NON-JUMPS IN HYPERGRAPHS

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ABSTRACT. We say that a density $\alpha \in [0, 1)$ is a jump for r if there is some $c > 0$ such that there does not exist a family of r-uniform hypergraphs $\mathcal{F}$ with Turán density $\pi(\mathcal{F})$ in $(\alpha, \alpha + c)$. Erdös conjectured [3] that all $\alpha \in [0, 1)$ are jumps for any r. This was disproven by Frankl and Rodl when they provided examples of non-jumps [4]. In this paper, we will provide a method for finding non-jumps for $r = 3$ using patterns. As a direct consequence, we find a few more examples of non-jumps for $r = 3$.

CONTENTS

1. INTRODUCTION

An r-uniform hypergraph is a pair (V, E) where V is a set of vertices and E is a set of r-tuples of vertices called edges. So, a 2-uniform hypergraph is a graph in the usual sense, where every edge is a pair of vertices. We will use r-graph in place of r-uniform hypergraph in this paper. For an r-graph G, let $V(G)$ and $E(G)$ denote the set of vertices and edges of G, respectively, and let $v(G) = |V(G)|$ and $e(G) = |E(G)|$. We define the density of an r-graph as follows.

Definition 1.1. The density of an r-graph G is

$$d(G) = \frac{e(G)}{\binom{n}{r}}.$$

For some r-graph G and some size n, the extremal number $\mathrm{ex}(n, G)$ denotes the maximimum number of edges in an r-graph on n vertices that does not contain G as an induced subgraph. The extremal number for a family of r-graphs is defined the same, except we forbid induced subgraphs isomorphic to any member of the family. We use the Turán density to describe the limiting behavior of the extremal number for a family of r-graphs.

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Definition 1.2. The Turán density of a family of r-graphs $\mathcal{F}$ is defined as

$$\pi(\mathcal{F}) = \lim_{n\to\infty} \frac{\mathrm{ex}(n,\mathcal{F})}{\binom{n}{r}}.$$

We know that such a limit exists by an averaging argument described later. With this, we can define a jump.

Definition 1.3. We say that $\alpha \in [0,1)$ is a jump for r if there exists some $c > 0$, depending on α and r, such that there does not exist any family of r-graphs $\mathcal{F}$ with $\pi(\mathcal{F}) \in (\alpha, \alpha + c)$.

Estimating Turán densities of hypergraphs is often difficult. Although jump densities alone cannot give us the Turán density of a particular hypergraph family, they allow us to describe intervals in which Turán densities may lie. By defnition, if α is not a jump, then there must exist a decreasing sequence of Turán densities approaching α. In this paper, we will provide a way to find non-jumps using patterns, introduced later. This comes from Shaw's work [9] about Frankl and Rödl's method of finding nonjumps. As a corollary, we find a few examples of non-jumps. Specifically, we will show the following.

Theorem 1.4. *The density* $64/81$ *is not a jump for* $r = 3$.

Theorem 1.5. *Let* $n \in \mathbb{Z}^+$ *and let* $k = \sqrt{3n-2}$. *Then,* $1 - \frac{3n^2-2n+k^3}{(n+k)^3}$ *is not a jump for* $r = 3$.

Before that, we highlight some of the most important results about Turán densities and hypergraph jumps. For this, we will need the following averaging argument. Consider a graph G with n vertices, and fix some integer m with $r \le m \le n$. We will use $G[S]$ to denote the subgraph of G induced by the vertex set S. Suppose $d(G[S]) < d(G)$ for $S \subset V(G)$ of size m. For any edge in G there are $\binom{v(G)-r}{m-r}$ sets S of size m such that $G[S]$ contains that edge. So,

$$\begin{aligned}
d(G) &= \frac{e(G)}{\binom{n}{r}} \\
&= \frac{1}{\binom{v(G)-r}{m-r}\binom{n}{r}} \sum_{\substack{S\subset E(G)\\|S|=m}} e(G[S]) \\
&= \frac{1}{\binom{v(G)}{m}} \sum_{\substack{S\subset E(G)\\|S|=m}} \frac{e(G[S])}{\binom{m}{r}} \\
&< \frac{1}{\binom{v(G)}{m}} \sum_{\substack{S\subset E(G)\\|S|=m}} d(G) \\
&= d(G).
\end{aligned}$$

This is a contradiction. So, there must exist some subgraph of G on m vertices with density at least $d(G)$. We refer to this averaging argument in the next proof.

Proposition 1.6. *The following are equivalent:*

(1) α *is a jump for* r

(2) *there exists some $d > 0$, depending on α and r, such that given any $\epsilon > 0$ and any integer $m \geq r$ there is some integer $N > 0$, depending on α, r, m, and ϵ, such that any r-graph on $n \geq N$ vertices with at least $(\alpha+\epsilon)\binom{n}{r}$ edges contains some subgraph on m vertices with at least $(\alpha+d)\binom{m}{r}$ edges*

Proof. We start with (1) $\implies$ (2). Fix $d > 0$. Then, let $\epsilon < d$ and $m \geq r$. Assume for contradiction that for all $N > 0$, there exists some r-graph on $n \geq N$ vertices with at least $(\alpha+\epsilon)\binom{n}{r}$ edges that does not contain a member of $\mathcal{F}$, defined as the set of all graphs on m vertices with at least $(\alpha+d)\binom{m}{r}$ edges. Then by definition, $\pi(\mathcal{F}) \geq \alpha+\epsilon$. Furthermore, by an averaging argument, $\pi(\mathcal{F}) < \alpha + d$. Since we are free to choose any d, this contradicts the assumption that α is a jump for r.

To prove the converse, assume for contradiction that for all $c > 0$ there exists a family of r-graphs $\mathcal{F}_c$ with $\pi(\mathcal{F}_c) \in (\alpha, \alpha+c)$. Fix $d > c$. Then, there exists some m such that any graph on m vertices with at least $(\alpha+d)\binom{m}{r}$ edges contains a member of $\mathcal{F}_c$. If we choose some positive $\epsilon < \pi(\mathcal{F}_c) - \alpha$, then (2) implies that any graph on $n \geq N$ vertices with at least $(\alpha+\epsilon)\binom{n}{r}$ edges contains a member of $\mathcal{F}_c$. But this implies $\pi(\mathcal{F}_c) \leq \alpha+\epsilon < \pi(\mathcal{F}_c)$, a contradiction. □

Often, the second definition is used in place of the first. To characterize jumps for $r = 2$, we recall the Erdös-Stone Theorem.

Theorem 1.7 (Erdös-Stone)**.** *Let G be a graph with chromatic number $\chi > 2$. Then,*

$$\pi(G) = 1 - \frac{1}{\chi - 1}.$$

From this, we can show $\pi(\mathcal{F}) = \inf_{F\in\mathcal{F}} \pi(\mathcal{F})$ for any family of graphs $\mathcal{F}$. Let $F \in \mathcal{F}$ be a graph with minimal chromatic number χ. Then, the Turán graph on n vertices with $\chi - 1$ parts, denoted $T(n, \chi-1)$, does not contain any member of $\mathcal{F}$. Furthermore, $\lim_{n\to\infty} d(T(n,\chi-1)) = 1 - \frac{1}{\chi-1}$. So, $\pi(\mathcal{F}) \geq 1 - \frac{1}{\chi-1}$. By Erdös-Stone, $\pi(F) = 1 - \frac{1}{\chi-1}$. Therefore, $\pi(\mathcal{F}) = 1 - \frac{1}{\chi-1}$. We need one more result from Erdös [2] to characterize jumps for $r = 2$.

Theorem 1.8 (Erdös)**.** *Let $K^{(r)}(\ell,\dots,\ell)$ denote an r-partite r-graph with ℓ vertices in each part. Then, for any $r \geq 2$ and $\ell \geq 1$,*

$$\pi(K^{(r)}(\ell,\dots,\ell)) = 0.$$

In particular, since $\lim_{\ell\to\infty} d(K^{(r)}(\ell,\dots,\ell)) = \frac{r!}{r^r}$, this proves that all $\alpha \in \left[0, \frac{r!}{r^r}\right)$ are jumps for r. Furthermore, this shows that $\pi(K_{m,n}) = 0$ for all bipartite graphs $K_{m,n}$. Combining this with Erdös-Stone, we see that $\pi(G) \in \{1 - \frac{1}{k} \mid k \in \mathbb{Z}^+\}$ for all graphs G. Since $\pi(\mathcal{F}) = \inf_{F\in\mathcal{F}} \pi(\mathcal{F})$, this means $\pi(\mathcal{F}) \in \{1 - \frac{1}{k} \mid k \in \mathbb{Z}^+\}$ for any family of graphs $\mathcal{F}$. So by definition, all $\alpha \in [0, 1)$ are jumps for $r = 2$. In the general case of r-graphs, few densities are known to be jumps or non-jumps. We often use a strategy involving blowups of r-graphs to find jumps. This is introduced in the next section.

2. Blowups and Lagrangians

We will refer to the notation used in Keevash's survey on the topic [6]. First, we define a blowup.

Definition 2.1. Let G be an r-graph on n vertices $v_1, \ldots, v_n$ and $t = (t_1, \ldots, t_n)$. A t-blowup of G is an r-graph $G(t)$ such that each vertex v_i of G is replaced by t_i copies and an r-tuple of vertices in $G(t)$ is an edge if and only if the corresponding r-tuple of vertices in G is an edge.

We will use the notation (v, i) for the ith copy of vertex v in $G(t)$. So, each vertex v of G corresponds to an independent set of t_i vertices $(v, 1), \ldots, (v, t_i)$ in $G(t)$, and each edge of G corresponds to an r-partite r-graph in $G(t)$. For example, a $(2, 2)$-blowup of a single edge on two vertices v and w is a square with vertices $(v, 1), (v, 2), (w, 1), (w, 2)$.

Now, let $p_G(t)$ denote the number of edges in $G(t)$. Then, we have $p_G(t) = \sum_{e \in E(G)} \prod_{i \in e} t_i$. Suppose we want to find the largest possible density for $G(t)$ as $|t| \to \infty$, where $|t| = \sum_{i \in n} t_i$. For any t, we have

$$\lim_{m \to \infty} d(G(mt)) = \lim_{m \to \infty} \frac{1}{\binom{m|t|}{r}} p_G(mt) = r! p_G(t_1/|t|, \ldots, t_n/|t|).$$

So, it suffices to maximize $r! p_G(x)$ over all x in the standard simplex $S = \{x \mid x_1 + \cdots + x_n = 1, x_i \geq 0 \quad \forall i \in [n]\}$. Since we perform this sort of optimization frequently, we give a name to the essential component to be maximized.

Definition 2.2. The Lagrangian of an r-graph G is defined as

$$\lambda(G) = \max_{x \in S} p_G(x).$$

We also give a name to the maximum density.

Definition 2.3. The blowup density of an r-graph G is defined as

$$b(G) = r! \lambda(G).$$

We can think of the x_is of x as optimal weights in $[0, 1]$ that give the whole graph G a weight $w(G) = \lambda(G)$.

3. Patterns

To guide our search for non-jumps, we will need patterns as defined by Hou, Li, Yang, and Zhang in [5]. A pattern follows all of the same rules as a hypergraph except edges are multisets of vertices, where a multiset is defined as a set with repetitions allowed.

Definition 3.1. An r-pattern P is a pair $(n, \mathcal{E})$, where $n \geq r$ is an integer representing the number of vertices of P and $\mathcal{E}$ is a collection of r-multisets on $[n]$ representing the edges of P.

For example, a 3-pattern on 3 vertices may have the edge set $\mathcal{E} = \{112, 123, 223\}$. We define a blowup of a pattern in the same way we defined it for hypergraphs. Let P be an r-pattern. If $t = (t_1, \ldots, t_n)$, then $P(t)$ is an r-pattern where each vertex i is replaced by t_i copies and an r-multiset in $P(t)$ is an edge if and only if the corresponding r-multiset in P is an edge. We define a simple blowup $P[t]$ of P to be the blowup $P(t)$ with all edges containing repeat vertices removed. For example, if $P = (3, \mathcal{E})$ and $t = (2, 2, 1)$ then $P[t]$ has edge set

$$\mathcal{E}' = \{(1,1)(2,1)3, (1,2)(2,1)3, (1,1)(2,2)3, (1,2)(2,2)3, (1,1)(1,2)3, (2,1)(2,2)3\}.$$

We can define the Lagrangian for an r-pattern in a similar way to how we defined it for hypergraphs. Let $m_e(i)$ denote the multiplicity of i in an edge e. Then for any pattern P and n-tuple t, we have

$$|\mathcal{E}(P[t])| = \sum_{e\in\mathcal{E}(P)} \prod_{i\in e} \binom{t_i}{m_e(i)}.$$

As $t_i \to \infty$ for all $i \in [n]$, we get

$$\sum_{e\in\mathcal{E}(P)} \prod_{i\in e} \binom{t_i}{m_e(i)} \to \sum_{e\in\mathcal{E}(P)} \prod_{i\in e} \frac{t_i^{m_e(i)}}{m_e(i)!}.$$

Let $p_P(t)$ denote the expression on the right. Then, for any t we have

$$\lim_{m\to\infty} d(P[mt]) = \lim_{m\to\infty} \frac{1}{\binom{m|t|}{r}} p_P(mt) = r! p_P(t_1/|t|, \ldots, t_n/|t|).$$

Just as before, if we wish to maximize the asymptotic density of $P[t]$, then it suffices to maximize $r!p_P(x)$ over the standard simplex S. So, we define the Lagrangian of P as $\lambda(P) = \max_{x\in S} p_P(x)$ and the blowup density as $b(P) = r!\lambda(P)$.

4. Jumps

To find jumps and non-jumps for $r \geq 3$, we use the following result proved by Frankl and Rödl. We provide a statement here, but omit the proof which can be found in [4].

Theorem 4.1. *The following are equivalent:*

(1) α is a jump for r

(2) there exists a finite family of r-graphs $\mathcal{F}$ such that $\pi(\mathcal{F}) \leq \alpha$ and $b(F) > \alpha$ for all $F \in \mathcal{F}$

In particular, we get Erdös' result [2] that all $\alpha \in \left[0, \frac{r!}{r^r}\right)$ are jumps for r by considering a graph with a single edge. A natural question to ask is whether $\frac{r!}{r^r}$ is a jump for r. Erdös conjectured that it is [3]. This problem remains open, but there has been some progress in its direction. For $r = 3$, we want to know whether $2/9$ is a jump. Erdös took a step in this direction by constructing a finite family of 3-graphs all with blowup density greater than $2/9$. Let

$$G_1 = \{123, 124, 134\}, G_2 = \{123, 124, 125, 345\}, G_3 = \{123, 124, 235, 145, 345\}.$$

Let $\mathcal{F} = \{G_1, G_2, G_3\}$. Erdös suggested that $\pi(\mathcal{F}) \leq 2/9$. As Baber and Talbot showed [1], we can construct an $\mathcal{F}$-free 3-graph on 7 vertices with blowup density greater than 0.2319 (G_4 defined below is one such 3-graph). This shows $\pi(\mathcal{F}) > 0.2319$. By adding more 3-graphs with blowup density greater than $2/9$ to the family, it is possible to decrease its Turán density. Let

$$G_4 = \{123, 135, 145, 245, 126, 246, 346, 356, 237, 147, 347, 257, 167\}$$
$$G_5 = \{123, 124, 135, 145, 236, 346, 256, 456, 247, 347, 257, 357, 157\}.$$

Let $\mathcal{F}' = \mathcal{F} \cup \{G_4, G_5\}$. Baber and Talbot showed that all of the 3-graphs in $\mathcal{F}'$ have blowup density greater than 0.2316 [1]. Using Razborov's flag algebra method, they showed $\pi(\mathcal{F}') \leq 0.2299$. Therefore, by Theorem 4.1, all $\alpha \in [0.2299, 0.2316)$ are jumps for $r = 3$. Baber and Talbot also showed that all $\alpha \in [0.2871, 8/27)$ are jumps for $r = 3$. Besides $\alpha \in \left[0, \frac{r!}{r^r}\right)$ for r, these are the only known jumps.

5. Non-jumps

Frankl and Rödl were the first to find examples of non-jumps [4]. They showed that for $r \geq 3$ and $\ell > 2r$ the density $1 - \frac{1}{\ell^{r-1}}$ is not a jump for r. Since then, a number of non-jumps and sequences of non-jumps have been found [8] [9] [10] [7]. Peng showed that if $\alpha\frac{r!}{r^r}$ is a non-jump for $r \geq 3$, then $\alpha\frac{p!}{p^p}$ is a non-jump for any $p \geq r$ [8]. Recently, Shaw showed that the smallest non-jumps we can find using the Frankl-Rödl method are $\frac{6}{121}\left(5\sqrt{5}-2\right)$ for $r = 3$ and $2 \cdot \frac{r!}{r^r}$ for $r \geq 4$ [9]. We will describe one way to find non-jumps using this method, and provide a few examples of the non-jumps we can find. First, we introduce the Frank-Rödl construction $\mathrm{FR}_v(P)$ from a pattern P, as defined by Shaw.

Definition 5.1. Suppose we are given an r-pattern $P = (n, \mathcal{E})$ and $v \in [n]$. Let t be defined such that $t_v = r$ and $t_i = 1$ for all other vertices $i \in [n]$. Let P' be an r-pattern on the vertices of $P(t)$ with

$$\mathcal{E}(P') = \{e \in \mathcal{E}(P(t)) \mid m_e((v,i)) \leq 1 \quad \forall i \in \{2, \ldots, r\}\},$$

Then, $\mathrm{FR}_v(P)$ is defined as

$$\mathrm{FR}_v(P) = P' \cup \{(v,1) \cdots (v,n)\}.$$

Now, we can describe a sufficient condition for α to be a non-jump for r. This comes from Shaw's work in [9].

Theorem 5.2. *Let P be an r-pattern, and v be a vertex of P such that a maximal weighting of P assigns v positive weight. Suppose*

$$\lambda(\mathrm{FR}_v(P)) = \lambda(P) < 1.$$

Then, $r!\lambda(P)$ is not a jump for r-graphs.

As the name suggests, this method was first used by Frankl and Rodl to show the existence of non-jumps. Using this technique, we will prove the densities given in Theorem 1.4 and Theorem 1.5 are non-jumps. First, we need a lemma. We say that two vertices in a pattern are equivalent if their labelings can be swapped to produce a pattern isomorphic to the original.

Lemma 5.3. *Let $P = (n, \mathcal{E})$ be a 3-pattern, and let i and j be vertices in P. If i and j are equivalent, then either $w_i = w_j$ or one of w_i and w_j is 0 for an optimal weighting of P, where w_v represents the weighting of vertex v.*

Proof. Suppose we know the optimal weights of all of the vertices in $[n] \setminus \{i, j\}$. Then, let $w(P)$ be the optimal weight of P and C_1, C_2, C_3 be values depending only on the weights of vertices in $[n] \setminus \{i, j\}$ such that

$$w(P) = (w_i + w_j)C_1 + (w_i^2 + w_j^2)C_2 + (w_i w_j)C_3.$$

By the Lagrange multiplier method, either at least one of w_i, w_j is 0 or $\frac{\partial}{\partial w_i}w(P) = \frac{\partial}{\partial w_j}w(P)$. In the second case,

$$\begin{aligned}2w_iC_2 + w_jC_3 &= 2w_jC_2 + w_iC_3 \\ (2C_2 - C_3)(w_i - w_j) &= 0.\end{aligned}$$

If $C_3 = 2C_2$, then

$$w(P) = (w_i + w_j)C_1 + (w_i + w_j)^2 C_2.$$

So, the relative weights of w_i and w_j does not affect the weight of the pattern. Therefore, $w_i = w_j$ gives us an optimal weighting. $\square$

We also need to know the blowup density of a specific small 3-pattern.

Lemma 5.4. *Let* $P = \{112, 122\}$. *Then,*

$$\lambda(\mathrm{FR}_1(P)) = \lambda(P) = \frac{1}{8}.$$

Proof. Let w be an optimal weighting of P and w_1 and w_2 be the weights of the vertices. Since both 1 and 2 are equivalent, by Lemma 5.3 either $w_1 = w_2$ or one of w_1 or w_2 is 0. Since we cannot have the second case, $w_1 = w_2 = \frac{1}{2}$. So,

$$\lambda(P) = \frac{w_1^2 w_2}{2} + \frac{w_1 w_2^2}{2} = \frac{1}{8}.$$

Now, let w' be an optimal weighting of $\mathrm{FR}_1(P)$ and $w_{(1,1)}$, $w_{(1,2)}$, $w_{(1,3)}$, and w_2 be the weights of the vertices. By Lemma 5.3, either $w_{(1,2)} = w_{(1,3)}$ or one of $w_{(1,2)}$ or $w_{(1,3)}$ is 0. The second case does not give us an optimal weighting, so we only consider the first. Let $a = w_{(1,1)}$, $b = 2w_{(1,2)} = w_{(1,2)} + w_{(1,3)}$, and $c = w_2$. Then,

$$w'(\mathrm{FR}_1(P)) = \frac{ab^2}{4} + \frac{a^2c}{2} + \frac{b^2c}{4} + abc + \frac{(a+b)c^2}{2}.$$

We are subject to the constraint $a + b + c = 1$. Assume a, b, and c are all positive. Then, by the Lagrange multiplier method we have

$$\frac{\partial}{\partial a} w'(\mathrm{FR}_1(P)) = \frac{\partial}{\partial b} w'(\mathrm{FR}_1(P)) = \frac{\partial}{\partial c} w'(\mathrm{FR}_1(P)).$$

We have

$$\begin{aligned}
\frac{\partial}{\partial a} w'(\mathrm{FR}_1(P)) &= \frac{b^2}{4} + ac + bc + \frac{c^2}{2} \\
\frac{\partial}{\partial b} w'(\mathrm{FR}_1(P)) &= \frac{ab}{2} + \frac{bc}{2} + ac + \frac{c^2}{2} \\
\frac{\partial}{\partial c} w'(\mathrm{FR}_1(P)) &= \frac{a^2}{2} + \frac{b^2}{4} + ab + (a+b)c.
\end{aligned}$$

The first and last equations give us

$$\begin{aligned}
\frac{a^2}{2} - \frac{c^2}{2} + ab = 0 \\
b &= \frac{c^2 - a^2}{2a}.
\end{aligned}$$

The first two equations give us

$$\begin{aligned}
\frac{ab}{2} - \frac{bc}{2} - \frac{b^2}{4} = 0 \\
b &= 2(a - c).
\end{aligned}$$

If $c < a$, we get $b < 0$ from the first and last equations, and if $c > a$ we get $b < 0$ from the first set of equations. So, we must have $a = c$. However, this implies that $b = 0$, a contradiction. So, one of a, b, or c must be 0. We cannot have $a = 0$ for an optimal weighting. If $c = 0$, then the optimal weight is $\frac{a^2b}{4} \leq \frac{1}{27}$. If $b = 0$, then we are left with the expression for $w(P)$. So, $\lambda(\mathrm{FR}_1(P)) = \lambda(P) = \frac{1}{8}$. This implies that $\frac{3}{4}$ is a non-jump for $r = 3$. $\square$

Now, we can prove what we want.

Theorem 5.5. *Let $P = (n, \mathcal{E})$ be a 3-pattern such that a maximal weighting of P assigns vertex 1 positive weight. Suppose $\{122\} \cup \{11i \mid i \in [n] \setminus \{1\}\}$ is a subset of $\mathcal{E}$. Then, P is a 3-pattern satisfying the conditions of Theorem 5.2, assuming $\lambda(P) < 1$.*

Proof. Consider $\mathrm{FR}_1(P)$. Suppose w is an optimal weighting of $\mathrm{FR}_1(P)$ and w_v is the weight of vertex v. Let $a = w_{(1,1)}$, $b = w_{(1,2)} + w_{(1,3)} = 2w_{(1,2)}$, and $k = \sum_{i=2}^{n} w_i$. Then, we have

$$w(\mathrm{FR}_1(P)) = \frac{ab^2}{4} + \frac{a^2k}{2} + \frac{b^2k}{4} + abk + \frac{1}{2}(a+b)\left(\sum_{(1,i,i)\in\mathcal{E}} w_i^2\right) + (a+b)\left(\sum_{\substack{(1,i,j)\in\mathcal{E}\\ i\neq j}} w_iw_j\right) + \sum_{\substack{(i,j,k)\in\mathcal{E}\\ i\neq j, j\neq k, i\neq k}} w_iw_jw_k + \sum_{(i,i,j)\in\mathcal{E}} \frac{w_i^2w_j}{2}.$$

Fix $w_2, \ldots, w_n$ to be any value. If $a \neq 0$ and $b \neq 0$, then by the Lagrange multiplier method

$$\frac{\partial}{\partial a} w(\mathrm{FR}_1(P)) = \frac{\partial}{\partial b} w(\mathrm{FR}_1(P)).$$

We have

$$\frac{\partial}{\partial a} w(\mathrm{FR}_1(P)) = \frac{b^2}{4} + ak + bk + \frac{1}{2}\sum_{(1,i,i)\in\mathcal{E}} w_i^2 + \sum_{\substack{(1,i,j)\in\mathcal{E}\\ i\neq j}} w_iw_j$$

$$\frac{\partial}{\partial b} w(\mathrm{FR}_1(P)) = \frac{ab}{2} + \frac{bk}{2} + ak + \frac{1}{2}\sum_{(1,i,i)\in\mathcal{E}} w_i^2 + \sum_{\substack{(1,i,j)\in\mathcal{E}\\ i\neq j}} w_iw_j.$$

So,

$$\begin{aligned}\frac{\partial}{\partial a} w(\mathrm{FR}_1(P)) - \frac{\partial}{\partial b} w(\mathrm{FR}_1(P)) &= 0\\ \frac{b^2}{4} + \frac{bk}{2} - \frac{ab}{2} &= 0\\ b + 2(k-a) &= 0.\end{aligned}$$

This can only be true if $a > k$. Now, we assume this is the case and show $b = 0$.

Let w' be an optimal weighting for $\mathrm{FR}_1(P)$ with this assumption and w'_v be the weight of vertex v. Also, let $a' = w'_{(1,1)}$, $b' = w'_{(1,2)} + w'_{(1,3)} = 2w'_{(1,2)}$, and $k' = \sum_{i=2}^{n} w'_i$. Fix some distribution of the weights $w'_2, \ldots, w'_n$ so that $w'_i = c'_i k'$ for constants $c'_2, \ldots, c'_n$ in $[0,1]$. Then, when expressing $w'(\mathrm{FR}_1(P))$ we can collect all of the constants $c'_2, \ldots, c'_n$ into two constants c_1 and c_2 to get

$$w'(\mathrm{FR}_1(P)) = \frac{a'b'^2}{4} + \frac{a'^2k'}{2} + \frac{b'^2k'}{4} + a'b'k' + c_1(a'+b')k'^2 + c_2k'^3.$$

By Lemma 5.3, we know that $w'(\mathrm{FR}_1(P)) \geq \frac{1}{8}$. Therefore, in an optimal weighting for $\mathrm{FR}_1(P)$ we have

$$c_2 \geq \frac{1}{k'^3}\left(\frac{1}{8} - \frac{a'b'^2}{4} - \frac{a'^2k'}{2} - \frac{b'^2k'}{4} - a'b'k' - c_1(a'+b')k'^2\right).$$

Assume a', b', and k' are nonzero. Then, the Lagrange multiplier method gives us

$$\frac{\partial}{\partial k'}w'(\mathrm{FR}_1(P)) - \frac{\partial}{\partial a'}w'(\mathrm{FR}_1(P)) = 0.$$

We want to show $k' > \frac{1}{2}$ in this equation, since this contradicts the assumption $a' > k'$. If c_1 or c_2 increases, then the k' that maximizes $w'(\mathrm{FR}_1(P))$ must not decrease. So it suffices to show $k' > \frac{1}{2}$ for the minimum possible c_2. If we plug this value for c_2 into the equation, we get

$$\frac{a'^2}{2} + a'b' + (2c_1-1)(a'+b')k' + \\ \frac{3}{k'}\left(\frac{1}{8} - \frac{a'b'^2}{4} - \frac{a'^2k'}{2} - \frac{b'^2k'}{4} - a'b'k' - c_1(a'+b')k'^2\right) - c_1k'^2 = 0.$$

Now, if we implicitly differentiate with respect to c_1 holding a' and b' constant we get

$$2(a'+b')k' + (2c_1-1)(a'+b')\frac{dk'}{dc_1} - \frac{3}{k'}\left(\frac{a'^2}{2} + \frac{b'^2}{4} + a'b' + 2c_1(a'+b')k'\right)\frac{dk'}{dc_1} - \\ 3(a'+b')k' + \frac{3}{k'^2}\left(\frac{a'b'^2}{4} - \frac{1}{8} + \frac{a'^2k'}{2} + \frac{b'^2k'}{4} + a'b'k' + c_1(a'+b')k'^2\right)\frac{dk'}{dc_1} - \\ k'^2 - 2c_1k'\frac{dk'}{dc_1} = 0.$$

Let

$$g(a',b',k',c_1) = (2c_1-1)(a'+b') - 2c_1k' + \frac{3}{k'^2}\left(\frac{a'b'^2}{4} - \frac{1}{8}\right) - 3c_1(a'+b').$$

Then,

$$\frac{dk'}{dc_1} = \frac{(a'+b')k' + k'^2}{g(a',b',k',c_1)}.$$

We have $g(a',b',k',c_1) < 0$. Therefore, given that c_2 is a function of c_1 and a' and b' are constant, the k' that maximizes $w'(\mathrm{FR}_1(P))$ increases as c_1 decreases. If we chose any c_2 other than the minimum, then k' would only increase more. However, this implies $k' > \frac{1}{2}$ for $c_1 < \frac{1}{2}$, since $k = \frac{1}{2}$ maximizes $w'(\mathrm{FR}_1(P))$ for $c_1 = \frac{1}{2}$ and $c_2 = 0$ by Lemma 5.4. If $c_1 > \frac{1}{2}$, then we also get $k' > \frac{1}{2}$, since k increases when c_1 increases and c_2 is non-decreasing. This contradicts the assumption that $a' > k'$.

Since $a \le k'$, one of a', b', k' must be zero for an optimal weighting of $\mathrm{FR}_1(P)$. We cannot have $a' = 0$. If $k' = 0$, then $w'(\mathrm{FR}_1(P)) = \frac{a'b'^2}{4} \le \frac{1}{27}$. Therefore, $b' = 0$ for an optimal weighting. □

Now, we prove Theorem 1.4.

Proof of Theorem 1.4. Let $P = \{123, 122, 112, 113, 223\}$. By Theorem 5.2 and Theorem 5.5, it suffices to show that vertex 1 receives a positive weight in an optimal weighting for P and

$$\lambda(P) = \frac{32}{243}.$$

This will show us that $3! \cdot \frac{32}{243} = \frac{64}{81}$ is not a jump for $r = 3$. Suppose w is a maximal weighting for P and w_1, w_2, w_3 are the weights of the vertices. Then,

$$w(P) = w_1w_2w_3 + \frac{w_1w_2^2}{2} + \frac{w_1^2w_2}{2} + \frac{w_1^2w_3}{2} + \frac{w_2^2w_3}{2}.$$

Suppose w_1, w_2, and w_3 are positive. Then, by Lemma 5.3 we have $w_1 = w_2$. Therefore,

$$w(P) = 2w_1^2 w_3 + w_1^3.$$

We are subject to the constraint

$$2w_1 + w_3 = 1.$$

So, using the method of Lagrange multipliers, we must have

$$\begin{aligned}\frac{\partial}{\partial w_1} w(P) &= 2\frac{\partial}{\partial w_3} w(P)\\ 4w_1w_3 + 3w_1^2 &= 4w_1^2\\ w_1 - 4w_3 &= 0.\end{aligned}$$

Then, using the constraint, we get $w_1 = w_2 = 4/9$ and $w_3 = 1/9$, which leaves us $w(P) = 32/243$. If $w_1 = 0$, then $w_2 = \frac{2}{3}$ and $w_3 = \frac{1}{3}$ maximizes $w(P)$. This leaves $w(P) = 1/27 < 32/243$. We get the same result when $w_2 = 0$. If $w_3 = 0$, then by Lemma 5.4 we have $w(P) \leq 1/8 < 32/243$. So, $\lambda(P) = 32/243$. $\square$

We also prove Theorem 1.5.

Proof of Theorem 1.5. Let n be fixed. Then, let P be the pattern on $n+1$ vertices with edges $[n+1]^{(3)} \cup \{1,2,2\} \cup \{1,3,3\} \cup \cdots \cup \{1, n+1, n+1\}$. By Theorem 5.2 and Theorem 5.5, it suffices to show that vertex 1 receives a positive weight in an optimal weighting for P and

$$\lambda(P) = \frac{1}{6} - \frac{3n^2 - 2n + k^3}{6(n+k)^3}.$$

Let w_1 be the weight of the first vertex. By Lemma 5.3, either all of the weights of the other vertices are equal, or some of them are 0 and the rest are equal. Let m be the number of these vertices that are assigned positive weight and w_2 equal m times the weight of one of these vertices. Then,

$$\begin{aligned} w(P) &= \binom{m}{2}\frac{w_1 w_2^2}{m^2} + \binom{m}{3}\frac{w_2^3}{m^3} + \frac{w_1^2 w_2}{2} + \frac{w_1 w_2^2}{2m}\\ &= \frac{w_1 w_2^2}{2} + \binom{m}{3}\frac{w_2^3}{m^3} + \frac{w_1^2 w_2}{2}.\end{aligned}$$

This is increasing in m, so we must have $m = n$. We are subject to the constraint

$$w_1 + w_2 = 1.$$

First, suppose both w_1 and w_2 are positive. Then, by the Lagrange multipliers method we have

$$\begin{aligned}\frac{\partial}{\partial w_1} w(P) &= \frac{\partial}{\partial w_2} w(P)\\ \frac{w_2^2}{2} + w_1 w_2 &= w_1 w_2 + \binom{n}{3}\frac{3w_2^2}{n^3} + \frac{w_1^2}{2}\\ \left(1 - \frac{(n-1)(n-2)}{n^2}\right) w_2^2 &= w_1^2\\ w_1 &= \frac{k}{n} w_2.\end{aligned}$$

We use the constraint equation to get

$$w_2 = \frac{1}{1+\frac{k}{n}} = \frac{n}{n+k}$$
$$w_1 = \frac{k}{n}w_2 = \frac{k}{n+k}.$$

Finally, plugging these values for w_1 and w_2 into the expression for $w(P)$ we get

$$w(P) = \frac{1}{6} - \frac{3n^2-2n+k^3}{6(n+k)^3}.$$

Now, suppose $w_1 = 0$, $w_2 = 1$. Then,

$$w(P) = \binom{n}{3}\frac{1}{n^3} = \frac{(n-1)(n-2)}{6n^2} = \frac{1}{6} - \frac{3n-2}{6n^2}.$$

However,

$$\frac{1}{6} - \frac{3n-2}{6n^2} < \frac{1}{6} - \frac{3(n+k)-2}{6(n+k)^2} < \frac{1}{6} - \frac{3n^2-2n+k^3}{6(n+k)^3}.$$

Clearly, $w_2 = 0$ does not maximize $w(P)$. Therefore, $\lambda(P) = \frac{1}{6} - \frac{3n^2-2n+k^3}{6(n+k)^3}$. □

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