

A Combinatorial Proof of Brouwer's Fixed Point Theorem via Sperner's Lemma

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Abstract

This paper presents a combinatorial proof of Brouwer's Fixed Point Theorem using Sperner's Lemma. We begin by establishing the necessary geometric foundations involving simplices and simplicial complexes, then prove Sperner's Lemma using a parity argument based on counting labeled faces. Finally, we demonstrate how this combinatorial result leads directly to a constructive proof of Brouwer's theorem. This approach provides an elegant alternative to traditional topological proofs and offers computational insights into the existence of fixed points.

1 Introduction

Brouwer's Fixed Point Theorem stands as one of the most celebrated results in topology, asserting that any continuous map from a convex compact set in $\mathbb{R}^n$ to itself must have at least one fixed point. This paper demonstrates how a purely combinatorial approach via Sperner's Lemma can yield the proof of this theorem. The connection between these seemingly disparate areas—combinatorial geometry and fixed point theory—illustrates the deep unity underlying mathematics. Our proof not only establishes the existence of fixed points but also provides a constructive method that can be implemented algorithmically to approximate such points.

2 Preliminaries: The Geometry of Simplices

2.1 Simplices and Simplicial Complexes

Definition 2.1 (Simplex). Let $\{u_0, u_1, \dots, u_k\}$ be a set of $k + 1$ affinely independent points in $\mathbb{R}^d$. The k -**simplex** σ is the convex hull of vertices given by:

$$\sigma = \text{conv}\{u_0, \dots, u_k\} = \left\{ \sum_{i=0}^k \lambda_i u_i \mid \sum_{i=0}^k \lambda_i = 1 \text{ and } \lambda_i \geq 0 \text{ for all } i \right\}$$

The points u_i are the **vertices** of the simplex. We refer to a 0-simplex as a vertex, a 1-simplex as an edge, and a 2-simplex as a triangle.

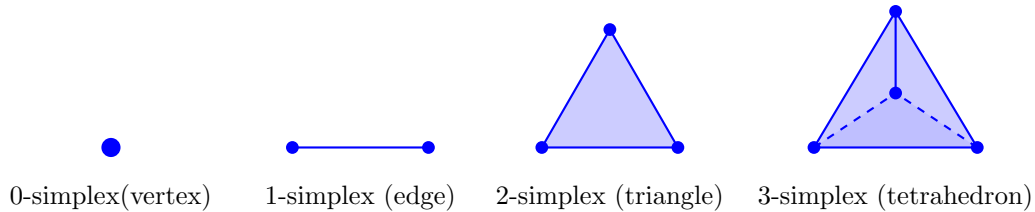


Figure 1: Examples of simplices in different dimensions. The shaded regions show the interior of each simplex.

Definition 2.2 (Face). A **face** of a simplex $\sigma = \text{conv}\{u_0, \dots, u_k\}$ is the convex hull of any non-empty subset of its vertices. A face is **proper** if it is not σ itself. The **boundary** of σ , denoted $\partial\sigma$, is the union of all its proper faces.

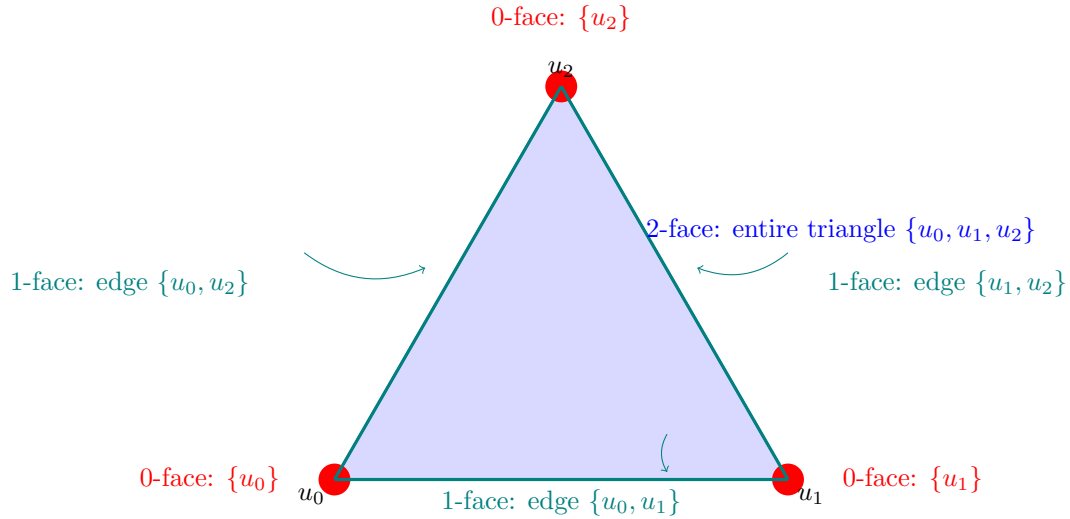
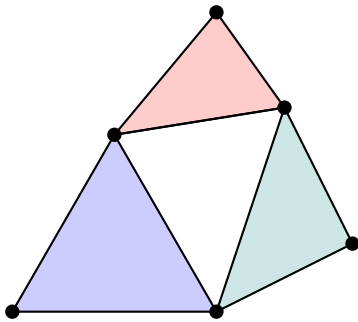


Figure 2: A 2-simplex (triangle) showing all its faces. The 0-faces are vertices (red), 1-faces are edges (teal), and the 2-face is the entire triangle (blue shaded region).

Definition 2.3 (Simplicial Complex). A **simplicial complex** K is a finite collection of simplices in $\mathbb{R}^d$ such that:

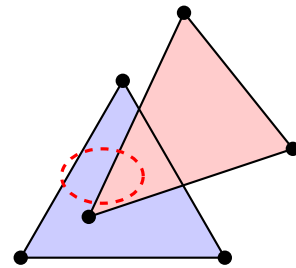
1. If a simplex σ is in K , then all of its faces are also in K .
2. The intersection of any two simplices $\sigma_1, \sigma_2 \in K$ is either empty or a face of both σ_1 and σ_2 .

A **triangulation** of a topological space X is a simplicial complex whose underlying space, $|K| = \bigcup_{\sigma \in K} \sigma$, is homeomorphic to X .



Valid Simplicial Complex

Faces intersect properly



NOT a Simplicial Complex

Triangles intersect improperly

Figure 3: Left: A valid simplicial complex where triangles share common edges or vertices. Right: An invalid configuration where triangles intersect in their interiors.

2.2 Subdivision and Refinement

Definition 2.4 (Subdivision). A simplicial complex L is a **subdivision** of a simplicial complex K if $|L| = |K|$ and every simplex of L is contained in some simplex of K .

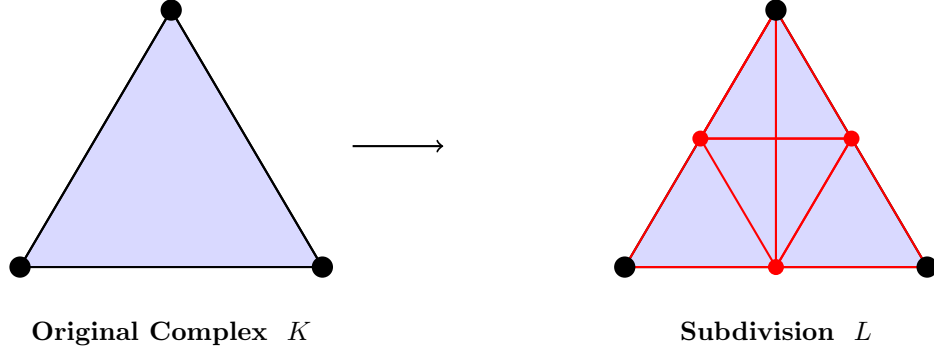


Figure 4: A subdivision where the original triangle is divided into smaller triangles. Every small triangle (in red) lies within the original triangle.

Definition 2.5 (Barycentric Subdivision). The **barycenter** of a simplex $\sigma = \text{conv}\{u_0, \dots, u_k\}$ is the point $\hat{\sigma} = \frac{1}{k+1} \sum_{i=0}^k u_i$. The **barycentric subdivision** of a simplicial complex K , denoted $Sd(K)$, has vertices at the barycenters of all simplices in K . A collection of vertices $\{\hat{\sigma}_0, \dots, \hat{\sigma}_j\}$ forms a simplex in $Sd(K)$ if and only if the corresponding simplices form a chain $\sigma_0 \subset \sigma_1 \subset \dots \subset \sigma_j$.

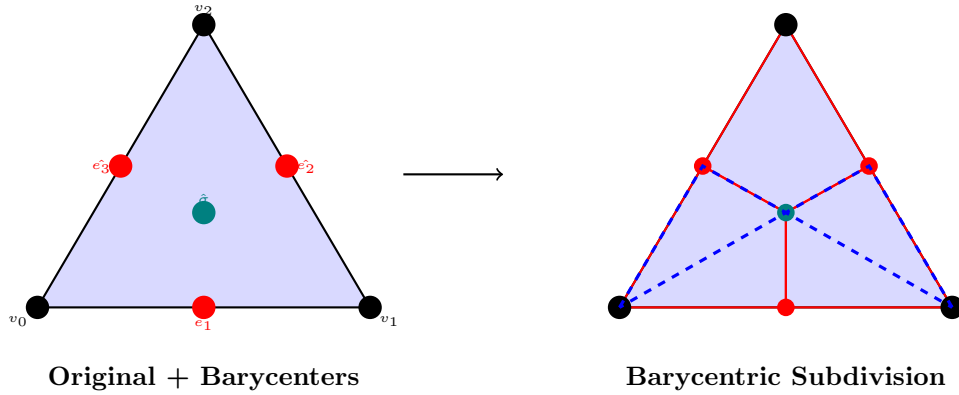


Figure 5: Barycentric subdivision of a triangle. Left: Original triangle with barycenters marked. Right: The resulting subdivision where each new simplex corresponds to a chain of faces from the original complex.

Lemma 2.6 (Mesh Lemma). For a d -dimensional simplicial complex K , the mesh of its barycentric subdivision, $Sd(K)$, satisfies:

$$\text{mesh}(Sd(K)) \leq \frac{d}{d+1} \text{mesh}(K)$$

where the mesh is the diameter of the largest simplex, equivalent to the length of the longest edge.

Proof. To establish the bound, we need to find the maximum possible length of an edge in the barycentric subdivision $Sd(K)$.

Step 1: Characterize an edge in $Sd(K)$. An edge in $Sd(K)$ connects the barycenters of two faces of an original simplex in K , where one face is a proper face of the other. Let σ be a simplex in K , and let τ_1 and τ_2 be two faces of σ such that τ_1 is a proper face of τ_2 . The new edge, e , connects their barycenters,

$b(\tau_1)$ and $b(\tau_2)$. The length of any edge in $\text{Sd}(K)$ can be expressed as $\|b(\tau_2) - b(\tau_1)\|$. The longest possible edge will connect the barycenter of a k -dimensional simplex to the barycenter of one of its m -dimensional faces, where $m < k$.

Step 2: Express the edge vector algebraically. Let τ_2 be a k -simplex with vertices $\{v_0, v_1, \dots, v_k\}$. Without loss of generality, let τ_1 be an m -face of τ_2 with vertices $\{v_0, v_1, \dots, v_m\}$, where $m < k$.

The barycenter of a simplex is the average of its vertices. So, we have:

$$b(\tau_2) = \frac{1}{k+1} \sum_{i=0}^k v_i$$

$$b(\tau_1) = \frac{1}{m+1} \sum_{i=0}^m v_i$$

The vector for the edge e is $e = b(\tau_2) - b(\tau_1)$.

Step 3: Decompose the vector to find its length. This is the crucial step where the ratio emerges. We can rewrite $b(\tau_2)$ in terms of $b(\tau_1)$:

$$\begin{aligned} e &= \left(\frac{1}{k+1} \sum_{i=0}^k v_i \right) - b(\tau_1) \\ &= \frac{1}{k+1} \left(\sum_{i=0}^m v_i + \sum_{i=m+1}^k v_i \right) - b(\tau_1) \\ &= \frac{1}{k+1} \left((m+1)b(\tau_1) + \sum_{i=m+1}^k v_i \right) - b(\tau_1) \\ &= \frac{m+1}{k+1} b(\tau_1) + \frac{1}{k+1} \sum_{i=m+1}^k v_i - \frac{k+1}{k+1} b(\tau_1) \\ &= \frac{1}{k+1} \sum_{i=m+1}^k v_i - \frac{k-m}{k+1} b(\tau_1) \\ &= \frac{1}{k+1} \left(\sum_{i=m+1}^k v_i - (k-m)b(\tau_1) \right) \end{aligned}$$

Since $\sum_{i=m+1}^k 1 = k - m$, we can rewrite this as:

$$e = \frac{1}{k+1} \sum_{i=m+1}^k (v_i - b(\tau_1))$$

Step 4: Bound the length of the vector. Now we take the norm and apply the triangle inequality:

$$\|e\| = \left\| \frac{1}{k+1} \sum_{i=m+1}^k (v_i - b(\tau_1)) \right\| \leq \frac{1}{k+1} \sum_{i=m+1}^k \|v_i - b(\tau_1)\|$$

Every vertex v_i is a point in τ_2 , and the barycenter $b(\tau_1)$ is also a point in τ_2 (since τ_1 is a face of τ_2). The distance between any two points in a simplex cannot exceed the diameter of that simplex, $\text{diam}(\tau_2)$. The diameter of τ_2 is, by definition, less than or equal to the mesh of the entire complex K .

$$\|v_i - b(\tau_1)\| \leq \text{diam}(\tau_2) \leq \text{mesh}(K)$$

Substituting this into our inequality:

$$\|e\| \leq \frac{1}{k+1} \sum_{i=m+1}^k \text{mesh}(K)$$

The sum has $k - m$ identical terms.

$$\|e\| \leq \frac{k-m}{k+1} \text{mesh}(K)$$

Step 5: Maximize the bound. The length of any edge in $\text{Sd}(K)$ is bounded by the expression above. To find the mesh of $\text{Sd}(K)$, we must find the maximum possible value of this bound over all possible choices of k and m . The dimensions satisfy $0 \leq m < k \leq d$. To maximize the fraction $\frac{k-m}{k+1}$:

1. We must choose the smallest possible value for m . The minimum dimension of a face is $m = 0$ (a vertex).
2. We must choose the largest possible value for k . The maximum dimension of a simplex in K is d .

The function $f(k) = \frac{k}{k+1}$ is an increasing function of k . Therefore, the fraction is maximized when k is maximized. Substituting $m = 0$ and the maximal value $k = d$:

$$\|e\| \leq \frac{k-0}{k+1} \text{mesh}(K) = \frac{k}{k+1} \text{mesh}(K) \leq \frac{d}{d+1} \text{mesh}(K)$$

Since this holds for any arbitrary edge e in $\text{Sd}(K)$, it must also hold for the longest edge. Therefore, the mesh of the barycentric subdivision is bounded by this value.

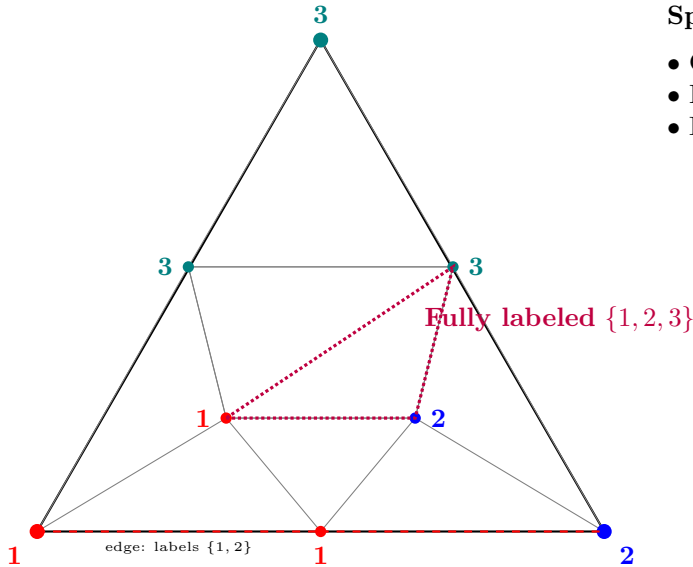
$$\text{mesh}(\text{Sd}(K)) \leq \frac{d}{d+1} \text{mesh}(K)$$

□

3 Sperner's Lemma

Definition 3.1 (Sperner Labeling). Let $\Delta^n = \text{conv}\{v_1, \dots, v_{n+1}\}$ be an n -simplex with a triangulation T . Similarly, let $V(T)$ be the set of vertices of the simplex. Then, **Sperner labeling** is a function $L : V(T) \rightarrow \{1, 2, \dots, n+1\}$ satisfying:

1. $L(v_i) = i$ for $i = 1, \dots, n+1$ (vertices labeled distinctly)
2. If vertex $x \in V(T)$ lies on a face spanned by $\{v_{i_1}, \dots, v_{i_k}\}$, then $L(x) \in \{i_1, \dots, i_k\}$



Sperner Rules:

- Corner vertices: fixed labels
- Edge vertices: labels from endpoints
- Interior vertices: any label

Figure 6: A Sperner labeling of a triangulated 2-simplex. The purple dotted triangle is fully labeled with all three colors $\{1, 2, 3\}$.

Theorem 3.2 (Sperner's Lemma). *Every Sperner-labeled triangulation of an n -simplex contains an odd number of fully labeled simplices (simplices whose vertices receive all $n + 1$ distinct labels). In particular, at least one such simplex exists.*

Proof. We use a parity argument by counting certain faces of the simplices in two different ways. The argument proceeds by induction on the dimension n .

Case $n = 1$

First, let us consider the 1-dimensional case. Here, our simplex is a line segment, let's say with vertices a and b . The segment is subdivided into smaller segments. The vertices of this subdivision are colored with 2 colors (say, 1 and 2). The condition for a Sperner labeling is that the endpoints a and b receive different colors.

Let's say a is colored 1 and b is colored 2. As we traverse the segment from a to b , the color of the vertices must switch at some point. To end up with color 2 at vertex b , the color must switch an odd number of times. Each time the color switches between adjacent vertices, we create a small segment whose endpoints have both colors, 1, 2. Therefore, there must be an odd number of small segments that are fully colored with 1, 2.

Case $n = 2$

We have a Sperner-labeled simplicial subdivision of a triangle T . Let the vertices of T be colored 1, 2, and 3. We are interested in the number of small triangles that have all three colors, 1, 2, 3. Let's call these "rainbow" cells.

Let's define a "door" as an edge in the subdivision whose endpoints are colored with 1, 2. We will count these doors in two ways.

Let R be the number of rainbow cells (colored 1, 2, 3). Let Q be the number of cells colored 1, 1, 2 or 1, 2, 2. Let X be the number of doors on the boundary of the large triangle T . Let Y be the number of doors in the interior of T .

Count 1: Over cells of the subdivision. An interior door (type Y) is an edge for two small triangles. A boundary door (type X) is an edge for only one. We sum the number of doors contributed by each small triangle.

- A cell of type R ($\{1,2,3\}$) has exactly one door.
- A cell of type Q ($\{1,1,2\}$ or $\{1,2,2\}$) has exactly two doors.
- A cell with colors $\{1,1,3\}$, $\{2,2,3\}$, $\{3,3,1\}$, etc., has zero doors.

Summing the doors over all cells gives the total number of doors, counting interior doors twice and boundary doors once. This gives the equation:

$$R + 2Q = X + 2Y$$

Count 2: Over the boundary of T . Doors on the boundary of T (type X) can only lie on the edge whose vertices are colored 1 and 2. This edge is a 1-dimensional simplex. From our argument for the case $n = 1$, we know that the number of segments colored 1,2 on this boundary edge must be odd. Thus, X is an odd number.

From the equation $R + 2Q = X + 2Y$, we can rearrange to $R = X + 2Y - 2Q = X + 2(Y - Q)$. Since X is odd and $2(Y - Q)$ is even, R must be odd.

General case

In the general n -dimensional case, we proceed by induction on n . We have a Sperner-labeled subdivision of an n -simplex S using $n + 1$ colors.

Let R be the number of rainbow cells (using all $n + 1$ colors). Let Q be the number of cells that use colors $\{1, 2, \dots, n\}$ such that exactly one of these colors is used twice and the other $n - 1$ colors are used once. (These are the cells that are "almost" rainbow faces of dimension $n - 1$).

We consider $(n-1)$ -dimensional faces that are colored with the set $\{1, 2, \dots, n\}$. Let's call these "special faces". Let X be the number of special faces on the boundary of S . Let Y be the number of special faces in the interior of S .

Again, we count the special faces in two ways.

Count 1: Over cells of the subdivision.

- Each cell of type R (a rainbow cell) contributes exactly one special face (the face opposite the vertex colored $n+1$).
- Each cell of type Q contributes exactly two special faces.

An interior special face is shared by two cells, while a boundary special face belongs to one. This gives the equation:

$$R + 2Q = X + 2Y$$

Count 2: On the boundary of S . The boundary of the n -simplex S consists of $n+1$ faces, each of which is an $(n-1)$ -simplex. A special face (colored $\{1, 2, \dots, n\}$) can only exist on the boundary face of S whose vertices are also colored $\{1, 2, \dots, n\}$. Let's call this boundary face F .

By the definition of a Sperner labeling, F itself is a properly colored $(n-1)$ -dimensional subdivision. We can now apply the inductive hypothesis to F . The hypothesis states that F contains an odd number of rainbow $(n-1)$ -simplices. These are precisely our special faces. Therefore, the number of special faces on the boundary, X , must be odd.

From the equation $R = X + 2Y - 2Q = X + 2(Y - Q)$, since X is odd, we conclude that R must be odd. This completes the induction. $\square$

4 Proof of Brouwer's Fixed Point Theorem

Theorem 4.1 (Brouwer Fixed Point Theorem). *Let Δ^n be an n -simplex and $f : \Delta^n \rightarrow \Delta^n$ be continuous. Then f has a fixed point, i.e. there exists $p \in \Delta^n$ such that $f(p) = p$.*

Proof. Assume for contradiction that f has no fixed point. Write points $x \in \Delta^n$ in barycentric coordinates $x = (x_1, \dots, x_{n+1})$, with $f(x) = (f_1(x), \dots, f_{n+1}(x))$. Define

$$g_i(x) := x_i - f_i(x) \quad (i = 1, \dots, n+1),$$

so $\sum_{i=1}^{n+1} g_i(x) = \sum_i x_i - \sum_i f_i(x) = 0$ for all $x \in \Delta^n$.

Labeling rule. For each $x \in \Delta^n$ set

$$L(x) := \min \left\{ i : g_i(x) = \max_{1 \leq j \leq n+1} g_j(x) \right\},$$

i.e. pick an index attaining the maximum of the $g_j(x)$ and break ties by choosing the smallest index. This is well-defined because $f(x) \neq x$ for every x , so the vector $g(x)$ is never the zero vector.

Verification of Sperner conditions.

- (Corner vertices) Let v_k be the k -th vertex of the standard simplex, so $(v_k)_k = 1$ and $(v_k)_j = 0$ for $j \neq k$. Then for $j \neq k$ we have $g_j(v_k) = -f_j(v_k) \leq 0$, while $g_k(v_k) = 1 - f_k(v_k) \geq 0$. Since $f(v_k) \neq v_k$ we actually have $g_k(v_k) > 0$, so the maximum of the $g_j(v_k)$ occurs at index k . Thus $L(v_k) = k$.
- (Face condition) Let F be a face spanned by indices $I \subset \{1, \dots, n+1\}$. If $x \in F$ then $x_j = 0$ for all $j \notin I$, hence $g_j(x) = -f_j(x) \leq 0$ for $j \notin I$. Because g is not identically zero we must have some $i \in I$ with $g_i(x) > 0$, and therefore the global maximum of the $g_j(x)$ is achieved at some index in I . Hence $L(x) \in I$.

Therefore L is a Sperner labeling on any triangulation of Δ^n .

Existence of fully labeled simplices and limit argument. For each $\varepsilon > 0$ choose a triangulation T_ε of Δ^n with mesh $< \varepsilon$. By Sperner's Lemma each triangulation contains at least one fully labeled n -simplex; denote such a simplex by σ_ε and let its vertices be

$$w_1^{(\varepsilon)}, w_2^{(\varepsilon)}, \dots, w_{n+1}^{(\varepsilon)},$$

with $L(w_i^{(\varepsilon)}) = i$. By definition we have

$$g_i(w_i^{(\varepsilon)}) = \max_j g_j(w_i^{(\varepsilon)}) \geq 0 \quad (i = 1, \dots, n+1).$$

Take a sequence $\varepsilon_k \downarrow 0$. From compactness of Δ^n and a diagonal/subsequence selection argument we may extract a subsequence (still denoted ε_k) such that, for each fixed i , the sequence $w_i^{(\varepsilon_k)}$ converges. Since the diameters of σ_{ε_k} tend to zero, all these limits coincide; denote the common limit by $x^* \in \Delta^n$.

By continuity of the g_i we have $g_i(x^*) \geq 0$ for every i . Summing gives $0 = \sum_i g_i(x^*) \geq 0$, so each $g_i(x^*) = 0$. Hence $f_i(x^*) = x_i^*$ for all i , i.e. $f(x^*) = x^*$, contradicting the assumption that f has no fixed point.

Therefore f must have a fixed point in Δ^n . □

Corollary 4.2. *Every continuous map $g : B^n \rightarrow B^n$ of the closed unit ball to itself has a fixed point.*

Proof. Let $h : B^n \rightarrow \Delta^n$ be a homeomorphism. For a continuous $g : B^n \rightarrow B^n$ define $f = h \circ g \circ h^{-1} : \Delta^n \rightarrow \Delta^n$. By the theorem f has a fixed point p , and then $x = h^{-1}(p)$ satisfies $g(x) = x$. □

5 Conclusion

We have demonstrated how Sperner's combinatorial lemma provides an elegant and constructive approach to proving Brouwer's Fixed Point Theorem.

This approach illustrates the power of discrete methods in continuous mathematics and demonstrates how combinatorial tools can resolve fundamental questions in analysis and topology. The extensive use of visual aids throughout this exposition shows how geometric intuition can guide rigorous mathematical reasoning.

Future research directions include extending these methods to more general topological spaces and developing efficient algorithms for computing approximate fixed points in high-dimensional settings, as explored in recent computational topology literature [2].

6 Acknowledgement:

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