SYMMETRY AND THE CAUCHY IDENTITY

ETHAN LI

ABSTRACT. In this paper, we begin by introducing several fundamental bases of symmetric functions, including the monomial, elementary, complete homogeneous, and Schur bases, and discuss how these functions form the foundation of the algebra of symmetric functions. We then describe the Robinson–Schensted–Knuth (RSK) algorithm, a combinatorial bijection that reveals deep connections between matrices and pairs of semistandard Young tableaux. Finally, we present the Cauchy identity, a central result that unites these ideas by expressing a generating function for products of symmetric functions.

CONTENTS

1. INTRODUCTION

Symmetric functions occupy a central role in algebraic combinatorics, connecting representation theory, geometry, and probability. At their core, they are polynomials invariant under variable permutations, but their structure is remarkably rich: different families of symmetric functions form natural bases that reveal distinct algebraic and combinatorial properties.

One interesting bridge between algebra and combinatorics arises through the Robinson–Schensted–Knuth (RSK) algorithm. Originally formulated to study permutations, the RSK correspondence produces pairs of Young tableaux that encode deep information about sequences and symmetries. Its connections to symmetric functions become apparent when one explores generating functions and identities. We will end with the Cauchy identity as an application. The identity weaves together orthogonality, duality between bases, and the combinatorial structure revealed by RSK. Together, these ideas illustrate how symmetric functions serve as a unifying language across mathematics, linking algorithms, combinatorial structures, and algebraic identities.

1.1. **Background.**

First, we begin by defining a symmetric function.

Definition 1.1. A homogenous symmetric function of degree n over a commutative ring R (with identity) is a formal power series

$$f(x) = \sum_{\alpha} c_\alpha x^\alpha$$

where:
a) α ranges over all weak compositions* $(\alpha_1, \alpha_2, \ldots)$ of n (of infinite length)
b) $c_\alpha \in R$
c) $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$, with $x_i^0 = 1$
d) $f(x_{w(1)}, x_{w(2)}, \ldots) = f(x_1, x_2, \ldots)$ for all permutations w of the positive integers.

*A weak composition of n is a sequence of nonnegative integers to sum to n.

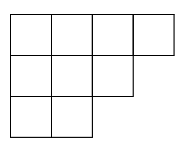
We denote the set of all symmetric functions of degree n over R by Λ_R^n. A concept central to indexing the bases of Λ_R^n is the partition.

Definition 1.2. A partition λ of n is a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{N}^k$, such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ and $\sum_i \lambda_i = n$. We also write this as $\lambda \vdash n$.

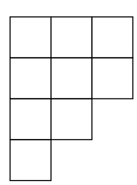
For example, the partitions of 4 are:

$$(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$$

We denote the set of all partitions of n by $\mathrm{Par}(n)$.
Partitions can be visualized using Young diagrams, where each component λ_i is represented by a row with λ_i squares, stacked in a decreasing order. The partition $(4,3,2)$ would look like



A natural way to get another partition from a given partition is to take the transpose of the diagram above. This is defined as a conjugate partition. For example, for the above partition, the conjugate would be $\lambda' = (3,3,2,1)$, and its corresponding Young diagram would be



Next, we want to define a dominance order on partitions in $\mathrm{Par}(n)$. If $\mu, \lambda \vdash n$, then $\mu \leq \lambda$ if for all $i \geq 1$,

$$\mu_1 + \mu_2 + \cdots + \mu_i \leq \lambda_1 + \lambda_2 + \cdots + \lambda_i$$

We can visualize the dominance ordering of partitions of 6 by

51

42

411 33

321

3111 222

2211

21111

111111

where $6 \geq 51 \geq 42$ and so on. 411 and 33 are incomparable, as are 3111 and 222. Thus, $42 \geq 411$ and $42 \geq 33$, and $411 \geq 321$ and $33 \geq 321$ (similarly for 3111 and 222).

From the above example, we observe that the dominance order is not a total order, i.e. not any two partitions can be compared.

From now on, we will consider $\Lambda^n_{\mathbb{Q}}$ and write it as Λ^n. Λ^n is a rational vector space, so to understand it, it is natural to find bases for it, which we will do in the next section.

2. Bases of Symmetric Functions

Monomial symmetric functions. The first, and most intuitive, basis we will define is the basis of monomial symmetric functions.

Definition 2.1. Given $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$, define a monomial symmetric function $m_\lambda(x) \in \Lambda^n$ as

$$m_\lambda = \sum_\alpha x^\alpha,$$

where the sum ranges over all distinct permutations $\alpha = (\alpha_1, \alpha_2, \ldots)$ of the entries of the vector $\lambda = (\lambda_1, \lambda_2, \ldots)$.

For example,

$$\begin{aligned} m_\emptyset &= 1 \\ m_1 &= \sum_i x_i \\ m_2 &= \sum_i x_i^2 \\ m_{11} &= \sum_{i<j} x_i y_j \end{aligned}$$

Proposition 2.2. *If $f = \sum_\alpha c_\alpha x^\alpha \in \Lambda^n$, then $f = \sum_{\lambda \vdash n} c_\lambda m_\lambda$.*

Proof. Let $f = \sum_\alpha c_\alpha x^\alpha \in \Lambda^n$. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be a weak composition of n such that $c_\alpha \neq 0$. Define $\lambda = (\lambda_1, \lambda_2, \ldots)$ to be the weakly decreasing rearrangement of the entries of α. Then, $\lambda \vdash n$ is a partition.

Let S be the set of all distinct permutations $\sigma = (\sigma_1, \sigma_2, \ldots)$ of the entries of the vector $\alpha = (\alpha_1, \alpha_2, \ldots)$, which is exactly the set of all distinct permutations of entries of the vector λ. By Definition 1.1, f is invariant under any permutation of variables, so x^σ must have the same coefficient as x^α and thus $c_\sigma = c_\alpha$, for all

$\sigma \in S$. Then, $\sum_{\sigma\in S} c_\sigma x^\sigma = \sum_{\sigma\in S} c_\alpha x^\sigma = c_\alpha \sum_{\sigma\in S} x^\sigma = c_\alpha m_\lambda$. Let $c_\lambda = c_\alpha$ for any α that has a weakly decreasing rearrangement of λ, which is well-defined since all such c_α are equivalent. Therefore, $f = \sum_\alpha c_\alpha x^\alpha = \sum_{\lambda\vdash n} c_\lambda m_\lambda$.

$\square$

Since $\{m_\lambda \mid \lambda \vdash n\}$ spans Λ^n and is linearly independent, $\{m_\lambda \mid \lambda \vdash n\}$ forms a basis for Λ^n. Thus, $\dim(\Lambda^n) = p(n)$. Furthermore, $\{m_\lambda \mid \lambda \in \mathrm{Par}\}$ is a basis for Λ.

Remark. In this paper, we define the bases of the symmetric functions as formal series in infinite variables. However, we can also picture these functions in finite variables. Consider Λ^3. $\dim(\Lambda^3) = 3$, since there are three partitions of 3. A basis for Λ^3 is $\{m_3, m_{21}, m_{111}\}$.

When we consider Λ^3 in three variables (i.e., $\Lambda^3(x_1, x_2, x_3)$), we have $\{m_3(x_1, x_2, x_3), m_{21}(x_1, x_2, x_3), m_{111}(x_1, x_2, x_3)\}$ as a basis. However, when we consider Λ^3 in two variables, we have $\{m_3(x_1, x_2, x_3), m_{21}(x_1, x_2, x_3)\}$ as a basis, since there are not enough variables for m_{111}. Therefore, $\dim(\Lambda^n(x_1, x_2, \ldots, x_k)) = \dim(\Lambda^n)$ if and only if $k \geq n$.

Elementary symmetric functions.

Definition 2.3. Define the elementary symmetric functions e_λ for $\lambda \in \mathrm{Par}$ by the formulas

$$e_n = m_{1^n} = \sum_{i_1<\cdots<i_n} x_{i_1}\cdots x_{i_n}, \quad n \geq 1 (\text{with } e_0 = m_\emptyset = 1)$$

$$e_\lambda = e_{\lambda_1} e_{\lambda_2}\cdots, \quad \text{if } \lambda = (\lambda_1, \lambda_2, \ldots).$$

Since $\{m_\lambda \mid \lambda \vdash n\}$ forms a basis for Λ^n, we can write

$$e_\lambda = \sum_{\mu\vdash n} M_{\lambda\mu} m_\mu \tag{2.4}$$

For example,

$$e_{111} = m_3 + 3m_{21} + 6m_{111}$$
$$e_{21} = m_{21} + 3m_{111}$$
$$e_3 = m_{111}$$

Proposition 2.5. *Let $\lambda \vdash n$, and let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be a weak composition of n. Then, the coefficient $M_{\lambda\alpha}$ of x^α in e_λ is equal to the number of $(0,1)$ matrices $A = (a_{ij})_{i,j\geq 1}$ satisfying* $\mathrm{row}(A) = \lambda$ *and* $\mathrm{col}(A) = \alpha$.

Proof. Consider a matrix

$$X = \begin{bmatrix} x_1 & x_2 & x_3 & \ldots \\ x_1 & x_2 & x_3 & \ldots \\ x_1 & x_2 & x_3 & \ldots \\ \ldots & & & \end{bmatrix}$$

Pick an arbitrary x^α term in the expansion of $e_\lambda = e_{\lambda_1} e_{\lambda_2}\cdots$. Then, each e_{λ_n} contributes a $x_{i_1}\cdots x_{i_{\lambda_n}}$ factor to x^α. We can represent this factor by choosing λ_n entries from row n. Convert each chosen entry in matrix X to 1's and convert all other entries to 0's. Thus, $\mathrm{row}(X) = \lambda$, $\mathrm{col}(X) = \alpha$ and X corresponds to the chosen x^α. Each unique $(0,1)$ matrix A satisfying $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \alpha$ represents a different way that x^α appears in the expansion of $e_\lambda = e_{\lambda_1} e_{\lambda_2}\cdots$, so therefore, the number of $(0, 1)$ matrices A satisfying $\text{row}(A) = \lambda$ and $\text{col}(A) = \alpha$ is equal to the coefficient $M_{\lambda\alpha}$ of x^α.

□

Since an arbitrary x^α is unique to some m_μ, $M_{\lambda\alpha} = M_{\lambda\mu}$.

For example,

$$e_{11} = m_2 + 2m_{11},$$

since (2) and $(1, 1)$ are the partitions of 2, and there is one $(0, 1)$ matrix with $\text{row}(A) = (1, 1)$ and $\text{col}(A) = (2)$:

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

and two $(0, 1)$ matrices with $\text{row}(A) = (1, 1)$ and $\text{col}(A) = (1, 1)$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For every $(0, 1)$ matrix that satisfies $\text{row}(A) = \lambda$ and $\text{col}(A) = \alpha$, its transpose satisfies $\text{row}(A) = \alpha$ and $\text{col}(A) = \lambda$. Therefore, $M_{\lambda\mu} = M_{\mu\lambda}$. The following theorem uses such $M_{\lambda\mu}$'s to form a transition matrix from the monomial symmetric functions to a new basis, ultimately proving that the elementary symmetric functions are indeed a basis as well.

Theorem 2.6. *Let $\lambda, \mu \vdash n$. Then:*
a) If $M_{\lambda\mu} \neq 0$, then $\mu \leq \lambda'$.
b) $M_{\lambda\lambda'} = 1$.
c) $\{e_\lambda \mid \lambda \vdash n\}$ is a basis for Λ^n.

Suppose $M_{\lambda\mu} \neq 0$. Then, by Proposition 2.4, there exists at least one $(0, 1)$ matrix A such that $\text{row}(A) = \lambda$ and $\text{col}(A) = \mu$. Let A' be the matrix A with its 1's left-adjusted. In other words, $A' = (a_{ij})$, where $a_{ij} = 1$ if $1 \leq j \leq \lambda_i$. Then, $\text{row}(A') = \lambda$ and $\text{col}(A') = \lambda'$. For each i, the sum of the first i columns of A' will be greater than or equal to the sum of the first i columns of A. Therefore, $\text{col}(A') \geq \text{col}(A)$ in the dominance order. Since $\text{col}(A') = \lambda'$ and $\text{col}(A) = \mu$, $\lambda' \geq \mu$. Furthermore, A' is the only matrix such that $\text{row}(A') = \lambda$ and $\text{col}(A') = \lambda'$, so $M_{\lambda\lambda'} = 1$.
Let $p(n)$ be the number of partitions of n. Consider an ordering $\lambda^1, \lambda^2, \ldots, \lambda^{p(n)}$ of $\text{Par}(n)$, where either $\lambda^i < \lambda^{i+1}$ or λ^i and λ^{i+1} are incomparable, for all $1 \leq i < p(n)$. Let $(M_{\lambda\mu}) = (m_{ij})$ be a matrix, where $m_{ij} = M_{(\lambda^j)'\lambda^i}$. Then, if $i > j$, either $\lambda^i > \lambda^j$ or λ^i and λ^j are incomparable, so by the contrapositive of Statement A, $M_{(\lambda^j)'\lambda^i} = 0$. If $i = j$, then by Statement B, $M_{(\lambda^j)'\lambda^i} = 1$. Therefore, $(M_{\lambda\mu})$ is an upper triangular matrix with 1's on its diagonal, so it is invertible. Thus, $(M_{\lambda\mu})$ is a transition matrix between two bases of Λ^n, $\{m_\lambda \mid \lambda \vdash n\}$ and $\{e_\lambda \mid \lambda \vdash n\}$, so $\{e_\lambda \mid \lambda \vdash n\}$ is a basis for Λ^n.

Complete homogeneous symmetric functions.

Definition 2.7. Define the complete homogeneous symmetric functions h_λ for $\lambda \in$ Par by the formulas

$$h_n = \sum_{\lambda \vdash n} m_\lambda = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n}, \quad n \geq 1 \text{ (with } h_0 = m_\emptyset = 1)$$

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots, \quad \text{if } \lambda = (\lambda_1, \lambda_2, \ldots).$$

Since $\{m_\lambda \mid \lambda \vdash n\}$ forms a basis for Λ^n, we can write

$$h_\lambda = \sum_{\mu \vdash n} N_{\lambda\mu} m_\mu \tag{2.8}$$

Proposition 2.9. *Let $\lambda \vdash n$, and let $\alpha = (\alpha_1, \alpha_2, \dots)$ be a weak composition of n. Then, the coefficient $N_{\lambda\alpha}$ of x^α in h_λ is equal to the number of $\mathbb{N}$ matrices $A = (a_{ij})_{i,j\geq 1}$ satisfying* $\mathrm{row}(A) = \lambda$ *and* $\mathrm{col}(A) = \alpha$.

Proof. Consider a matrix

$$X = \begin{bmatrix} x_1 & x_2 & x_3 & \dots \\ x_1 & x_2 & x_3 & \dots \\ x_1 & x_2 & x_3 & \dots \\ \dots & & & \end{bmatrix}$$

Pick an arbitrary x^α term in the expansion of $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$. Each h_{λ_n} contributes a $x_{i_1}^{\alpha_{k_1}} \cdots x_{i_m}^{\alpha_{k_m}}$ factor to x^α, where $m \leq \lambda_n$ and $\alpha_{k_1} + \cdots + \alpha_{k_m} = \lambda_n$. We can represent this factor by choosing m entries from row n. Convert each chosen entry x_{i_j} in matrix X to α_{k_j}'s and convert all other entries to 0's. Thus, $\mathrm{row}(X) = \lambda$, $\mathrm{col}(X) = \alpha$ and X corresponds to the chosen x^α. Each unique $\mathbb{N}$ matrix A satisfying $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \alpha$ represents a different way that x^α appears in the expansion of $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$, so therefore, the number of $\mathbb{N}$ matrices A satisfying $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \alpha$ is equal to the coefficient $N_{\lambda\alpha}$ of x^α. □

The following identity proves that $(1-x)^{-1}$ is equal to the formal power series $\sum_{n=0} x^n$, which will be immediately useful in Proposition 2.11.

$$\frac{1}{1-x} = \sum_{n=0} x^n \tag{2.10}$$

Proof.

$$\begin{aligned} &(1-x)\sum_{n=0} x^n \\ &= \sum_{n=0} x^n - x\sum_{n=0} x^n \\ &= \sum_{n=0} x^n - \sum_{n=1} x^n \\ &= 1 \end{aligned}$$

□

The following proposition will be relevant in proving the Cauchy identity.

Proposition 2.11. $\displaystyle\prod_{i,j}(1 - x_i y_j)^{-1} = \sum_{\lambda,\mu} N_{\lambda\mu} m_\lambda(x) m_\mu(y) = \sum_\lambda m_\lambda(x) h_\lambda(y)$

Proof. By Equation 2.10, $\displaystyle\prod_{i,j}(1 - x_i y_j)^{-1} = \prod_{i,j}\sum_{n=0}(x_i y_j)^n$. Pick an arbitrary term $x^\alpha y^\beta = x_1^{\alpha_1} x_2^{\alpha_2} \cdots y_1^{\beta_1} y_2^{\beta_2} \cdots$ in this expansion. We can write $x^\alpha y^\beta = \displaystyle\prod_{i,j}(x_i y_j)^{a_{ij}}$, where a_{ij} is a natural number or 0. Such a_{ij}'s form a $\mathbb{N}$-matrix A with $\mathrm{row}(A) = \alpha$

and $\mathrm{col}(A) = \beta$. By Proposition 2.9, $\prod_{i,j}(1 - x_i y_j)^{-1} = \sum_{\lambda,\mu} N_{\lambda\mu} m_\lambda(x) m_\mu(y)$, and by Equation 2.8, $\sum_{\lambda,\mu} M_{\lambda\mu} m_\lambda(x) m_\mu(y) = \sum_{\lambda} m_\lambda(x) h_\lambda(y)$. □

Definition 2.12. Define an endomorphism $\omega : \Lambda \to \Lambda$ by $\omega(e_n) = h_n$, $n \geq 1$.

Theorem 2.13. *The endomorphism ω is an involution, i.e. $\omega^2 = 1$, or equivalently, $\omega(h_n) = e_n$.*

Proof. Consider the formal power series

$$H(t) := \sum_{n \geq 0} h_n t^n \in \Lambda[[t]]$$

$$E(t) := \sum_{n \geq 0} e_n t^n \in \Lambda[[t]]$$

First, we want to show that $H(t) = \prod_n (1 - x_n t)^{-1}$. From (2.10), $\prod_n (1 - x_n t)^{-1} = \prod_n \sum_{i=0} x_n^i t^i$.

Fix $n \geq 0$. The coefficient of t^n in $\prod_n \sum_{i=0} x_n^i t^i$ is $\sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n} = h_n$. Therefore, $\prod_n (1 - x_n t)^{-1} = \prod_n \sum_{i=0} x_n^i t^i = \sum_{n \geq 0} h_n t^n = H(t)$.
Similarly, $E(t) = \prod_n (1 + x_n t)$.
Thus, $H(t)E(-t) = 1$.

$$\Big(\sum_{n \geq 0} h_n t^n\Big)\Big(\sum_{n \geq 0} e_n (-t)^n\Big) = 1$$

Fix $n \geq 1$. A term in the expansion of $(\sum_{n \geq 0} h_n t^n)(\sum_{n \geq 0} e_n (-t)^n)$ containing t^n is a product of $h_{n-i} t^{n-i}$ and $e_i (-t)^i$. Therefore, the coefficient of t^n in $(\sum_{n \geq 0} h_n t^n)(\sum_{n \geq 0} e_n (-t)^n)$ is $\sum_{i=0}^{n} (-1)^i e_i h_{n-i}$. It follows that $\sum_{i=0}^{n} (-1)^i e_i h_{n-i} = 0$.

We want to show that if $\sum_{i=0}^{n} (-1)^i u_i h_{n-i} = 0$ for all $n \geq 1$, and if $u_i \in \Lambda$ with $u_0 = 1$, then $u_i = e_i$.
We prove by induction. Let $n = 1$. Then, $u_0 h_1 - u_1 h_0 = 0$. Since u_0 and h_0 equal 1, then $h_1 - u_1 = 0$ and $h_1 = u_1$. Since $h_1 = e_1$, $u_1 = e_1$.
Inductive Step: Fix $n \geq 1$. Suppose that $u_0 = e_0$, $u_1 = e_1$, $\ldots$, $u_{n-1} = e_{n-1}$.

$$\sum_{i=0}^{n} (-1)^i u_i h_{n-i} = \sum_{i=0}^{n} (-1)^i u_{n-i} h_i = 0$$

$$\sum_{i=1}^{n} (-1)^i u_{n-i} h_i - u_n h_0 = 0$$

$$u_n = \sum_{i=1}^{n}(-1)^i u_{n-i}h_i = \sum_{i=1}^{n}(-1)^i e_{n-i}h_i$$

Since we know that

$$\sum_{i=0}^{n}(-1)^i e_i h_{n-i} = 0,$$

we can similarly rearrange like shown above to get

$$e_n = \sum_{i=1}^{n}(-1)^i e_{n-i}h_i,$$

so therefore $u_n = e_n$.

$$\sum_{i=0}^{n}(-1)^i e_i h_{n-i} = (-1)^n \sum_{i=0}^{n}(-1)^i e_{n-i}h_i$$

Applying ω gives

$$0 = \sum_{i=0}^{n}(-1)^i h_i \omega(h_{n-i}) = (-1)^n \sum_{i=0}^{n}(-1)^i h_{n-i}\omega(h_i) = 0.$$

Then,

$$\sum_{i=0}^{n}(-1)^i h_{n-i}\omega(h_i) = 0 = \sum_{i=0}^{n}(-1)^i e_i h_{n-i},$$

so therefore, $\omega(h_n) = e_n$.

$\square$

Since $\{e_\lambda \mid \lambda \vdash n\}$ is a basis for Λ^n, and $\omega : \Lambda \to \Lambda$, $\omega(e_n) = h_n$ is invertible, $\{h_\lambda \mid \lambda \vdash n\}$ is a basis for Λ^n.

Schur functions. Now, we will define another interesting basis for the symmetric functions, the Schur functions.
First, we introduce semistandard Young tableux (SSYT). Let λ be a partition. An SSYT of shape λ is an array $T = (T_{ij})$, where $1 \leq i \leq l(\lambda)$ and $1 \leq j \leq \lambda_i$, which is weakly increasing in every row and strictly increasing in every column.

For example, an SSYT of shape $(5, 3, 2, 1)$ is given by

$$\begin{array}{ccccc} 1 & 1 & 2 & 3 & 5 \\ 2 & 3 & 3 & & \\ 3 & 5 & & & \\ 6 & & & & \end{array}$$

An SSYT T has type $\alpha = (\alpha_1, \alpha_2, \ldots)$ if T has $\alpha_i = \alpha_i(T)$ parts equal to i. We can also write

$$x^T = x_1^{\alpha_1(T)} x_2^{\alpha_2(T)} \cdots .$$

Thus, the SSYT above has type $\alpha(T) = (2, 2, 4, 0, 2, 1)$ and $x^T = x_1^2 x_2^2 x_3^4 x_5^2 x_6$.
Let μ be a partition. $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all i. A semistandard Young tableux of skew shape λ/μ is an array $T = (T_{ij})$, where $1 \leq i \leq l(\lambda)$ and $\mu_i \leq j \leq \lambda_i$, which is weakly increasing in every row and strictly increasing in every column.

Definition 2.14. Let λ/μ be a skew shape. The skew Schur function $s_{\lambda/\mu}$ is defined as

$$s_{\lambda/\mu}(x) = \sum_T x^T$$

summed over all SSYT T of shape λ/μ. If $\mu = \emptyset$, then $\lambda/\mu = \lambda$, and we define s_λ as the Schur function of shape λ.

For example, consider $\lambda = (2,1)$ and $\mu = (1)$. Then,

$$s_\lambda(x_1,x_2,x_3) = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + 2x_1x_2x_3$$

and

$$s_{\lambda/\mu}(x_1,x_2,x_3) = x_1^2 + x_2^2 + x_3^2 + 2\,(x_1x_2 + x_1x_3 + x_2x_3).$$

Theorem 2.15. *For any skew shape λ/μ, the skew Schur function $s_{\lambda/\mu}$ is a symmetric function.*

Proof. It suffices to show that $s_{\lambda/\mu}$ is invariant under interchanging x_i and x_{i+1}, since any permutation of the positive integers is a composition of transpositions of arbitrary i and $i+1$. Suppose that $|\lambda/\mu| = n$ and that $\alpha = (\alpha_1, \alpha_2, \ldots)$ is a weak composition of n. Let

$$\tilde{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_{i+1}, \alpha_i, \alpha_{i+2}, \ldots).$$

Let $\mathcal{T}_{\lambda/\mu,\alpha}$ denote the set of all SSYT of shape λ/μ and type α. We want to show that there exists a bijection $\varphi : \mathcal{T}_{\lambda/\mu,\alpha} \to \mathcal{T}_{\lambda/\mu,\tilde{\alpha}}$.
Let $T \in \mathcal{T}_{\lambda/\mu,\alpha}$. Each column of T either contains both i and $i+1$, neither i nor $i+1$, or only one or the other, since columns are strictly increasing. We ignore the columns of the first two kinds because each column already contains the same number of i and $i+1$. Denote all entries of i and $i+1$ in the columns of the last kind by i' and $(i+1)'$ respectively. In each row, there will be m number of i' and n number of $(i+1)'$, i.e. each row will contain a sequence

$$i_1' \quad i_2' \quad \cdots \quad i_m' \quad (i+1)_1' \quad (i+1)_2' \quad \cdots \quad (i+1)_n'$$

for $m, n \geq 0$. We replace this sequence with n number of i and m number of $i+1$, so that each row instead contains

$$i_1' \quad i_2' \quad \cdots \quad i_n' \quad (i+1)_1' \quad (i+1)_2' \quad \cdots \quad (i+1)_m'$$

Then, the resulting array $\varphi(T) \in \mathcal{T}_{\lambda/\mu,\tilde{\alpha}}$, so thus we establish our bijection.
For all $T, T' \in \mathcal{T}_{\lambda/\mu,\alpha}$, $x^T = x^{T'} = x^\alpha$. Since there is a bijection between $\mathcal{T}_{\lambda/\mu,\alpha}$ and $\mathcal{T}_{\lambda/\mu,\tilde{\alpha}}$, $|\mathcal{T}_{\lambda/\mu,\alpha}| = |\mathcal{T}_{\lambda/\mu,\tilde{\alpha}}|$, so the coefficient of x^α in $s_{\lambda/\mu}$ is equal to the coefficient of $x^{\tilde{\alpha}}$ in $s_{\lambda/\mu}$. This is true for all α and $\tilde{\alpha}$, so therefore, $s_{\lambda/\mu}$ is invariant under interchanging x_i and x_{i+1} and is thus symmetric. □

Define the skew Kostka number $K_{\lambda/\nu,\alpha}$ as the number of SSYT of shape λ/ν and type α. Then,

$$s_{\lambda/\nu} = \sum_\alpha K_{\lambda/\nu,\alpha} x^\alpha$$

Since $s_{\lambda/\nu}$ is symmetric, we can write it in terms of the monomial symmetric functions:

$$s_{\lambda/\nu} = \sum_{\mu \vdash n} K_{\lambda/\nu,\mu} m_\mu$$

3. RSK Algorithm

First, we want to define the row insertion $P \leftarrow k$ of a positive integer k into a nonskew SSYT $P = (P_{ij})$:
Let r be the largest positive integer such that $P_{1,r-1} \leq k$. If $P_{11} > k$, then $r = 0$. If P_{1r} does not exist (i.e. k is greater than or equal to all entries in the first row), then insert k at the end of the row. Then, the insertion stops and the resulting SSYT is $P \leftarrow k$. If P_{1r} does exist, then replace it with k and insert P_{1r} into the next row as we just did with k. The insertion process stops when an entry is inserted at the end of a row, and the resulting SSYT is $P \leftarrow k$.
For example, let

$$P = \begin{matrix} 1 & 2 & 3 & 5 & 8 \\ 2 & 3 & 6 & & \\ 3 & 5 & & & \\ 6 & & & & \end{matrix}$$

Then,

$$P \leftarrow 3 = \begin{matrix} 1 & 2 & 3 & \mathbf{3} & 8 \\ 2 & 3 & \mathbf{5} & & \\ 3 & 5 & \mathbf{6} & & \\ 6 & & & & \end{matrix}$$

with the inserted elements shown in bold.

Corollary 3.1. *If P is an SSYT and $k \geq 1$, then $P \leftarrow k$ is also an SSYT.*

Let $A = (a_{ij})_{i,j \geq 1}$ be an $\mathbb{N}$-matrix of finite support, i.e. with finitely many zero entries. A can be uniquely determined by a two-line array:

$$w_A = \begin{pmatrix} i_1 & i_2 & \dots & i_m \\ j_1 & j_2 & \dots & j_m \end{pmatrix}$$

where a) $i_1 \leq i_2 \leq \ldots \leq i_m$, b) if $i_r = i_s$ and $r \leq s$, then $j_r \leq j_s$, and c) for each entry (a_{ij}) in A, there are a_{ij} values of n such that $(i_n, j_n) = (i, j)$. For example, let

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then,

$$w_A = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 1 & 2 & 2 & 2 & 3 \end{pmatrix}$$

Now we define the RSK algorithm,

$$A \xrightarrow{\text{RSK}} (P, Q),$$

where P and Q are a pair of SSYT of the same shape.
Let

$$w_A = \begin{pmatrix} i_1 & i_2 & \dots & i_m \\ j_1 & j_2 & \dots & j_m \end{pmatrix}$$

Start with $(P(0), Q(0)) = (\emptyset, \emptyset)$. Then, if $t < m$ and $(P(t), Q(t)$ is defined,

a) $P(t+1) = P(t) \leftarrow j_{t+1}$

b) $Q(t+1)$ is obtained by keeping all parts of $Q(t)$ unchanged and inserting i_{t+1}

such that $Q(t+1)$ has the same shape as $P(t+1)$.

The algorithm ends at $(P(m), Q(m)) = (P, Q)$.

For example, let

$$w_A = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 1 & 2 & 2 & 2 & 3 \end{pmatrix}$$

Then,

$$P(1) = \begin{array}{c} 2 \end{array}$$

$$P(2) = \begin{array}{cc} 2 & 3 \end{array}$$

$$P(3) = \begin{array}{ccc} 2 & 3 & 3 \end{array}$$

$$P(4) = \begin{array}{ccc} 1 & 3 & 3 \\ 2 & & \end{array}$$

$$P(5) = \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & \end{array}$$

$$P(6) = \begin{array}{ccc} 1 & 2 & 2 \\ 2 & 3 & 3 \end{array}$$

$$P(7) = \begin{array}{cccc} 1 & 2 & 2 & 2 \\ 2 & 3 & 3 & \end{array}$$

$$P = P(8) = \begin{array}{ccccc} 1 & 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & & \end{array}$$

and

$$Q(1) = \begin{array}{c} 1 \end{array}$$

$$Q(2) = \begin{array}{cc} 1 & 1 \end{array}$$

$$Q(3) = \begin{array}{ccc} 1 & 1 & 1 \end{array}$$

$$Q(4) = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & & \end{array}$$

$$Q(5) = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & \end{array}$$

$$Q(6) = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}$$

$$Q(7) = \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & \end{array}$$

$$Q = Q(8) = \begin{array}{ccccc} 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & & \end{array}$$

Given this definition of the RSK algorithm, we seek to observe a bijection, which we can use to connect the generating function seen earlier with Schur functions.

Theorem 3.2. *The RSK algorithm is a bijection between* $\mathbb{N}$*-matrices* $A = (a_{ij})_{i,j\geq 1}$ *of finite support and ordered pairs* (P, Q) *of SSYT of the same shape.*

Proof. Consider a pair of SSYT (P, Q). We can identify the last element inserted into Q by finding the rightmost entry of the largest value in Q. This element would be j_m. Suppose j_m is in Q at row r and column s. Then, denote the entry in P at row r and column s as P_{rs}. P_{rs} was either inserted there or bumped by an element from the previous row, so we can reverse the insertion process to recover i_m. We can do this for all $(P(n), Q(n))$ to uniquely recover w_A, and thus A, so the RSK algorithm is bijective. □

4. Cauchy Identity

An important application of the RSK algorithm is the Cauchy Identity, which equates a generating function with the sum of Schur functions, connecting combinatorics and algebra. It also has many other applications in symmetric function theory, such as Hall-Littlewood polynomials.

Theorem 4.1.

$$\prod_{i,j}(1-x_iy_j)^{-1} = \sum_{\lambda} s_\lambda(x)s_\lambda(y)$$

Proof. By (2.11),

$$\prod_{i,j}(1-x_iy_j)^{-1} = \prod_{i,j}\sum_{n=0}(x_iy_j)^n$$

By Proposition 2.12, the coefficient of a term $x^\alpha y^\beta$ in the expansion of $\prod_{i,j}\sum_{n=0}(x_iy_j)^n$ is equal to the number of $\mathbb{N}$-matrices A that satisfy $\text{row}(A) = \alpha$ and $\text{col}(A) = \beta$. By Definition 2.15, the coefficient of a term $x^\alpha y^\beta$ in the expansion of $\sum_\lambda s_\lambda(x)s_\lambda(y)$ is the number of pairs of SSYT (P, Q), where $\text{type}(P) = \alpha$ and $\text{type}(Q) = \beta$. Since the RSK algorithm is a bijection between $\mathbb{N}$-matrices A that satisfy $\text{row}(A) = \alpha$ and $\text{col}(A) = \beta$ and pairs of SSYT (P, Q) with $\text{type}(P) = \alpha$ and $\text{type}(Q) = \beta$, $\prod_{i,j}(1-x_iy_j)^{-1} = \sum_\lambda s_\lambda(x)s_\lambda(y)$. □

In this paper, we identified the symmetric functions as a vector space, and as such, we defined several important bases. The final basis we defined, and a key concept in symmetric function theory, is the Schur function. It allowed us to set up the RSK algorithm, ultimately resulting in the Cauchy Identity, which is an application prevalent in representation theory, combinatorics, and algebraic geometry.

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