AN INTRODUCTION TO THE STEINBERG VARIETY

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Abstract. The purpose of this paper is to provide an introduction to the Steinberg variety, which is an important notion in geometric representation theory. It is used to study the representations of Weyl groups. Throughout the paper, various geometric objects associated to a complex semisimple algebraic group G, including the flag variety $\mathcal{B}$ and the Springer resolution of the nilpotent cone variety $\mathcal{N}$, are studied. In particular, the fundamental example of $SL_2(\mathbb{C})$ will be developed, in the hope of providing a concrete idea behind the geometric notions.

Contents

1. Introduction

The irreducible representations of the Weyl group can be studied via the Steinberg variety. More precisely, the group algebra of the Weyl group can be constructed

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(via the so-called "Lagrangian construction") as a convolution algebra of the top Borel-Moore homology of the Steinberg variety (see [2]). The Steinberg variety is a geometric object associated to a semisimple algebraic group G, in the sense that it is the disjoint union of conormal bundles to all G-diagonal orbits in $\mathcal{B} \times \mathcal{B}$, the product of flag varieties $\mathcal{B}$ of G (see Corollary 6.5). Our goal is to give a brief introduction of this variety and use it to study the Springer fibers $\mathcal{B}_x$ and nilpotent orbits $\mathbb{O} \subset \mathcal{N}$.

This paper closely follows the exposition of [2]. We first introduce the geometric flag variety $\mathcal{B}$ and use it to give a geometric interpretation of the Bruhat decomposition, which can be viewed as a pure algebraic statement about double-cosets in G. The course of the proof is based on the Bialynicki-Birula decomposition (Theorem 7.5), which we briefly explain in the appendix. After that, we associate the double-coset decomposition of G with the abstract Weyl group $\mathbb{W}$ of G and elaborate on Springer's diagram (3.30). We make a digression towards symplectic geometry, with the aim of introducing the notion of Lagrangian subvarieties and moment maps. Then we turn to the nilpotent cone variety $\mathcal{N}$, which is another geometric object associated to G. The Springer resolution $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ turns out to be a moment map with respect to the Hamiltonian G-action on the cotangent bundle $\mathrm{T}^* \mathcal{B}$ of the flag variety. It is, moreover, a resolution of singularities. Finally, we introduce the Steinberg variety $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$ as a triple variety, and study its geometric properties. The appendix attempts to make the exposition a self-contained one. It is devoted to reviewing some geometric notions, e.g., the formulation of vector bundles in algebraic varieties and the statement of the Bialynicki-Birula decomposition. Some useful lemmas in algebraic varieties are also discussed at the end of the appendix.

Convention 1.1. We will be working on algebraic varieties over the complex number field $\mathbb{C}$. Some statements, in particular in the section on symplectic geometry, are formulated in the smooth category. But we expect similar statements to hold in the setting of algebraic geometry. If not stated explicitly, we assume G to be a connected semisimple algebraic group over $\mathbb{C}$ and $\mathfrak{g}$ be its (semisimple) Lie algebra. We denote $B\backslash G$ (resp. G/B) the left (resp. right) orbit space of the subgroup $B \subset G$. The right orbit of $g \in G$ is denoted by gB. All vector spaces in this paper are assumed to be finite-dimensional over $\mathbb{C}$. The dualization of the vector space V is denoted by V^*. We denote $\mathbb{P}(V)$ to be the projectivization of the vector space V. Given a finitely generated $\mathbb{C}$-algebra A, we let $\operatorname{Specm}(A)$ denote the set of all maximal ideals in A. The permutation group of n elements is denoted by S_n.

2. Flag Varieties

Definition 2.1. Let V be a complex vector space of dimension n. A *flag* in V is a chain $0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_k = V$ of subspaces of V, each properly included in the next. A *full flag* is one for which $k = n$, i.e., $\dim V_{i+1}/V_i = 1$. Let $\mathcal{F}(V)$ denote the collection of all full flags in V.

We will give it the structure of a projective variety, to be called the *flag variety*. To this end, we will use the notion of Grassmannians (c.f., [5], [8]), whose construction will be briefly recalled in the following.

Construction 2.2 (Grassmannians)**.** Let V be an n-dimensional $\mathbb{C}$-vector space. The Grassmannian variety, denoted as $\mathrm{Gr}_d(V)$, is the collection of all d-dimensional vector subspaces in V. Let U be a linear subspace of V of codimension d, and let

$\mathrm{Gr}_d(V)_U$ be the collection of d-dimensional linear subspaces of V contained in $\mathbb{P}(V) \smallsetminus \mathbb{P}(U)$, i.e., the direct sum complemenet of U in V. Observe that $\mathrm{Gr}_d(V)_U$ has a natural structure of an affine space of dimension $(n-d)d$. Indeed, we note that every element in $\mathrm{Gr}_d(V)_U$ can be identified with the image of a section to the natural projection $\pi : V \to V/U$. The set of sections of π is naturally a principal homogeneous space over $\mathrm{Hom}_{\mathbb{C}}(V/U, U)$ via addition. Upon fixing a base point in $\mathrm{Gr}_d(V)_U$, we have the following identification $\mathrm{Gr}_d(V)_U \cong \mathrm{Hom}_{\mathbb{C}}(V/U, U) \cong \mathbb{C}^{d(n-d)}$. It can be shown that $\mathrm{Gr}_d(V)$ admits a unique structure of variety for which every $\mathrm{Gr}_d(V)_U$ is affine open and its identification with $\mathbb{C}^{d(n-d)}$ is an isomorphism.

We illustrate an alternative construction of the Grassmannian, using the so-called Plücker embeddings. It is a fact (c.f. [8]) that the two constructions coincide.

Construction 2.3. The injective map $\delta : \mathrm{Gr}_d(V) \hookrightarrow \mathbb{P}(\bigwedge^d V)$, called the Plücker embedding, is given by $W \mapsto [w_1 \wedge \cdots \wedge w_d]$, where $\{w_i\}_{i=1}^d$ is a basis of $W \in \mathrm{Gr}_d(V)$. It sends the Grassmannian $\mathrm{Gr}_d(V)$ bijectively onto a closed subset of $\mathbb{P}(\bigwedge^d V)$. Choose a basis $(v_1, \ldots, v_n)$ of V, and choose U to the span of $(v_{d+1}, \ldots, v_n)$. Denote $\mathbb{P}(\bigwedge^d V)_U$ to be a standard affine open chart of $\mathbb{P}(\bigwedge^d V)$ consisting of points whose homogeneous coordinate relative to $v_1 \wedge \cdots \wedge v_d$ is nonzero. Then $\delta : \mathbb{C}^{(n-d)\times d} \cong \mathrm{Gr}_d(V)_U \to \mathbb{P}(\bigwedge^d V)_U$ is an isomorphism onto its image. We now give an explicit expression of δ over this affine chart. Fix $W := \mathrm{span}_{\mathbb{C}}\{v_1, \ldots, v_d\} \in \mathrm{Gr}_d(V)_U$ and the base point $Id|_W \in \mathrm{Gr}_d(V)$ (viewed as a section of $\pi : V \to W \cong V/U$), the identification $\mathbb{C}^{(n-d)\times d} \cong \mathrm{Hom}_{\mathbb{C}}(W, U) \cong \mathrm{Gr}_d(V)_U$ is given by

$$(a_{ji})_{1\le i\le d<j\le n} \in \mathbf{M}_{(n-d)\times d}(\mathbb{C}) \longmapsto \mathrm{span}_{\mathbb{C}}\{v_i + \sum_{j=d+1}^{n} a_{ji} v_j\}_{i=1}^d. \tag{2.4}$$

So that δ is given by

$$(a_{ji})_{1\le i\le d<j\le n} \mapsto \left[\left(v_1 + \sum_{j=d+1}^{n} a_{j1} v_j\right) \wedge \cdots \wedge \left(v_d + \sum_{j=d+1}^{n} a_{jd} v_j\right)\right]. \tag{2.5}$$

It follows that δ defines a closed immersion of $\mathrm{Gr}_d(V)_U$ onto its image. As a consequence, the Grassmannian $\mathrm{Gr}_d(V)$ is an irreducible closed subvariety of $\mathbb{P}(\bigwedge^d V)$.

We return to the discussion on flag varieties.

Proposition 2.6 (Flag varieties)**.** Let V be an n-dimensional complex vector space and $\mathcal{F}(V)$ the collection of all full flags in V. Identify $\mathcal{F}(V)$ with a subset of the Cartesian product $\mathbb{G} = \mathrm{Gr}_1(V) \times \mathrm{Gr}_2(V) \times \cdots \times \mathrm{Gr}_n(V)$. Then $\mathcal{F}(V)$ is closed $\mathbb{G}$. In particular, $\mathcal{F}(V)$ is a projective variety.

Proof. When $n = 1$, there is nothing to prove. So we assume $n \ge 2$. Let $1 \le d \le n-1$, we claim that the set F_d of pairs $(W_1, W_2) \in \mathrm{Gr}_d(V) \times \mathrm{Gr}_{d+1}(V)$ for which $W_1 \subset W_2$ is closed. The statement follows from the claim via taking a finite intersection of F_d (identified as a subset of $\mathbb{G}$) for $1 \le d \le n-1$. We proceed to prove the claim. Fix a basis $(v_1, \ldots, v_n)$ of V and consider various affine open subsets of $\mathbb{P}(\bigwedge^d V)$ (resp. $\mathbb{P}(\bigwedge^{d+1} V)$). Without loss of generality, we restrict ourselves to the open chart $\mathrm{Gr}_d(V)_U$ (resp. $\mathrm{Gr}_{d+1}(V)_{U'}$) (c.f. construction 2.3), which corresponds to (via the Plücker embedding) the restriction of an affine chart of $\mathbb{P}(\bigwedge^d V)$ consisting of points whose homogeneous coordinate relative to $v_1 \wedge \cdots \wedge v_d$ (resp. $v_1 \wedge \cdots \wedge v_{d+1}$) is nonzero.

Indeed, F_d is already covered by products of the form $\mathrm{Gr}_d(V)_U \times \mathrm{Gr}_{d+1}(V)_{U'}$. Therefore, the embedding

$$i : F_d \hookrightarrow \mathrm{Gr}_d(V) \times \mathrm{Gr}_{d+1}(V) \hookrightarrow \mathbb{P}(\bigwedge^d V) \times \mathbb{P}(\bigwedge^{d+1} V)$$

restricted to such an affine open chart is given explicitly by

$$\begin{aligned} \mathbb{C}^{(n-d)\times d} \times \mathbb{C}^{(n-d-1)\times(d+1)} &\longrightarrow \mathbb{P}(\bigwedge^d V)_U \times \mathbb{P}(\bigwedge^{d+1} V)_{U'} \\ ((a_{ji})_{1\le i\le d<j\le n}, (b_{ji})_{1\le i\le d+1<j\le n}) &\longmapsto \left(\bigwedge_{i=1}^{d} \left(v_i + \sum_{j=d+1}^{n} a_{ji} v_j \right), \bigwedge_{i=1}^{d+1} \left(v_i + \sum_{j=d+2}^{n} b_{ji} v_j \right) \right). \end{aligned}$$

The condition that $W_1 \subset W_2$ is precisely when the matrix formed by $v_i + \sum_{j=d+1}^n a_{ji} v_j$ ($1 \le i \le d < j \le n$) together with $v_i + \sum_{j=d+2}^n b_{ji} v_j$ ($1 \le i \le d+1 < j \le n$) has rank $d+1$, which is precisely cut out by all $(d+2)$-minors. Therefore F_d intersects $\mathrm{Gr}_d(V)_U \times \mathrm{Gr}_{d+1}(V)_{U'}$ in a closed set. □

Proposition 2.7. The flag variety $\mathcal{F}(\mathbb{C}^n)$, upon fixing a coordinate flag: $\mathcal{F} = (0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^n)$, is acted naturally by $GL_n(\mathbb{C})$. In particular, $\mathcal{F}(\mathbb{C}^n)$ is irreducible and smooth.

Proof. Define the $GL_n(\mathbb{C})$-action on the projective space $\mathbb{P}(\bigwedge^N \mathbb{C}^n)$ (for some fixed $N \in \mathbb{Z}_{\ge 1}$) by

$$GL_n(\mathbb{C}) \times \mathbb{P}(\bigwedge^N \mathbb{C}^n) \longrightarrow \mathbb{P}(\bigwedge^N \mathbb{C}^n), \quad (g, [v]) \longmapsto [\bigwedge^N g \cdot v].$$

This is an algebraic action by construction. The action map restricts to an $GL_n(\mathbb{C})$-action on the Grassmannian $\mathrm{Gr}_N(\mathbb{C}^n)$. It follows that the natural $GL_n(\mathbb{C})$-action on the flag variety $\mathcal{F}(V)$ is a transitive algebraic action. By Proposition 7.23 in the appendix, $\mathcal{F}(V)$ is smooth and irreducible, as $GL_n(\mathbb{C})$ is irreducible. □

We give an identification of the flag variety with the collection $\mathcal{B}$ of Borel subalgebras of $\mathfrak{sl}_n(\mathbb{C}) = \mathrm{Lie}(SL_n(\mathbb{C})) = \{x : \mathbb{C}^n \to \mathbb{C}^n \in \mathbf{M}_n(\mathbb{C}) : \mathbf{tr}(x) = 0\}$. Note that $\mathcal{B}$ is an algebraic variety. Indeed, by the conjugation theorem (c.f. [4]), $\mathcal{B}$ consists of maximal solvable subalgebras of $\mathfrak{sl}_n(\mathbb{C})$ whose dimensions are the same. The solvability condition is an algebraic condition. We will see in the next subsection that $\mathcal{B}$ is a closed smooth subvariety of a Grassmannian variety.

Proposition 2.8. Using the notations above, we have $\mathcal{B} \cong \mathcal{F}(\mathbb{C}^n)$ as projective varieties.

Proof. This is Lemma 3.1.15 of [2]. □

Example 2.9.
(1) When $n = 2$, $\mathcal{F}(\mathbb{C}^2) = \{(0 \subset V_1 \subset V_2 = \mathbb{C}^2) : \dim V_1 = 1\} = \mathbb{P}^1$.
(2) When $n = 3$, $\mathcal{F}(\mathbb{C}^3) = \{(0 \subset V_1 \subset V_2 \subset V_3 = \mathbb{C}^3) : \dim V_1 = 1, \dim V_2 = 2\}$. There is a map $\mathcal{F}(\mathbb{C}^3) \to \mathbb{P}^1$ sending a flag $(0 \subset V_1 \subset V_2 \subset V_3 = \mathbb{C}^3)$ to $V_1 \subset \mathbb{C}^3$, i.e., to a point in $\mathbb{P}^2$. This is a $\mathbb{P}^1$-fibration.

In fact, flag varieties can be defined generally as a quotient of algebraic groups. This will be made precise in the next subsection.

2.1. **Semisimple Lie Algebras and Semisimple Groups.** We will briefly review some facts about semisimple Lie algebras and semisimple algebraic groups. For proofs and further information, the reader is referred to [4], [5], and [12].

Fix G as a complex semisimple connected Lie group with Lie algebra $\mathfrak{g}$, often viewed as a G-module by means of the adjoint action. Let B be a Borel subgroup - a maximal solvable (connected) subgroup of G. There is a one-to-one correspondence between Borel subgroups of G and Borel subalgebras of $\mathfrak{b}$ which sends a Borel subgroup B to its Lie algebra. A unipotent radical is the largest connected normal unipotent subgroup of G. If U is the unipotent radical $U = [B, B]$ of B, and T is a maximal torus, then the Levi decomposition gives $B = T \cdot U$.

Example 2.10. Let $G = SL_n(\mathbb{C})$,

(1) $B = \{$upper triangular matrices in $SL_n(\mathbb{C})\}$;
(2) $U = \{$upper triangular matrices in $SL_n(\mathbb{C})$ with diagonal elements are $1\}$;
(3) $T = \{$diagonal matrices in $SL_n(\mathbb{C})\}$.

Fact 2.1. (1) Any two Borel subgroups are conjugate, and each Borel subgroup is its own normalizer, i.e., $B = N_G(B)$.
(2) Each maximal torus is its own centralizer, and its normalizer gives a finite group $N_G(T)/T$, called the Weyl group of the pair (G, T). The number of Borel subgroups B containing a maximal torus T is finite, and the Weyl group permutes them principally (i.e., simply transitively).
(3) Any orbit of a unipotent group action on an affine algebraic variety is closed in Zariski topology.
(4) For any element $x \in \mathfrak{g}$, we write $Z_G(x)$ and $Z_{\mathfrak{g}}(x)$ for the centralizer of x in G and $\mathfrak{g}$ respectively. The Lie algebra of the stabilizer subgroup $Z_G(x) \subset G$ is precisely $Z_{\mathfrak{g}}(x) = \ker \operatorname{ad} x$.
(5) Cartan subalgebras in $\mathfrak{g}$ are conjugate to each other. Its dimension is defined to be the rank $\mathrm{rk}\mathfrak{g}$ of $\mathfrak{g}$. Furthermore, for any $x \in \mathfrak{g}$, $\dim Z_{\mathfrak{g}}(x) \geq \mathrm{rk}\mathfrak{g}$. In particular, a semisimple element $x \in \mathfrak{g}$ is called *regular* if the equality holds. More generally, an element $x \in \mathfrak{g}$ is called *regular* precisely when the dimension of the generalized eigenspace $\mathfrak{g}_{0,x}$ of $\operatorname{ad} x$ for eigenvalue zero is equal to $\mathrm{rk}\mathfrak{g}$.

We will use the following properties of semisimple elements without proof in this paper.

Lemma 2.11. For $\mathfrak{h} \subset \mathfrak{g}$, Cartan and Borel subalgebras of $\mathfrak{g}$.

(1) Any element of $\mathfrak{h}$ is semisimple and any semisimple element of $\mathfrak{g}$ is G-conjugate to an element of $\mathfrak{h}$.
(2) The centralizer of a semisimple regular element is a Cartan subalgebra.
(3) If $x \in \mathfrak{b}$ is a semisimple regular element then $Z_{\mathfrak{g}}(x) \subset \mathfrak{b}$.
(4) The set of regular semisimple elements of $\mathfrak{g}$, denoted by $\mathfrak{g}^{sr}$, is a G-stable subset of $\mathfrak{g}$. Furthermore, $\mathfrak{g}^{sr}$ is a Zariski-open affine subset of $\mathfrak{g}$ (c.f. Proposition 7.22).

Construction 2.12 (Flag varieties)**.** Let $\mathcal{B}$ be the set of all Borel subalgebras in $\mathfrak{g}$. By definition, $\mathcal{B}$ is the closed subvariety of the Grassmannian of $\mathfrak{b}$-dimensional subspaces in $\mathfrak{g}$ formed by all solvable Lie subalgebras. Hence $\mathcal{B}$ is a projective variety. Recall that all Borel subalgebras are conjugate under the adjoint action of G, and that $G_{\mathfrak{b}}$ is the isotropy subgroup of $\mathrm{Lie}(B) =: \mathfrak{b}$ in G, which is equal to

B as $N_G(B) = B$. Thus, the assignment $[g] \mapsto \mathrm{Ad}(g)\mathfrak{b}$ gives a bijective morphism (which can also be viewed as the orbit map of the algebraic G-action on $\mathcal{B}$ with base point $\mathfrak{b}$):

$$G/B \xrightarrow{\cong} \mathcal{B}, \tag{2.13}$$

where G/B is a projective variety (c.f. [5]). Furthermore, since G/B is irreducible as G is a connected algebraic group, $\mathcal{B}$ is irreducible. As the G-action on $\mathcal{B}$ is transitive, $\mathcal{B}$ is smooth (c.f. Proposition 7.23). It follows from Proposition 7.20 that this bijection is a G-equivariant isomorphism of algebraic varieties.

3. Bruhat Decomposition

In linear algebra, we know that there is a generalization of LU-decomposition, i.e., the LPU-decomposition.

Theorem 3.1. For any $g \in GL_n(\mathbb{C})$, there exists $L \in B_-$, $U \in B$, and a permutation matrix $w \in S_n \hookrightarrow GL_n(\mathbb{C})$ such that $g = LwU$, where B_- (resp. B) denotes the subgroup of upper (resp. lower) triangular matrices in $GL_n(\mathbb{C})$. There is a stratification

$$GL_n(\mathbb{C}) = \bigsqcup_{w \in S_n} B_- w B. \tag{3.2}$$

Observe that $B_- = w_0 B w_0$, $w_0 = \begin{pmatrix} & & 1 \\ & \vdots & \\ 1 & & \end{pmatrix}$ is the permutation matrix with skew-diagonal entries that are one. The above decomposition can be rewritten as

$$GL_n(\mathbb{C}) = \bigsqcup_{w \in S_n} BwB. \tag{3.3}$$

The Bruhat decomposition generalizes the theorem above for connected semisimple algebraic groups, despite the fact that $GL_n(\mathbb{C})$ is not semisimple.

Let G be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$. Fix a Borel subgroup $B \subset G$ with Lie algebra $\mathfrak{b}_o$. Recall that we might identify $\mathcal{B}$, the collection of Borel subalgebras in $\mathfrak{g}$, with the flag variety G/B as in (2.13). In the rest of the exposition, we do not distinguish the two.

Theorem 3.4. Choose a maximal torus $T \subset B$, write $W_T := N_G(T)/T$ for the Weyl group of G respect to T. We have the following maps
(3.5)

$$W_T \xrightarrow{(1)} B\backslash G/B \xrightarrow{(2)} \{B\text{-orbits on } \mathcal{B} \cong G/B\} \xrightarrow{(3)} \{G\text{-diagonal orbits on } \mathcal{B} \times \mathcal{B}\},$$

given explicitly by

(1) $W_T \xrightarrow{(1)} B\backslash G/B$, $[w] \longmapsto BwB$;
(2) $B\backslash G/B \xrightarrow{(2)} \{B\text{-orbits on } \mathcal{B} \cong G/B\}$, $BgB \longmapsto B(gB) = B \cdot \mathrm{Ad}(g)\mathfrak{b}_o$;
(3) $\{B\text{-orbits on } \mathcal{B} \cong G/B\} \xrightarrow{(3)} \{G\text{-diagonal orbits on } \mathcal{B} \times \mathcal{B}\}$, $B \cdot \mathfrak{b} \longmapsto G \cdot (\mathfrak{b}_o, \mathfrak{b})$.

Furthermore, all the above maps are bijections.

Remark 3.6. Observe that the rightmost set in (3.5) depends neither on the choice of T nor on the choice of B, while the leftmost set obviously does. We will see later in section 3.1 that the set of G-orbits in $\mathcal{B}\times\mathcal{B}$ is in fact in canonical bijection with an abstract Weyl group $\mathbb{W}$.

Proof of theorem 3.4. We begin by observing that the bijection $B\backslash G/B \xrightarrow{(2)} \{B\text{-orbits on } \mathcal{B}\cong G/B\}$ is immediate from the definition. The last arrow in (3.5) follows from a general fact in group theory: for any group G and its subgroup H, one has a canonical bijection

$$H\backslash G/H = \{H\text{-orbits on } G/H\} \xrightarrow{\cong} \{G\text{-diagonal orbits on } G/H\times G/H\} = G\backslash(G/H\times G/H)$$
$$HgH \mapsto G\cdot(eH, gH), \quad e\in G \text{ is the identity element.}$$

It remains to prove that the leftmost arrow in (3.5) is a bijection. To that end, we will use the Bialynicki-Birula theorem 7.5.

Lemma 3.7. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{b}\supset\mathfrak{t}$ a Borel subalgebra. Any Borel subalgebra containing $\mathfrak{t}$ has the form $\mathrm{Ad}(w)\mathfrak{b}$, $[w]\in W_T := N_G(T)/T$, for a unique $[w]\in W_T$. Furthermore, the T-fixed points in $\mathcal{B}$ are in natural bijection with W_T, which is given explicitly by sending $[w]\in W_T$ to $\mathrm{Ad}(w)\mathfrak{b}$.

Proof. This is Lemma 3.1.10 of [2] or Lemma 3.9 of [7]. □

Proposition 3.8. Every B-orbit on $\mathcal{B}$ contains a unique fixed point T-fixed point.

Proof. We first claim that the set of $\mathbb{C}^*$-fixed points in $\mathcal{B}$ is precisely the T-fixed points. We know that in general, one can decompose $\mathfrak{g}$ into

$$\mathfrak{g} = \mathfrak{h}\oplus\bigoplus_{\alpha\in\mathfrak{h}^*}\mathfrak{g}_\alpha, \quad \text{where } \mathfrak{g}_\alpha = \{x\in\mathfrak{g} : \mathrm{ad}\,h(x) = \alpha(h)x \text{ for all } h\in\mathfrak{h}\}. \tag{3.9}$$

For a regular semisimple element $h\in\mathfrak{h} = \mathrm{Lie}(T)$ that spans the Lie algebra of a one-parameter subgroup $\mathbb{C}^*\hookrightarrow T$, we deduce from the decomposition (7.3) that $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}}\mathfrak{g}_n$ where $\mathfrak{g}_n$ is the eigenspace of n with respect to $\mathrm{ad}\,h$. Note further that we have $\mathfrak{g}_0 = \ker\mathrm{ad}\,h = Z_{\mathfrak{g}}(h) = \mathfrak{h}$ as h is semisimple regular. Recall $\dim\mathfrak{g}_\alpha = 1$ if $\alpha\in\mathfrak{g}^*$ is a root and $\dim\mathfrak{g}_n = 1$, we have

$$\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{n}^+\oplus\mathfrak{n}^-, \tag{3.10}$$

where $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) denotes the positive (resp. negative) eigenspaces. Our choice of h and $\mathbb{C}^*\hookrightarrow T$ can be made so that the eigenvalues of $\mathrm{ad}\,h$ on $\mathrm{Lie}(B)$ are nonnegative (by choosing an appropriate Weyl chamber). Therefore we have $\mathrm{Lie}(B) = \mathfrak{h}\oplus\mathfrak{n}^+$.

We observe that the collection of $\mathbb{C}^*$-fixed points in $\mathcal{B}$ is precisely the set of T-fixed points in $\mathcal{B}$, and hence corresponds to the Weyl group W_T by the previous lemma. Indeed, $\mathfrak{b}\in\mathcal{B}$ is fixed by the $\mathbb{C}^*$-action if and only if $h\in\mathfrak{b}$ by fact 2.1. The latter condition is precisely that $\mathfrak{h} = Z_{\mathfrak{g}}(h)\subset\mathfrak{b}$ by Lemma 2.11. Hence, the set of $\mathbb{C}^*$-fixed points in $\mathcal{B}$ is precisely the T-fixed points by the lemma above. The claim is proved.

We now apply the Bialynicki-Birula decomposition to the above $\mathbb{C}^*$-action on $\mathcal{B}$ with the finite fixed point set $\cong W_T$. We obtain a decomposition

$$\mathcal{B}\cong\bigsqcup_{w\in W_T}\mathcal{B}_w, \quad \text{where } \mathcal{B}_w = \{x\in\mathcal{B} : \lim_{z\to 0} z\cdot x = w\},$$

and that $\mathcal{B}_w\cong\mathrm{T}_w(\mathcal{B}_w) = \mathrm{T}_w^+\mathcal{B}$ for all $w\in W_T$. Now fix $w\in W_T$ and view it as a fixed point of the $\mathbb{C}^*$-action on $\mathcal{B}$. The orbit map $G\to G/B\cong\mathcal{B}$ sending g to $g\cdot w$ gives rise to a surjective linear map $\varphi:\mathfrak{g}\to\mathrm{T}_w\mathcal{B}$ with kernel being the Lie algebra of the isotropy group of $w\in\mathcal{B}$. As $\mathfrak{h} = \mathrm{Lie}(T)$ is contained in the isotropy group, we know $\mathrm{T}_w\mathcal{B} = \mathrm{T}_w^+\mathcal{B} \oplus \mathrm{T}_w^-\mathcal{B}$ is a quotient of $\mathfrak{g}/\mathfrak{h} = \mathfrak{n}^+ \oplus \mathfrak{n}^-$ under φ, respecting the direct sum. Therefore, $\varphi(\mathfrak{n}^+) = \mathrm{T}_w^+\mathcal{B} = \mathrm{T}_w(\mathcal{B}_w)$.

We claim that $U \cdot w \subset \mathcal{B}_w$, where U is the unipotent group corresponding to the Lie algebra $\mathfrak{n}^+$. Indeed, for any $u \in U$, and $t \in \mathbb{C}^* \subset T$, we have $tut^{-1} \to 1$ when $t \to 0$ using the exponential map over unipotent groups and that t is in the general position. Therefore $tu \cdot (w) = tut^{-1} \cdot t(w) = tut^{-1} \cdot w \xrightarrow{t\to 0} w$. The differential of the action map $U \to \mathcal{B}_w \cong \mathrm{T}_w(\mathcal{B}_w)$ sending u to $u \cdot w$ is precisely the restriction of φ to $\mathfrak{n}^+$, which is surjective. It follows from the "implicit function theorem" (Proposition 7.28) that $U \cdot w$ is open dense in $\mathcal{B}_w$. We deduce from (3) of fact 2.1 that U-orbit on the affine variety $\mathcal{B}_w \cong \mathrm{T}_w(\mathcal{B}_w)$ is closed. Hence $\mathcal{B}_w = U \cdot w$. Since w is fixed by T, $U \cdot w = U \cdot Tw = B \cdot w$, where the last equality is the Levi decomposition of B. Thus, we have proved that B-orbits in $\mathcal{B}$ are precisely the *Bruhat cells* $\mathcal{B}_w$. It follows that each B-orbit contains a unique point of the form $wB \in G/B \cong \mathcal{B}$. □

Proof of theorem 3.4 (cont'd). As a consequence of Proposition 3.8 above, we know that we can identify $B\backslash\mathcal{B} = B\backslash G/B$ with the T-fixed points in $\mathcal{B}$, which corresponds precisely to the Weyl group W_T. So (1) in (3.5) is a bijection, as desired. □

Example 3.11. Let $G = SL_2(\mathbb{C}) = \{M \in GL_2(\mathbb{C}) : \det(M) = 1\}$,

$$B = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} : a \in \mathbb{C}^*, \beta \in \mathbb{C}\right\} \subset SL_2(\mathbb{C})$$

be the subgroup of upper triangular matrices. Let

$$T := \left\{\begin{pmatrix} x & 0 \\ 0 & x^{-1}\end{pmatrix} : x \in \mathbb{C}^*\right\} \subset SL_2(\mathbb{C}).$$

The normalizer $N_G(T)$ consists of elements of the following form

$$N_G(T) = T \sqcup \left\{\begin{pmatrix} 0 & y \\ -y^{-1} & 0\end{pmatrix} : y \in \mathbb{C}^*\right\}.$$

It follows that the Weyl group $W_T = \{eT, sT\}$ is a group of order two, where e is the identity matrix and $s = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} \in SL_2(\mathbb{C})$. Note that sT is a generator of $W_T \cong S_2$. We observe that by the Bruhat decomposition

$$SL_2(\mathbb{C}) = B \sqcup BsB, \tag{3.12}$$

where in fact BsB is an open dense Bruhat cell in $SL_2(\mathbb{C})$ (c.f. Corollary 7.17). Observe that the natural $SL_2(\mathbb{C})$ action on the flag variety $\mathcal{F}(\mathbb{C}^2)$ induces an isomorphism of projective varieties (c.f. Proposition 2.8):

$$SL_2(\mathbb{C})/B \cong \mathcal{F}(\mathbb{C}^2) = \mathbb{P}^1.$$

In particular the Bruhat decomposition (3.12) of $SL_2(\mathbb{C})$ gives

$$\mathbb{P}^1 \cong B/B \sqcup BsB/B = \{*\} \sqcup BsB/B, \tag{3.13}$$

where BsB/B is an open affine subset of $\mathbb{P}^1$ isomorphic to $\mathbb{A}^1$. More precisely, we observe that every element $M \in BsB/B$ can be represented by

$$M = \begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} \cdot sB$$

for a unique $x \in \mathbb{C}$. Indeed, since $s \in N_G(T)$ and $T \subset B$, a direct computation yields

$$\begin{aligned}\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} sB &= \begin{pmatrix} 1 & ab \\ 0 & 1 \end{pmatrix} ss^{-1} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} sB \\ &= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} sB,\end{aligned}$$

for any $a \in \mathbb{C}^*$, $b \in \mathbb{C}$ and $x \coloneqq ab$. In fact, (3.13) is a special case of the Schubert decomposition.

3.1. **Abstract Weyl Group.** Recall that a finite reduced root system R in a complex vector space H (c.f. [2],[4],[5]) can be decomposed into a disjoint union $R = R^+ \sqcup R^-$, where R^+ is the set of positive roots and $R^- = -R^+$ is the set of negative roots. Further, there is a uniquely determined subset $S \subset R^+$, called the set of simple roots, such that any positive root is a sum of a certain number of simple roots. Write $\mathbb{W}$ for the group generated by the reflection map $s_\alpha : H \to H$, $\alpha \in R$, called the *Weyl group of the root system* R. In fact, $\mathbb{W}$ is generated by the set $\{s_\alpha : \alpha \in S\}$ of *simple reflections* subject to the braid relation (c.f. [2]), and it is isomorphic to the abstract Coxeter group $(\mathbb{W}, S)$.

Let G be a semisimple group and T be a maximal torus in G. We would like to relate the abstract Weyl group $\mathbb{W}$ associated to a root system of G to $W_T = N_G(T)/T$, the Weyl group of the pair (G,T). To that end, we will need to define an abstract Cartan subalgebra $\mathfrak{H}$ and relate it with a Cartan subalgebra in $\mathfrak{g}$. This follows from the lemma below.

Lemma 3.14. For any Borel subalgebras $\mathfrak{b}, \mathfrak{b}' \in \mathcal{B}$, there is a canonical isomorphism

$$\mathfrak{b}/[\mathfrak{b},\mathfrak{b}] \cong \mathfrak{b}'/[\mathfrak{b}',\mathfrak{b}'].$$

Proof. We know that all Borel subalgebras are conjugate and self-normalizing. Therefore $\mathfrak{b}' = \mathrm{Ad}(g)\mathfrak{b}$ for some $g \in G$. The isomorphism can be given by $\mathrm{Ad}(g)$, as $\mathrm{Ad}(g)[\mathfrak{b},\mathfrak{b}] = [\mathrm{Ad}(g)\mathfrak{b}, \mathrm{Ad}(g)\mathfrak{b}] = [\mathfrak{b}',\mathfrak{b}']$. If we let $g' = gb$ for some $b \in B$, we get a new map $\mathrm{Ad}(g') : \mathfrak{b} \to \mathfrak{b}'$. This new map is the same on the quotients $\mathfrak{b}/[\mathfrak{b},\mathfrak{b}] \cong \mathfrak{b}'/[\mathfrak{b}',\mathfrak{b}']$ as one induced by g since the adjoint action of B on $\mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$ is trivial. Therefore, all quotients $\mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$ are canonically isomorphic. □

Definition 3.15 (Abstract Cartan subalgebras)**.** We identify all quotient spaces $\mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$, for all Borel subalgebras $\mathfrak{b}$ by means of the canonical isomorphism of Lemma 3.14, and call the resulting vector space $\mathfrak{H}$ the abstract Cartan subalgebra.

We would like to give a root system to the abstract Cartan subalgebra $\mathfrak{H}$.

Construction 3.16 (Root system on $\mathfrak{H}$)**.** Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The weights of the adjoint $\mathfrak{h}$-action on $\mathfrak{g}$ form a set R of roots in $\mathfrak{h}^*$ (c.f. 3.9). Using the Killing form κ (c.f. Proposition 7.18) as the inner product, we define the set of coroots $R^\vee \subset \mathfrak{h}$. Thus, we obtain a root system, $R_{\mathfrak{g},\mathfrak{h}}$ in $\mathfrak{h}$. However, this root system does not have a preferred set of simple roots. To get the latter, we choose a Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$, and take the weights of the adjoint $\mathfrak{h}$-action on $\mathfrak{b}$ to be positive roots. This specifies the set $S = S_{\mathfrak{b}} \subset R_{\mathfrak{g},\mathfrak{h}}$ of simple roots. Consider the composition of natural maps

$$\mathfrak{h} \hookrightarrow \mathfrak{b} \twoheadrightarrow \mathfrak{b}/[\mathfrak{b},\mathfrak{b}] =: \mathfrak{H}, \tag{3.17}$$

which gives an isomorphism of the Cartan subalgebra $\mathfrak{h}$ with the abstract Cartan subalgebra $\mathfrak{H}$. Transferring the root system $(R_{\mathfrak{g},\mathfrak{h}}, S_{\mathfrak{b}})$ to (R, S) in $\mathfrak{H}$ via the isomorphism in (3.17). As a consequence, (3.17) (which depends on the choice of $\mathfrak{b}$) gives an isomorphism of root systems

$$(\mathfrak{h}, R_{\mathfrak{g},\mathfrak{h}}, S_{\mathfrak{g}}) \xrightarrow{(3.17),\cong} (\mathfrak{H}, R, S). \tag{3.18}$$

Observe that we have the following diagram for any $w \in W_T$, $\mathrm{Lie}(T) = \mathfrak{h}$:

$$\begin{tikzcd}
\mathfrak{h} \arrow[r, "\mathrm{Ad}(w)"] \arrow[d, "\cong"'] & \mathfrak{h} \arrow[d, "\cong"] \\
\mathfrak{b}/[\mathfrak{b},\mathfrak{b}] \arrow[r, "\mathrm{Ad}(w)"] & \mathfrak{b}'/[\mathfrak{b}',\mathfrak{b}'],
\end{tikzcd}$$

where the vertical arrows are the isomorphisms in (3.17). By identifying $\mathfrak{b}'/[\mathfrak{b}',\mathfrak{b}']$ with $\mathfrak{H}$ via the bottom arrow, we have a natural action of W_T on $\mathfrak{H}$. It follows that $W_T \cong \mathbb{W}$ as groups, where $\mathbb{W}$ is the abstract Weyl group associated to the root system $(\mathfrak{H}, R, S)$. So far, the construction depends on the choice of a Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$ as well as the Cartan subalgebra $\mathfrak{h}$.

We now introduce the notion of the relative position of two Borel subalgebras of $\mathfrak{g}$. We need a lemma.

Lemma 3.19. Given two Borel subalgebras $\mathfrak{b}, \mathfrak{b}' \subset \mathfrak{g}$. There exists a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b} \cap \mathfrak{b}'$.

Proof. This is Lemma 3.20. of [7]. □

Let W be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. Let $\mathfrak{b}$ and $\mathfrak{b}'$ be two Borel subalgebras, and pick a Cartan subalgebra $\mathfrak{h}$ contained in their intersection. There is a unique element $w \in W$ such that $\mathfrak{b}' = \mathrm{Ad}(w)\mathfrak{b}$ (c.f. Lemma 3.7). Let $w(\mathfrak{b}, \mathfrak{b}') \in \mathbb{W}$ be the element of the abstract Weyl group corresponding to w under the isomorphism $W \cong \mathbb{W}$ induced by (3.17).

Lemma 3.20 (See [2])**.** The element $w(\mathfrak{b}, \mathfrak{b}') \in \mathbb{W}$ is independent of the choice of the Cartan subalgebra $\mathfrak{h}$.

Definition 3.21. Two Borel subalgebras $\mathfrak{b}, \mathfrak{b}'$ are said to be in relative position $w \in \mathbb{W}$ if $w(\mathfrak{b}, \mathfrak{b}') = w$.

We are now ready to state the main theorem of this subsection.

Theorem 3.22. Two pairs of Borel subalgebras $(\mathfrak{b}_1, \mathfrak{b}_1')$ and $(\mathfrak{b}_2, \mathfrak{b}_2')$ are in the same relative position $w \in \mathbb{W}$ if and only if the points $(\mathfrak{b}_1, \mathfrak{b}_1') \in \mathcal{B} \times \mathcal{B}$ and $(\mathfrak{b}_2, \mathfrak{b}_2') \in \mathcal{B} \times \mathcal{B}$ belong to the same G-orbit under the G-action on $\mathcal{B} \times \mathcal{B}$. In other words, the assignment $(\mathfrak{b}, \mathfrak{b}') \mapsto w(\mathfrak{b}, \mathfrak{b}')$ give a canonical bijection

$$\{G\text{-diagonal orbits on } \mathcal{B} \times \mathcal{B}\} \cong \mathbb{W}$$

Proof. We only show the "only if" direction. Choose a Borel subalgebra $\mathfrak{b}$ and let T be the maximal torus corresponding to a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. Let $w \in W_T$. Identify $w \in W_T$ with an element of $\mathbb{W}$ given by $\mathfrak{b}$. Then $\mathfrak{b}$ and $\mathrm{Ad}(w)\mathfrak{b}$ are in relative position $w \in \mathbb{W}$. It suffices to show that each G-orbit on $\mathcal{B} \times \mathcal{B}$ contains a single point of the form $(\mathfrak{b}, \mathrm{Ad}(w)\mathfrak{b})$, $w \in W_T$. We observe that this follows directly from the Bruhat decomposition. More precisely, the last bijection of (3.5) provides what we want. □

3.2. Chevelley Restriction Theorem.

Definition 3.23. Let $\tilde{\mathfrak{g}} := \{(x, \mathfrak{b}) \in \mathfrak{g} \times \mathcal{B} : x \in \mathfrak{b}\}$, let $\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$ and $\pi : \tilde{\mathfrak{g}} \to \mathcal{B}$ be the first and the second projection, respectively.

Recall the definition of vector bundles and associated bundles (see Appendix), we have the following.

Proposition 3.24. The projection $\pi : \tilde{\mathfrak{g}} \to \mathcal{B}$ makes $\tilde{\mathfrak{g}}$ a G-equivariant vector bundle over $\mathcal{B} \cong G/B$ with fiber $\mathfrak{b}$. Here G-action on $\tilde{\mathfrak{g}}$ is give explicitly by sending $g \cdot (x, \mathfrak{b})$ to $(gxg^{-1}, \mathrm{Ad}(g)\mathfrak{b})$ for any $g \in G$ and $(x, \mathfrak{b}) \in \tilde{\mathfrak{g}}$. The assignment $(g, x) \mapsto (\mathrm{Ad}(g)x, \mathrm{Ad}(g)\mathfrak{b}) = (\mathrm{Ad}(g)x, g{\cdot}eB)$ gives a G-equivariant isomorphism $G \times_B \mathfrak{b} \xrightarrow{\cong} \tilde{\mathfrak{g}}$. Here $G \times_B \mathfrak{b} \to \mathcal{B}$ is a G-equivariant vector bundle by Corollary 7.17

Proposition 3.25. The morphism $\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$ is proper with respect to the Zariski topology.

Proof. The morphism μ is the restriction of the first projection $\mathfrak{g} \times \mathcal{B} \to \mathfrak{g}$, which is proper as $\mathcal{B}$ is a projective variety (and hence quasi-compact). □

Definition 3.26. For a general semisimple Lie algebra $\mathfrak{g}$, we define a map $\nu : \tilde{\mathfrak{g}} \to \mathfrak{H}$ as the projection

$$(x, \mathfrak{b}) \mapsto x \mod [\mathfrak{b}, \mathfrak{b}] \in \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] = \mathfrak{H}.$$

Set $\tilde{\mathfrak{g}}^{sr} := \mu^{-1}(\mathfrak{g}^{sr})$, the preimage of regular semisimple elements. We have the following.

Proposition 3.27. For any $x \in \tilde{\mathfrak{g}}^{sr}$, there is a canonical free $\mathbb{W}$-action on $\mu^{-1}(x)$ making the projection $\tilde{\mathfrak{g}}^{sr} \to \mathfrak{g}^{sr}$ a principal $\mathbb{W}$-bundle.

Proof. This is Proposition 3.1.36 of [2]. □

Let $\mathbb{C}[\mathfrak{g}]$ be the polynomial functions on $\mathfrak{g}$, and let $\mathbb{C}[\mathfrak{g}]^G$ be the subalgebra of G-invariant polynomials with respect to the adjoint action. Given a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, we have the restriction map $\mathbb{C}[\mathfrak{g}] \to \mathbb{C}[\mathfrak{h}]$. Let W be the Weyl group for the pair $(\mathfrak{g}, \mathfrak{h})$, we have $\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}]^W$. We are now ready to state and prove the Chevalley restriction theorem, which asserts that the above map is an isomorphism of algebras.

Theorem 3.28. For any Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, the restriction map gives a canonical graded algebra isomorphism

$$\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\cong} \mathbb{C}[\mathfrak{h}]^W. \tag{3.29}$$

Proof. We first prove the easy part: the restriction map is injective. Let $P \in \mathbb{C}[\mathfrak{g}]^G$ be such that $P|_{\mathfrak{h}} = 0$. Then $P|_{\mathfrak{g}^{sr}} = 0$ because of (1) in Lemma 2.11. But this implies $P \equiv 0$ as $\mathfrak{g}^{sr}$ is dense in $\mathfrak{g}$.

We now prove surjectivity. Fix a W-invariant polynomial P on $\mathfrak{h}$, our goal is to construct a G-invariant polynomial R on $\mathfrak{g}$ which restricts to P. To that end, we choose a Borel subalgebra $\mathfrak{b}$ containing the fixed Cartan subalgebra $\mathfrak{h}$. Via the isomorphism in (3.17), we identify $\mathfrak{h}$ with $\mathfrak{H}$ and $\mathbb{C}[\mathfrak{h}]^W$ with $\mathbb{C}[\mathfrak{H}]^{\mathbb{W}}$. Therefore we may canonically identify P with $P_{\mathfrak{H}} \in \mathbb{C}[\mathfrak{H}]^{\mathbb{W}}$.

Recall the maps defined at the beginning of this subsection:

$$\begin{tikzcd} & \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathfrak{g} & & \mathfrak{H}. \end{tikzcd}$$

Let $\tilde{P} := \nu^* P_{\mathfrak{H}} \in \mathcal{O}(\tilde{\mathfrak{g}})$ be the pullback of $P_{\mathfrak{H}}$ along $\nu : \tilde{\mathfrak{g}} \to \mathfrak{H}$, i.e., $\tilde{P}(x) = P_{\mathfrak{H}}(\nu(x))$ for all $x \in \tilde{\mathfrak{g}}$. We claim that $\tilde{P}$ is G-invariant regular function. Indeed, we had a G-equivariant isomorphism $\tilde{\mathfrak{g}} \cong G \times_B \mathfrak{b}$ and thus ν can be identified as $G \times_B \mathfrak{b} \to \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ sending (g, x) to x mod $[\mathfrak{b}, \mathfrak{b}]$. In particular, we observe that all points in a G-orbit in the associated bundle $G \times_B \mathfrak{b}$ are sent to the same point in $\mathfrak{H} = \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$. It follows that $\tilde{P}$ is G-invariant.

We next claim that the restriction of $\tilde{P}$ to the Zariski open subset $\tilde{\mathfrak{g}}^{sr} := \mu^{-1}(\mathfrak{g}^{sr})$ is a $\mathbb{W}$-invariant rational function, i.e., $\tilde{P}|_{\tilde{\mathfrak{g}}^{sr}} \in \mathbb{C}(\tilde{\mathfrak{g}}^{sr})^{\mathbb{W}}$. Indeed, there is a canonical $\mathbb{W}$-action on $\tilde{\mathfrak{g}}^{sr}$ by the previous proposition. Furthermore, the map ν commutes with $\mathbb{W}$-action and $P_{\mathfrak{H}}$ is invariant under $\mathbb{W}$ action. The claim follows.

Recall that a morphism $\tilde{X} \to X$ of normal algebraic varieties is said to be a Galois covering if it is a quotient map by the free action of a finite group $\mathbb{W}$. There is a general fact that $\mathbb{C}(\tilde{X})/\mathbb{C}(X)$ is a Galois extension of fields of rational functions with Galois group $\mathbb{W}$. In particular, we see that $\mu : \tilde{\mathfrak{g}}^{sr} \to \mathfrak{g}^{sr}$ is a Galois covering with Galois group the abstract Weyl group $\mathbb{W}$. It follows that the induced inclusion $\mu^* : \mathbb{C}(\mathfrak{g}^{sr}) \to \mathbb{C}(\tilde{\mathfrak{g}}^{sr})$ gives rise to an isomorphism $\mathbb{C}(\mathfrak{g}^{sr}) \xrightarrow{\cong} \mathbb{C}(\tilde{\mathfrak{g}}^{sr})^{\mathbb{W}}$, for which we also call μ^*. Therefore we find a rational function $R \in \mathbb{C}(\mathfrak{g}^{sr})$ such that $\mu^*(R) = \tilde{P}|_{\tilde{\mathfrak{g}}^{sr}}$.

We claim that the rational function R is a polynomial on $\mathfrak{g}$ which is G-invariant. To see this, we observe that μ is G-equivariant and $\tilde{P}$ is G-invariant. For any relatively compact set $D \subset \mathfrak{g}$ (in the sense of Zariski topology), $D \cap \mathfrak{g}^{sr}$ is relatively compact since μ is proper. The function $R|_{D \cap \mathfrak{g}^{sr}} = \tilde{P}|_{\mu^{-1}(D \cap \mathfrak{g}^{sr})} = P_{\mathfrak{H}} \circ \nu|_{\mu^{-1}(D \cap \mathfrak{g}^{sr})}$ is bounded as $\nu(\mu^{-1}(D \cap \mathfrak{g}^{sr}))$ is contained in a compact set and $P_{\mathfrak{H}}$ is a polynomial. Therefore, R has no pole away from a hypersurface $\mathfrak{g} \setminus \mathfrak{g}^{sr}$ (c.f. Proposition 7.22) and is locally bounded on relatively compact subsets of $\mathfrak{g}^{sr}$. It cannot have a pole on the hypersurface, because if otherwise, one can find a convergent sequence in $\mathfrak{g}^{sr}$ to the pole on which R admits bounded values, a contradiction! Thus R is a polynomial on $\mathfrak{g}$ which is G-invariant. It remains to check that R is as desired, which is a routine verification. $\square$

3.3. **Grothendieck's Simultaneous Resolution.** The $\mathbb{W}$-vector space $\mathfrak{H}$ induces the topological orbit space $\mathfrak{H}/\mathbb{W}$ under the quotient topology. We have by Proposition 7.26 in the Appendix an isomorphism of algebraic varieties:

$$\mathrm{Specm}(\mathbb{C}[\mathfrak{H}]^{\mathbb{W}}) = \mathfrak{H}/\mathbb{W}.$$

By a general fact, called the Chevalley-Shephard-Todd theorem, $\mathbb{C}[\mathfrak{H}]^{\mathbb{W}}$ is a free polynomial algebra. It follows that $\mathfrak{H}/\mathbb{W}$ is isomorphic to some affine space.

Now given a pair $(\mathfrak{h}, \mathfrak{b})$ of a Cartan and a Borel subalgebra such that $\mathfrak{h} \subset \mathfrak{b}$, we have a diagram of map $\mathfrak{g} \hookleftarrow \mathfrak{h} \xrightarrow{(3.17)} \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] = \mathfrak{H}$, which induces a diagram of isomorphisms by Chevalley restriction theorem 3.28:

$$\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}]^W \leftarrow \mathbb{C}[\mathfrak{H}]^{\mathbb{W}}.$$

Observe that the map $\mathbb{C}[\mathfrak{H}]^{\mathbb{W}} \xrightarrow{\cong} \mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[\mathfrak{g}]$ is given precisely by a morphism of affine varieties $\rho : \mathfrak{g} \to \mathfrak{H}/\mathbb{W}$.

Proposition 3.30. The following diagram

$$\begin{tikzcd} & \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathfrak{g} \arrow[dr, "\rho"'] & & \mathfrak{H} \arrow[dl, "\pi"] \\ & \mathfrak{H}/\mathbb{W} & \end{tikzcd}$$

called Grothendieck's simultaneous resolution (or the Springer diagram), is commutative.

Proof. By Proposition 7.26 in the Appendix, we know that $\mathbb{W}$-invariant poloynomials on $\mathfrak{H}$ separate points of $\mathfrak{H}/\mathbb{W}$. It suffices to show the following diagram of algebras commutes:

$$\begin{tikzcd} & \mathcal{O}(\tilde{\mathfrak{g}}) & \\ \mathcal{O}(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}] \arrow[ur, "\mu^*"] & & \mathcal{O}(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}] \arrow[ul, "\nu^*"'] \\ & \mathcal{O}(\mathfrak{H}/\mathbb{W}) = \mathbb{C}[\mathfrak{H}]^{\mathbb{W}}. \arrow[ul, "\rho^*"] \arrow[ur, "\pi^*"'] & \end{tikzcd}$$

Let $P \in \mathcal{O}(\mathfrak{H}/\mathbb{W}) = \mathbb{C}[\mathfrak{H}]^{\mathbb{W}}$. Then $\nu^* \circ \pi^*(P) = \mu^*(R)$ for a polynomial $\mathbb{C}[\mathfrak{g}]^G$ by the course of proof of Theorem 3.28. Furthermore, by the definition of ρ^*, we have $\rho^*(P) = R$, so the diagram above commutes. $\square$

3.4. **Example: The $SL_n(\mathbb{C})$ Case.** We will give an explicit construction for the Springer diagram in Proposition 3.30 in the case of $G = SL_n(\mathbb{C})$.

Consider the variety $\mathbb{C}^n/S_n$, the orbit space of the symmetric group S_n acting on the vector space $\mathbb{C}^n$ by permutation of coordinates. We have the following identification.

Lemma 3.31. The orbit space $\mathbb{C}^n/S_n$ is isomorphic to $\mathbb{C}[\lambda]_{n-1}$, the n-dimensional vector space of polynomials over $\mathbb{C}$ in λ with degree less than or equal to $n-1$.

Proof. Consider the map

$$\psi : \mathbb{C}^n \to \mathbb{C}[\lambda]_{n-1} \cong \mathbb{C}^n, \quad (x_1, \dots, x_n) \mapsto -\lambda^n + \prod_{i=1}^{n} (\lambda - x_i). \tag{3.32}$$

The map descends to a bijection $\mathbb{C}^n/S_n \cong \mathbb{C}[\lambda]_{n-1} \cong \mathbb{C}^n$. $\square$

Consider the $(n-1)$-dimensional hyperplane of $\mathbb{C}^n$:

$$H \coloneqq \{(x_1, \dots, x_n) \in \mathbb{C}^n : \sum x_i = 0\} \subset \mathbb{C}^n,$$

which is clearly stable under the action of S_n. We may view H as the diagonal matrices in $\mathfrak{sl}_n(\mathbb{C})$, which is a Cartan subalgebra. And S_n is the Weyl group of $SL_n(\mathbb{C})$. Notice that $\mathbb{C}[H]^W = \mathbb{C}[H]^{S_n}$ is generated by elementary symmetric polynomials. Under the isomorphism $\mathbb{C}[H]^W \cong \mathbb{C}[\mathfrak{sl}_n(\mathbb{C})]^{SL_n(\mathbb{C})}$ in Chevalley restriction theorem 3.28, the elementary symmetric polynomials pull back to the coefficients of the characteristic polynomial. Concretely, for any $x \in \mathfrak{sl}_n(\mathbb{C})$, its characteristic polynomial $\chi_x(t)$ is given by

$$\chi_x(\lambda) = \lambda^n + 0 \cdot \lambda^{n-1} + p_1(x) \cdot \lambda^{n-2} + \cdots + p_{n-1}(x), \quad p_i(x) \in \mathbb{C}[\mathfrak{sl}_n(\mathbb{C})]^{SL_n(\mathbb{C})},$$

where $p_i|_H$ are precisely the elementary symmetric polynomial in $\mathbb{C}[H]^{S_n}$. In particular we have $\mathbb{C}[\mathfrak{sl}_n(\mathbb{C})]^{SL_n(\mathbb{C})} = \mathbb{C}[p_1, \ldots, p_{n-1}]$. Therefore, we have a map $\rho : \mathfrak{sl}_n(\mathbb{C}) \to \mathbb{C}^{n-1}$ given by $x \mapsto (p_1(x), \ldots, p_{n-1}(x))$. Under the identification in (3.32) upon fixing a basis $(\lambda^{n-1}, \ldots, \lambda, 1)$ in $\mathbb{C}[\lambda]_{n-1}$ and viewing $\mathbb{C}^{n-1}$ as a hyperplane in $\mathbb{C}[\lambda]_{n-1}$ by ignoring the first coordiante, we have a map sending each element of $\mathfrak{sl}_n(\mathbb{C})$ to its eigenvalues, counted with multiplicities:

$$\begin{aligned} \rho : \mathfrak{sl}_n(\mathbb{C}) &\to H/S_n \subset \mathbb{C}^n \\ x &\mapsto \{x_1, \ldots, x_n\}, \quad x_i \text{ are eigenvalues of } x. \end{aligned}$$

Recall that in Proposition 2.8, we have shown a natural identification of $\mathcal{B}$ in the case of $G = SL_n(\mathbb{C})$ with the flag variety $\mathcal{F}(\mathbb{C}^n)$. Therefore, the incidence variety is given by

$$\tilde{\mathfrak{g}} \coloneqq \{(x, F) \in \mathfrak{sl}_\mathfrak{n} \times \mathcal{F}(\mathbb{C}^n) : x(F_i) \subset F_i, \forall i\},$$

where $F = (0 = F_0 \subset F_1 \subset \cdots F_n = \mathbb{C}^n)$ stands for a complete flag in $\mathbb{C}^n$. We construct a map $\nu : \tilde{\mathfrak{g}} \to H$ which would be thought of as assigning $(x, F) \in \tilde{\mathfrak{g}}$ to an ordered tuple of eigenvalues (counted with multiplicities) of $x \in \mathfrak{sl}_n(\mathbb{C})$. More precisely, assigning to any $(x, F) \in \mathfrak{sl}_n(\mathbb{C})$ the ordered tuple $(x_1, \ldots, x_n) \in \mathbb{C}^n$, where x_i is the scalar associated to the linear operator $x : F_i/F_{i-1} \to F_i/F_{i-1}$ of a 1-dimensional vector space. Observe that $\nu(\tilde{\mathfrak{g}}) \subset H$.

Finally, define $\mu : \tilde{\mathfrak{g}} \to \mathfrak{sl}_n(\mathbb{C})$ to be the first projection $(x, F) \mapsto x$ and $\pi : H \to H/S_n$ the natural projection. We have the following

Corollary 3.33. The following diagram commutes:

$$\begin{tikzcd} & \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathfrak{sl}_n(\mathbb{C}) \arrow[dr, "\rho"'] & & H \arrow[dl, "\pi"] \\ & H/S_n. & \end{tikzcd}$$

4. Symplectic Geometry

Let X be a C^∞-manifold over $\mathbb{R}$ (resp. smooth holomorphic, or algebraic variety over $\mathbb{C}$). Let $\mathcal{O}(X)$ denote the algebra of C^∞ (holomorphic, algebraic regular) functions on X, and call it the algebra of regular functions on X. Although we will be mainly working on differential manifolds in this section, most of the results also hold in the category of algebraic varieties.

4.1. Examples of Symplectic Structures.

Definition 4.1. A symplectic structure on X is a non-degenerate regular 2-form ω such that $d\omega = 0$.

Example 4.2. Let $X = \mathbb{C}^{2n}$ with coordinates $q_1, \dots, q_n, p_1, \dots, p_n$. Then the 2-form $\omega := \sum_{i=1}^n dp_i \wedge dq_i$ gives X an symplectic structure. Indeed, $d\omega = \sum_i ddp_i \wedge dq_i - dp_i \wedge ddq_i = 0$.

We shall explain two fundamental examples of symplectic structures in the following propositions.

Proposition 4.3. Let M be any manifold. Then the cotangent bundle $\mathrm{T}^* M = X$ has a canonical symplectic structure.

Proof. Our goal is to construct a 1-form λ on $\mathrm{T}^* M$ and define the symplectic structure to be $\omega = d\lambda$, which is automatically a closed form. For any fixed $x \in M$ and $\alpha \in \mathrm{T}^*_x M$, we have a surjective linear map $\pi_{(x,\alpha)} : \mathrm{T}_{(x,\alpha)}(\mathrm{T}^* M) \to \mathrm{T}_x M$ induced from the standard projection $\pi : \mathrm{T}^* M \to M$. Let $\xi \in \mathrm{T}_{(x,\alpha)}(\mathrm{T}^* M)$ be a tangent vector to the cotangent bundle at (x, α). Then we define the 1-form λ on $\mathrm{T}^* M$ by stipulating its evaluation of ξ at (x, α) to be $\alpha_x(\pi_{(x,\alpha)}\xi)$.

To see that ω is non-degenerate, we examine its local expression explicitly. Let $(q_1, \dots, q_n)$ be local coordinates on M around x, and $(p_1, \dots, p_n)$ the additional dual coordinates in $\mathrm{T}^* M$ around $\alpha \in \mathrm{T}^*_x M$. The tangent vector $\xi \in \mathrm{T}_{(x,\alpha)}(\mathrm{T}^* M)$ has the form $\xi = \sum_{i=1}^n b_i \frac{\partial}{\partial p_i} + \sum_{i=1}^n c_i \frac{\partial}{\partial q_i}$ for some $b_i, c_i \in \mathbb{C}$. The tangent map of the standard projection π sends ξ to $\sum_{i=1}^n b_i \frac{\partial}{\partial p_i}$ in $\mathrm{T}_x M$. It follows that

$$\lambda_{(x,\alpha)}(\xi) = \alpha_x(\pi_{(x,\alpha)}\xi) = \alpha_x\left(\sum_{i=1}^n c_i \frac{\partial}{\partial q_i}\right) = \left\langle \sum_{i=1}^n p_i(\alpha) dq_i, \sum_{i=1}^n c_i \frac{\partial}{\partial q_i} \right\rangle = \sum_{i=1}^n p_i(\alpha) c_i,$$

where $\langle -, - \rangle$ is the natural evaluation pairing $\mathrm{T}^*_x M \times \mathrm{T}_x M \to \mathbb{C}$. Therefore, we conclude that in this local coordinate, $\lambda = \sum_{i=1}^n p_i\, dq_i$ and thus by constructiion $\omega := d\lambda = \sum dp_i \wedge dq_i$ is non-degenerate. □

Definition 4.4. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{g}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathbb{C})$, the dual of $\mathfrak{g}$. The adjoint G-action on $\mathfrak{g}$ gives rise to the *coadjoint G-action* on $\mathfrak{g}^*$, to be denoted by Ad^*. Explicitly, for any $f \in \mathfrak{g}^*$, $x \in \mathfrak{g}$ and $g \in G$, one has $\langle \mathrm{Ad}^*(g)(f), x \rangle = \langle f, \mathrm{Ad}(g^{-1})x \rangle$, where $\langle -, - \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{C}$ is the evaluation pairing. Moreover, differentiating the coadjoint action map $\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*)$ at $g = e$, we obtain a $\mathfrak{g}$-action, $\mathrm{ad}^* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$, on $\mathfrak{g}^*$.

We first recall the following standard fact from Lie theory (c.f. Section 20 of [6]):

Fact 4.1. Let $\varphi : G \to \mathrm{Diff}(M)$ be a smooth action, $m \in M$, then

(1) The orbit $G \cdot m$ is an immersed submanifold of M, and its tangent space at m is
$$\mathrm{T}_m(G \cdot m) = \{\xi_X(m) : X \in \mathfrak{g}\}.$$
(2) The stabilizer subgroup (isotropy group) G_m of m is a closed Lie subgroup of G, with Lie algebra
$$\mathfrak{g}_m = \{X \in \mathfrak{g} : \xi_X(m) = 0\}.$$

Here $\xi_X : M \to \mathrm{T} M$ is the vector field generated by $X \in \mathfrak{g}$. Explicitly, $\xi_X(m) := \frac{d}{dt}|_{t=0} \exp(-tX) \cdot m \in \mathrm{T}_m M$. Note that the minus sign here makes sure the infinitesimal G-action

$$\mathfrak{g} \to \mathrm{Vect}(M) := \{\text{Vector fields on } M\}, \quad X \mapsto \xi_X$$

is a Lie algebra homomorphism.

Proposition 4.5. Any coadjoint orbit $\mathbb{O} \subset \mathfrak{g}^*$ has a natural symplectic structure.

Proof. Fix an arbitrary $\alpha \in \mathbb{O} \subset \mathfrak{g}^*$, we have an injective immersion $G/G^\alpha \hookrightarrow \mathfrak{g}^*$ with image $\mathbb{O}$, where G^α is the isotropy group of α. Stipulating $G/G^\alpha \cong \mathbb{O}$ gives $\mathbb{O}$ a manifold structure. Therefore $\mathrm{T}_\alpha \mathbb{O} \cong \mathrm{T}_\alpha(G/G^\alpha) = \mathfrak{g}/\mathfrak{g}^\alpha$, where $\mathrm{Lie}(G^\alpha) = \mathfrak{g}^\alpha$. We define a skew symmetric 2-form on $\mathrm{T}_\alpha \mathbb{O} \cong \mathfrak{g}/\mathfrak{g}^\alpha$ by first defining such a form on $\mathfrak{g}$:

$$\omega_\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, \quad (x, y) \mapsto \alpha([x, y]).$$

One observes that $\omega_\alpha(x, y) = 0$ for any $y \in \mathfrak{g}$ and $x \in \mathfrak{g}^\alpha$. Indeed, by fact 4.1, one has

$$0 = \left(\frac{d}{dt}|_{t=0} \exp(-tx) \cdot \alpha\right)(y) = \alpha\left(\frac{d}{dt}|_{t=0} \mathrm{Ad}\exp(-tx)^{-1} \cdot y\right) = \alpha\left(\mathrm{ad}(x)y\right) = \omega_\alpha(x, y).$$

Then ω_α descends to a non-degenerate form on $\mathfrak{g}/\mathfrak{g}^\alpha$. Whence the assignment $\alpha \mapsto \omega_\alpha$ gives a regular 2-form ω on $\mathbb{O}$. It remains to show that ω is closed. We direct the reader to the calculation in the proof of Proposition 1.1.5 of [2], which utilizes Cartan's formula for exterior differentials. That finishes the proof. □

4.2. Poisson Structures on $\mathcal{O}(M)$.

Let A be a commutative, associative unital $\mathbb{C}$-algebra with multiplication $\cdot : A \times A \to A$.

Definition 4.6. A commutative algebra $(A, \cdot)$ endowed with an additional $\mathbb{C}$-bilinear anti-symmetric bracket $\{-,-\} : A \times A \to A$ is called a *Poisson algebra* if the following conditions hold:

(1) A is a Lie algebra with respect to $\{-,-\}$;
(2) Leibniz rule: $\{a, b \cdot c\} = \{a, b\} \cdot c + b \cdot \{a, c\}$ for any $a, b, c \in A$.

The Lie bracket $\{-,-\}$ will be called a *Poisson bracket* on A. The bracket gives a Poisson structure on the commutative algebra $(A, \cdot)$.

Construction 4.7. Let (M, ω) be a symplectic manifold. The non-degenerate 2-form ω gives a canonical identification $\mathrm{T}\, M \cong \mathrm{T}^*\, M$. Define a unique $\mathbb{C}$-linear map $\mathcal{O}(M) \to \mathrm{Vect}(M) := \{\text{Vector fields on } M\}$, $f \mapsto \xi_f$, by requiring $\omega(-, \xi_f) = df$, i.e., $-df = i_{\xi_f}\omega$, where $i : \Omega^\bullet(M) \to \Omega^{\bullet-1}(M)$ is the internel contraction. In particular, for any vector field η, $\omega(\eta, \xi_f) = df(\eta) = \eta(f)$. We define a bracket on $\mathcal{O}(M)$ by the following equivalent expressions

$$\{f, g\} := \omega(\xi_f, \xi_g) = \xi_f(g) = -\xi_g(f), \quad \text{for any } f, g \in \mathcal{O}(M). \tag{4.8}$$

Definition 4.9. A vector field ξ on a symplectic manifold (M, ω) is called *symplectic* if it preserves the symplectic form, i.e., $\mathcal{L}_\xi(\omega) = 0$, where $\mathcal{L}_\xi$ is the Lie derivative with respect to ξ. The collection of symplectic vector fields on (M, ω) is denoted by $\mathrm{Vect}^{sp}(M)$.

We will show that the $\mathbb{C}$-algebra of regular functions $(\mathcal{O}(M), \cdot)$ together with the bracket $\{-,-\}$ defined in (4.8) is a Poisson structure. To that end, we will first prove the following proposition.

Proposition 4.10. For any $f \in \mathcal{O}(M)$, ξ_f is a symplectic vector field, i.e., $\mathcal{L}_{\xi_f}(\omega) = 0$. Furthermore, we have a Lie bracket preserving map

$$(\mathcal{O}(M), \{-,-\}) \longrightarrow (\mathrm{Vect}^{sp}(M), [-,-]), \quad f \mapsto \xi_f.$$

Proof. We notice that symplectic vector fields are closed under brackets: $\mathcal{L}_{[X,Y]}\omega = \mathcal{L}_X\mathcal{L}_Y\omega - \mathcal{L}_Y\mathcal{L}_X\omega = 0$, where $X, Y \in \mathrm{Vect}^{sp}(M)$. The first assertion follows from Cartan's magic formula and that ω is a closed form. To verify the second assertion, we need to show for any $f, g \in \mathcal{O}(M)$, $[\xi_f, \xi_g] = \xi_{\{f,g\}}$. Recall that in general, for $X \in \mathrm{Vect}(M)$ and $\alpha \in \Omega^k(M)$, one has

$$(\mathcal{L}_X\alpha)(X_1,\dots,X_k) = X(\alpha(X_1,\dots,X_k)) - \sum_{i=1}^{k}\alpha(X_1,\dots,X_{i-1},[X,X_i],X_{i+1},\dots,X_k),$$

where $X_i \in \mathrm{Vect}(M)$. In particular, we have for any $\eta \in \mathrm{Vect}(M)$,

$$\xi_f(\omega(\xi_g,\eta)) = (\mathcal{L}_{\xi_f}\omega)(\xi_g,\eta) + \omega([\xi_f,\xi_g],\eta) + \omega(\xi_g,[\xi_f,\eta]) = \omega([\xi_f,\xi_g],\eta) + \omega(\xi_g,[\xi_f,\eta]),$$

where the second equality follows from the first assertion. Rephrasing the above equation using the definition of ξ_f (resp. ξ_g), we obtain

$$\begin{aligned} -\xi_f\eta(g) = \xi_f(\omega(\xi_g,\eta)) &= \omega([\xi_f,\xi_g],\eta) - [\xi_f,\eta](g) \\ &= \omega([\xi_f,\xi_g],\eta) - \xi_f\eta(g) + \eta\xi_f(g) \\ &= \omega([\xi_f,\xi_g],\eta) - \xi_f\eta(g) + \eta(\{f,g\}) \\ &= \omega([\xi_f,\xi_g],\eta) - \xi_f\eta(g) + \omega(\eta,\xi_{\{f,g\}}). \end{aligned}$$

The latter equality holds for all η, we conclude that $\xi_{\{f,g\}} = [\xi_f,\xi_g]$. ∎

We are now ready to prove the main theorem of this subsection.

Theorem 4.11. The algebra $\mathcal{O}(M)$ of regular functions (with pointwise multiplication) on a symplectic manifold (M,ω) together with $\{,\}$ is a Poisson algebra.

Proof. The Jacobi identity holds for $\{,\}$ because for any $f, g, h \in \mathcal{O}(M)$

$$\{\{f,g\},h\} = \xi_{\{f,g\}}h \overset{\text{Prop.4.10}}{=\!=\!=} [\xi_f,\xi_g]h = \{f,\{g,h\}\} - \{g,\{f,h\}\}.$$

The Leibniz rule follows immediately from the Leibniz rule for vector fields:

$$\{f, g\cdot h\} = \xi_f(g\cdot h) = \xi_f(g)\cdot h + \xi_f(h)\cdot g = \{f,g\}\cdot h + \{f,h\}\cdot g.$$

∎

4.3. **Isotropic, Coisotropic and Lagrangian subvarieties.** We need more definitions.

Notation 4.12. Let (V,ω) be a symplectic vector space. Given a vector subspace W, we let $W^{\perp_\omega} := \{v \in V : \omega(v,u) = 0 \text{ for all } u \in W\}$, to be distinguished from $W^\perp$, the annihilator of W in V^*.

Definition 4.13. Given (V,ω) a symplectic vector space. A linear subspace $W \subset V$ is called

(1) *Isotropic* if $\omega|_W = 0$, equivalently $W \subset W^{\perp_\omega}$;
(2) *Coisotropic* if $W^{\perp_\omega}$ is isotropic, equivalently, $W^{\perp_\omega} \subset W$;
(3) *Lagrangian* if W is both isotropic and coisotropic, i.e., $W = W^{\perp_\omega}$.

Definition 4.14. A (possibly singular) subvariety Z of a symplectic manifold M is called an isotropic (resp. coisotropic, lagrangian) subvariety of M, if for any smooth point $z \in Z$, $\mathrm{T}_x Z$ is an isotropic (resp. coisotropic, lagrangian) subspace of $\mathrm{T}_x M$.

Definition 4.15 (Conormal bundles)**.** Let X be a manifold. Given a submanifold $Y \subset X$, define $\mathrm{T}^*_Y X$, the *conormal bundle* of Y, to be the set of all covectors over Y which annihilate the subbundle $\mathrm{T}\,Y \subset \mathrm{T}\,X$. Explicitly,

$$\mathrm{T}^*_Y X \coloneqq \{(y, \alpha_y) \in (\mathrm{T}^* X)|_Y : \alpha_y(T_y Y) = 0\}.$$

In particular, we have a natural diagram

$$(\mathrm{T}^* X)|_Y \supset \mathrm{T}^*_Y X \twoheadrightarrow Y.$$

Proposition 4.16. The total space of the conormal bundle $\mathrm{T}^*_Y X$ is a Lagrangian submanifold of $\mathrm{T}^* X$ stable under dilations along the fibers of $\mathrm{T}^* X$.

Proof. This is Proposition 1.3.26 of [2]. □

4.4. **Moment Maps.** Let (M, ω) be a symplectic manifold. Suppose that a Lie group G acts smoothly on M, preserving the symplectic form, that is $\omega_p(x, y) = \omega(g \cdot x, g \cdot y)$ for all $x, y \in \mathrm{T}_p M$, $p \in M$ and $g \in G$. In other words, for any $g \in G$, the diffeomorphism $g : M \to M$ is a *symplectomorphism* (such a notion is an analogue of isometry in the setting of Riemannian manifolds), i.e., $g^*\omega = \omega$. The infinitesimal G-action on M gives a Lie algebra homomorphism (c.f. Fact 4.1)

$$\mathfrak{g} \coloneqq \mathrm{Lie}(G) \longrightarrow \mathrm{Vect}^{sp}(M), \quad X \mapsto \xi_X.$$

Indeed, this map make sense because $\mathcal{L}_{\xi_X}(\omega)|_p = \frac{d}{dt}|_{t=0}\phi_t^*\omega$, where $\phi_t(m) \coloneqq \exp(-tX)\cdot m$, and $\phi_t^*\omega = \omega$.

Theorem 4.10 gives us a Lie algebra homomorphism $\mathcal{O}(M) \to \mathrm{Vect}^{sp}(M)$. We now state a key definition.

Definition 4.17. A symplectic G-action on (M, ω) is said to be *Hamiltonian* if a Lie algebra homomorphism $H : \mathfrak{g} \to \mathcal{O}(M)$, $x \mapsto H_x$ is given so that we have a lifting of Lie algebra homomorphisms:

$$\begin{tikzcd} & \mathrm{Vect}^{sp}(M) & \\ \mathfrak{g} \arrow[ur] \arrow[rr, dashed, "H"] & & \mathcal{O}(M). \arrow[ul] \end{tikzcd}$$

In the case of a Hamiltonian G-action, we fix the map $H : \mathfrak{g} \to \mathcal{O}(M)$ once and for all and refer to it as the *Hamiltonian*. The *moment map* $\mu : M \to \mathfrak{g}^*$ is defined by $\mu(m)(x) \coloneqq H_x(m)$ for any $x \in \mathfrak{g}$ and $m \in M$.

Given a diffeomorphism $f : X \to X$, we have a pullback lift

$$f^* : \mathrm{T}^* X \to \mathrm{T}^* X, \quad (x, \alpha_x) \mapsto \big(f(x), (f^{-1})^*\alpha|_{f(x)}\big) = (f(x), \alpha_x \circ d_{f(x)} f^{-1}).$$

It is a routine to verify that $f^*\lambda = \lambda$ for the canonical one-form λ on $\mathrm{T}^* X$, and hence $f^*\omega = \omega$. Therefore, given a G-action on X, one obtains a G-action on $\mathrm{T}^* X$ which is symplectic. We will show in Proposition 4.20 that any manifold with a smooth Lie group G-action gives its cotangent bundle a natural Hamiltonian G-action.

We start by discussing a few lemmas related to the canonical symplectic structure on the cotangent bundle of an arbitrary smooth manifold. With the canonical symplectic structure on $\mathrm{T}^* X$, one can associate $h \in \mathcal{O}(\mathrm{T}^* X)$ with a vector field $\xi_h \in \mathrm{Vect}(\mathrm{T}^* X)$. In particular, let u be a vector field on X, we can associate it with a regular function $h_u \in \mathcal{O}(\mathrm{T}^* X)$ on the cotangent bundle, which assigns every $(x, \alpha_x) \in \mathrm{T}^*_x X$ to $\alpha_x(u_x)$. Therefore, we have in particular a vector field ξ_{h_u} on the cotangent bundle.

Let π be the standard projection $\pi : \mathrm{T}^* X \to X$. Denote $\tilde{u}$ to be the induced vector field on the cotangent bundle of X along π, whose construction can be made explicitly as follows. Let $H : \Omega \to X$ be a local flow whose infinitesimal generator is u. Here Ω is a neighborhood of $\{0\} \times X \subset \mathbb{R} \times X$ with the property that Ω meets every line $\mathbb{R} \times \{p\}$ in an interval $\Omega_p \times \{p\}$. We define $\tilde{H} : \tilde{\Omega} \to \mathrm{T}^* X$ to be $\tilde{H}(t, (p, \alpha)) := (H_t(p), \alpha \circ (d_{H_t(p)} H_t^{-1}))$, where $\tilde{\Omega}$ meets every line $\mathbb{R} \times \{(p, \alpha \in \mathrm{T}_p^* X)\}$ in $\Omega_p \times \{(p, \alpha)\}$. It is routine to check that $\tilde{H}$ defines a local flow on the cotangent bundle over X, whence its infinitesimal generator is the vector field $\tilde{u}$. In particular, for any $(x, \alpha) \in \mathrm{T}^* M$, we have $d_{(x,\alpha)}\pi(\tilde{u}_{(x,\alpha)}) = u_x$.

Our goal is to clarify the relationship between objects $u, \tilde{u}$ and h_u defined above. To that end, we need a lemma.

Lemma 4.18. For any vector field u on X, the induced vector field $\tilde{u}$ is a symplectic vector field on $\mathrm{T}^* X$.

Proof. Recall that the symplectic structure ω on $\mathrm{T}^* X$ is given by a canonical 1-form λ (c.f. Proposition 4.3). Observe that $\mathcal{L}_{\tilde{u}}\lambda|_{(x,\alpha)} = \frac{d}{dt}|_{t=0}\tilde{H}_t^*\lambda|_{(x,\alpha)} = \frac{d}{dt}|_{t=0}\lambda = 0$ by a routine calculation. It follows that $\mathcal{L}_{\tilde{u}}\omega = \mathcal{L}_{\tilde{u}} d\lambda = d\mathcal{L}_{\tilde{u}}\lambda = 0$ as desired. □

Proposition 4.19. With the notation introduced in the previous paragraphs, we have $\tilde{u} = \xi_{h_u} \in \mathrm{Vect}(\mathrm{T}^* X)$, and $h_u = \lambda(\tilde{u})$.

Proof. By Cartan's magic formula and the lemma above, $0 = \mathcal{L}_{\tilde{u}}\lambda = i_{\tilde{u}}\omega + d i_{\tilde{u}}\lambda$. Let $h = i_{\tilde{u}}\lambda \in \mathcal{O}(\mathrm{T}^* X)$, the preceeding statement implies $\omega(-, \tilde{u}) = dh$. Then $\xi_h = \tilde{u}$. It is enough to show $h_u = h = \lambda(\tilde{u})$. To this end, we observe that

$$h(x, \alpha) = \lambda(\tilde{u})(x, \alpha) = \alpha(\pi_{(x,\alpha)}\tilde{u}_{(x,\alpha)}) = \alpha(u_x) = h_u(x, \alpha).$$

The proposition follows. □

Proposition 4.20. For any G-manifold X, the G-action on $\mathrm{T}^* X$ is always Hamiltonian with Hamiltonian

$$x \mapsto H_x = \lambda(\tilde{u}_x) \in \mathcal{O}(\mathrm{T}^* X),$$

where λ is the canonical 1-form on $\mathrm{T}^* X$, u_x is the vector field on X given by infinitesimal $\mathfrak{g}$-action of $x \in \mathfrak{g}$ and $\tilde{u}_x$ is the induced vector field on the cotangent bundle. Such a Hamiltonian gives rise to a moment map.

Corollary 4.21. Let G be a Lie group and $P \subset G$ a closed Lie subgroup. Let $\mathfrak{p} := \mathrm{Lie}(P)$ and write $\mathfrak{p}^\perp$ for the annihilator of $\mathfrak{p}$ in $\mathfrak{g}^*$. By Proposition 4.20, the left G-action on G/P induces a Hamiltonian G-action on $\mathrm{T}^*(G/P)$. The latter gives rise to the moment map

$$\mu : \mathrm{T}^*(G/P) \longrightarrow \mathfrak{g}^*.$$

We now proceed to give an explicit description of this moment map. To this end, we first describe the cotangent bundle $\mathrm{T}^*(G/P)$, which will be very useful in the future.

Lemma 4.22. There is a natural G-equivariant isomorphism

$$\mathrm{T}^*(G/P) \cong G \times_P \mathfrak{p}^\perp, \tag{4.23}$$

where P acts on $\mathfrak{p}^\perp$ by the coadjoint action.

Proof. We recall that $G \times_P \mathfrak{p}^\perp$ is the associated vector bundle to the principal P-bundle $G \to G/P$ (see Definition 7.10 in the Appendix, despite that we are in the smooth category). Now consider G-action on the homogeneous space G/P. The stabilizer of the point $gP \in G/P$ is the subgroup gPg^{-1}, whose Lie algebra is precisely $\mathrm{Ad}(g)\mathfrak{p}$. The bundle map $G \to G/P$ decomposes into the following:

$$G \to G/gPg^{-1} \xrightarrow{\cong} G/P, \tag{4.24}$$

where the second map is given by the Orbit-stabilizer theorem (and the quotient manifold theorem). Taking differentials at $e \in G$ yields a chain of linear maps between tangent spaces:

$$\mathfrak{g} \longrightarrow \mathfrak{g}/\operatorname{Ad}(g)\mathfrak{p} \xrightarrow{\cong} \mathrm{T}_{gP}(G/P). \tag{4.25}$$

We define a bundle morphism

$$\varphi : G \times_P \mathfrak{g}/\mathfrak{p} \longrightarrow \mathrm{T}(G/P), \quad [g,v] \longmapsto (gP, \mathrm{Ad}(g)v).$$

This is independent of the choice of $v \in \mathfrak{g}$ by the identification in (4.25). Note that $(gpP, \mathrm{Ad}(gp)\,\mathrm{Ad}(p^{-1})v) = (gP, \mathrm{Ad}(g)v)$, the map is well-defined. It is an isomorphism because on each fiber over gP, φ coincides with $\mathfrak{g}/\mathfrak{p} \xrightarrow{\mathrm{Ad}(g),\cong} \mathfrak{g}/\operatorname{Ad}(g)\mathfrak{p} \xrightarrow{4.25,\cong} \mathrm{T}_{gP}(G/P)$, an isomorphism of vector spaces. Since $(\mathfrak{g}/\mathfrak{p})^* = \mathfrak{p}^\perp \subset \mathfrak{g}^*$, we obtain an isomorphism of vector bundles $G \times_P \mathfrak{p}^* \to \mathrm{T}^*(G/P)$. ☐

Proposition 4.26. Under the isomorphism

$$G \times_P \mathfrak{p}^* \to \mathrm{T}^*(G/P),$$

the moment map $\mu : \mathrm{T}^*(G/P) \to \mathfrak{g}^*$ is given explicitly by $[g,\alpha] \mapsto \mathrm{Ad}^*(g)\alpha$ for any $g \in G$, $\alpha \in \mathfrak{p}^\perp$.

Proof. Recall that the moment map sends $[g,\alpha]$ to the linear functional $\mu(g,\alpha) : \mathfrak{g} \to \mathbb{C}$ given by $x \mapsto H_x(g,\alpha)$, $x \in \mathfrak{g}$, where H_x is the Hamiltonian for x. Write u_x and $\tilde{u}_x$ the induced vector fields on G/P and $\mathrm{T}^*(G/P)$ as before. We have

$$\mu(g,\alpha)(x) = \lambda_{[g,\alpha]}(\tilde{u}_x) = \lambda_{(gP,\mathrm{Ad}^*(g)\alpha)}(\tilde{u}_x) = \left(\mathrm{Ad}^*(g)\alpha\right)(x),$$

as we want. ☐

In particular, in the algebraic group setting, we have the following (c.f. Example 7.17)

Corollary 4.27. Let G be a connected semisimple algebraic group and $B \subset G$ a Borel subgroup with Lie algebra $\mathfrak{b}$. Then there is a G-equivariant isomorphism

$$\mathrm{T}^*(G/B) \cong G \times_B \mathfrak{n}, \quad \mathfrak{n} = [\mathfrak{b}, \mathfrak{b}], \tag{4.28}$$

where we identify $\mathfrak{b}^\perp$ with $\mathfrak{n}$ via $\mathfrak{g} \cong \mathfrak{g}^*$ (c.f. Proposition 7.18), the isomorphism induced from the Killing form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, $\kappa(x,y) := \mathbf{tr}(\operatorname{ad} x \circ \operatorname{ad} y)$, which is nondegenerate when $\mathfrak{g}$ is semisimple.

Finally, we state a theorem which will play an important role in the discussion of the Steinberg variety.

Theorem 4.29 (c.f. Theorem 1.5.7 of [2])**.** Let A be a solvable algebraic group with a Hamiltonian action on a symplectic algebraic variety M. Let $\mathfrak{a} := \mathrm{Lie}(A)$ and let μ be the moment map

$$\mu : M \longrightarrow \mathfrak{a}^*.$$

Then for any coadjoint orbit $\mathbb{O} \subset \mathfrak{a}^*$, the set $\mu^{-1}(\mathbb{O})$ is either empty or is a coisotropy subvariety of M.

5. Nilpotent Cones

Definition 5.1. The set of nilpotent elements, denoted by

$$\mathcal{N} := \{x \in \mathfrak{g} : x \text{ is nilpotent}\}$$

is a closed, G-stable cone subvariety (i.e., invariant under $\mathbb{C}^*$-dilations) of $\mathfrak{g}$.

The projection $\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$ give rise to a subvariety of the source:

$$\tilde{\mathcal{N}} := \mu^{-1}(\mathcal{N}) = \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} : x \in \mathfrak{b}\}.$$

Observe that the fiber over $\mathfrak{b}$ of the second projection $\pi : \tilde{\mathcal{N}} \to \mathcal{B}$ is the nilradical $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$ of $\mathfrak{b}$, by the Jordan decomposition. This projection makes $\tilde{\mathcal{N}}$ into a vector bundle over $\mathcal{B}$ with fiber $\mathfrak{n}$. Furthermore, since any nilpotent element of $\mathfrak{g}$ is G-conjugate into $\mathfrak{n}$, we get a G-equivariant vector bundle isomorphism:

$$\tilde{\mathcal{N}} = \mu^{-1}(\mathcal{N}) \cong G \times_B \mathfrak{n},$$

where B is such that $\mathrm{Lie}(B) = \mathfrak{b}$. In particular, $\tilde{\mathcal{N}}$ is a smooth variety as a vector bundle of a smooth variety $\mathcal{B}$, while $\mathcal{N}$ itself is singular at the origin as it is a cone variety (but not a vector space).

Lemma 5.2. There is a natural G-equivariant vector bundle isomorphism

$$\tilde{\mathcal{N}} \cong \mathrm{T}^* \mathcal{B}.$$

Proof. By Corollary 4.27, we have an identification

$$\mathrm{T}^* \mathcal{B} = \mathrm{T}^*(G/B) \cong G \times_B \mathfrak{b}^\perp \cong G \times_B \mathfrak{n} \cong \tilde{\mathcal{N}},$$

from which the Lemma follows. □

Corollary 5.3. The projection $\mu : \mathrm{T}^* \mathcal{B} \cong \tilde{\mathcal{N}} \to \mathcal{N}$ is the moment map with respect to the Hamiltonian G-action on $\mathrm{T}^* \mathcal{B}$ arising form the G-action on $\mathcal{B}$. Furthermore, this moment map is surjective.

Proof. The first assertion follows from Proposition 4.26. The second assertion is true because any nilpotent element is contained in a Borel subalgebra. □

Example 5.4. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, the nilpotent cone $\mathcal{N}$ is isomorphic to a quadratic cone in $\mathbb{C}^3$,

$$\mathcal{N} = \left\{ x = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C}) : \det(x) = -a^2 - bc = 0 \right\}, \tag{5.5}$$

and $\tilde{\mathcal{N}}$ is a line bundle over $\mathbb{P}^1 \cong \mathcal{B}$, given explicitly by (c.f. Example 3.11)

$$\begin{aligned} \pi : \tilde{\mathcal{N}} &\to \mathcal{B} \cong SL_2(\mathbb{C})/B = B/B \sqcup BsB/B \cong \mathbb{P}^1 \cong \mathcal{F}(\mathbb{C}^2) \\ (n, \mathfrak{b}) &\mapsto \mathfrak{b}. \end{aligned}$$

Indeed, consider the following diagram:

$$\begin{tikzcd} & \tilde{\mathcal{N}} \arrow[dl, "\mu"'] \arrow[dr, "\pi"] & \\ \mathcal{N} & & \mathcal{B}. \end{tikzcd}$$

Every point $\mathfrak{b} \in \mathcal{B} \cong \mathbb{P}^1$ corresponds to a line $\ell \in \mathbb{C}^2$, which is preserved by $\mathfrak{b}$-action (c.f. Proposition 2.8). The fiber of $\mathfrak{b}$ along π can be identified naturally as the collection of nilpotent matrices in $\mathfrak{sl}_2(\mathbb{C})$, which preserve ℓ, or equivalently, the collection of nilpotent operators with image contained in ℓ. Such operators form a line in $\mathcal{N}$. Thus, points in $\mathcal{B}$ correspond to lines in $\mathcal{N}$. These lines are depicted in the picture.

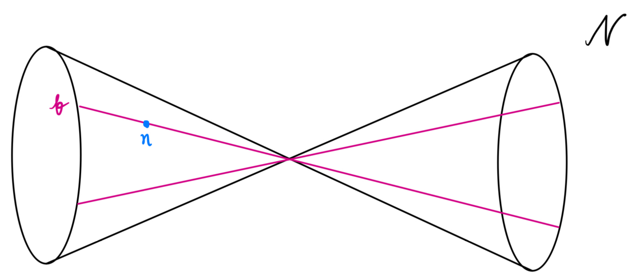

Figure 1. The nilpotent cone of $\mathfrak{sl}_2(\mathbb{C})$.

A point of $\tilde{\mathcal{N}}$ is represented by a pair $(n, \mathfrak{b})$ where $\mathfrak{b}$ corresponds to a line on the picture and n is a point on the line.

We would like to give an intrinsic characterization of nilpotent elements in $\mathfrak{g}^*$, even though one can require that elements in $\mathfrak{g}^*$ are nilpotent if and only if it is so in $\mathfrak{g}$ under the G-equivariant identification $\mathfrak{g}^* \cong \mathfrak{g}$ given by the Killing form. The idea generalizes the classical result saying that an $n \times n$-matrix A is nilpotent if and only if $\det(\lambda - A) = \lambda^n$.

Proposition 5.6. Let $\mathbb{C}[\mathfrak{g}]^G_+$ denote the set of G-invariant polynomials on $\mathfrak{g}$ without constant term. An element $x \in \mathfrak{g}$ is nilpotent if and only if for every $P \in \mathbb{C}[\mathfrak{g}]^G_+$, we have $P(x) = 0$.

Remark 5.7. This proposition shows that nilpotent elements in $\mathfrak{g}^*$ defined using the Killing form are the same as the following intrinsic definition: An element $\lambda \in \mathfrak{g}^*$ is nilpotent if and only if any G-invariant polynomial on $\mathfrak{g}^*$ without constant term vanishes at λ.

Proof of Proposition 5.6. We adopt the notations in the Springer diagram (3.30). Let $\rho : \mathfrak{g} \to \mathfrak{H}/\mathbb{W}$ and $\pi : \mathfrak{H} \to \mathfrak{H}/\mathbb{W}$ as in the Springer diagram. Since we have $\rho^* : \mathbb{C}[\mathfrak{H}]^W \xrightarrow{\cong} \mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[\mathfrak{g}]$ by construction, proving the statement amounts to show that

$$\rho^{-1}(0) = \mathcal{N}. \tag{5.8}$$

Indeed, $\{0\} \subset \mathfrak{H}/\mathbb{W}$ corresponds to the ideal $\mathbb{C}[\mathfrak{g}]^G_+ \cong \mathbb{C}[\mathfrak{H}]^{\mathbb{W}}_+$. The fiber of $\{0\}$ in $\mathfrak{g}$ corresponds to the vanishing locus of the ideal generated by $\mathbb{C}[\mathfrak{g}]^G_+$ in $\mathbb{C}[\mathfrak{g}]$. The verification of (5.8) is (3.2.7) of [2]

□

Corollary 5.9. We have an extension of the Springer diagram in Proposition 3.30, i.e., the diagram

$$\begin{tikzcd}
& \tilde{\mathcal{N}} \arrow[rr, hook] \arrow[dl, two heads, "\mu"'] & & \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\
\mathcal{N} \arrow[rr, hook] \arrow[dr] & & \mathfrak{g} \arrow[dr, "\rho"'] & & \mathfrak{H} \arrow[dl, "\pi"] \\
& \{0\} \arrow[rr, hook] & & \mathfrak{H}/\mathbb{W} &
\end{tikzcd}$$

is commutative.

Corollary 5.10. The nilpotent cone $\mathcal{N}$ is an irreducible variety of dimension $2\dim\mathfrak{n}$.

Proof. Since the cotangent bundle $\mathrm{T}^*\mathcal{B}$ is smooth and connected because the base space is so, it is irreducible. Surjectivity of the moment map $\mu : \mathrm{T}^*\mathcal{B} \to \mathcal{N}$ implies that $\mathcal{N}$ is irreducible and $\dim\mathcal{N} \leq \dim\mathrm{T}^*\mathcal{B} = 2\dim\mathcal{B} = 2\dim\mathfrak{n}$. On the other hand, the extended Springer diagram (5.9) illustrates that the nilpotent cone $\mathcal{N}$ is the zero fiber of the algebraic morphism $\rho : \mathfrak{g} \to \mathfrak{H}/\mathbb{W}$. By the fiber dimension theorem in algebraic varieties, we have

$$\dim(\text{fiber}) \geq \dim\mathfrak{g} - \dim(\mathfrak{H})/\mathbb{W} = \dim\mathfrak{g} - \mathrm{rk}\mathfrak{g} = 2\dim\mathfrak{n}.$$

Therefore, the statement follows. □

Proposition 5.11. The number of nilpotent conjugacy classes (G-orbits) of $\mathfrak{g}$ is finite.

Proof. We postpone the argument to Corollary 6.16. □

Proposition 5.12. The regular nilpotent elements form a single Zariski-open, dense conjugacy class in $\mathcal{N}$.

Proof. Since $\mathcal{N}$ is irreducible and is the union of finitely many G-orbits, it contains a unique open dense conjugacy class $\mathbb{O}$ by Corollary 7.24. Then $\dim\mathcal{N} = \dim\mathbb{O} = \dim G - \dim Z_G(x)$ for any $x \in \mathbb{O}$. Hence $\dim Z(x) = \dim G - \dim\mathcal{N} = \dim\mathfrak{g} - 2\dim\mathfrak{n} = \mathrm{rk}\mathfrak{g}$, which implies x is regular. For other orbits, which are contained in $\overline{\mathbb{O}} \setminus \mathbb{O}$ have dimension strictly less than $\dim\mathbb{O} = \dim\mathcal{N}$. A similar argument shows that $x \notin \mathbb{O}$ cannot be regular. □

Example 5.13. In $\mathfrak{sl}_n(\mathbb{C})$, there is only one regular nilpotent element, and it has the form

$$n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & \cdots & 0 \\ & & \ddots & & 1 \\ & & & & 0 \end{pmatrix},$$

and its centralizer is the linear span of the matrices $(x, x^2, \ldots, x^{n-1})$ which has dimension $n - 1 = \mathrm{rk}\ \mathfrak{sl}_n(\mathbb{C})$.

One can show that the Springer resolution $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ is indeed a resolution of singularities. More precisely, μ is an isomorphism over the open dense subset $\mathbb{O}$ consisting of regular nilpotent elements. We refer readers to Proposition 3.2.14 of [2].

6. The Steinberg Variety

We are now ready to introduce an important notion in geometric representation theory, which is closely related to the representation of Weyl groups (c.f. [2]).

Definition 6.1. Let $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ be the Springer resolution. The following subvariety in the product $\tilde{\mathcal{N}} \times \tilde{\mathcal{N}}$ is called the *Steinberg variety*:

$$Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} := \{(x, \mathfrak{b}), (x', \mathfrak{b}') \in \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} : x = x'\}.$$

Observe that Z is isomorphic to the variety of triples:

$$Z \cong \{(x, \mathfrak{b}, \mathfrak{b}') \in \mathcal{N} \times \mathcal{B} \times \mathcal{B} : x \in \mathfrak{b} \cap \mathfrak{b}'\}.$$

Restricting the Springer diagram to the Steinberg variety, we obtain two maps:

$$\mu : Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} \to \mathcal{N}, \quad (x, \mathfrak{b}, \mathfrak{b}') \mapsto x;$$

$$\pi^2 : Z \hookrightarrow \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \xrightarrow{\pi \times \pi} \mathcal{B} \times \mathcal{B}.$$

We will study the interplay of these two maps. Recall that in Lemma 5.2, we have shown there is a G-equivariant bundle isomorphism $\mathrm{T}^* \mathcal{B} \cong \tilde{\mathcal{N}}$. Combining this with the isomorphism

$$\mathrm{T}^* \mathcal{B} \times \mathrm{T}^* \mathcal{B} \xrightarrow{\cong} \mathrm{T}^*(\mathcal{B} \times \mathcal{B})$$
$$((x, \mathfrak{b}), (x', \mathfrak{b}')) \mapsto ((x, -x'), (\mathfrak{b}, \mathfrak{b}')),$$

we have a chain of isomorphisms

$$\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} \cong \mathrm{T}^* \mathcal{B} \times \mathrm{T}^* \mathcal{B} \xrightarrow{\cong} \mathrm{T}^*(\mathcal{B} \times \mathcal{B}). \tag{6.2}$$

This sign convention induces an isomorphism $\tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \cong \tilde{\mathcal{N}} \times \tilde{\mathcal{N}}$ given by the formula involving a minus sign

$$((x, \mathfrak{b}), (x', \mathfrak{b}')) \mapsto ((x, \mathfrak{b}), (-x', \mathfrak{b}')). \tag{6.3}$$

The following result provides a geometric interpretation for the Steinberg variety.

Proposition 6.4. The Steinberg variety Z is the union of the conormal bundles to all G-orbits in $\mathcal{B} \times \mathcal{B}$, which are parameterized by the Weyl group (c.f. Theorem 3.4).

Proof. For $(\mathfrak{b}_1, \mathfrak{b}_2) \in \mathcal{B} \times \mathcal{B}$, let $\alpha \in \mathrm{T}^*_{(\mathfrak{b}_1, \mathfrak{b}_2)}(\mathcal{B} \times \mathcal{B})$ be such that α annihilates the tangent space of the G-orbit in $\mathcal{B} \times \mathcal{B}$ through $(\mathfrak{b}_1, \mathfrak{b}_2)$. Recall that in the proof of $\mathrm{T}^* \mathcal{B} \cong G \times_B \mathfrak{n} \cong G \times_B \mathfrak{b}^{\perp}$ in Lemma 4.22, where $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ is the nilradical of $\mathfrak{b}$, we have shown an identification

$$\mathrm{T}^*_{\mathfrak{b}_1} \mathcal{B} = \mathrm{T}^*_{gB}(G/B) \cong (\mathfrak{g}/\operatorname{Ad}(g)\mathfrak{b}_o)^* = (\mathfrak{g}/\mathfrak{b}_1)^* = \mathfrak{b}_1^{\perp} \subset \mathfrak{g}^*,$$

where $\mathfrak{b}_1$ correspondes to gB and $\mathfrak{b}_o = \operatorname{Lie}(B)$. Then

$$\alpha = (\mathfrak{b}_1, \mathfrak{b}_2, x_1, x_2) \in \mathcal{B} \times \mathcal{B} \times \mathfrak{g}^* \times \mathfrak{g}^*,$$

where $x_1 \in \mathfrak{b}_1^{\perp}$, and $x_2 \in \mathfrak{b}_2^{\perp}$. Note that the tangent space at $(\mathfrak{b}_2, \mathfrak{b}_2)$ to the G-orbit through this point consists of points $(u, u) \in \mathfrak{g}/\mathfrak{b}_1 \times \mathfrak{g}/\mathfrak{b}_2$. Let $\langle -, - \rangle$ denote the nondegenerate pairing $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{C}$. Then α being annihilated by the tangent space of the G-orbit is precisely saying

$$\langle x_1, u \rangle + \langle x_2, u \rangle = 0 \quad \text{for all } u \in \mathfrak{g}.$$

Equivalently, $x_1 = -x_2$ so that $\alpha = ((x_1, -x_1), (\mathfrak{b}_1, \mathfrak{b}_2))$. This is precisely the Steinberg variety under the identification (6.3). $\square$

Corollary 6.5. Write Y_w for the orbit corresponding to an element $w \in \mathbb{W}$ in the abstract Weyl group $\mathbb{W}$. The Bruhat decomposition asserts that $G\backslash(\mathcal{B}\times\mathcal{B}) = \sqcup_{w\in\mathbb{W}} Y_w$. Then

(1) $Z = \sqcup_{w\in\mathbb{W}} \mathrm{T}^*_{Y_w}(\mathcal{B}\times\mathcal{B})$.
(2) Irreducible components of Z are parameterized by elements of $\mathbb{W}$. Every irreducible component is the closure of $\mathrm{T}^*_{Y_w}(\mathcal{B}\times\mathcal{B})$ for a uniquely determined $w \in \mathbb{W}$.

Proof. Notice that (1) is just a restatement of the above proposition. To see (2), we observe that for any $w \in \mathbb{W}$, Y_w is irreducible and smooth as an orbit (c.f. Proposition 7.23), so the conormal bundle (in particular a vector bundle) $\mathrm{T}^*_{Y_w}(\mathcal{B}\times\mathcal{B})$ over it is smooth and irreducible. By Proposition 4.16, each $\mathrm{T}^*_{Y_w}(\mathcal{B}\times\mathcal{B})$ is a Lagrangian subvariety of $\mathrm{T}^*(\mathcal{B}\times\mathcal{B})$, it has the same dimension as $\mathcal{B}\times\mathcal{B}$. Thus the closure of $\mathrm{T}^*_{Y_w}(\mathcal{B}\times\mathcal{B})$ are irreducible and have the same dimension. It follows from Proposition 7.4 in the Appendix that the closure of $\mathrm{T}^*_{Y_w}(\mathcal{B}\times\mathcal{B})$ is precisely an irreducible component of Z. To see that each irreducible component corresponds to a unique $w \in \mathbb{W}$, we assume for contradiction that $w_1 \neq \omega_2 \in \mathbb{W}$ are such that $\overline{\mathrm{T}^*_{Y_{w_1}}(\mathcal{B}\times\mathcal{B})} = \overline{\mathrm{T}^*_{Y_{w_2}}(\mathcal{B}\times\mathcal{B})} =: C$. Since each conormal bundle $\mathrm{T}^*_{Y_{w_i}}(\mathcal{B}\times\mathcal{B})$ is open in the irreducible subvariety C, they must have a nonempty intersection, which is a contradiction. □

Fix a Borel subalgebra $\mathfrak{b}\subset\mathfrak{g}$ with nilradical $\mathfrak{n} = [\mathfrak{b},\mathfrak{b}]$. The following theorem is one of the key results of this section.

Theorem 6.6. Let $\mathbb{O}$ be a coadjoint orbit in $\mathfrak{g}^*$. Let $x \in \mathbb{O}$ be such that $x|_{\mathfrak{n}} = 0$. Then $\mathbb{O}\cap(x+\mathfrak{b}^\perp)$ is a (possibly singular) Lagrangian subvariety in $\mathbb{O}$ with respect to the natural symplectic structure on coadjoint orbits.

By the identification $\mathfrak{g}^* \cong \mathfrak{g}$ using the Killing form, we have the following restatement of the above theorem (c.f. Proposition 7.18).

Theorem 6.7. For any G-adjoint orbit $\mathbb{O}\subset\mathfrak{g}$ and any $x\in\mathbb{O}\cap\mathfrak{b}$, the set $\mathbb{O}\cap(x+\mathfrak{n})$ is a Lagrangian subvariety in $\mathbb{O}$.

Proof. Recall that we have defined $\mu : Z \to \mathcal{N}$ at the beginning of this section. Given a nilpotent G-orbit $\mathbb{O}\subset\mathcal{N}$, put $Z_o := \mu^{-1}(\mathbb{O}) \subset Z$. Let B be a Borel subgroup of G so that $\mathrm{Lie}(B) = \mathfrak{b}$. We will only prove the case where $x \in \mathfrak{n}$ is nilpotent.

First, we claim that
$$\dim(\mathbb{O}\cap\mathfrak{n}) \leq 1/2\cdot\dim\mathbb{O}.$$
Let $n = \dim\mathfrak{n} = \dim\mathcal{B}$. Then $\dim \mathrm{T}^*\mathcal{B} = 2n = \dim Z$ by the Proposition 7.4. Note that we have a fiber bundle $Z_o \to \mathbb{O}$ with the fiber over $x\in\mathbb{O}$ equal to $\mathcal{B}_x\times\mathcal{B}_x$, where $\mathcal{B}_x$ is the fiber of the Springer resolution $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$, i.e., the collection of $\mathfrak{b}\in\mathcal{B}$ containing x. We can find a local trivialization on $U\subset\mathbb{O}$ so that $\mu^{-1}(U) \cong U\times(\mathcal{B}_x\times\mathcal{B}_x)$. Therefore, we have
$$\dim\mathbb{O} + 2\cdot\dim\mathcal{B}_x = \dim U + 2\cdot\dim\mathcal{B}_x = \dim(\mu^{-1}(U)) \leq \dim Z_o \leq \dim Z = 2n. \tag{6.8}$$
It follows that
$$\dim\mathcal{B}_x + 1/2\dim\mathbb{O} \leq n. \tag{6.9}$$
Let $S = \{g\in G : \mathrm{Ad}(g)x\in\mathfrak{b}\} = \{g\in G : x\in\mathrm{Ad}(g^{-1})\mathfrak{b}\}$, where $x\in\mathfrak{n}\subset\mathfrak{b}$. Observe that the left action of B on S induces an isomorphism $B\backslash S \cong \mathcal{B}_x$ given by $Bg \mapsto \mathrm{Ad}(g)\mathfrak{b}$

as B normalizes $\mathfrak{b}$. Hence

$$\dim S - \dim B + 1/2 \cdot \dim \mathbb{O} \le n. \tag{6.10}$$

Note that $\mathbb{O} \cap \mathfrak{n} = \{\mathrm{Ad}(g)x : g \in S\}$. It follows that $S/Z_G(x) \xrightarrow{\cong} \mathbb{O} \cap \mathfrak{n}$ via $gZ_G(x) \mapsto \mathrm{Ad}(g)x$. Thus

$$\dim S + \dim \mathbb{O} - \dim G = \dim S - \dim Z_G(x) = \dim(\mathbb{O} \cap \mathfrak{n}). \tag{6.11}$$

Combining with (6.10), and recall that $\dim G - \dim B = \dim \mathcal{B} = n$, we have

$$\dim(\mathbb{O} \cap \mathfrak{n}) \le 1/2 \cdot \dim \mathbb{O}. \tag{6.12}$$

The claim follows.

Due to the dimension estimate of the claim, proving the theorem amounts to showing $\mathbb{O} \cap \mathfrak{n}$ is a coisotropic subvariety. Now view $\mathbb{O}$ as a symplectic manifold. More precisely, it is considered as a coadjoint orbit in $\mathfrak{g}^*$ with a Hamiltonian B-action. The moment map $\mu : \mathbb{O} \to \mathfrak{b}^*$ has a factorization

$$\mathbb{O} \hookrightarrow \mathfrak{g}^* \xrightarrow{p} \mathfrak{b}^*,$$

where the second map is the dualization of the inclusion $\mathfrak{b} \subset \mathfrak{g}$. Notice that $p^{-1}(0) = \mathfrak{b}^\perp = \mathfrak{n}$ and $\mu^{-1}(0) = \mathbb{O} \cap \mathfrak{n}$. Since $\{0\}$ is a B-orbit in $\mathfrak{b}^*$ and B is a solvable group. It follows form Theorem 4.29 that $\mathbb{O} \cap \mathfrak{n}$ is coisotropic, as desired. $\square$

We now give some corollaries of the theorem above.

Corollary 6.13. For any nilpotent G-orbit $\mathbb{O} \subset \mathcal{N}$, each irreducible component of $Z_o = \mu^{-1}(\mathbb{O}) \subset Z$ has the same dimension $\dim Z_o$, and $\dim Z_o = \dim Z = m := 2\dim \mathfrak{n}$.

Proof. Denote $\tilde{O} := \mu^{-1}(\mathbb{O}) \subset \mathcal{N}$, where here μ is the Springer resolution (by an abuse of notation). Then we obtain $Z_o = \tilde{O} \times_{\mathbb{O}} \tilde{O}$. We have the following commutative diagram:

$$\begin{tikzcd}
 & \tilde{\mathcal{N}} \cong \mathrm{T}^* \mathcal{B} \arrow[dl, "\pi"'] \arrow[dr, "\mu"] & \tilde{O} \arrow[l, hook'] \arrow[dr] & \\
\mathcal{B} = G/B & & \mathcal{N} & \mathbb{O}. \arrow[l, hook']
\end{tikzcd}$$

Observe that π restricted to $\tilde{O} \cong G \times_B (\mathbb{O} \cap \mathfrak{n})$ is a fiber bundle with fiber $\mathbb{O} \cap \mathfrak{n}$ over G/B. By Theorem 6.7, $\mathbb{O} \cap \mathfrak{n}$ is Lagrangian. Hence, every irreducible component contains an open dense subset of smooth points with dimension $1/2 \cdot \dim \mathbb{O}$. Then by Proposition 7.4, we know it is a pure variety with dimension $1/2 \cdot \dim \mathbb{O}$ and thus each irreducible component of $\tilde{O}$ has dimension equal to $\dim(G/B) + 1/2 \cdot \dim \mathbb{O}$. It follows that each irreducible component of Z_o has dimension

$$2\dim(\tilde{O}) - \dim \mathbb{O} = 2\dim(G/B) = m. \tag{6.14}$$

The corollary follows. $\square$

Remark 6.15. The above corollary implies that the decomposition of the Steinberg variety

$$Z = \bigsqcup_{\mathbb{O} \subset \mathcal{N}} Z_O = \bigsqcup_{\mathbb{O} \subset \mathcal{N}} \mu^{-1}(\mathbb{O})$$

gives a partition of Z into equidimensional locally closed subvarieties of the top dimension. Hence, the closure of each irreducible component of $\mu^{-1}(\mathbb{O})$ is an irreducible component of Z, which is precisely the closure of the total space $\mathrm{T}^*_{Y_w}(\mathcal{B} \times \mathcal{B})$

of a conormal bundle for a unique $w \in \mathbb{W}$ (c.f. Corollary 6.5). By the same argument in Corollary 6.5, we note that each irreducible component of Z corresponds to the closure of an irreducible component of a unique nilpotent orbit $\mathbb{O} \subset \mathcal{N}$.

Corollary 6.16. In particular, the number of nilpotent orbits in $\mathfrak{g}$ is finite.

Corollary 6.17. All irreducible components of the Springer fiber $\mathcal{B}_x$ have the same dimension $\dim \mathcal{B}_x$, and

$$1/2 \cdot \dim \mathbb{O} + \dim \mathcal{B}_x = \dim \mathcal{B}. \tag{6.18}$$

Proof. Given $x \in \mathbb{O}$, we identify the adjoint orbit $\mathbb{O}$ with the quotient $G/Z_G(x)$ where $Z_G(x)$ is the centralizer of x in G. Note that $Z_G(x)$ acts on the Borel subalgebras $\mathcal{B}_x$ containing x, we have an G-equivariant isomorphism

$$G \times_{Z_G(x)} \mathcal{B}_x \xrightarrow{\cong} \tilde{O} = \mu^{-1}(\mathbb{O}), \quad (g, \mathfrak{b}) \mapsto (\mathrm{Ad}(g)x, \mathrm{Ad}(g)\mathfrak{b}).$$

From this, we deduce that

$$Z_o = \tilde{O} \times_{\mathbb{O}} \tilde{O} \cong G \times_{Z_G(x)} (\mathcal{B}_x \times \mathcal{B}_x).$$

Hence irreducible component of Z_o is of the form $G \times_{Z_G(x)} (\mathcal{B}_1 \times \mathcal{B}_2)$, where $\mathcal{B}_1, \mathcal{B}_2$ are irreducible component of $\mathcal{B}_x$. It follows that

$$\dim \mathbb{O} + \dim \mathcal{B}_1 + \dim \mathcal{B}_2 = \dim Z_o = 2 \dim \mathcal{B}. \tag{6.19}$$

In particular, taking $\mathcal{B}_1 = \mathcal{B}_2$ yields the desired statement. □

We conclude this section by giving a concrete example.

Example 6.20. Let $G = SL_n(\mathbb{C})$ and $\mathbb{O} \subset \mathfrak{sl}_n(\mathbb{C})$ the variety of rank 1 nilpotent matrices. It is a standard fact in linear algebra that every rank 1 matrix A can be written in the form

$$(a_{ij}) = A = \alpha \cdot \beta^t, \quad \text{where } \alpha = (\alpha_1, \dots, \alpha_n)^t, \beta = (\beta_1, \dots, \beta_n)^t.$$

In particular, $a_{ij} = \alpha_i \beta_j$. It follows that

$$\mathbb{O} = \{(a_{ij}) \in \mathfrak{sl}_n(\mathbb{C}) : a_{ij} = \alpha_i \beta_j, \sum \alpha_i \beta_i = 0\}. \tag{6.21}$$

Let $\mathfrak{n} \subset \mathfrak{sl}_n(\mathbb{C})$ be the Lie subalgebra of upper triangular matrices. We consider the variety $\mathbb{O} \cap \mathfrak{n}$. Observe that for $A = \alpha \cdot \beta^t \in \mathbb{O} \cap \mathfrak{n}$, one must have

$$\begin{aligned} \alpha &= (\alpha_1, \dots, \alpha_k, 0, \dots, 0), \\ \beta &= (0, \dots, 0, \beta_{k+1}, \dots, \beta_n), \quad 1 \le k \le n-1. \end{aligned}$$

Irreducible components of the variety are parameterized by k above. Notice that each of these components has dimension $n-1$, which is half of $\dim \mathbb{O} = 2n-2$.

7. Appendix

7.1. **$\mathbb{C}^*$-actions on a Projective Variety.** Let X be a smooth projective variety with an algebraic $\mathbb{C}^*$-action $\mathbb{C}^* \times X \to X$. We embed the torus $\mathbb{C}^*$ into the Riemann sphere $\mathbb{CP}^1$ so that $\mathbb{CP}^1 \smallsetminus \mathbb{C}^* = \{0\} \cup \{\infty\}$. The following lemma is a special case of the Borel fixed point theorem for solvable Lie group actions (see [5]).

Lemma 7.1. Let X be a projective variety with an algebraic $\mathbb{C}^*$-action $\mathbb{C}^* \times X \to X$. For any point $x \in X$, the map $z \mapsto z \cdot x$ has a limit point as $z \in \mathbb{C}^*$ approaches $0 \in \mathbb{CP}^1$ (resp. ∞). Furthermore, the limit points $\lim_{z \to 0} z \cdot x$ (resp. $\lim_{z \to \infty} z \cdot x$) are fixed points of the $\mathbb{C}^*$-action. In particular, the fixed point set of an algebraic $\mathbb{C}^*$-action on a projective variety is always non-empty.

Proof. Fix $x \in X$. Denote $G = \{(z, z \cdot x) \in \mathbb{C}^* \times X : z \in \mathbb{C}^*\}$ to be the graph of the morphism $\varphi : \mathbb{C}^* \to X$ sending z to $z \cdot z \cdot x$. It is a closed subvariety of $\mathbb{C}^* \times X$ and is isomorphic to $\mathbb{C}^*$. Therefore, it is a 1-dimensional irreducible closed subvariety of $\mathbb{C}^* \times X$. Now, denote $Y = \overline{G} \subset \mathbb{CP}^1 \times X$ to be the closure of the graph in the product. We observe that Y is again a 1-dimensional irreducible variety. Furthermore, one observes that the restriction p of the first projection $p_1 : \mathbb{CP}^1 \times X \to \mathbb{CP}^1$ to Y is an isomorphism over $\mathbb{C}^* = \mathbb{CP}^1 \smallsetminus \{0, \infty\}$. Indeed, we know from point-set topology that $\overline{G} \cap (\mathbb{C}^* \times X) = Y \cap (\mathbb{C}^* \times X)$ is precisely the closure of G in $\mathbb{C}^* \times X$, which is just G itself as a closed subvariety. It follows that $p|_G$ is an isomorphism onto the image. We claim that $p : Y \to \mathbb{CP}^1$ is an isomorphism. Assuming this, the existence of the limit points will follow from explicit formulae $\lim_{z\to 0} z \cdot x = p_2 \circ p^{-1}(0)$ (resp. $\lim_{z\to\infty} z \cdot x = p_2 \circ p^{-1}(\infty)$), where p_2 is the projection onto the second component of $\mathbb{CP}^1 \times X$.

We proceed to prove the claim. Consider the normalization (c.f. section 1.10 of [8]) $\pi : \tilde{Y} \twoheadrightarrow Y$ of the irreducible projective curve Y. Here Y is projective because it is an irreducible closed subvariety of $\mathbb{CP}^1 \times X$, which is projective as a product of projective spaces. The normalization of an irreducible projective curve is projective, irreducible, and smooth (c.f. Lemma 4.1.3 of [8]). We notice that $p \circ \pi : \tilde{Y} \to \mathbb{CP}^1$ is a birational equivalence. As birational isomorphisms preserve genus, $\tilde{Y} \cong \mathbb{CP}^1$, and any birational automorphism of $\mathbb{CP}^1$ is an isomorphism, we deduce that $p \circ \pi$ is an isomorphism. Hence π becomes a bijective finite (and hence closed) morphism between two irreducible smooth varieties, which is an isomorphism. It follows that p is an isomorphism, as desired.

Now that we have $p : Y \cong \mathbb{CP}^1$, the restriction of $p_2 : \mathbb{CP}^1 \times X \to X$ to Y is a closed map. In particular, $p_2(Y) \subset X$ is closed. Observe that the orbit $\mathbb{C}^* \cdot x = p_2(G) \subset p_2(Y)$, we have $\overline{\mathbb{C}^* \cdot x} \subset p_2(Y)$. The reverse inclusion also holds. Indeed, $p_2(Y) = p_2(\overline{G}) \subset \overline{p_2(G)} = \overline{\mathbb{C}^* \cdot x}$. Whence $p_2(Y) = \overline{\mathbb{C}^* \cdot x}$. As a consequence, the closure of the orbit $\mathbb{C}^* \cdot x$ in X is obtained by adding to the orbit at most two points, the images of 0 and ∞. These points form a $\mathbb{C}^*$-stable subset. Finally, any $\mathbb{C}^*$-orbit is connected, so that each of the points must be a fixed point. □

7.2. **Bialynicki-Birula Decomposition.** Assume we are in the setting of subsection 7.1. Let $\mathbb{W}$ denote the fixed point set of the $\mathbb{C}^*$-action on X, which we will assume to be **finite**. For each $w \in \mathbb{W}$, we define the *attracting set*

$$X_w \coloneqq \{x \in X : \lim_{z\to 0} z \cdot x = w\}.$$

Since $\mathbb{C}^*$ fixes w, there is a natural $\mathbb{C}^*$-action on $\mathrm{T}_w X$, the tangent space of X at w. We first recall the classification of representations of $\mathbb{C}^* = GL_1(\mathbb{C})$ in the following proposition.

Lemma 7.2. Every algebraic group homomorphism $\mathbb{C}^* \to \mathbb{C}^*$ is given by $z \mapsto z^n$ for some integer $n \in \mathbb{Z}$. In other words, we have an identification $\mathbb{Z} \cong \mathrm{Hom}_{\mathrm{grp}}(\mathbb{C}^*, \mathbb{C}^*)$ via $n \mapsto (z \mapsto z^n)$.

Proof. Given an algebraic group morphism $\chi : \mathbb{C}^* \to \mathbb{C}^*$, we consider the corresponding homomorphism of algebra of regular functions: $\chi^* : \mathcal{O}(\mathbb{C}^*) = \mathbb{C}[w, w^{-1}] \to \mathcal{O}(\mathbb{C}^*) = \mathbb{C}[z, z^{-1}]$, which is determined by $\chi^*(w) = p(z)$ for a Laurant polynomial $p(z) \in \mathbb{C}[z, z^{-1}]$. Observe that $\chi(z) = p(z)$. The condition that χ is a group homomorphism forces that $\chi(z) = z^n$ for some $n \in \mathbb{Z}$, as desired. □

Proposition 7.3. Let (V,ρ) be a finite-dimensional rational representation of $\mathbb{C}^*$. We have a direct sum decomposition $V = \oplus_{n\in\mathbb{Z}} V_n$, where

$$V_n = \{v \in V : \rho(z)v = z^n v \text{ for all } z \in \mathbb{C}^*\}.$$

This is also called a *weight space decomposition* of V.

Proof. We will use the following assertion without proof (c.f. Theorem 8.3.5 of [3]): any rational representation of a connected complex reductive group $G \subset GL_n(\mathbb{C})$ is completely reducible. In particular, we know $\mathbb{C}^*$ satisfies the condition of the assertion. By the previous lemma, it suffices to show that any irreducible representation of $\mathbb{C}^*$ is 1-dimensional. Let (χ, W) be an irreducible representation of $\mathbb{C}^*$. For any $a \in W$, $\chi(a)$ is a scalar operator by Schur's lemma. Since $\mathbb{C}^*$ is abelian, $\{\chi(a)\}_{a\in\mathbb{C}^*}$ forms a collection of commuting diagonalizable operators. Whence they are simultaneously diagonalizable. In particular, they share an eigenvector, whose $\mathbb{C}$-span forms an invariant subspace of W. Irreducibility of W forces W to be 1-dimensional. □

Corollary 7.4. The natural $\mathbb{C}^*$-action on $\mathrm{T}_w X$ gives the weight space decomposition

$$\mathrm{T}_w X = \bigoplus_{n\in\mathbb{Z}} \mathrm{T}_w X(n), \quad \text{where } \mathrm{T}_w X(n) := \{v \in \mathrm{T}_w X : z\cdot v = z^n v \text{ for all } z \in \mathbb{C}^*\}.$$

Since w is an isolated fixed point of the $\mathbb{C}^*$-action, $n = 0$ is not an eigenvalue. The above decomposition can be rewritten as

$$\mathrm{T}_w X = \left(\bigoplus_{n>0, n\in\mathbb{Z}} \mathrm{T}_w X(n)\right) \oplus \left(\bigoplus_{n<0, n\in\mathbb{Z}} \mathrm{T}_w X(n)\right) = \mathrm{T}_w^+ X \oplus \mathrm{T}_w^- X.$$

We are now ready to state the main theorem without proof.

Theorem 7.5 (Bialynicki-Birula Decomposition)**.**
(1) The attracting sets form a decompoisition $X = \bigsqcup_{w\in\mathbb{W}} X_w$ into smooth locally closed algebraic subvarieties;
(2) There are natural $\mathbb{C}^*$-equivariant isomorphisms of algebraic varieties

$$X_w \cong \mathrm{T}_w(X_w) = \mathrm{T}_w^+ X.$$

7.3. **Vector Bundles and Principal G-Bundles.** The notion of a vector bundle in differential topology carries over to the setting of algebraic varieties in a rather straightforward manner.

Definition 7.6. A rank r vector bundle over a variety X is given by a morphism of varieties $\xi : E \to X$ such that each fiber $E_x := \xi^{-1}(x)$ has the structure of a $\mathbb{C}$-vector space of dimension r in the following sense: we can cover X by open subsets U for which there exists a morphisms $\varphi_U : E_U := \xi^{-1}(U) \to \mathbb{A}^r = \mathbb{C}^r$ which is an isomorphism of vector spaces when restricted to every fiber E_x, $x \in U$, and together with the projection $\xi_U : E_U \xrightarrow{\xi} U$ which yields an isomorphism (called a local trivialization over U) $E_U \cong U \times \mathbb{A}^r$ sending $e \mapsto (\xi(e), \varphi_U(e))$.

Recall that for a fixed ringed space $(X, \mathcal{O})$ (e.g., a variety X with the sheaf of $\mathbb{C}$-valued regular functions), an $\mathcal{O}$-module $\mathcal{F}$ on X is said to be *free* of rank n if $\mathcal{F} \cong \mathcal{O}^{\oplus n}$ and is said to be *locally free* of rank n if we can cover X by open subsets U such that $\mathcal{F}|_U \cong \mathcal{O}^{\oplus n}|_U$. Now let $\mathcal{F}$ be a locally free sheaf on $(X, \mathcal{O}_X)$. We may think of $\mathcal{F}$ as a sheaf of sections of an object that may be thought of as a vector bundle. This will be made precise in the following proposition.

Proposition 7.7. Let $(X, \mathcal{O}_X)$ be a variety with a sheaf of regular functions. Let $\xi : E \to X$ be a rank r vector bundle over X. For every open $U \subset X$, denote by $\mathcal{O}_X(\xi)(U)$ the set of morphisms $s : U \to E_U = \xi^{-1}(U)$ that are sections of ξ_U: $\xi_U \circ s = id_U$. Pointwise addition and multiplication with an element of $\mathcal{O}_X(U)$ turns $\mathcal{O}_X(\xi)(U)$ into an $\mathcal{O}(U)$-module. Then

(1) $\mathcal{O}_X(\xi)$ is a coherent $\mathcal{O}_X$-module that is locally free of rank r;
(2) Conversely, for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ that is locally free of rank r, there exists a rank r vector bundle $\xi : E \to X$ such that $\mathcal{F}$ is isomorphic to $\mathcal{O}_X(\xi)$ and that ξ is unique up to isomorphism.

Example 7.8 (Tautological line bundle over $\mathbb{P}^n$)**.** The tautological line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$ over $\mathbb{P}^n$ is defined as $\xi : E \to \mathbb{P}^n$, where

$$E := \{(\ell, v) \in \mathbb{P}^n \times \mathbb{C}^{n+1} : v \in \ell\},$$

where ℓ is a 1-dimensional linear subspace (a line) in $\mathbb{C}^{n+1}$. Its local trivialization over a standard affine chart $\mathbb{C}^n \cong U_i := \{[x_0 : \ldots : x_n] \in \mathbb{P}^n : x_i \neq 0\}$ is given by

(7.9)
$$\phi_i : E_{U_i} = \left\{\left(\ell = [x_0 : \ldots : x_n], \lambda \cdot (\frac{x_0}{x_i}, \ldots, 1, \ldots, \frac{x_n}{x_i})\right) : x_i \neq 0, \lambda \in \mathbb{C}\right\} \longrightarrow U_i \times \mathbb{C}, \quad (\ell, \lambda x_\ell) \mapsto (\ell, \lambda),$$

where $x_\ell \in \mathbb{C}^{n+1}$ is the dehomogenization of ℓ in U_i. In particular, the line bundle over $\mathbb{CP}^1$ is nothing but the Mobius strip (without boundary). From a straightforward calculation, one observes that the transition function $g_{ij} := \phi_j \circ \phi_i^{-1} \in \mathcal{O}_{\mathbb{P}^n}(-1)^\times$ sends $(\ell = [x_0 : \ldots : x_n], \lambda \in \mathbb{C})$ to $(\ell, \frac{x_j}{x_i}\lambda)$.

We now return to the category of topological spaces.

Definition 7.10 (Principal G-bundle)**.** Let $\pi : E \to B$ be a fiber bundle with fiber X, and let there be a continuous right topological group action $E \times G \to E$ that preserves the fibers of π and acts principally (i.e., transitively and freely) on fibers. For each fiber $\pi^{-1}(b)$ over $b \in B$, the restriction of the action gives a map $\pi^{-1}(b) \times G \to \pi^{-1}(b)$. For a fixed $x \in \pi^{-1}(b)$, the bijective continuous orbit map $\phi_x : G \to \pi^{-1}(b)$ is a homeomorphism. In particular, each fiber (called a G-torsor) of the bundle is homeomorphic to the group G itself, but without a group structure, as there is no preferred choice of an identity element.

In fact, we have a more general construction.

Definition 7.11. Let G be a topological group. Suppose G is acting faithfully on a space F on the left. A (G, F)-fiber bundle is a fiber bundle $\pi : E \to B$ such that there exists an open cover $\{U_\alpha\}$ of B and local trivializations $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ such that the corresponding transition functions $g^\alpha_\beta : U_\alpha \cap U_\beta \to \mathrm{Homeo}(F)$ factor (as a set map) through $G \subset \mathrm{Homeo}(F)$.

Remark 7.12. Principal G bundles can be shown to be a (G, G)-bundle with G acts on itself by left multiplication. One can modify the definition of principal G-bundle to the setting of smooth manifolds with a smooth Lie group action, or in the setting of an algebraic fiber bundle with an algebraic group action. If F is a module and G-action preserves this structure, this (G, F)-bundle becomes a vector bundle.

Definition 7.13 (Associated bundles)**.** If $\pi : E \to X$ is a principal G-bundle, and G acts on F, a topological space, on the left, then the associated bundle, denoted by $E \times_G F$, is given by the quotient space $E \times F/\sim$ with the equivalence relation $(eg, f) \sim (e, gf)$ for any $e \in E$, $g \in G$ and $f \in F$.

Remark 7.14. Again, one has similar constructions in the category of smooth manifolds (as we have the quotient manifold theorem, c.f. [6]) and algebraic varieties. However, one needs to be careful when dealing with algebraic varieties, as the orbit space of an algebraic action is in general not an algebraic variety (and that is why we have the notion of a *categorical quotient*). In some particular cases, as in the proposition below, such a construction makes sense.

Lemma 7.15. Let G be a linear algebraic group and H a closed subgroup, and let $\pi : G \to G/H$ be a canonical morphism such that it is locally trivial in the following sense: G/H is covered by open subsets on each of which π admits a section $s : U \to \pi^{-1}U$. In this case, $(x, h) \mapsto \sigma(x)h$ defines an isomorphism of varieties $U \times H \cong \pi^{-1}U$ for every such open $U \subset G/H$. Assume that V is a variety with a left algebraic H-action. Then the quotient $G \times_H V$ exists.

Proof. This is Lemma 5.5.8 of [11]. □

Corollary 7.16. Let G be a linear algebraic group and H a closed subgroup, and let $\pi : G \to G/H$ be a canonical morphism such that it is locally trivial. Let V be a finite-dimensional H-module. Then the associated bundle $G \times_H V$ is a vector bundle over G/H of rank $\dim_{\mathbb{C}}(V)$.

Corollary 7.17. If G is a connected semisimple algebraic group, and B is a Borel subgroup, then $G \to G/B$ is locally trivial. Identifying $G/B \cong \mathcal{B}$ (2.13), we observe that $G \times_B \mathfrak{b} \to \mathcal{B}$ is a vector bundle of rank $\dim \mathfrak{b}$, where B acts on its Lie algebra $\mathfrak{b}$ by adjoint action.

Proof. We show that the G-equivariant morphism $\pi : G \to G/B$ is locally trivial. It suffices to show there is an open subset U of G/B on which π admits a section, as G acts transitively on $G/B \cong \mathcal{B}$. By the Bruhat decomposition in Theorem 3.4,

$$\mathcal{B} \cong G/B = \sqcup_{w \in W_T} \mathcal{B}_w,$$

where for any $w \in W_T$, $\mathcal{B}_w$ is a smooth locally closed subvariety which is precisely a B-orbit of $\mathcal{B}$. Since $\mathcal{B}$ is irreducible, and the Weyl group W_T is finite, there exists $w_0 \in W_T$ such that $\mathcal{B}_{w_0}$ is dense (otherwise $\mathcal{B}$ will be a finite union of proper closed subsets and contradict the irreducibility). Since $\mathcal{B}_{w_0} = U \cap C$ for an open subset U and closed subset C in $\mathcal{B}$, $\mathcal{B} = \overline{\mathcal{B}_{w_0}} \subset C$, so $C = \mathcal{B}$, i.e., $\mathcal{B}_{w_0}$ is open. Consider the morphism $\sigma : \mathcal{B}_{w_0} \cong B(w_0 B) \to G$ defined by $b(w_0 B) \mapsto b w_0 \in G$. It is a routine to verify that σ is a well-defined section. □

7.4. **Miscellaneous Results.**

Proposition 7.18. Let $\mathfrak{g}$ be a semisimple Lie algebra. The nondegerate Killing form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ defined by $\kappa(x, y) = \mathbf{tr}(\operatorname{ad} x \circ \operatorname{ad} y)$ induces an identification $\phi : \mathfrak{g} \cong \mathfrak{g}^*$. If $\mathfrak{g}$ is the Lie algebra of a connected semisimple Lie group G, then the isomorphism is G-equivariant, where $\mathfrak{g}$ (resp. $\mathfrak{g}^*$) is viewed as a G-module via adjoint (resp. coadjoint) action. Moreover, let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ be a Borel subalgebra, where $\mathfrak{h}$ is a Cartan subalgebra and $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$ is the nilradical of $\mathfrak{b}$. Then φ restricts to an isomorphism between $\mathfrak{n}$ and $\mathfrak{b}^\perp \subset \mathfrak{g}^*$. Here $\mathfrak{b}^\perp := \{f \in \mathfrak{g}^* : f|_{\mathfrak{b}} \equiv 0\}$.

Proof. We only prove the last assertion. By the root space decomposition, we know $\mathfrak{n} = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$, where Φ^+ denotes the set of positive roots. We claim that $\phi(\mathfrak{n}) \subset \mathfrak{b}^\perp$. Indeed, for $\alpha, \beta \in \mathfrak{h}^*$ and $\alpha + \beta \neq 0$, $\mathfrak{g}_\alpha$ and $\mathfrak{h}_\beta$ are orthogonal relative to the Killing form (c.f. Proposition 8.1 of [4]). In particular,

$$\kappa(\mathfrak{n}, \mathfrak{b}) = \kappa\left(\bigoplus_{\alpha\in\mathfrak{h}^*} \mathfrak{g}_\alpha, \mathfrak{h} \oplus \bigoplus_{\alpha\in\mathfrak{h}^*} \mathfrak{g}_\alpha\right) = 0.$$

The claim follows. Since κ is a nondegenerate symmetric bilinear form,

$$\dim(\mathfrak{b}^\perp) = \dim\mathfrak{g} - \dim\mathfrak{b} = \dim\mathfrak{n}.$$

It follows that $\phi|_{\mathfrak{n}} : \mathfrak{n} \xrightarrow{\cong} \mathfrak{b}^\perp$. □

Theorem 7.19 (Zariski's main theorem)**.** (c.f. [10]) Let $f : X' \to X$ be a morphism of varieties over an algebraically closed field k with finite fibers. Then f factors as $X' \xrightarrow{i} Y \xrightarrow{g} X$, where Y is a variety, i is an open immersion, and g is a finite morphism.

Corollary 7.20. Let $f : X' \to X$ be a morphism of irreducible varieties over $\mathbb{C}$. If f is bijective and X is normal, then the morphism f is an isomorphism. In particular, every bijective morphism between smooth complex varieties is an isomorphism.

Proof. By Zariski's main theorem, f factors as $X' \xrightarrow{i} Y \xrightarrow{g} X$, where i is an open immersion and g is finite. The statement follows immediately if we can show g is an isomorphism. Identify X' with its image under i. We observe that $Y \smallsetminus X'$ is a proper closed subset of Y. Since g is finite, it is closed. It follows that $g(Y \smallsetminus X')$ is a proper closed subset as $\dim g(Y \smallsetminus X') \leq \dim Y \smallsetminus X' < \dim Y = \dim X$. So over the complement of this proper closed subset, which is an open dense irreducible subset, g is one-to-one. Observe that $g^* : \mathbb{C}(X) \hookrightarrow \mathbb{C}(Y)$ is a finite algebraic extension of fields with extension degree d, which is the number of elements in a generic fiber (this is where we use that the ground field is $\mathbb{C}$!). Therefore, $d = 1$, i.e., g is a birational map with finite fibers. By an alternative form of Zariski's main theorem (c.f. [10]), we conclude that g is an isomorphism. □

Proposition 7.21. Let X be a constructible algebraic variety, i.e., it can be written as a finite union of finitely many locally closed subvarieties V_i of X. If all of V_i are irreducible and have the same dimension, then their closure will be top-dimensional irreducible closed subvarieties of X. In particular, $Z_i := \overline{V_i}$ are irreducible components of X.

Proof. Assume to the contrary, there exists an irreducible closed subvariety Y of X with $\dim Y \geq \dim Z_i + 1$. Consider $Y = \cup_i (Y \cap V_i)$, we observe that there exists i so that $\overline{Y \cap V_i} = Y$ because Y is irreducible and cannot be written as a finite union of proper closed subsets. It follows that

$$\dim Y = \dim(Y \cap V_i) \leq \dim V_i = \dim Z_i \leq \dim Y - 1,$$

which is absurd. So Z_i are of top dimension. In other words, X is a pure variety with pure dimension $\dim Z_i$. □

The following proposition will be useful in the proof of Chevalley's restriction theorem.

Proposition 7.22. Let G be a connected semisimple Lie group, $\mathfrak{g}$ its (semisimple) Lie algebra. Let $\mathfrak{g}^{sr}$ be the set of semisimple regular elements in $\mathfrak{g}$. Then there exists a G-invariant polynomial $P(x) \in \mathbb{C}[\mathfrak{g}]$ such that $x \in \mathfrak{g}^{sr}$ if and only if $P(x) \neq 0$. In particular, $\mathfrak{g} \setminus \mathfrak{g}^{sr}$ is a hypersurface and $\mathfrak{g}^{sr}$ is a Zariski open affine subset of $\mathfrak{g}$, with ring of regular function $\mathbb{C}[\mathfrak{g}^{sr}] = \mathbb{C}[\mathfrak{g}][1/P]$.

Proof. This is Lemma 3.1.5 of [2]. □

Proposition 7.23 (c.f. [5])**.** Let an algebraic group G act algebraically on the nonempty variety X. Then each orbit is a smooth, locally closed subset of X, whose boundary is a union of orbits of strictly lower dimension. In particular, each orbit is open in its closure, and orbits of minimal dimension are closed (so closed orbits exist).

Corollary 7.24. Let G be a connected algebraic group acting on an algebraic variety X with finite orbits. Then any irreducible closed G-stable algebraic subvariety of X is the closure of a G-orbit.

Proof. Let Y be this irreducible G-stable subvariety, let $\mathbb{O}$ be an orbit of maximal dimension contained in Y. Since $\mathbb{O}$ cannot be contained in the closure of any other orbit $\mathbb{O}' \subset Y$ (c.f. Proposition 7.23), and there are only finitely many orbits in Y, $\mathbb{O}$ is open in the irreducible closed subvariety Y. Its closure is irreducible and contained in Y, so $\overline{\mathbb{O}} = Y$ as desired. □

Proposition 7.25. Let $\mathbb{W}$ be a finite group and V a finite-dimensional vector space with an algebraic $\mathbb{W}$-action. Given two distinct $\mathbb{W}$-orbits, one can find a $\mathbb{W}$-invariant polynomial f in V (i.e., $f(x) = f(w^{-1} \cdot x)$ for all $x \in V$) which is identically one on one orbit and identically zero on the other.

Corollary 7.26. The quotient space $V/\mathbb{W}$ has a natural identification as a topological space:

$$\operatorname{Specm}(\mathbb{C}[V]^{\mathbb{W}}) \cong V/\mathbb{W}. \tag{7.27}$$

This isomorphism gives the quotient space the structure of a variety. In fact, this is the so-called categorical quotient.

Proposition 7.28. Let G be an algebraic group with Lie algebra $\mathfrak{g}$. Let V be a finite-dimensional representation of G and $E \subset V$ be a G-stable linear subspace. Then

(1) If G is connected, then for all $v \in V$, the following conditions are equivalent:
  (a) the affine linear subspace $v + E \subset V$ is G-stable;
  (b) We have $\mathfrak{g} \cdot v \subset E$, that is the image of $\mathfrak{g}$ under the induced infinitesimal Lie algebra action map $\mathfrak{g} \to V$, $x \mapsto x \cdot v$ is contained in E.
(2) Moreover, if the linear map $\mathfrak{g} \to E$, $x \mapsto x \cdot v$ is surjective, then $G \cdot v$, the G-orbit of v is a Zariski open dense subset of $v + E$.

Proof. This is Lemma 1.4.12 of [2]. □

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