*(A) \simeq \pi_0(A)[\tau]$, where the latter is the graded algebra in $\mathcal{C}^\heartsuit$ given by $(\pi_0(A)[\tau])_k \simeq \pi_0(A)[-k]$.

Definition 3.3. For a fixed shift algebra A and $1 \leq n \leq \infty$, we say M is a *periodic n-module* if $M \in \mathcal{M}od_{A_{\leq n}}$ satisfies that $\pi_0(A) \otimes_{A_{\leq n}} M$ is discrete. When $n = \infty$, we also refer to it as a *periodic module*.

Now we fix a shift algebra A and the modules mentioned below are all modules over proper truncations of A. First we show a special property enjoyed by periodic modules, which says the left tensoring with truncation of A is t-truncation itself, making the computation easier.

Proposition 3.4. *Given a periodic n-module M, there is a map induced by extension of scalars:*

$$M \simeq A_{\leq n} \otimes_{A_{\leq n}} M \to A_{\leq m} \otimes_{A_{\leq n}} M \simeq \tau_{\leq m} M.$$

Then this map coincides with the t-structure truncation of M itself.

Proof. The cofiber sequence

$$A_{\leq n} \otimes_{A_{\leq n}} M \to A_{\leq m} \otimes_{A_{\leq n}} M \to \Sigma\tau_{[m+1,n]} A \otimes_{A_{\leq n}} M \simeq \Sigma\tau_{[m+1,n]} M$$

proves the claim, where the last equivalence uses the property of periodic n-modules. □

Then we want to be more concrete about the functor Θ. Instead of computing from the definition, we associate it with an intrinsic map of modules.

Definition 3.5. Given a periodic m-module M, we can compute the cofiber of π:

$$\Theta_m^{n-m} M \xrightarrow{\pi} M \xrightarrow{t} \Sigma^{m+3}\tau_{[0,n-m-1]} M[-(m+1)].$$

We will refer to the map t as the *twisting map* of M.

Remark 3.6. The twisting map is only defined up to an automorphism of M. However, we do not need to worry about the choice since any choice works for the computation of ΘM.

Remark 3.7. In [PV22], the authors showed the equivalence of Θ_m^{n-m}-Sect and $\mathcal{M}od_{A_{\leq n}}$ to show that a periodic m-module M can be lifted to a periodic n-module if and only if the twisting map of M is nullhomotopic, since the existence of the lifting is equivalent to the existence of a section $M \to \Theta_m^{n-m} M$. Here we emphasize the nullhomotopic map t since this enables us to compute the section $M \to \Theta_m^{n-m} M$ and this nullhomotopy prevents $\Theta_m^{n-m} M$ from being $M \oplus \Sigma\tau_{[m+1,n]} M$.

Proposition 3.8. *Given a periodic n-module M, put $M' := A_{\leq m} \otimes_{A_{\leq n}} M$. Then there is an equivalence $\Theta M' \simeq M' \oplus \Sigma\tau_{[m+1,n]} M$ as left $A_{\leq n}$-modules. Suppose M corresponds to the triangle*

$$\begin{tikzcd} M' \arrow[rr, "s"] \arrow[dr, "id"'] & & \Theta M' \arrow[dl, "\pi"] \\ & M' & \end{tikzcd}.$$

Then the composition $M' \xrightarrow{s} \Theta M' \to \Sigma\tau_{[m+1,n]} M$ coincides with the map in the fiber-cofiber sequence

$$M \to M' \to \Sigma\tau_{[m+1,n]} M,$$

where the first map is t-structure truncation. The map from M' to $\tau_{[m+1,n]} M$ will be called the differential map.

Proof. Consider the square as in Proposition 2.1:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & A_{\leq m} \otimes_{A_{\leq n}} M \arrow[d] \\ A_{\leq m} \otimes_{A_{\leq n}} M \arrow[r] & (A_{\leq m} \oplus \Sigma F) \otimes_{A_{\leq n}} M \end{tikzcd}$$

which is a pushout-pullback square in $\mathcal{M}od_{A_{\leq n}}$. Unwinding the definition, $\Theta M'$ is exactly $(A_{\leq m} \oplus \Sigma F) \otimes_{A_{\leq n}} M$ when treated as a left $A_{\leq n}$-module. Since $A_{\leq m} \oplus \Sigma F$

is the trivial square-zero extension, this is the equivalence we want. For the second part, it suffices to pass to the underlying objects and notice that the lower horizontal map $M' \to M' \oplus \Sigma\tau_{[m+1,n]}M$ is $(id, 0)$. $\square$

Now we can do elementwise computation and try to write down the representation of $M \in \mathcal{M}od_{A_{\leq n}}$. Let us start with the simplest case when $n = 1$.

Example 3.9. Given an $A_{\leq 0}$-module M, regarded as a chain complex of projective objects

$$\cdots \longrightarrow M_2 \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} M_0 \xrightarrow{\partial_0} M_{-1} \longrightarrow \cdots$$

Then ΘM can be written as the fiber of $M \xrightarrow{t^{(2)}} \Sigma^3 M[-1]$:

$$\begin{tikzcd}
\cdots \arrow[r] & M_4 \arrow[r, "\partial_4"] \arrow[drr, "t_4^{(2)}"] & & M_3 \arrow[r, "\partial_3"] \arrow[drr, "t_3^{(2)}"] & & M_2 \arrow[r, "\partial_2"] \arrow[drr, "t_2^{(2)}"] & & M_1 \arrow[r] & \cdots \\
 & \oplus & & \oplus & & \oplus & & \oplus & \\
\cdots \arrow[r] & M_2[-1] \arrow[rr, "\partial_2[-1]"'] & & M_1[-1] \arrow[rr, "\partial_1[-1]"'] & & M_0[-1] \arrow[rr, "\partial_0[-1]"'] & & M_{-1}[-1] \arrow[r] & \cdots
\end{tikzcd}$$

where $t_n^{(2)}$ are the twisting maps.

Then any $A_{\leq 1}$-module $\bar{M}$ satisfying $A_{\leq 0} \otimes_{A_{\leq 1}} \bar{M} = M$, regarded as an object in Θ_0^1-Sect, can be written as

$$\begin{tikzcd}
M_{k+2} \arrow[d] & & M_{k+2} \oplus M_k[-1] \arrow[d] \\
M_{k+1} \arrow[d] \arrow[rr, "{(id,d^{(2)}_{k+1})}"] \arrow[dr, "id"] & & M_{k+1} \oplus M_{k-1}[-1] \arrow[d] \arrow[dl, "{(id,0)}"'] \\
M_k \arrow[d] \arrow[rr, dashed] \arrow[dr, "id"] & M_{k+1} \arrow[d] & M_k \oplus M_{k-2}[-1] \arrow[d] \arrow[dl, "{(id,0)}"'] \\
M_{k-1} \arrow[d] \arrow[rr, dashed] \arrow[dr, "id"] & M_k \arrow[d] & M_{k-1} \oplus M_{k-3}[-1] \arrow[d] \arrow[dl, "{(id,0)}"'] \\
M_{k-2} & M_{k-1} \arrow[d] & M_{k-2} \oplus M_{k-4}[-1] \\
 & M_{k-2} &
\end{tikzcd}$$

where the map from $M_{k+2} \oplus M_k[-1]$ to $M_{k+1} \oplus M_{k-1}[-1]$ is exactly the one in the illustration of ΘM. In particular, we have

$$d_k^{(2)} \circ \partial_{k+1} = \partial_{k-1}[-1] \circ d_{k+1}^{(2)} + t_{k+1}^{(2)} \circ id.$$

Remark 3.10. The section map is named $d^{(2)}$ because it is exactly the d_2-differential of the Adams spectral sequence if we take $\mathcal{C}$ to be the category of synthetic spectra, which we will discuss in Section 5. In applications, we usually consider periodic 1-modules. Then $M_{-k} \simeq 0$ for all $k \geq 1$, so all twisting maps and the first nontrivial $d^{(2)}$ from M_2 to $M_0[-1]$ determine all the $d^{(2)}$ maps.

For larger n, we have similar results:

Theorem 3.11. *Given a periodic n-module M, under the equivalence*

$$\mathcal{M}od_{A_{\leq n}} \simeq \Theta_{(0,1,\ldots,n-1)}^{(1,1,\ldots,1)}\text{-Sect},$$

we can inductively write down the representation of M. First we determine $\pi_0(M)$ and compute its projective resolution as a $\pi_0(A)$-module, then we determine the first twisting map

$$t^{(2)} : \pi_0(M) \to \Sigma^3\pi_0(M)[-1]$$

and compute the section

$$s^{(2)} : \pi_0(M) \to \mathrm{Fib}(t^{(2)})$$

so that we have the description of $A_{\leq 1} \otimes_{A_{\leq n}} M$:

$$\begin{tikzcd} A_{\leq 0} \otimes_{A_{\leq n}} M \arrow[rr, "s^{(2)}"] \arrow[dr, "id"'] & & \mathrm{Fib}(t^{(2)}) \arrow[dl, "\pi"] \\ & A_{\leq 0} \otimes_{A_{\leq n}} M & \end{tikzcd}.$$

With this description, we can write down the second twisting map as

$$t^{(3)} : A_{\leq 1} \otimes_{A_{\leq n}} M \to \Sigma^4\pi_0(M)[-2],$$

regarded as a map in Θ^1_0-Sect, and organize $A_{\leq 2} \otimes_{A_{\leq n}} M$ as

$$\begin{tikzcd} A_{\leq 1} \otimes_{A_{\leq n}} M \arrow[rr, "s^{(3)}"] \arrow[dr, "id"'] & & \mathrm{Fib}(t^{(3)}) \arrow[dl, "\pi"] \\ & A_{\leq 1} \otimes_{A_{\leq n}} M & \end{tikzcd}.$$

Inductively, we can write down the representation of M as an object of

$$\Theta^{(1,1,\dots,1)}_{(0,1,\dots,n-1)}\text{-Sect} \subset \mathrm{Fun}((\Delta^2)^n, \mathrm{N}_{dg}(\mathrm{Ch}^-(\mathcal{M}od^{\mathrm{proj}}_{\pi_0(A)}(\mathcal{C}^\heartsuit)))).$$

To determine $\mathrm{Fib}(t^{(k)})$, we should first write down the representation of $\Sigma^{k+1}\pi_0[-(k-1)]$ as an $A_{\leq k-2}$-module. This requires the computation of pullbacks along the Postnikov tower, namely the tower of $\mathbb{E}_k$-algebras:

$$\cdots \to A_{\leq 3} \to A_{\leq 2} \to A_{\leq 1} \to A_{\leq 0},$$

where each map is a square-zero extension. A slightly more general result is given as follows:

Proposition 3.12. *Given a periodic n-module M, then $\pi_0(M) \simeq A_{\leq 0} \otimes_{A_{\leq n}} M$ can be computed as the colimit of the chain of modules*

$$\cdots \xrightarrow{\tau^n} \Sigma^{n+2}M[-(n+2)] \xrightarrow{\tau} \Sigma^{n+1}M[-(n+1)] \xrightarrow{\tau^n} \Sigma M[-1] \xrightarrow{\tau} M,$$

where the $(2m+1)$-th term X_{2m+1} is $\Sigma^{m(n+1)+1}M[-m(n+1)-1]$ and the $(2m)$-th term X_{2m} is $\Sigma^{m(n+1)}M[-m(n+1)]$.

Proof. First one notices that the truncation map $\tau_{\leq 0} : \mathcal{M}od_{A_{\leq n}} \to \mathcal{M}od_{A_{\leq n}}$ gives the map from M to $\pi_0(M)$, inducing the map from the colimit of this chain to $\pi_0(M)$. Since the t-structure is separated, we only need to check the homotopy groups.

The chain is designed to satisfy that $\operatorname{colim} \pi_k(X_m) \simeq 0$ unless $k = 0$ and $\operatorname{colim} \pi_0(X_m) \simeq \pi_0(M)$. Since the t-structure is compatible with filtered colimits, the homotopy group of the colimit is the one we want. $\square$

Remark 3.13. The representation of $A_{\leq m} \otimes_{A_{\leq n}} M$ can be computed similarly by canceling the t-structure homotopy groups with τ.

In algebraic topology, we are always concerned about the mapping space between two objects. We will illustrate how to understand the mapping space for two periodic modules.

Proposition 3.14. *For any two periodic modules M and N, there is a sequence of mapping spaces*

$$\mathrm{Map}_{A_{\le 0}}(M_{\le 0}, N_{\le 0}) \leftarrow \mathrm{Map}_{A_{\le 1}}(M_{\le 1}, N_{\le 1}) \leftarrow \cdots \leftarrow \mathrm{Map}_{A_{\le k}}(M_{\le k}, N_{\le k}) \leftarrow \cdots$$

such that $\mathrm{Map}_{A_{\le m}}(M_{\le n-1}, N_{\le n-1}) \leftarrow \mathrm{Map}_{A_{\le n}}(M_{\le n}, N_{\le n})$ *has fiber*

$$\mathrm{Map}_{A_{\le 0}}(\pi_0(M), \Sigma^n \pi_0(N)[-n]).$$

Furthermore, if the t-structure is complete, the limit of this sequence is exactly $\mathrm{Map}_A(M, N)$.

Proof. Actually, there are equivalences

$$\mathrm{Map}_A(M, N_{\le n}) \simeq \mathrm{Map}_A(M_{\le n}, N_{\le n}) \simeq \mathrm{Map}_{A_{\le n}}(A_{\le n} \otimes_A M_{\le n}, N_{\le n}).$$

Tensoring with $M_{\le n}$ as an A-$A_{\le n}$-bimodule, the fiber-cofiber sequence $A_{\ge n+1} \to A \to A_{\le n}$ induces a fiber-cofiber sequence

$$A_{\ge n+1} \otimes_A M_{\le n} \to M_{\le n} \to A_{\le n} \otimes_A M_{\le n}.$$

Furthermore,

$$\begin{aligned} &\mathrm{Map}_{A_{\le n}}(A_{\le n} \otimes_A M_{\le n}, N_{\le n}) \to \mathrm{Map}_{A_{\le n}}(M_{\le n}, N_{\le n}) \\ \to &\mathrm{Map}_{A_{\le n}}(A_{\ge n+1} \otimes_A M_{\le n}, N_{\le n}) \simeq 0 \end{aligned}$$

is a fiber-cofiber sequence. Hence, $\mathrm{Map}_{A_{\le n}}(M_{\le n}, N_{\le n}) \simeq \mathrm{Map}_A(M, N_{\le n})$. Now the sequence can be interpreted as

$$\mathrm{Map}_A(M, N_{\le 0}) \leftarrow \mathrm{Map}_A(M, N_{\le 1}) \leftarrow \cdots \leftarrow \mathrm{Map}_A(M, N_{\le k}) \leftarrow \cdots$$

where the map $\mathrm{Map}_A(M, N_{\le n-1}) \leftarrow \mathrm{Map}_A(M, N_{\le n})$ has fiber

$$\mathrm{Map}_A(M, \Sigma^n \pi_n(N)) \simeq \mathrm{Map}_{A_{\le 0}}(\pi_0(M), \Sigma^n \pi_0(N)[-n]).$$

□

Remark 3.15. The above proposition can be promoted to a spectral sequence; see [PV22, Corollary 4.13].

For shift algebras, we have a simpler way to determine the category over its zero-truncation.

Proposition 3.16. [PV22, Theorem 3.11] *Let A be a shift algebra. Suppose $\mathcal{M}od_A$ is generated under colimits by periodic modules. Then the left modules over the zero-truncated ring $A_{\le 0}$ are given by the derived ∞-category over $\pi_0(A)$ in the 1-category $\mathbb{M}od_{\pi_0(A)}(\mathcal{C}^{\heartsuit})$, which is*

$$\mathcal{M}od_{A_{\le 0}} \simeq \mathcal{D}^-(\mathbb{M}od_{\pi_0(A)}(\mathcal{C}^{\heartsuit})).$$

Therefore, the mapping space $\mathrm{Map}_{A_{\le 0}}(\pi_0(M), \Sigma^n \pi_0(N)[-n])$ can be computed as the derived mapping space in the derived category $\mathcal{D}^-(\mathbb{M}od_{\pi_0(A)}(\mathcal{C}^{\heartsuit}))$.

4. Application in the category of graded spectra

In this section, we try to use our framework to study spectra. As mentioned before, we prefer to deal with a shift algebra, so we will not work in the category $\mathcal{S}p = \mathcal{M}od_{\mathbb{S}}$ but work in the category of graded spectra $\mathcal{S}p^{gr}$. We will first embed $\mathcal{S}p$ into $\mathcal{S}p^{gr}$ and identify them with their image as periodic modules over the image of $\mathbb{S}$. However, the embedding is merely an $\mathbb{E}_1$-monoidal functor, which prevents us from comparing the symmetric monoidal structure.

Definition 4.1. We regard $\mathbb{Z}$ as a discrete symmetric monoidal category. Then the *category of graded spectra* $\mathcal{S}p^{gr}$ is defined to be $\mathrm{Fun}(\mathbb{Z}, \mathcal{S}p)$.

Proposition 4.2. *We can endow* $\mathrm{Fun}(\mathbb{Z}, \mathcal{S}p)$ *with a symmetric monoidal structure by Day convolution.*

Proof. This is the combination of [Gla16, Definition 2.9] and [Gla16, Proposition 2.11]. Furthermore, as mentioned in [Gla16, Proposition 2.12], commutative monoids in $\mathrm{Fun}(\mathbb{Z}, \mathcal{S}p)^{\otimes}$ can be identified with lax monoidal functors from $\mathbb{Z}^{\otimes}$ to $\mathcal{S}p^{\otimes}$. □

Definition 4.3. We define the t-structure of $\mathcal{S}p^{gr}$ to be given by the homotopy group in each degree. The grading $[k]$ is given by $\mathbb{S}^{0,k} \otimes (-)$.

One can check that $\mathcal{S}p^{gr}$ satisfies the conditions in Notations 1.1.

Notations 4.4. We will use $\{X_n\}_n$ to denote the graded spectrum whose n-th degree is a spectrum X_n. We use $\mathbb{S}^{a,b}$ to denote the graded spectrum with $\mathbb{S}^a$ in degree b and 0 in other degrees.

Lemma 4.5. [CPH22, Lemma 4.4] *There is a cocontinuous* $\mathbb{E}_1$*-monoidal functor* $\Phi : \mathcal{S}p^{gr} \to \mathcal{S}p^{gr}$*, sending* $\{X_k\}_k$ *to* $\{\Sigma^{-k} X_k\}_k$*.*

Definition 4.6. We define lax $\mathbb{E}_1$-monoidal functors $\Phi^{(n)} : \mathcal{S}p \to \mathcal{S}p^{gr}$ for all $0 \le n \le \infty$, sending X to $\{\tau_{[0,n-1]}\Sigma^{-k} X\}_k$.

Now we take A to be $\mathbb{S}^{(\infty)} := \Phi^{(\infty)}(\mathbb{S})$, which is a shift algebra with τ given by t-structure truncation in each degree:

$$\tau : \Sigma\mathbb{S}^{(\infty)}[-1] = \{\Sigma^{-k}\tau_{\ge k+1}\mathbb{S}\}_k \to \{\Sigma^{-k}\tau_{\ge k}\mathbb{S}\}_k = \mathbb{S}^{(\infty)}.$$

Recall that a spectrum is uniquely determined by its homotopy groups and the k-invariants between the homotopy groups. Classically, given a spectrum X, the n-th k-invariant is defined as the connecting map in the fiber-cofiber sequence

$$\Sigma^{n+1}\pi_{n+1}(X) \to \tau_{[n,n+1]}X \to \Sigma^n\pi_n(X) \xrightarrow{k} \Sigma^{n+2}\pi_{n+1}(X).$$

However, there is no way to be more specific about this map since this is not an $H\mathbb{Z}$-module map but a $\mathbb{S}_{\le 1}$-module homomorphism. Working in the graded sense, we define the graded k-invariant as follows:

Definition 4.7. Given a spectrum X, we define the *graded k-invariant* of X to be the connecting map in the fiber-cofiber sequence

$$\Sigma\Phi^{(0)}(X)[-1] \to \Phi^{(1)}(X) \to \Phi^{(0)}(X) \xrightarrow{k} \Sigma^2\Phi^{(0)}(X)[-1].$$

Now the graded k-invariant is a $\mathbb{S}^{(1)}$-module homomorphism, so that we can write it concretely in $\mathcal{M}od_{\mathbb{S}^{(1)}} \simeq \Theta_0^1\text{-Sect}$. However, as in Proposition 3.8, this map should have been seen when we try to determine $\Phi^{(1)}$ before writing down the map $\Phi^{(0)}(X) \to \Sigma^2\Phi^{(0)}[-1]$. First we identify the category $\mathcal{M}od_{\mathbb{S}^{(0)}}$.

Lemma 4.8. *There is an equivalence of ∞-categories:*

$$\mathcal{M}od_{\mathbb{S}(0)} \simeq \mathcal{D}^-(\mathbb{M}od^{gr}_{\pi_*(\mathbb{S})}).$$

Proof. This is a corollary of Proposition 3.16. $\square$

Then we can invoke Example 3.9 to be specific about the graded k-invariant:

Theorem 4.9. *Given a spectrum X, we can write the projective resolution of $\pi_*(X)$ as a graded $\pi_*(\mathbb{S})$-module as*

$$\cdots \longrightarrow X_2 \xrightarrow{\partial_2} X_1 \xrightarrow{\partial_1} X_0 \longrightarrow 0 \longrightarrow \cdots$$

Then the graded k-invariant is exactly $d^{(2)}$ in the illustration:

$$\begin{tikzcd}
X_{k+2} \arrow[r, "{(id,d^{(2)}_{k+2})}"] \arrow[d] & X_{k+2} \oplus X_k[-1] \arrow[d] \\
X_{k+1} \arrow[r, "{(id,d^{(2)}_{k+1})}"] \arrow[d] & X_{k+1} \oplus X_{k-1}[-1] \arrow[d] \\
X_k \arrow[r, "{(id,d^{(2)}_{k})}"] \arrow[d] & X_k \oplus X_{k-2}[-1] \arrow[d] \\
X_{k-1} \arrow[r, "{(id,d^{(2)}_{k-1})}"] & X_{k-1} \oplus X_{k-3}[-1].
\end{tikzcd}$$

Remark 4.10. The twisting maps $t^{(2)}$ prevent the maps $d^{(2)}$ from being a $\pi_*(\mathbb{S})$-module homomorphism and almost determine $d^{(2)}$. However, little is known about the twisting maps in the category of spectra, so we can only give a formal result here and wait for further study.

Fortunately, we do have some known computations if we replace $\mathbb{S}$ by the $\mathbb{E}_1$-ring $\mathrm{End}(\mathbb{F}_p)$, and the result is known as the *secondary Steenrod algebra*. In [Bau06], Baues introduced and computed the structure of the secondary Steenrod algebra. We will not dive into the details of that book but only extract from [CPH22] the way to compute the twisting map, which will be discussed in Section 6.

5. Application in the category of synthetic spectra

In this section, we take $\mathcal{C} = \widehat{\mathcal{S}yn}_{\mathbb{F}_p}$ and $A = \nu\mathbb{S} = \mathbb{S}^{0,0}$, which is the synthetic analogue of $\mathbb{S}$ in the category $\mathcal{S}yn$. See [Pst18] for the definition and properties of synthetic spectra. The main feature of the synthetic category we use here is that its t-structure truncation gives the information of (co)homology and the organization of the t-structure homotopy groups can be viewed as the organization of different pages of the Adams spectral sequence. For the convenience of the reader, we list the basic properties we use:

Proposition 5.1. *(1) $\widehat{\mathcal{S}yn}_{\mathbb{F}_p}$ admits a natural separated t-structure whose heart is isomorphic to the abelian category $\mathcal{C}omod_{\mathcal{A}_p}$.*

(2) $\widehat{\mathcal{S}yn}_{\mathbb{F}_p}$ is a presentable stable symmetric monoidal ∞-category with the unit $\mathbb{S}^{0,0} \in (\widehat{\mathcal{S}yn}_{\mathbb{F}_p})_{\geq 0}$.

(3) *There is a canonical functor $\nu : \mathcal{S}p \hookrightarrow \widehat{\mathcal{S}yn}_{\mathbb{F}_p}$ whose images are periodic $\nu(\mathbb{S})$-modules.*
(4) *$\widehat{\mathcal{S}yn}_{\mathbb{F}_p}$ is generated under colimits by the periodic $\nu(\mathbb{S})$-modules.*
(5) *There is a canonical map $\tau : \mathbb{S}^{0,-1} \to \mathbb{S}^{1,1}$, which is the 1-connective covering. We will denote the cofiber of this map by $C\tau$.*

By [Lur17, Theorem 7.4.1.26], we have a tower of square-zero extensions of $\mathbb{E}_\infty$-algebras:

$$\nu(\mathbb{S}) \to \cdots \to C\tau^n \to \cdots \to C\tau^2 \to C\tau,$$

inducing the symmetric monoidal equivalences

$$\mathcal{M}od_{C\tau^n} \simeq \Theta^{(1,1,\ldots,1)}_{(0,1,\ldots,n-1)}\text{-Sect}.$$

As a corollary of (1) and (3), we have the following identification:

$$\mathcal{M}od^{\otimes}_{C\tau} \simeq \mathcal{D}^-(\mathcal{C}omod_{\mathcal{A}_p})^{\otimes},$$

and we shall not distinguish these two categories in this chapter.

Corollary 5.2. *Given any spectrum X, $X_n := C\tau^n \otimes \nu(X)$ is a periodic n-module.*

Warning 5.3. When we try to apply Construction 2.8 to the category of synthetic spectra, there is a small subtlety. Since $\mathcal{C}omod_{\mathcal{A}_p}$ does not admit enough projective objects, we do not have $\mathcal{D}^-(\mathcal{C}omod_{\mathcal{A}_p}) \simeq \mathrm{N}_{dg}(\mathrm{Ch}^-(\mathcal{C}omod^{\mathrm{proj}}_{\mathcal{A}_p}))$ but the equivalence $\mathcal{D}^+(\mathcal{C}omod_{\mathcal{A}_p}) \simeq \mathrm{N}_{dg}(\mathrm{Ch}^+(\mathcal{C}omod^{\mathrm{inj}}_{\mathcal{A}_p}))$. However, what we will compute is mainly restricted to the two-sided bounded case, so we can still use the injective resolutions to represent the maps in $\mathcal{D}^-(\mathcal{C}omod_{\mathcal{A}_p})$.

Notations 5.4. Given a spectrum X, we can apply ν to get its synthetic analogue νX, which is a periodic module over $\nu\mathbb{S}$. Then we have $C\tau^n \otimes \nu X \simeq \tau_{\leq n-1}(\nu X)$, which we will simply denote by X_n.

In the following, there will be two different ways to denote the bigraded shifting. One is given as $\Sigma^{s,t}$, indicating the tensor product with $\mathbb{S}^{s,t}$; the other is given as $\Sigma^u X[v]$, which we use to denote the categorical suspension and the grading. These two notations can be converted to each other by the following formula: $\Sigma^{s,t}X = \Sigma^{s-t}X[t]$.

The main purpose of this section is to apply the framework we developed in the previous chapter to the category of synthetic spectra and try to understand how various classical homotopy phenomena combine and interplay with each other. Then we shall realize why the explicit computation of Θ should lie at the center of the study of the Adams spectral sequence. Before doing so, let us recall a variation of the Adams spectral sequence, which is called the *τ-Bockstein spectral sequence*.

Construction 5.5. Given a spectrum X, we define the *τ-Bockstein spectral sequence* of X to be

$$\mathfrak{E}^{s,t}_k(X) := \pi_{t,t+s}(X_{k-1}) \simeq \pi_0\mathrm{Map}_{\mathcal{M}od_{C\tau^{k-1}}}(\Sigma^{t,t+s}C\tau^{k-1}, X_{k-1}).$$

In particular, $\mathfrak{E}^{s,t}_2 = \pi_{t,t+s}(X_1) = \mathrm{Ext}^{s,t+s}_{\mathcal{C}omod_{\mathcal{A}_p}}(\mathbb{F}_p, H_*(X))$ is the E_2 page of the classical Adams spectral sequence.

As we have shown before, X_k can be written as a triple (X_{k-1}, s, h), where s is a $C\tau^{k-1}$-module map $X_{k-1} \to \Theta^1_{k-2} X_{k-1}$. Taking the homotopy of this map, we get a morphism

$$\pi_{*,*}(X_{k-1}) \to \pi_{*,*}(\Theta^1_{k-2} X_{k-1}) = \pi_{*,*}(X_{k-1}) \ltimes \pi_{*-1,*+k-1}(X_1),$$

which is the identity on $\pi_{*,*}(X_{k-1})$ and induces differentials

$$\overline{d_k} : \mathfrak{E}^{s,t}_k(X) = \pi_{t,t+s}(X_{k-1}) \to \pi_{t-1,t+s+k-1}(X_1) = \mathfrak{E}^{s+k,t-1}_2(X).$$

Remark 5.6. This can be viewed as a variation of the Adams spectral sequence. Informally, this spectral sequence has the following characters:

(1) The synthetic differentials are essential geometric information of the spectrum and can be recovered to give the differentials in the classical Adams spectral sequence.
(2) The $\mathfrak{E}_k$ page can be viewed as a bigraded ring over $\mathbb{Z}/p^{k-1}$, containing the information of the extension problem in the classical Adams spectral sequence.
(3) $\mathfrak{E}^{*,*}_k(X)$ can be viewed as a (left) module over $\mathfrak{E}^{*,*}_k(\mathbb{S})$.

Variation 5.7. To simplify the computation, we restrict our attention to the (2^k+1)-pages $\mathfrak{E}_{2^k+1}$. We have

$$\mathfrak{E}^{s,t}_{2^k+1}(X) = \pi_{t,t+s}(X_{2^k}).$$

Under the identification

$$\mathcal{M}od^{\otimes}_{C\tau^{2^{(k+1)}}} \simeq \Theta^{1,2,4,\dots,2^k}_{0,1,3,\dots,2^k-1}\text{-Sect}^{\otimes},$$

there is a section map of $C\tau^{2^k}$-modules

$$X_{2^k} \to \Theta^{2^k}_{2^k-1} X_{2^k}.$$

Similarly, passing to bigraded homotopy groups gives the differentials

$$d^{\mathcal{S}yn}_{2^{k+1}} : \mathfrak{E}^{s,t}_{2^k+1}(X) = \pi_{t,t+s}(X_{2^k}) \to \pi_{t-1,t+s+2^k}(X_{2^k}) = \mathfrak{E}^{s+2^k+1,t-1}_{2^k+1}(X).$$

Remark 5.8. This variation has three advantages:

(1) The source and target of the differential lie on the same page.
(2) The differential $d^{\mathcal{S}yn}_{2^{k+1}}$ encodes the information of $d_{2^k+1}, d_{2^k+2}, \dots, d_{2^{k+1}}$ in the classical Adams spectral sequence. As a result, this spectral sequence converges much faster than the classical one.
(3) $\Theta^{2^k}_{2^k-1} X_{2^k}$ is the fiber of $t^{(2^k)} : X_{2^k} \to \Sigma^{2^k+2} X_{2^k}[-2^k]$, so we no longer need to compute the pullback along the Postnikov tower as in Proposition 3.12.

The following theorem relates the synthetic spectral sequence to the classical Adams spectral sequence.

Theorem 5.9. *Given a spectrum X, let $\mathfrak{E}^{*,*}_k$ and $E^{*,*}_k$ denote the k-th pages of its synthetic and classical Adams spectral sequences, respectively. Then:*

(1) $E^{s,t}_2 = \mathrm{Ext}^{s,s+t}_{\mathcal{A}^\vee_p}(\mathbb{F}_p, H_*(X)) = \pi_{t,t+s}(X_1) = \mathfrak{E}^{s,t}_2$*, and* $-d_2 = d^{\mathcal{S}yn}_2$.
(2) There are $\mathbb{Z}$-linear maps $\tau_* : \mathfrak{E}^{s+1,t}_* \to \mathfrak{E}^{s,t}_*$ induced by $\tau : \Sigma^{0,-1}X \to X$. Moreover, $\tau^{k-1}_* = 0$ on the $\mathfrak{E}_k$-page.
(3)

$$E^{s,t}_k = \mathfrak{E}^{s,t}_k / \ker(\tau^{k-2}_k).$$

(4) *If X is $\mathbb{F}_p$-nilpotent complete, then $\varprojlim \mathfrak{E}_k^{s,t} \simeq \pi_t(X)_p^\wedge$ for all $s \le 0$.*

Proof. (1) The first statement is contained in [BHS19, Theorem 9.19].

(2) Concretely, the map on bigraded homotopy groups is

$$\pi_{t,t+s}(\Sigma^{0,-k} X_k) \to \pi_{t,t+s}(X_k),$$

induced by $\tau^k : \Sigma^{0,-k} C\tau^k \to C\tau^k$. It suffices to show that $\pi_{0,-k}(C\tau^k) = \mathfrak{E}_{k+1}^{-k,0}(\mathbb{S}) = 0$, which follows by induction.

(3) This follows from [BHS19, Theorem 9.19]. The vanishing of the differentials $d_2, \dots, d_{k-1}$ on $x \in \mathfrak{E}_2^{s,t}$ is equivalent to the existence of a lift of x to $\mathfrak{E}_k^{s,t}$. Thus $\mathfrak{E}_k^{s,t}$ can be described as the extension of such liftings by τ-multiples. Since $\tau_k^{k-1} = 0$ on the k-th page, all τ-multiples lie in $\ker(\tau_k^{k-2})$. To identify the elements hit by differentials, note from (3a) of [BHS19, Theorem 9.19] that if x is hit by d_r $(1 \le r \le k-1)$, then any lifting of x in $\mathfrak{E}_k^{s,t}$ is a τ^{r-1}-torsion element. By (2b), if x is not hit by any differential, then $\tau_k^{k-2} x \neq 0$. This proves the claim.

(4) By [BHS19, Proposition A.13], X is $\mathbb{F}_p$-nilpotent complete if and only if the canonical map

$$\nu(X) \to \varprojlim C\tau^n \otimes \nu(X)$$

is an equivalence. Hence

$$\pi_t(X)_p^\wedge \simeq \pi_{t,t+s}(\nu X) \simeq \varprojlim \pi_{t,t+s}(C\tau^n \otimes \nu(X)) \simeq \varprojlim \mathfrak{E}_{n+1}^{s,t}(X)$$

for all $s \le 0$, where the first equivalence follows from [Pst18, Corollary 4.12].

∎

For

$$X_{2^{k+1}} = \begin{tikzcd} X_{2^k} \arrow[rr, "s^{(2^k)}"] \arrow[dr, "id"'] & & \Theta^{2^k}_{2^k-1} X_{2^k} \arrow[dl, "\pi"] \\ & X_{2^k} & \end{tikzcd}$$

we may understand the map $s^{(2^k)}$ as a variation of the classical Adams differential. However, the module structure provides more information, such as the p-multiplication.

Lemma 5.10. *In the category $\mathcal{M}od_{C\tau^k}$, multiplication by p is induced by the p-fold multiple of the identity map of $C\tau^k$. Concretely, for any map*

$$f : M \to N \in \mathcal{M}od_{C\tau^k},$$

the map $p \cdot f$ is given by

$$p \cdot f : C\tau^k \otimes_{C\tau^k} M \xrightarrow{(p,f)} C\tau^k \otimes_{C\tau^k} N.$$

Proof. This follows directly from the linearity of the tensor product. ∎

Suppose the map $p^{2^k} : C\tau^{2^{k+1}} \to C\tau^{2^{k+1}}$ is expressed as

$$\begin{tikzcd}
C\tau^{2^k} \arrow[r, "s^{(2^k)}"] \arrow[d, "0"'] \arrow[dr, phantom, "h_0^{(2^k)}"] & \Theta C\tau^{2^k} \arrow[d, "0"] \\
C\tau^{2^k} \arrow[r, "s^{(2^k)}"] & \Theta C\tau^{2^k}
\end{tikzcd}$$

This corresponds to an element $h_0^{(2^k)} \in \pi_{0,2^k}(C\tau^{2^k}) = \mathfrak{E}^{2^k,0}_{2^k+1}(\mathbb{S})$.

Now, for any $C\tau^{2^{k+1}}$-module homomorphism $f : M \to N$ lifting $\overline{f}$, the essential information of f can be represented by

$$\begin{tikzcd}
\overline{M} \arrow[r, "s^{(2^k)}_{\overline{M}}"] \arrow[d, "\overline{f}"'] \arrow[dr, phantom, "h"] & \Theta\overline{M} \arrow[d, "\Theta\overline{f}"] \\
\overline{N} \arrow[r, "s^{(2^k)}_{\overline{N}}"] & \Theta\overline{N}
\end{tikzcd}$$

Then $p^{2^k} \cdot f : M \to N$ is determined by the square

$$\begin{tikzcd}
\overline{M} \arrow[rr, "s^{(2^k)}_{\overline{M}}"] \arrow[dd, "0"'] \arrow[dr, "h'"] & & \Theta\overline{M} \arrow[dd, "0"] \\
& \Theta C\tau^{2^k} \otimes_{C\tau^{2^k}} \Theta\overline{N} \arrow[dr] & \\
\overline{N} \arrow[rr, "s^{(2^k)}_{\overline{N}}"] & & \Theta\overline{N}
\end{tikzcd}$$

where

$$h' := h_0^{(2^k)} \otimes_{C\tau^{2^k}} \left(s^{(2^k)}_{\overline{N}} \circ \overline{f}\right).$$

This calculation provides a way to solve the extension problem in the τ-Bockstein spectral sequence.

Theorem 5.11. *There is a long exact sequence*

$$\pi_{t,t+s+2^k}(X_{2^k}) \to \pi_{t,t+s}(X_{2^{k+1}}) \to \pi_{t,t+s}(X_{2^k}) \xrightarrow{d^{\mathbb{S}yn}_{2^{k+1}}} \pi_{t-1,t+s+2^k}(X_{2^k}),$$

which reduces to a short exact sequence

$$0 \to \operatorname{coker}(d^{\mathbb{S}yn}_{2^{k+1}})_{s+2^k,t} \to \mathfrak{E}^{s,t}_{2^{k+1}} \to \ker(d^{\mathbb{S}yn}_{2^{k+1}})_{s,t} \to 0.$$

The extension is detected by $h_0^{(2^k)}$.

Proof. All abelian groups under consideration are p-groups, and both $\ker(d^{\mathbb{S}yn}_{2^{k+1}})$ and $\operatorname{coker}(d^{\mathbb{S}yn}_{2^{k+1}})$ are naturally $\mathbb{Z}/(p^{2^k})$-modules. To determine the extension of these two modules, it suffices to compute the p^{2^k}-multiple of a lifting $f \in \pi_{t,t+s}(X_{2^{k+1}})$ of an element $\overline{f} \in \ker(d^{\mathbb{S}yn}_{2^{k+1}})$.

Taking $M = \Sigma^{t,t+s}C\tau^{2^{k+1}}$ and $N = X_{2^{k+1}}$, we view any diagram

$$\begin{tikzcd}
\Sigma^{t,t+s}C\tau^{2^k} \arrow[r, "s^{(2^k)}"] \arrow[d, "\overline{f}"'] \arrow[dr, phantom, "h"] & \Sigma^{t,t+s}\Theta C\tau^{2^k} \arrow[d, "\Theta\overline{f}"] \\
X_{2^k} \arrow[r, "s_X^{2^k}"] & \Theta X_{2^k}
\end{tikzcd}$$

as representing an element $f \in \pi_{t,t+s}(X_{2^{k+1}})$ lifting $\overline{f} \in \ker(d^{\mathbb{S}yn}_{2^k+1})$. Then the p^{2^k}-multiple of f is given by

$$\begin{tikzcd}
\Sigma^{t,t+s}C\tau^{2^k} \arrow[r] \arrow[d, "0"'] \arrow[dr, phantom, "h_0^{(2^k)}\cdot(s_X^{(2^k)}\circ\overline{f})"] & \Sigma^{t,t+s}\Theta C\tau^{2^k} \arrow[d, "0"] \\
X_{2^k} \arrow[r] & \Theta X_{2^k}
\end{tikzcd}$$

Since the p^{2^k}-multiple of f is independent of the homotopy h, we obtain a well-defined map

$$h_0^{(2^k)} \cdot (-) : \ker(d^{\mathbb{S}yn}_{2^{k+1}})_{s,t} \to \pi_{t,t+s+2^k}(\Theta X_{2^k}).$$

This map is a restriction of the natural action

$$\pi_{*,*}(\Theta C\tau^{2^k}) \times \pi_{*,*}(\Theta X_{2^k}) \to \pi_{*,*}(\Theta X_{2^k}).$$

Forgetting the module structure and passing to the underlying object, the product $h_0^{(2^k)} \cdot (s_X^{(2^k)} \circ \overline{f})$ yields an element

$$h_0^{(2^k)} \cdot \overline{f} \in \pi_{t,t+s+2^k}(X_{2^k}).$$

□

Remark 5.12. For $k = 0$, this recovers the classical Adams case, showing that $h_0 = h_0^{(1)}$ detects $\mathbb{Z}/p$-extensions.

Example 5.13. Now we show the 14-th and 15-th columns of the $\mathfrak{E}^3$-page of the sphere as an example to show how the product structure in the $\mathfrak{E}^3$-page helps us to determine higher differentials. The product structure can still be computed by the secondary Steenrod algebra as shown in [CPH22]. Here we directly use the result in [BHS19, Proposition A.20]. We will use $\widetilde{x}$ to denote any chosen lifting of $x \in \mathfrak{E}_2^{*,*}(\mathbb{S})$ in $\mathfrak{E}_3^{*,*}(\mathbb{S})$, and the elements listed are the generators of the corresponding abelian groups. Under such notation, we have $h_0^{(2)} = \widetilde{h_0}^2$ and a hidden extension

$$\widetilde{h_0}^2\widetilde{h_3^2} = 2\widetilde{d_0} = \tau h_0 d_0.$$

The full diagram is given as:

$$
\begin{array}{cccc}
 & & \mathbb{Z}/2 & \\
 & & \mathbb{Z}/4 & \\
\widetilde{h_0}^2\widetilde{d_0} & \mathbb{Z}/2 & \mathbb{Z}/4 & \widetilde{h_0}^4\widetilde{h_0h_4} \\
\widetilde{h_0}\widetilde{d_0} & \mathbb{Z}/4 & \mathbb{Z}/4 & \widetilde{h_0}^3\widetilde{h_0h_4} \\
\widetilde{d_0} & \mathbb{Z}/4 & \mathbb{Z}/4 & \widetilde{h_0}^2\widetilde{h_0h_4} \\
\widetilde{h_0h_3^2}, \tau d_0 & \mathbb{Z}/2\oplus\mathbb{Z}/2 & \mathbb{Z}/4 & \widetilde{h_0}\widetilde{h_0h_4} \\
\widetilde{h_3^2} & \mathbb{Z}/2 & \mathbb{Z}/4 & \widetilde{h_0h_4} \\
 & & \mathbb{Z}/2 & \tau h_0h_4
\end{array}
$$

Invoking Theorem 5.11, $\widetilde{h_0}^2\widetilde{h_3^2}$ is either the generator or zero in $\mathrm{coker}(d_3^{\mathcal{S}yn})$, so there must be a differential hitting $2\widetilde{d_0} = \tau h_0 d_0$. By degree reasons, the only possibility is $d_3^{\mathcal{S}yn}(\tau h_0 h_4) = \tau h_0 d_0$. This differential is completely hidden but can be easily recovered to give the classical Adams differential $d_3(h_0h_4) = h_0d_0$ by the Leibniz rule.

So far, we have seen how the twisting maps nearly determine the differentials. This highlights the importance of explicit computations for the Θ-functor. More concretely, let us unpack the essential data needed.

Given a spectrum X, to emphasize the homological feature of X_1, write

$$H_*(X) := H_*(X_1; \mathbb{F}_p) \in \mathcal{D}(\mathcal{C}omod_{\mathcal{A}_p^\vee})$$

for $C\tau \otimes X$. Since we must keep track of actual maps rather than homotopy classes, we regard $H_*(X)$ as the injective resolution of $H_*(X)$ as a comodule. Then the Θ-functor is determined by:

(1) A map $H_*(X) \to \Sigma^3 H_*(X)[-1]$, corresponding to $\Theta_0^1 H_*(X)$.
(2) Maps $H_*(X) \to \Sigma^4 H_*(X)[-2]$ and $H_*(X) \to \Sigma^3\Theta_0^1 H_*(X)[-2]$ witnessing the homotopy

$$
\begin{tikzcd}
H_*(X) \arrow[r] \arrow[d, "f"] & \Theta_0^1(H_*(X)) \arrow[d, "\Theta_0^1 f"] \\
\Sigma^4 H_*(X)[-2] \arrow[r] & \Sigma^4\Theta_0^1 H_*(X)[-2]
\end{tikzcd}
$$

for $\Theta_{(0,1)}^{(1,2)} H_*(X)$.

(3) A map $H_*(X) \to \Sigma^6 H_*(X)[-4]$, together with all homotopies fitting into the diagram

$$\begin{tikzcd}
& H_*(X) \arrow[rr] \arrow[dl] \arrow[dd] & & \Theta_0^1(H_*(X)) \arrow[dl] \arrow[dd] \\
\mathrm{Fib}(f) \arrow[rr] \arrow[dd] & & \mathrm{Fib}(\Theta_0^1 f) \arrow[dd] & \\
& \Sigma^6 H_*(X)[-4] \arrow[rr] \arrow[dl] & & \Sigma^6 \Theta_0^1(H_*(X))[-4] \arrow[dl] \\
\Sigma^6 \,\mathrm{Fib}(f)[-4] \arrow[rr] & & \Sigma^6 \,\mathrm{Fib}(\Theta_0^1(f))[-4] &
\end{tikzcd}$$

(4) In general, maps $H_*(X) \to \Sigma^{2^k+2} H_*(X)[-2^k]$ for all $k \geq 0$, together with the higher cells witnessing the corresponding homotopies.

6. Relationship between Theta-functor and n-ary Steenrod algebra

In this section, we first recall the definitions and basic properties of the n-ary Steenrod algebra (see [CPH22, Section 4] for details). We then derive a formula for the first twisting map from the computation of the secondary Steenrod algebra.

Definition 6.1. For all $1 \leq n \leq \infty$, the *n-ary Steenrod algebra* $\mathcal{A}^{(n)}$ is defined as $\Phi^{(n)}\mathrm{End}(\mathbb{F}_p)$, which is an $\mathbb{E}_1$-algebra in $\mathcal{S}p^{gr}$. The *n-ary cohomology* functor

$$\mathcal{H}^{(n)} : \mathcal{S}p \longrightarrow \mathcal{M}od_{\mathcal{A}^{(n)}}$$

sends a spectrum X to $\Phi^{(n)}\mathrm{Map}^{\mathcal{S}p}(X, \mathbb{F}_p)$.

Notations 6.2. We use $\mathrm{Map}^{\mathcal{S}p}$ to denote the mapping spectrum, and $\mathrm{End}(\mathbb{F}_p)$ for the $\mathbb{E}_1$-ring of endomorphisms of $\mathbb{F}_p$.

As computed by [Bau06], the secondary Steenrod algebra is an algebra over $\mathbb{S}_{\leq 1}^{\mathbb{F}_p} \simeq \mathbb{Z}/p^2$. Consequently, it can be expressed as a differential graded $\mathbb{Z}/p^2$-algebra

$$\begin{tikzcd}
\mathcal{B}_1 \arrow[d, "d^{\mathcal{B}}"] \\
\mathcal{B}_0
\end{tikzcd}$$

together with a short exact sequence

$$0 \longrightarrow \mathcal{A}_p[+1] \longrightarrow \mathcal{B}_1 \xrightarrow{d^{\mathcal{B}}} \mathcal{B}_0 \xrightarrow{\pi_{\mathcal{B}}} \mathcal{A}_p \longrightarrow 0.$$

For $\mathcal{B}_\bullet$ to be a differential graded algebra, $\mathcal{B}_0$ must be a $\mathbb{Z}/p^2$-algebra and $\mathcal{B}_1$ a $\mathcal{B}_0$-bimodule. As a right $\mathcal{B}_0$-module, $\mathcal{B}_1$ decomposes as

$$\mathcal{B}_1 \;\cong\; \ker(\pi_{\mathcal{B}}) \;\oplus\; \mathcal{A}_p[+1].$$

The left action is given by

$$a \cdot (r, p) = \big(ar,\, A(\pi^{\mathcal{B}}(a), r) + \pi^{\mathcal{B}}(a)p\big),$$

where

$$A\colon \mathcal{A}_p \otimes_{\mathbb{Z}} \ker \pi_{\mathcal{B}} \;\longrightarrow\; \mathcal{A}_p[+1]$$

is a nontrivial bilinear map.

In [CPH22], Chua provides explicit generators and relations for this differential graded algebra in the case $p = 2$. These presentations are sufficient to compute all

d_2-differentials. Here we recall only the material relevant for computing twisting maps.

The dual of $\mathcal{B}_0$ can be identified with

$$(\mathcal{B}_0)_* \;=\; \mathbb{Z}/4[\xi_k,\, 2\xi_{k,l} \mid 0 \le k < l,\ \xi_0 = 1],$$

whose coproduct is encoded by the power series

$$f(x) = \sum_{k\ge 0} \xi_k x^{2^k} \;+\; \sum 2\xi_{k,l} x^{2^k+2^l}.$$

Accordingly, $\mathcal{B}_0$ itself is described as follows:

Definition 6.3 ([CPH22], Definition 8.6)**.** Define $Sq(R)$ and $Y_{k,l}$ to be the dual elements of ξ^R and $2\xi_{k,l}$, respectively, where R denotes a multi-index in Milnor's basis.

Remark 6.4. Chua uses the notation $Sq(R)$ both for elements of $\mathcal{B}_0$ and for those of $\mathcal{A}_p$. We follow this convention, but note that the products differ: in general, the multiplication of $Sq(R)Sq(S)$ depends on whether it is taken in $\mathcal{B}_0$ or in $\mathcal{A}_p$.

Lemma 6.5 ([CPH22], Lemma 8.7, 8.8, 8.10)**.** *The generators satisfy the following relations:*

(1) $\pi_{\mathcal{B}}(Y_{k,l}) = 0$ *and* $\pi_{\mathcal{B}}(Sq(R)) = Sq(R)$.
(2) $Y_{a,b}Y_{c,d} = 0$ *for all* a, b, c, d, *and* $2Y_{*,*} = 0$.

Definition 6.6. [CPH22, Definition 8.11] Let $X = (x_{ij})$ be a matrix indexed on the non-negative integers. Define

$$r_i(X) = \sum_j 2^j x_{ij}, \quad s_j(X) = \sum_i x_{ij}, \quad t_n(X) = \sum_{i+j=n} x_{ij},$$
$$R(X) = (r_1(X), r_2(X), \ldots), \quad S(X) = (s_1(X), s_2(X), \ldots), \quad T(X) = (t_1(X), \ldots),$$
$$b(X) = \frac{\prod t_n!}{\prod x_{ij}!} \in \mathbb{Z}$$

We put $\mathcal{T} : \mathcal{A}_p^\vee \otimes \mathcal{A}_p \to \mathcal{A}_p$ to be the contraction operator, then in Milnor basis, we have

$$\mathcal{T}(\xi^R, Sq(S)) = Sq(S - R),$$

where $Sq(S - R)$ is zero if any entry is negative.

Theorem 6.7. [CPH22, Theorem 8.13]

$$\begin{aligned} Sq(R)Sq(S) = &\sum_{k\ge 0}\sum_{0\le m<n} Y_{m+k,n+k}\mathcal{T}(\xi_m^{2^k}\xi_n^{2^k}, Sq(R))\mathcal{T}(\xi_{k+1}, Sq(S)) \\ &+ \sum_{R(X)=R, S(X)=S} b(X)Sq(T(X)). \end{aligned}$$

Remark 6.8. As $\mathcal{B}_0 \to \mathcal{A}_p$ is a map of algebras, we have

$$\pi_{\mathcal{B}}(Sq(R)Sq(S)) = \pi_{\mathcal{B}}(Sq(R))\pi_{\mathcal{B}}(Sq(S)),$$

which recovers Milnor's result in [Mil58, Theorem 4b].

Finally, we describe the function A:

Lemma 6.9. [CPH22, Lemma 8.14] *We have*

$$\begin{aligned} A(a,2) &= \mathcal{T}(\xi_1, a), \\ A(a, Y_{k,l}) &= \sum_{i,j\geq 0} Z_{k+i,l+j}\mathcal{T}(\xi_i^{2^k}\xi_j^{2^l}, a) \\ A(a, rSq(R)) &= A(a,r)Sq(R), \end{aligned}$$

where $Z_{k,l} = Sq(\Delta_k + \Delta_l)$ *if* $k \geq l$ *and* $Z_{k,l} = 0$ *otherwise.*

When $n = 2$, we have described $\mathcal{A}_p^{(2)}$ as a differential graded algebra, so the category $\mathcal{M}od_{\mathcal{A}_p^{(2)}}$ is purely algebraic. This allows us to reinterpret computations in $\mathcal{M}od_{C\tau^2}$, which is a topological category, as computations in $\mathcal{M}od_{\mathcal{A}_p^{(2)}}$, which is algebraic.

To carry this out, one must construct a cofibrant replacement of $\mathcal{H}^{(2)}(X)$ in $\mathcal{M}od_{\mathcal{A}_p^{(2)}}$. This can be achieved by lifting a free resolution of $H^*(X)$ as an $\mathcal{A}_p$-module.

More generally, suppose we are given an $\mathcal{A}_p$-module $\overline{M}$. We aim to find a cofibrant $\mathcal{A}_p^{(2)}$-module M such that

$$\mathcal{A}_p \otimes_{\mathcal{A}_p^{(2)}} M \;\cong\; \overline{M}.$$

Let

$$\cdots \longrightarrow \overline{M^{(2)}} \xrightarrow{\overline{\partial^{(2)}}} \overline{M^{(1)}} \xrightarrow{\overline{\partial^{(1)}}} \overline{M^{(0)}} \longrightarrow \overline{M}$$

be a free resolution of $\overline{M}$. As mentioned in Section 1, to make this lift homotopically meaningful, we must also specify the homotopies, i.e. a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & M_1^{(3)} \arrow[r, "\partial_1^{(3)}"] \arrow[d, "d^{(3)}"] & M_1^{(2)} \arrow[r, "\partial_1^{(2)}"] \arrow[d, "d^{(2)}"] & M_1^{(1)} \arrow[r, "\partial_1^{(1)}"] \arrow[d, "d^{(1)}"] & M_1^{(0)} \arrow[d, "d^{(0)}"] \\
\cdots \arrow[r] \arrow[urr, "h^{(4)}"] & M_0^{(3)} \arrow[r, "\partial_0^{(3)}"'] \arrow[urr, "h^{(3)}"] & M_0^{(2)} \arrow[r, "\partial_0^{(2)}"'] \arrow[urr, "h^{(2)}"] & M_0^{(1)} \arrow[r, "\partial_0^{(1)}"'] & M_0^{(0)}
\end{tikzcd}$$

satisfying the relations

$$\begin{aligned} d^{(s-1)}\partial_1^{(s)} &= \partial_0^{(s)}d^{(s)}, \\ \partial_0^{(s-1)}\partial_0^{(s)} &= d^{(s-2)}h^{(s)}, \\ \partial_1^{(s-1)}\partial_1^{(s)} &= h^{(s)}d^{(s)}, \\ h^{(s-1)}\partial_0^{(s)} &= \partial_1^{(s-2)}h^{(s)}. \end{aligned}$$

Given such homotopies h, we obtain a cofibrant replacement of $\overline{M}$ in the form

$$M_0^{(3)} \oplus M_1^{(2)} \xrightarrow{\begin{pmatrix} \partial_0^{(3)} & d^{(2)} \\ -h^{(3)} & -\partial_1^{(2)} \end{pmatrix}} M_0^{(2)} \oplus M_1^{(1)} \xrightarrow{\begin{pmatrix} \partial_0^{(2)} & d^{(1)} \\ -h^{(2)} & -\partial_1^{(1)} \end{pmatrix}} M_0^{(1)} \oplus M_1^{(0)} \xrightarrow{\begin{pmatrix} \partial_0^{(1)} & d^{(0)} \end{pmatrix}} M_0^{(0)}.$$

One verifies directly that this is a cofibrant object with the correct homotopy type. For our purposes, however, it suffices to analyze the diagram to understand the meaning of the homotopies h.

By decomposing $M_1^{(n)}$ as

$$M_1^{(n)} \;\cong\; \ker(\pi_{(0)}) \;\oplus\; \overline{M^{(n)}}[+1],$$

the map $h^{(n+2)}$ separates into two components. The commutativity of the diagram forces the ker-part to be determined by $\partial_1^{(n+2)}\partial_1^{(n+1)}$, so the essential contribution is the map

$$h_\tau \colon M_0^{(n+1)} \longrightarrow \overline{M^{(n)}}[+1].$$

In fact, since $\partial_1^{(n+2)}\partial_1^{(n+1)}(\ker d^{(n+2)}) = 0$, the map h_τ factors through

$$h_\tau \colon \overline{M^{(n+2)}} \longrightarrow \overline{M^{(n)}}[+1].$$

The degree of h_τ coincides precisely with the degree of the d_2-differential and the connecting map $X \to \Sigma^2 X[-1]$. This is no accident: in [CPH22, Section 5], Dexter Chua constructs a comparison functor

$$\mathcal{H}^{(n)}_{\mathcal{S}yn} \colon \mathcal{M}od_{C\tau^n} \longrightarrow \mathcal{M}od^{op}_{\mathcal{A}^{(n)}}$$

extending the n-ary cohomology functor on $\mathcal{S}p$, fitting into the commutative diagram

$$\begin{tikzcd} & \mathcal{S}p \arrow[dl, "C\tau^n\otimes\nu(-)"'] \arrow[dr, "\mathcal{H}^{(n)}"] & \\ \mathcal{M}od_{C\tau^n} \arrow[rr, "\mathcal{H}^{(n)}_{\mathcal{S}yn}"] & & \mathcal{M}od^{op}_{\mathcal{A}^{(n)}} \end{tikzcd}$$

with $\mathcal{H}^{(n)}$ almost fully faithful.

Theorem 6.10 ([CPH22], Theorem 5.6)**.** *For any $X \in \mathcal{M}od_{C\tau^n}$ and $Y \in \mathcal{M}od^{ft}_{C\tau^n}$, the map*

$$\mathcal{H}^{(n)}_{\mathcal{S}yn} \colon \mathrm{Map}_{\mathcal{M}od_{C\tau^n}}(X, Y) \;\longrightarrow\; \mathrm{Map}_{\mathcal{M}od^{op}_{\mathcal{A}^{(n)}}}(\mathcal{H}^{(n)}_{\mathcal{S}yn}(Y), \mathcal{H}^{(n)}_{\mathcal{S}yn}(X))$$

is an equivalence. Here $\mathcal{M}od^{ft}_{C\tau^n}$ is the full stable subcategory generated by $\{C\tau^n \otimes \nu P\}$ for finite type spectra P.

Applying this when $n = 2$, one sees that h_τ detects the d_2-differential as in [CPH22, Lemma 9.1]. Indeed, if the resolution of $H^*(X)$ is chosen to be minimal, then each E_2-class is represented by a generator of $\mathcal{A}_p$, and a nontrivial d_2 occurs precisely when h_τ maps one generator to another, up to nullhomotopies.

Finally, this leads to the formula for the first twisting map:

Theorem 6.11. *Let $\overline{M}^\vee$ be a $\mathcal{A}_p^\vee$-comodule, with dual $\mathcal{A}_p$-module $\overline{M}$. Suppose the free resolution*

$$\cdots \longrightarrow \overline{M^{(2)}} \xrightarrow{\overline{\partial^{(2)}}} \overline{M^{(1)}} \xrightarrow{\overline{\partial^{(1)}}} \overline{M^{(0)}} \longrightarrow \overline{M}$$

is dual to an injective resolution in $\mathcal{C}omod_{\mathcal{A}_p^\vee}$, and that $\overline{M}^\vee$ lifts to a periodic $C\tau^2$-module. Then the first twisting map

$$t^{(2)} \colon \overline{M^{(n)}}^\vee \longrightarrow \overline{M^{(n+3)}}^\vee[-1]$$

has dual

$$(t^{(2)})^\vee \colon \overline{M^{(n+3)}} \longrightarrow \overline{M^{(n)}}[+1],$$

sending each generator g to

$$\sum A\big(\alpha^i,\ \partial_0^{(n+2)}\partial_0^{(n+1)}(g_i)\big),$$

where $\overline{\partial^{(n+3)}}(g) = \sum \alpha^i g_i$ with $\{g_i\}$ generators of $\overline{M^{(n+2)}}$.

Proof. From the discussion above, the dual of

$$d^{(2)}_n\colon \overline{M^{(n)}}^{\vee} \to \overline{M^{(n+2)}}^{\vee}[-1]$$

is identified with

$$h_\tau^{(n+2)}\colon \overline{M^{(n+2)}} \to \overline{M^{(n)}}[+1].$$

Commutativity of the chain map $\overline{M}^{\vee} \to \Theta_0^1\overline{M}^{\vee}$ then implies

$$\overline{\partial^{(n+1)}}h_\tau^{(n+3)}(g) = h_\tau^{(n+2)}\big(\overline{\partial^{(n+2)}}(g)\big) + (t^{(2)})^{\vee}(g).$$

By [CPH22, Corollary 8.29], the left-hand side expands as

$$\sum \alpha^i h_\tau^{(n+2)}(g_i) + \sum A\big(\alpha^i,\ \partial_0^{(n+2)}\partial_0^{(n+1)}(g_i)\big),$$

hence we identify

$$(t^{(2)})^{\vee}(g) \ = \ \sum A\big(\alpha^i,\ \partial_0^{(n+2)}\partial_0^{(n+1)}(g_i)\big).$$

□

Remark 6.12. We expect that the above result should remain valid without assuming that $\overline{M}^{\vee}$ can be lifted to a periodic $C\tau^2$-module. However, since the function

$$g \mapsto \sum A(\alpha^i, \partial_0^{(n+2)}\partial_0^{(n+1)}(g_i))$$

lacks a functorial definition, at present it can only be compared in an *ad hoc* manner.

Remark 6.13. This result clarifies what the secondary Steenrod algebra is actually computing. As pointed out by Chua, $\mathcal{B}_0$ is not canonical; it is chosen large enough to support a nontrivial function A. Our result explains how this enlargement interacts with the function A to recover the canonical functor Θ. Consequently, our construction provides a practical method for performing elementwise computations in higher Steenrod algebras.

We conclude with an example that illustrates how to compute the first d_2-differential on the sphere by hand. (Further computations can be found in [Chu22].)

Example 6.14. We recover the differential $d_2(h_4) = h_0h_3^2$ in the $\mathbb{F}_2$-Adams spectral sequence. To this end, we consider the diagram of $\mathcal{A}_2^{\vee}$-comodules. Here we use $\langle a\rangle$ to denote one copy of $\mathcal{A}_2^{\vee}$, with Steenrod algebra elements written in the Milnor basis:

$$\begin{tikzcd}
\langle 1\rangle \arrow[r, "(id{,}0)"] \arrow[d] & \langle 1\rangle \oplus (\langle h_0^2\rangle \oplus \langle h_1^2\rangle \oplus \langle h_0h_2\rangle \oplus \cdots)[-1] \arrow[d] \\
\langle h_0\rangle \oplus \langle h_1\rangle \oplus \langle h_2\rangle \oplus \cdots \arrow[r, "(id{,}d^{(2)})"'] & (\langle h_0\rangle \oplus \langle h_1\rangle \oplus \cdots) \oplus (\langle h_0^3\rangle \oplus \langle h_0^2h_2\rangle \oplus \cdots)[-1]
\end{tikzcd}$$

To detect $d^{(2)}(\langle h_4\rangle)$ via $\langle h_0h_3^2\rangle$, we compute the image of $\langle h_4\rangle$ under $t^{(2)}$. Since direct comodule calculations are cumbersome, we instead dualize to $\mathcal{A}_2$-modules and apply the formula in Theorem 6.11. This yields

$$\begin{aligned}(t^{(2)})^\vee(\langle h_0h_3^2\rangle) =& A(Sq^{(9,2)} + Sq^{(6,3)}, Y_{0,1} + 2Sq^2)+\\ & A(Sq^{(7,2)}, Y_{0,1}Sq^2 + 2Sq^4 + 2Sq^{(1,1)})+\\ & A(Sq^{(0,4)}, Y_{0,2}Sq^1 + 2Sq^5 + 2Sq^{(2,1)})+\\ & A(Sq^9 + Sq^{(0,3)}, Y_{0,1}Sq^6 + 2Sq^8 + 2Sq^{(2,2)})+\\ & A(Sq^8 + Sq^{(2,2)}, Y_{0,1}(Sq^7 + Sq^{(4,1)} + Sq^{(1,2)}) + Y_{0,2}Sq^5+\\ & Y_{0,3}Sq^1 + 2(Sq^9 + Sq^{(6,1)} + Sq^{(0,3)} + Sq^{(2,0,1)} + Sq^{(3,2)}))+\\ & A(Sq^7 + Sq^{(4,1)} + Sq^{(0,0,1)}, Y_{0,1}(Sq_8 + Sq^{(2,2)}) + Y_{0,2}Sq^{(3,1)}+\\ & Y_{1,2}Sq^5 + 2(Sq^10 + Sq^{(7,1)} + Sq^{(1,3)} + Sq^{(3,0,1)} + Sq^{(4,2)}))+\\ & A(Sq^1, Y_{0,1}Sq^{14} + 2(Sq^{16} + Sq^{(7,3)} + Sq^{(4,4)}))\end{aligned}$$

Although this expansion already contains many terms, further application of the defining formula for A produces nearly 60 terms. A direct calculation shows that the result simplifies to

$$Sq^{16} + Sq^{(6,1,1)} + Sq^{(10,2)} \in \langle 1\rangle.$$

Since the following square commutes:

$$\begin{tikzcd}
\langle 1\rangle & \langle 1\rangle \oplus (\langle h_0^2\rangle \oplus \langle h_1^2\rangle \oplus \langle h_0h_2\rangle \oplus \cdots)[-1] \arrow[l, "{(id,0)}"'] \\
\langle h_0\rangle \oplus \langle h_1\rangle \oplus \langle h_2\rangle \oplus \cdots \arrow[u] & (\langle h_0\rangle \oplus \langle h_1\rangle \oplus \cdots) \oplus (\langle h_0^3\rangle \oplus \langle h_0^2h_2\rangle \oplus \cdots)[-1] \arrow[l, "{(id,d^{(2)})}"] \arrow[u]
\end{tikzcd}$$

we conclude that

$$d_2 : \mathfrak{E}_2^{1,15}(\mathbb{S}) \to \mathfrak{E}_2^{3,14}(\mathbb{S})$$

sends h_4 to $h_0h_3^2$, recovering the first d_2-differential on the sphere.

Remark 6.15. Although the explicit computation is lengthy, one can quickly verify that the formula for $(t^{(2)})^\vee(h_0h_3^2)$ contains only a single term, Sq^{16}, since no other products contribute in this degree. As Sq^{16} is the unique generator in degree 16, this suffices to detect the first differential.

Appendix A. Square-zero extension

In this Appendix, we introduce the basic definitions and properties of the square-zero extension we use in Section 2

Definition A.1 (Stable envelope)**.** Let $\mathcal{C}$ be a presentable ∞-category . A *stable envelope* of $\mathcal{C}$ is a categorical fibration $u : \mathcal{C}' \to \mathcal{C}$ with the following properties:

- The category $\mathcal{C}'$ is stable and presentable.
- The functor u admits a left adjoint.
- For every presentable stable (pointed) ∞-category $\mathcal{E}$, composition with u induces an equivalence of ∞-categories $\mathrm{RFun}(\mathcal{E}, \mathcal{C}') \to \mathrm{RFun}(\mathcal{E}, \mathcal{C})$. Here $\mathrm{RFun}(\mathcal{E}, \mathcal{C}')$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{E}, \mathcal{C}')$ spanned by those functors which admit left adjoints. and $\mathrm{RFun}(\mathcal{E}, \mathcal{C})$ is defined similarly.

More generally, suppose that $p : \mathcal{C} \to \mathcal{D}$ is a presentable fibration. A stable envelope of p is a categorical fibration $u : \mathcal{C}' \to \mathcal{C}$ with the following properties:

- The composition $p \circ u$ is a presentable fibration.
- The functor u carries $p \circ u$-Cartesian morphisms of $\mathcal{C}'$ to p-Cartesian morphisms of $\mathcal{C}$.
- For every object $D \in \mathcal{D}$, the induced map $\mathcal{C}'_D \to \mathcal{C}_D$ is a stable envelope of $\mathcal{C}_D$.

Definition A.2 (Tangent bundle)**.** Let $\mathcal{C}$ be a presentable ∞-category. A *tangent bundle* to $\mathcal{C}$ is a functor $T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ which exhibits $T_{\mathcal{C}}$ as the stable envelope of the presentable fibration $\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{1\}, \mathcal{C}) \simeq \mathcal{C}$.

Definition A.3 (relative version of adjoint functors)**.** Suppose we are given a commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[dr, "q"'] & & \mathcal{D} \arrow[ll, "G"'] \arrow[dl, "p"] \\ & \mathcal{E} & \end{tikzcd}$$

of ∞-categories, where the maps p and q are categorical fibrations. The following conditions are equivalent:

- The functor G admits a left adjoint F. Moreover, for every object $X \in \mathcal{C}$, the functor q carries the unit map $u_X : X \to GFX$ to an equivalence in $\mathcal{E}$.
- There exists a functor $F : \mathcal{C} \to \mathcal{D}$ and a natural transformation $u : id_{\mathcal{C}} \to G \circ F$ which exhibits F as a left adjoint to G, and has the property that $q(u)$ is the identity transformation from q to itself.

We will say that G admits a left adjoint relative to $\mathcal{E}$ if the equivalent conditions above are satisfied.

Proof. This is [Lur17, Proposition 7.3.2.1] □

Proposition A.4. *Let $\mathcal{C}$ be a presentable ∞-category, and consider the associated diagram*

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[dr, "q"'] & & T_{\mathcal{C}} \arrow[ll, "G"'] \arrow[dl, "p"] \\ & \mathcal{E} & \end{tikzcd}$$

where q is given by evaluation at $\{1\} \subset \Delta^1$. Then G admits a left adjoint F relative to $\mathcal{E}$.

Proof. By the definition of the tangent bundle $T_{\mathcal{C}}$, the functor G carries p-Cartesian morphisms to q-Cartesian morphisms. Since for each object $A \in \mathcal{C}$, the fibers $Sp(\mathcal{C}_{/A})$ and $\mathcal{C}_{/A}$ admit final object, so p and q are locally Cartesian categorical fibrations. The induced functor $G_A : Sp(\mathcal{C}_{/A}) \to \mathcal{C}_{/A}$ admits a left adjoint Σ^∞, so we can apply [Lur17, Proposition 7.3.2.6] to get a left adjoint of G relative to $\mathcal{E}$, which we will denote by F. □

Definition A.5 (Absolute cotangent complex functor)**.** Let $\mathcal{C}$ be a presentable ∞-category, the *absolute cotangent complex functor* $L : \mathcal{C} \to T_{\mathcal{C}}$ is defined to be the composition

$$\mathcal{C} \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \xrightarrow{F} T_{\mathcal{C}},$$

where the first map is given by the diagonal embedding. We will denote the value of L on an object $A \in \mathcal{C}$ by $L_A \in Sp(\mathcal{C}_{/A})$, and will refer to L_A as the *cotangent complex* of A.

Definition A.6 (trivial square-zero extension)**.** Let $\mathcal{C}^{\otimes}$ be a stable symmetric monoidal ∞-category (such that the tensor product on $\mathcal{C}$ is exact in each variable) and let A be an $\mathcal{O}$-algebra object of $\mathcal{C}$, then the *trivial square-zero extension* of an A-module is defined to be the composition

$$\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{M}od_A^{\mathcal{O}}) \simeq \mathcal{S}p(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})/_A) \xrightarrow{\Omega^\infty} \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})/_A.$$

This functor associates to each $M \in \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{M}od_A^{\mathcal{O}})$ a algebra which we will denote by $A \oplus M$.

Remark A.7. Since the diagonal embedding admits a right adjoint $\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{0\}, \mathcal{C}) \simeq \mathcal{C}$, we have a right adjoint of L

$$\mathcal{C} \xleftarrow{ev_0} \mathrm{Fun}(\Delta^1, \mathcal{C}) \xleftarrow{G} T_{\mathcal{C}}.$$

When $\mathcal{C}$ is the full subcategory spanned by all $\mathcal{O}$-monoidal algebra object in a symmetric ∞-category $\mathcal{E}$ and let $A \in \mathcal{C}$ with M in its fiber $\mathcal{S}p(\mathcal{E}/_A) \simeq \mathcal{M}od_A^{\mathcal{O}}(\mathcal{E})$. Then this right adjoint sends M to its trivial square-zero extension $A \oplus M$.

Definition A.8 (Tangent correspondence)**.** Let $\mathcal{C}$ be a presentable ∞-category and let $G : T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ be a tangent bundle to $\mathcal{C}$. We define the *tangent correspondence* $s : M^T(\mathcal{C}) \to \Delta^1$ to be the Cartesian fibration associated to the functor

$$T_{\mathcal{C}} \xrightarrow{G} \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{0\}, \mathcal{C}) \simeq \mathcal{C}.$$

Proposition A.9. *Let $\mathcal{C}$ be a presentable ∞-category. Then:*

1. *$M^T(\mathcal{C})$ equips with a functor $q : M^T(\mathcal{C}) \to \mathcal{C}$, restricting to $(M^T(\mathcal{C}))_0 \simeq \mathcal{C} \xrightarrow{id_{\mathcal{C}}} \mathcal{C}$ and $(M^T(\mathcal{C}))_1 \simeq T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{1\}, \mathcal{C}) \simeq \mathcal{C}$.*
2. *The projection $M^T(\mathcal{C}) \xrightarrow{(s,q)} \Delta^1 \times \mathcal{C}$ is a categorical fibration.*
3. *The map p is also a coCartesian fibration, accociated to the cotangent complex functor $L : \mathcal{C} \to T_{\mathcal{C}}$.*

Definition A.10. Let $\mathcal{C}$ be a presentable ∞-category, and let $p : M^T(\mathcal{C}) \to \Delta^1 \times \mathcal{C}$ denote a tangent correspondence to $\mathcal{C}$. A derivation in $\mathcal{C}$ is a map $f : \Delta^1 \to M^T(\mathcal{C})$ such that $p \circ f$ coincides with the inclusion $\Delta^1 \times \{A\} \subset \Delta^1 \times \mathcal{C}$, for some $A \in \mathcal{C}$. In this case, we will identify f with a morphism $\eta : A \to M$ in $M^T(\mathcal{C})$, where $M \in T_{\mathcal{C}} \times_{\mathcal{C}} \{A\} \simeq Sp(\mathcal{C}/_A)$. We will say that $\eta : A \to M$ is a derivation of A into M.

We let $\mathrm{Der}(\mathcal{C})$ denote the fiber product $\mathrm{Fun}(\Delta^1, M^T(\mathcal{C})) \times_{\mathrm{Fun}(\Delta^1, \Delta^1 \times \mathcal{C})} \mathcal{C}$. We will refer to $\mathrm{Der}(\mathcal{C})$ as the ∞-category of derivations in $\mathcal{C}$.

Remark A.11. A derivation $\eta : A \to M$ can be identified with a map $d : L_A \to M$ in the fiber $T_{\mathcal{C}} \times_{\mathcal{C}} \{A\} \simeq \mathcal{S}p(\mathcal{C}/_A)$ since $A \to L_A$ is a coCartesian map in $M^T(\mathcal{C})$.

Definition A.12. Let $\mathcal{C}$ be a presentable ∞-category, and let $p : M^T(\mathcal{C}) \to \Delta^1 \times \mathcal{C}$ be a tangent correspondence for $\mathcal{C}$. An extended derivation is a diagram σ

$$\begin{tikzcd} \tilde{A} \arrow[r, "f"] \arrow[d] & A \arrow[d, "\eta"] \\ 0 \arrow[r] & M \end{tikzcd}$$

in $M^T(\mathcal{C})$ with the following properties:

(1) The diagram is a pullback square.

(2) The objects $\tilde{A}$ and A belongs to $\mathcal{C} \subset M^T(\mathcal{C})$, while 0 and M belong to $T_{\mathcal{C}} \subset M^T(\mathcal{C})$.
(3) Let $\bar{f} : \Delta^1 \to \mathcal{C}$ be the map which classifies the morphism f appearing in the diagram above, and let $e : \Delta^1 \times \Delta^1 \to \Delta^1$ be the unique map such that $e^{-1}\{0\} = \{0\} \times \{0\}$. Then the diagram

$$\begin{tikzcd}
\Delta^1 \times \Delta^1 \arrow[r, "\sigma"] \arrow[d, "e"] & M^T(\mathcal{C}) \arrow[r, "p"] & \Delta^1 \times \mathcal{C} \arrow[d] \\
\Delta^1 \arrow[rr, "\bar{f}"] & & \mathcal{C}
\end{tikzcd}$$

is commutative.
(4) The object $0 \in T_{\mathcal{C}}$ is a zero object of $\mathcal{S}p(\mathcal{C}/_A)$.

We let $\widetilde{\mathrm{Der}}$ denote the full subcategory of

$$\mathrm{Fun}(\Delta^1 \times \Delta^1, M^T(C)) \times_{\mathrm{Fun}(\Delta^1 \times \Delta^1, \Delta^1 \times C)} \mathrm{Fun}(\Delta^1, \mathcal{C})$$

spanned by the extended derivations.

Proposition A.13. *Let $\mathcal{C}$ be a presentable ∞-category. Then the forgetful $\phi : \widetilde{\mathrm{Der}}(\mathcal{C}) \to \mathrm{Der}(\mathcal{C})$ is a trivial Kan fibration.*

Definition A.14. Let $\mathcal{C}$ be a presentable ∞-category, and let $\Phi : \mathrm{Der}(\mathcal{C}) \to \widetilde{\mathrm{Der}}(\mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$, where the first map is any section of the trivial fibration and the second map is induced by the inclusion $\Delta^1 \times \{0\} \subset \Delta^1 \times \Delta^1$. We will denote the image of a derivation $(\eta : A \to M) \in \mathrm{Der}(\mathcal{C})$ under Φ by $(A^\eta \to A)$.

Let $f : \tilde{A} \to A$ be a morphism in $\mathcal{C}$. We will say f is a square-zero extension if there exists a derivation $\eta : A \to M$ in $\mathcal{C}$, and an equivalence $\tilde{A} \simeq A^\eta$ in the ∞-category $\mathcal{C}/_A$. In this case, we will also say that $\tilde{A}$ is a square-zero extension of A by $M[-1]$.

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