CONSTRUCTION OF IRREDUCIBLE REPRESENTATIONS OF THE SYMMETRIC GROUP

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Abstract. In this paper we develop some basic results in representation theory of the symmetric group S_n. In particular, we describe the construction of Specht modules using polytabloids, and prove their irreducibility. We also prove the standard basis theorem for Specht modules by introducing Garnir elements and relations.

Contents

1. Representation theory preliminaries

The main focus of this paper, the representation theory of S_n, requires several fundamental definitions and theorems. They are given below for the sake of completeness, but without proofs (which could be found, for example, in [1] or [3]).

1.1. Basic definitions and theorems.

Definition 1.1 (Representation)**.** Let G be a finite group and let V be a finite-dimensional complex vector space. A (complex) *representation* of G on V is a group homomorphism

$$\rho : G \longrightarrow GL(V).$$

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Equivalently, it is a map $G \times V \to V$, $(g, v) \mapsto g \cdot v$, such that $e \cdot v = v$, $(gh) \cdot v = g \cdot (h \cdot v)$, and $g \cdot (\alpha v + \beta w) = \alpha(g \cdot v) + \beta(g \cdot w)$. We write (V, ρ) or simply V when ρ is clear.

Definition 1.2 (Subrepresentation)**.** If (V, ρ) is a representation of G, a subspace $W \subseteq V$ is a *subrepresentation* if $g \cdot W \subseteq W$ for all $g \in G$.

Definition 1.3 (Intertwiner; isomorphism)**.** Given representations (V, ρ_V) and (W, ρ_W) of G, a linear map $f : V \to W$ is a *G-homomorphism* (or *intertwiner*) if

$$f(\rho_V(g)v) = \rho_W(g)f(v) \text{ for all } g \in G,\ v \in V.$$

If f is also bijective, then f is an *isomorphism of representations.*

Definition 1.4 (Irreducible representation)**.** A nonzero representation V of G is *irreducible* if its only subrepresentations are 0 and V.

Lemma 1.5. *The number of conjugacy classes of a finite group is equal to the number of its irreducible complex representations (up to isomorphism).*

Definition 1.6 (Group algebra)**.** The *group algebra* $\mathbb{C}[G]$ is the complex vector space with basis $\{\, \mathbf{g} \mid g \in G \,\}$ and multiplication extended linearly from the rule $\mathbf{g} \cdot \mathbf{h} = \mathbf{gh}$. We identify $g \in G$ with its basis element $\mathbf{g} \in \mathbb{C}[G]$ and write

$$\mathbb{C}[G] = \Big\{ \sum_{g \in G} a_g\, g \ :\ a_g \in \mathbb{C} \Big\}.$$

Proposition 1.7 (Group algebra viewpoint)**.** *A representation of G on V is equivalent to a (left) $\mathbb{C}[G]$-module structure on V. Moreover, under this correspondence,*

$$\mathrm{Hom}_G(V, W) = \mathrm{Hom}_{\mathbb{C}[G]}(V, W).$$

Proof. ($\Rightarrow$) Let $\rho : G \to GL(V)$ be a representation. Define, for $x = \sum_g a_g g \in \mathbb{C}[G]$,

$$x \cdot v \ := \ \sum_g a_g\, \rho(g)v \qquad (v \in V).$$

Then $(xy) \cdot v = \sum_{g,h} a_g b_h\, \rho(gh)v = \sum_{g,h} a_g b_h\, \rho(g)\big(\rho(h)v\big) = x \cdot (y \cdot v)$ and $1 \cdot v = v$, so V is a left $\mathbb{C}[G]$-module.

($\Leftarrow$) Conversely, let V be a left $\mathbb{C}[G]$-module. For each $g \in G$ define $\rho(g) : V \to V$ by $\rho(g)v := g \cdot v$. Then

$$\rho(gh)v = (gh) \cdot v = g \cdot (h \cdot v) = \rho(g)\rho(h)v, \qquad \rho(e) = \mathrm{id}_V,$$

so $\rho : G \to GL(V)$ is a representation.

For morphisms, if $f : V \to W$ is G-equivariant then, for $x = \sum_g a_g g \in \mathbb{C}[G]$,

$$f(x \cdot v) = f\Big(\sum_g a_g\, g \cdot v\Big) = \sum_g a_g\, g \cdot f(v) = x \cdot f(v),$$

so f is $\mathbb{C}[G]$-linear. Conversely, if f is $\mathbb{C}[G]$-linear, then in particular $f(g \cdot v) = g \cdot f(v)$ for all $g \in G$, so f is G-equivariant. □

Theorem 1.8 (Maschke)**.** *Let G be a finite group and V a finite-dimensional complex G-representation. Then every G-stable subspace $W \subseteq V$ admits a G-stable complement: there exists a G-subrepresentation $U \subseteq V$ with $V = W \oplus U$. Equivalently, every complex representation of G is completely reducible.*

Lemma 1.9 (Schur's Lemma). *Let V, W be finite-dimensional complex irreducible G-representations.*

(1) If $V \ncong W$, then $\mathrm{Hom}_G(V, W) = 0$.
(2) If $V \cong W$, then $\mathrm{Hom}_G(V, W)$ is one-dimensional; in particular $\mathrm{End}_G(V) \cong \mathbb{C}$.

1.2. **Restriction, induction, and permutation modules.**

Definition 1.10 (Restriction). If $H \leq G$ and V is a G-representation, the *restriction* of V to H, denoted $\mathrm{Res}^G_H V$, is V regarded as an H-representation via the inclusion $H \hookrightarrow G$.

Definition 1.11 (Induction). Let $H \leq G$ and let W be an H-representation. The *induced* G-representation is

$$\mathrm{Ind}^G_H W \ := \ \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W,$$

with G acting on the left factor by left multiplication. Equivalently, $\mathrm{Ind}^G_H W$ is the space of functions $f : G \to W$ satisfying $f(gh) = h^{-1} f(g)$ for all $g \in G$, $h \in H$, with $(g_0 \cdot f)(g) = f(g_0^{-1} g)$.

Definition 1.12 (Permutation modules). If G acts on a finite set X, the *permutation representation* $\mathbb{C}[X]$ is the $\mathbb{C}$-vector space with basis $\{\, \mathbf{x} : x \in X \,\}$ and G-action $g \cdot \mathbf{x} = (\mathbf{g} \cdot \mathbf{x})$.

Proposition 1.13 (Cosets as an induced trivial module). *Let $H \leq G$. For the left G-set $X = G/H$, there is a natural G-isomorphism*

$$\mathbb{C}[G/H] \ \cong \ \mathrm{Ind}^G_H \xi,$$

where ξ is the trivial H-representation (see definition just below).

Proof. Define $\Phi : \mathbb{C}[G] \otimes_{\mathbb{C}[H]} \xi \to \mathbb{C}[G/H]$ by $\Phi(g \otimes 1) = (\mathbf{gH})$. If $h \in H$, then $gh \otimes 1 = g \otimes (h \cdot 1) = g \otimes 1$, so Φ is well-defined. It is G-equivariant since $g_0 \cdot (g \otimes 1) = (g_0 g) \otimes 1$ maps to $(\mathbf{g_0 gH}) = g_0 \cdot (\mathbf{gH})$. Surjectivity and dimension count give bijectivity. □

2. Partitions and Young Subgroups

We can begin by considering some examples of basic complex representations of S_n:

- *Trivial representation.* The one-dimensional module on which S_n acts trivially. We denote it ξ.
- *Sign representation.* The homomorphism $\mathrm{sgn} : S_n \to \{\pm 1\} \subset \mathbb{C}^\times$ gives a one-dimensional representation which we denote η.
- *Permutation representation on $\mathbb{C}^n$.* Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{C}^n$. Define
$$\sigma \cdot e_i \ = \ e_{\sigma(i)} \qquad (\sigma \in S_n).$$
Let $\varepsilon : \mathbb{C}^n \to \mathbb{C}$ be the augmentation $\varepsilon(e_i) = 1$ (with the trivial S_n-action). Then ε is S_n-equivariant, so $\ker(\varepsilon)$ is an S_n-subrepresentation and
$$\mathbb{C}^n \ = \ \ker(\varepsilon) \ \oplus \ \mathbb{C} \cdot \Big(\sum_{i=1}^n e_i \Big),$$
exhibiting $\mathbb{C}^n \simeq \ker(\varepsilon) \oplus \xi$.

Proposition 2.1. *If $n > 1$, then* $\ker(\varepsilon)$ *is irreducible.*

Proof. Let $0 \neq W \subseteq \ker(\varepsilon)$ be an S_n-subrepresentation. Take $v = \sum_{i=1}^{r} a_i e_i \in W$ with r minimal among nonzero vectors in W, whilst permuting basis vectors such that $a_i \neq 0$ for $1 \leq i \leq r$ and $a_i = 0$ for $i > r$. Since $v \in \ker(\varepsilon)$, we have $\sum_{i=1}^{r} a_i = 0$.

If $r = 1$, then $a_1 = 0$, a contradiction. If $r = 2$, then $a_1 + a_2 = 0$, so v is a scalar multiple of $e_1 - e_2$. Acting by transpositions shows $e_i - e_j \in W$ for all i, j, hence $W = \ker(\varepsilon)$.

Suppose $r \geq 3$. Then some pair $a_{r-1} \neq a_r$. Consider

$$w \;=\; a_{r-1}\, v \;-\; a_r\, (r\ r-1) \cdot v \;\in\; W.$$

A direct computation yields

$$w \;=\; (a_{r-1} - a_r) \sum_{i=1}^{r-2} a_i\, e_i \;+\; (a_{r-1}^2 - a_r^2)\, e_{r-1},$$

so $w \neq 0$ and has support contained in $\{1, \ldots, r-1\}$, contradicting the minimality of r. Hence no such $r \geq 3$ exists, and the only nonzero subrepresentation is $\ker(\varepsilon)$ itself. $\square$

Definition 2.2. The representation $\ker(\varepsilon)$ is called the *standard representation* of S_n and is denoted ω.

We can manually check that, for example, S_3 only has the ξ, η and ω irreducible representations. To find more irreducibles, we can further examine permutation representations. Notice that if we have a subgroup $H \subseteq S_n$, we can look at the permutation representation associated to the action of S_n on the cosets S_n/H, which is why we need a good way to construct subgroups of S_n. This brings along the following two definitions, fundamental to the topic.

Definition 2.3. A partition of a natural number n is a non-increasing sequence of positive integers

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r), \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0,$$

such that $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$. We write $\lambda \vdash n$ to indicate that λ is a partition of n. We also write λ_i^m for m equal parts, e.g. $\lambda = (1^n)$ to indicate splitting $n = 1 + 1 + \cdots + 1$.

To motivate this definition, we recall that every element $\sigma \in S_n$ can be expressed uniquely as a product of disjoint cycles. The cycle decomposition of σ has a well-defined cycle type, the list of lengths of its disjoint cycles. Since the order of cycles is irrelevant, the cycle type of σ is essentially a partition of n. For instance, the permutation $(1\,2\,3)(4\,5) \in S_5$ has cycle type $(3, 2)$.

Now we can use partitions to produce subgroups of S_n.

Definition 2.4 (Young subgroup)**.** Let $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$. Choose an ordered set-partition

$$P_\lambda = (I_1, \ldots, I_r) \quad \text{of } \{1, \ldots, n\} \quad \text{with } |I_i| = \lambda_i.$$

For a subset $J \subset \{1, \ldots, n\}$, let $S_J \leq S_n$ be the subgroup of permutations that fix $\{1, \ldots, n\} \setminus J$ pointwise and permute the elements of J. The *Young subgroup* associated to λ (relative to P_λ) is

$$S_\lambda \;=\; S_{I_1} \times \cdots \times S_{I_r} \;\hookrightarrow\; S_n,$$

embedded blockwise so that S_{I_i} acts on I_i and trivially on I_j for $j \neq i$.

If P_λ and P'_λ are two choices of ordered set-partitions of type λ, then the corresponding Young subgroups are conjugate in S_n. Thus S_λ is well-defined up to conjugacy.

Before investigating Young subgroups as a source of representations, it will be nice to know the number of irreducible representations.

Lemma 2.5. *Conjugacy classes in S_n are in bijection with partitions of n. Moreover, two permutations are conjugate in S_n if and only if they have the same cycle type.*

Proof. Suppose $\sigma, \tau \in S_n$. If they are conjugate, then $\tau = g\sigma g^{-1}$ for some $g \in S_n$, so the cycle structure of τ is obtained from that of σ by relabeling the underlying set $\{1, 2, \ldots, n\}$. Hence σ and τ have the same cycle type. Conversely, if σ and τ have the same cycle type, then one can construct an element $g \in S_n$ which maps each cycle of σ to the corresponding cycle of τ by relabeling entries. Thus σ and τ are conjugate. $\square$

Corollary 2.6. *The number of irreducible representations of S_n over $\mathbb{C}$ is equal to the number of partitions of n.*

Proof. This immediately follows from Theorem 2.5 and Lemma 1.5 $\square$

We have now seen that partitions count irreducible representations of S_n, so there is hope in moving forward with Young subgroups.

Definition 2.7. A *λ-partition* of $\{1, \ldots, n\}$ is an ordered r-tuple

$$(A_1, \ldots, A_r) \quad \text{with} \quad \{1, \ldots, n\} = A_1 \sqcup \cdots \sqcup A_r \quad \text{and} \quad |A_i| = \lambda_i \text{ for all } i.$$

Denote by X_λ the set of all λ-partitions. The group S_n acts on X_λ by

$$g \cdot (A_1, \ldots, A_r) \; = \; (gA_1, \ldots, gA_r).$$

Theorem 2.8. *Fix $P_\lambda = (I_1, \ldots, I_r)$ as in Definition 2.4. Then:*

(1) The S_n-action on X_λ is transitive.
(2) The stabilizer of P_λ is S_λ.
(3) The map

$$\theta : \; S_n/S_\lambda \longrightarrow X_\lambda, \qquad \theta(gS_\lambda) = (gI_1, \ldots, gI_r),$$

is a well-defined S_n-equivariant bijection. In particular, $X_\lambda \simeq S_n/S_\lambda$ as S_n-sets.

(4) The associated permutation module satisfies

$$\mathbb{C}[X_\lambda] \; \simeq \; \mathbb{C}[S_n/S_\lambda] \; \simeq \; \mathrm{Ind}_{S_\lambda}^{S_n} \xi,$$

Proof. (1) Let $(A_1, \ldots, A_r) \in X_\lambda$. For each i, take a bijection $\phi_i : I_i \to A_i$ and define $g \in S_n$ by $g|_{I_i} = \phi_i$. Notice that these agree on disjoint blocks and cover $\{1, \ldots, n\}$. Then $g \cdot P_\lambda = (A_1, \ldots, A_r)$, so the action is transitive.

(2) If $g \in S_\lambda$, then by definition $gI_i = I_i$ for all i, which means that g fixes P_λ. On the other hand, if $g \cdot P_\lambda = P_\lambda$, then $gI_i = I_i$ for all i, so $g \in S_{I_1} \times \cdots \times S_{I_r} = S_\lambda$.

(3) Well-definedness: if $gS_\lambda = g'S_\lambda$ then $g^{-1}g' \in S_\lambda$, hence $(g^{-1}g')I_i = I_i$ for all i, so $gI_i = g'I_i$, i.e. $\theta(gS_\lambda) = \theta(g'S_\lambda)$. S_n-equivariance is immediate from the definitions. Surjectivity follows from (1): given $(A_1, \ldots, A_r)$, pick g as there;

then $\theta(gS_\lambda) = (A_1, \ldots, A_r)$. For injectivity, notice that if $\theta(gS_\lambda) = \theta(g'S_\lambda)$, then $gI_i = g'I_i$ for all i, so $h := (g')^{-1}g$ fixes each I_i, hence $h \in S_\lambda$ and $gS_\lambda = g'S_\lambda$.

(4) The first isomorphism is the canonical identification of the permutation module of an S_n-set with the permutation module of its isomorphic coset space. The second is the standard identification $\mathbb{C}[S_n/S_\lambda] \simeq \mathrm{Ind}_{S_\lambda}^{S_n} \xi$ (as in Proposition 1.13). $\square$

We can now use this result to consider some simple Young subgroups. Let $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$ and let $S_\lambda \leq S_n$ be the Young subgroup. The S_n-set of cosets S_n/S_λ yields the permutation module

$$\mathbb{C}[S_n/S_\lambda] \;\simeq\; \mathrm{Ind}_{S_\lambda}^{S_n} \xi.$$

In particular:

- For $\lambda = (n)$, one has $S_\lambda = S_n$, so $\mathbb{C}[S_n/S_\lambda] \simeq \xi$.
- For $\lambda = (1^n)$, one has $S_\lambda = \{1\}$, so $\mathbb{C}[S_n/S_\lambda] \simeq \mathbb{C}[S_n]$ (the regular representation).
- For $\lambda = (n-1, 1)$, one has $S_\lambda \simeq S_{n-1}$. Identifying S_n/S_λ with $\{1, \ldots, n\}$ via $gS_\lambda \mapsto g(n)$, the permutation module $\mathbb{C}[S_n/S_\lambda]$ identifies with the permutation representation on $\mathbb{C}^n$. Hence

$$\mathrm{Ind}_{S_{n-1}}^{S_n} \xi \;\simeq\; \mathbb{C}^n \;\simeq\; \xi \;\oplus\; \omega.$$

To consider $\mathbb{C}[S_n/S_\lambda]$ more generally for any λ, it turns out that the following definition is useful:

Definition 2.9 (Dominance order)**.** Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ be partitions of n (extend by zeros as needed). We say that λ *dominates* μ, and write $\lambda \trianglerighteq \mu$, if

$$\sum_{i=1}^{k} \lambda_i \;\geq\; \sum_{i=1}^{k} \mu_i \qquad \text{for all } k \geq 1.$$

If $\lambda \trianglerighteq \mu$ and $\lambda \neq \mu$, we write $\lambda \triangleright \mu$. This defines a partial order on the set of partitions of n.

To understand why we define ordering on partitions in such a way it is useful to consider the following example. The partitions of $n = 4$ are

$$(4),\ (3,1),\ (2,2),\ (2,1,1),\ (1,1,1,1).$$

The dominance order $\trianglerighteq$ on which is

$$(4) \;\trianglerighteq\; (3,1) \;\trianglerighteq\; (2,2) \;\trianglerighteq\; (2,1,1) \;\trianglerighteq\; (1,1,1,1)$$

For each partition $\lambda \vdash 4$, let $M_\lambda = \mathbb{C}[S_4/S_\lambda]$ be the corresponding Young permutation module. It turns out that the irreducible representations of S_4 (over $\mathbb{C}$) are

$$\xi,\ \eta,\ \omega,\ \omega \otimes \eta,\ V,$$

with dimensions $1, 1, 3, 3, 2$, respectively (we can manually verify the first 4, and know that there exists some fifth V by 2.6)[1]. One has the following decompositions:

$$\begin{aligned} M_{(4)} &\cong \xi, \\ M_{(3,1)} &\cong \xi \ \oplus \ \omega, \\ M_{(2,2)} &\cong \xi \ \oplus \ \omega \ \oplus \ V, \\ M_{(2,1,1)} &\cong \xi \ \oplus \ 2 \cdot \omega \ \oplus \ V \ \oplus \ (\omega \otimes \eta), \\ M_{(1,1,1,1)} &\cong \mathbb{C}[S_4] \quad \text{(the regular representation)}. \end{aligned}$$

In particular, proceeding down the dominance order:

- (4) contributes the new irreducible ξ.
- $(3, 1)$ contributes the new irreducible ω; ξ already appeared above.
- $(2, 2)$ contributes the new irreducible V; ξ and ω already appeared above.
- $(2, 1, 1)$ contributes the new irreducible $\omega \otimes \eta$; the other summands have appeared above.
- $(1, 1, 1, 1)$ yields the regular representation, in which every irreducible appears with multiplicity equal to its dimension; in particular η appears here for the first time.

Such pattern is not a coincidence, as it turns out that in general for each λ, all composition factors of M_λ are indexed by partitions μ with $\mu \trianglerighteq \lambda$, and exactly one new irreducible L_λ (corresponding to λ, e.g. $L_{(3,1)} = \omega$) occurs for the first time in M_λ

We now give an alternative way to look at partitions, which will turn out to be very useful when exploring M_λ.

2.1. **Young diagrams.**

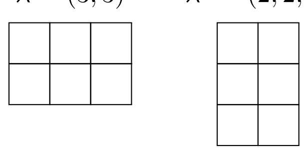
Definition 2.10 (Young diagram)**.** For a partition $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, the *Young diagram* of λ is the left-centered array of boxes with λ_i boxes in row i

Definition 2.11 (Transpose (conjugate) partition)**.** Given $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, its transpose (also called the conjugate partition) $\lambda^\dagger$ is defined by

$$(\lambda^\dagger)_i \ = \ \#\{\, j \mid \lambda_j \geq i \,\} \qquad (i \geq 1).$$

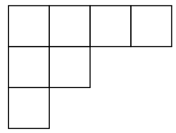
Equivalently, the Young diagram of $\lambda^\dagger$ is obtained from that of λ by interchanging rows and columns. One has $(\lambda^\dagger)^\dagger = \lambda$.

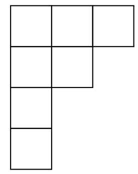
$\lambda = (3, 3)$ $\qquad$ $\lambda^\dagger = (2, 2, 2)$



[1] We do not explain what V is, for the sake of cohesion in this argument. Although it turns out V is an interesting object related to graph matching, more detail could be found in [2].

$\lambda = (4, 2, 1)$ $\lambda^\dagger = (3, 2, 1, 1)$





Definition 2.12 (Tableau and tabloid)**.** Fix a partition $\lambda \vdash n$ and its Young diagram with rows indexed from top.

- A λ-*tableau* is a filling of the boxes of the diagram with the numbers $1, \ldots, n$, each used exactly once. We usually write t for such a tableau.
- λ-tableaux are *row-equivalent* if one can be obtained from the other by permuting entries *within each row* independently. A λ-*tabloid* is a row-equivalence class of λ-tableaux; the tabloid of a tableau t is denoted $\{t\}$.

Definition 2.13 (Actions and permutation modules)**.** The group S_n acts on λ-tableaux by left relabeling:

$(g \cdot t)$ is the tableau obtained from t by replacing each entry i by $g(i)$.

Thus for λ-tabloids:

$$g \cdot \{t\} \;=\; \{g \cdot t\}.$$

Remark 2.14 (Standard identifications)**.** There are $n!$ λ-tableaux, so $\mathbb{C}[\lambda\text{-tableaux}]$ is the regular representation of S_n. The S_n-set of λ-tabloids is naturally identified with the coset space S_n/S_λ, and hence

$$M_\lambda \;:=\; \mathbb{C}[\lambda\text{-tabloids}] \;\cong\; \mathbb{C}[S_n/S_\lambda] \;\cong\; \mathrm{Ind}_{S_\lambda}^{S_n} \xi.$$

The modules $M_\lambda = \mathrm{Ind}_{S_\lambda}^{S_n} \xi$ provide a systematic source of S_n-representations enumerated by partitions. Empirically we find that the composition factors of M_λ are precisely those L_μ with $\mu \trianglerighteq \lambda$ in dominance order, and L_λ occurs with multiplicity one; the other factors L_μ with $\mu \triangleright \lambda$ have appeared already in M_μ for more dominant μ.

We also introduce the "opposite" permutation module

$$N_\lambda \;:=\; \mathrm{Ind}_{S_{\lambda^\dagger}}^{S_n} \eta \;\cong\; M_{\lambda^\dagger} \otimes \eta,$$

We can think of N_λ as tableux of shape λ which switches sign based upon permutation of elements in a column, by the sign of such permutation. It turns out that, for example for $n = 4$, $N_{(1,1,1,1)} = \eta = L_{(1,1,1,1)}$ and $N_{(2,1,1)} = 2 \cdot \eta \oplus \omega = L_{(1,1,1,1)} \oplus L_{(2,1,1)}$. Heuristically, the composition factors of N_λ are L_μ with $\mu \trianglelefteq \lambda$, while the opposite holds for M_λ, i.e. $M_\lambda = L_\lambda \oplus (\bigoplus L_\mu)$ for $\lambda \trianglelefteq \mu$. This suggests that M_λ and N_λ share exactly one "new" common irreducible, the L_λ, and motivates seeking a canonical S_n-map

$$\Phi_\lambda : \; N_\lambda \longrightarrow M_\lambda$$

where the image would isolate the L_λ.

We now explicitly construct such a map, using an averaging procedure over columns.

3. Specht Modules

Definition 3.1 (Row and column groups)**.** For a λ-tableau t, let

$$R_t \ = \ \{\text{permutations that permute entries only within each row of } t\} \ \leq S_n$$

also called a row stabilizer, and

$$C_t \ = \ \{\text{permutations that permute entries only within each column of } t\} \ \leq S_n.$$

which is also called a column stabilizer.

Definition 3.2 (Polytabloid)**.** For a λ-tableau t, define the

$$\kappa_t = \sum_{c \in C_t} \operatorname{sgn}(c), \quad c \in \mathbb{C}[S_n].$$

The *polytabloid* attached to t is the vector

$$e_t := \kappa_t \cdot \{t\} \in M_\lambda.$$

Lemma 3.3. *For $r \in R_t$ and $c \in C_t$ it holds that*

$$r \cdot e_t = e_t, \qquad c \cdot e_t = \operatorname{sgn}(c) e_t.$$

Consequently, the span of $\{e_t \mid t \text{ a } \lambda\text{-tableau}\}$ is an S_n-submodule of M_λ.

Proof. If $r \in R_t$, then $r\{t\} = \{t\}$ and $rC_tr^{-1} = C_t$, so $r\kappa_t = \kappa_t r$ and $r \cdot e_t = \kappa_t(r \cdot \{t\}) = \kappa_t\{t\} = e_t$. If $c \in C_t$, then $c\kappa_t = \operatorname{sgn}(c)\kappa_t$ by multiplicativity of the sign on the subgroup C_t, hence $c \cdot e_t = \operatorname{sgn}(c)e_t$. Finally, we get S_n-stability from $g \cdot e_t = e_{g \cdot t}$. $\square$

Definition 3.4. Define the *Specht module* of shape λ as

$$S^\lambda := \operatorname{span}_{\mathbb{C}}\{e_t \mid t \text{ a } \lambda\text{-tableau}\} \ \subseteq \ M_\lambda.$$

Proposition 3.5 (Averaging map)**.** *There is an S_n-equivariant surjection*

$$\alpha_\lambda \colon N_\lambda \twoheadrightarrow S^\lambda \subseteq M_\lambda$$

given on tableaux by $t \mapsto e_t$ (viewing a tableau both as a basis vector of N_λ and, via its tabloid, of M_λ). In particular, S^λ is a quotient of N_λ and a submodule of M_λ.

Proof. As a vector space, $N_\lambda \cong \mathbb{C}[\lambda\text{-tableaux}]$ with the C_t-action twisted by sgn (coming from induction from $S_{\lambda^\dagger}$ with sign). Lemma 3.3 shows that the assignment $t \mapsto e_t$ respects the C_t-sign and is invariant under R_t. Thus $t \mapsto e_t$ is a universal map removing the row-symmetry and imposing column-antisymmetry, hence it is S_n-equivariant and factors through N_λ with image S^λ. $\square$

Now we aim to show that S^λ is irreducible, $S^\lambda \cong L_\lambda$, and $\{S^\lambda\}_{\lambda \vdash n}$ exhaust all irreducible S_n modules up to isomorphism.

Lemma 3.6. *Let $\lambda, \mu \vdash n$, t a λ-tableau and t' a μ-tableau. Suppose that for every pair a, b lying in the same row of t', the entries a and b lie in* different *columns of t. Then $\lambda \trianglerighteq \mu$.*

Proof. Let the first row of t' contain μ_1 entries. By hypothesis, they lie in μ_1 distinct columns of t. Applying C_t we can independently within each column move each of these entries into the top box of its column. Hence the first row of t has space for all μ_1 entries, so $\lambda_1 \geq \mu_1$.

Now, consider the first two rows of t', which have $\mu_1 + \mu_2$ entries. Again, within each column of t we can accommodate at most two chosen entries into the first two rows (at most one in each of the first two positions). Since entries from a fixed row of t' occupy distinct columns of t, we can perform these placements simultaneously, meaning $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$. We can continue analogously to get that $\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i$ for all k. Which by definition is equivalent to $\lambda \trianglerighteq \mu$. □

Lemma 3.7. *Let $\lambda, \mu \vdash n$, t a λ-tableau and t' a μ-tableau. If $\kappa_t\{t'\} \neq 0$, then $\lambda \trianglerighteq \mu$. Furthermore, if $\lambda = \mu$, then $\kappa_t\{t'\} = \pm e_t$.*

Proof. If a, b lie in the same row of t', then $(ab)\{t'\} = \{t'\}$. If a, b also lie in the same column of t, then $(ab) \in C_t$, which means that

$$\kappa_t\{t'\} = (ab)\,\kappa_t\{t'\} = \operatorname{sgn}(ab)\,\kappa_t\{t'\} = -\,\kappa_t\{t'\},$$

which would imply $\kappa_t\{t'\} = 0$, contradiction. As such, any two entries in a row of t' must be in distinct columns of t, and by Lemma 3.6 it follows that $\lambda \trianglerighteq \mu$.

If $\lambda = \mu$, the same argument shows $t' \in C_t \cdot t$; so $t' = ct$ with $c \in C_t$. Then $\kappa_t\{t'\} = \kappa_t c\{t\} = \operatorname{sgn}(c)\,\kappa_t\{t\} = \pm e_t$. □

Corollary 3.8. *For any $m \in M_\lambda$ and λ-tableau t, it holds that $\kappa_t m \in \mathbb{C}e_t$.*

Proof. Consider $m = \sum_{t'} a_{t'}\{t'\}$ as a linear combination of λ-tabloids. By Lemma 3.7, each term is such that $\kappa_t\{t'\} \in \{0, \pm e_t\}$, meaning that $\kappa_t m$ is in fact a scalar multiple of e_t. □

3.1. Irreducibility of S^λ.

Definition 3.9. For $\lambda \vdash n$, define a symmetric bilinear form $\langle -, - \rangle$ on M_λ by

$$\langle \{t\}, \{t'\} \rangle \;=\; \delta_{\{t\},\{t'\}},$$

extended bilinearly. This form is S_n-invariant.

Lemma 3.10. *If $V \subseteq M_\lambda$ is an S_n-subrepresentation, then either $V \supseteq S^\lambda$ or $V \subseteq (S^\lambda)^\perp$*

Proof. Let $v \in V$ and let t be any λ-tableau. If $\kappa_t v \neq 0$, then by Corollary 3.8 we have $\kappa_t v \in \mathbb{C}\,e_t$, meaning that $e_t \in V$. Since the S_n-orbit of e_t spans S^λ, we get $S^\lambda \subseteq V$. Otherwise $\kappa_t v = 0$ for all t. Then for every t we have

$$0 = \langle \kappa_t v, \{t\} \rangle \;=\; \langle v, \kappa_t\{t\} \rangle \;=\; \langle v, e_t \rangle,$$

meaning that v is orthogonal to e_t for all t, i.e. $v \in (S^\lambda)^\perp$, so $V \subseteq (S^\lambda)^\perp$. □

Proposition 3.11. *The Specht module S^λ is irreducible.*

Proof. Let $W \subsetneq S^\lambda$ be a nonzero S_n-subrepresentation (inside M_λ). By Lemma 3.10, $W \subseteq (S^\lambda)^\perp$. Then

$$W \;\subseteq\; S^\lambda \cap (S^\lambda)^\perp \;=\; 0,$$

since the form from Definition 3.9 can be taken positive definite (e.g. as a Hermitian inner product) and thus has trivial radical on S^λ. Therefore $W = 0$, a contradiction. Hence S^λ has no nontrivial subrepresentations, i.e. is irreducible. □

We have now proved that the Specht modules S^λ are irreducible, but their construction is not perfect, as they are defined as a span of linearly dependent tableux. It would be useful to find a basis for each Specht module to better understand their structure (which could then be used, for example, to analyze characters)

4. Standard basis

Lemma 4.1. *Let $\lambda, \mu \vdash n$ and let $f : M_\lambda \to M_\mu$ be an S_n-equivariant linear map. Assume that $S^\lambda \not\subseteq \ker f$. Then it holds that $\lambda \trianglerighteq \mu$. Moreover, if $\lambda = \mu$, then $f|_{S^\lambda}$ is a scalar multiple of the identity.*

Proof. Consider any λ-tableau t. Since $S^\lambda \not\subseteq \ker f$, there exists some (hence every) nonzero $v \in S^\lambda$ with $f(v) \neq 0$. In particular, we can take $v = e_t = \kappa_t\{t\}$, so

$$0 \neq f(e_t) = f(\kappa_t\{t\}) = \kappa_t f(\{t\}),$$

where we used S_n-equivariance to commute κ_t out of f. Write $f(\{t\}) = \sum_{t'} a_{t'}\{t'\}$ as a linear combination of μ-tabloids. Then

$$0 \neq \kappa_t f(\{t\}) = \sum_{t'} a_{t'}\, \kappa_t\{t'\}.$$

By Lemma 3.7, each term satisfies $\kappa_t\{t'\} \in \{0, \pm e_t\}$. Furthermore, the sum is not zero, meaning some term is nonzero, hence $\lambda \trianglerighteq \mu$.

Now consider the case of $\mu = \lambda$. Since S_λ is irreducible by Proposition 3.11, Shur's Lemma 1.9 gives $\mathrm{End}_{S_n}(S_\lambda) \cong \mathbb{C}$. Thus any nonzero map $S_\lambda \to S_\lambda$ (for instance, the one obtained by composing the inclusion $S_\lambda \hookrightarrow M_\lambda$ with the projection $M_\lambda \twoheadrightarrow S_\lambda$) must be a scalar multiple of the identity. ◻

Corollary 4.2. *We have an orthogonal direct sum decomposition*

$$M_\lambda \;=\; S^\lambda \;\oplus\; (S^\lambda)^\perp,$$

and $(S^\lambda)^\perp$ is a (finite) direct sum of Specht modules S^μ with $\mu \trianglerighteq \lambda$ and $\mu \neq \lambda$. Furthermore, every irreducible S_n-representation is isomorphic to some S^λ.

Proof. The bilinear form $\langle -, - \rangle$ on M_λ from Definition 3.9 is S_n-invariant and positive definite (over $\mathbb{R}$, or take the Hermitian version), so $(S^\lambda)^\perp$ is an S_n-submodule and $M_\lambda = S^\lambda \oplus (S^\lambda)^\perp$. By complete reducibility (Maschke's Theorem 1.8), $(S^\lambda)^\perp$ is a direct sum of irreducibles. Since each S^λ, by Proposition 3.11, is irreducible, it follows that

$$(S^\lambda)^\perp \;\cong\; \bigoplus_{\mu \neq \lambda} m_{\lambda\mu}\, S^\mu.$$

Now, remember that $\mu \trianglerighteq \lambda$ holds for any term, which is compatible with the column–dominance mechanism of Lemmas 3.6 and 3.7 (see also Corollary 3.8), which forces nonzero intertwiners out of S^λ only into shapes μ dominating λ.

It remains to show that distinct Specht modules are pairwise non-isomorphic, and that every irreducible is a Specht. Suppose $S^\lambda \simeq S^\mu$. Over characteristic 0, there must exist an S_n-equivariant projection and inclusion maps making this isomorphism a term inside M_λ and M_μ. In other words, we can extend the isomorphism to an S_n-map $f : M_\lambda \to M_\mu$ with $f|_{S^\lambda} : S^\lambda \xrightarrow{\sim} S^\mu$. By Lemma 4.1, $\lambda \trianglerighteq \mu$, and applying the same argument to the inverse isomorphism extended to $M_\mu \to M_\lambda$ gives $\mu \trianglerighteq \lambda$, hence $\lambda = \mu$. Therefore the family $\{S^\lambda\}_{\lambda \vdash n}$ consists of pairwise non-isomorphic irreducibles.

Finally, by 2.6 the number of partitions of n equals the number of conjugacy classes of S_n, so the number of irreducibles (over $\mathbb{C}$). Since there are that many pairwise non-isomorphic irreducibles S^λ, they exhaust all irreducibles of S_n. ◻

4.1. **Garnir relations.** Let us now consider some simple generic examples of Specht modules. For $\lambda = (n)$ it holds that $S^\lambda = \xi$, while for $\lambda = (1^n)$ the column group is all of S_n, so e_t is the projection of $\{t\}$ to the sign isotypic piece and $S^\lambda = \eta$. For $\lambda = (n-1,1)$, identifying $M_\lambda \simeq \mathbb{C}^n$, the polytabloids e_t correspond to vectors of the form $v_j - v_i \in \ker(\varepsilon)$, so $S^\lambda = \omega$. Already here we see a linear relation

$$(v_k - v_j) + (v_j - v_i) = (v_k - v_i),$$

which translate into more complicated linear relations between the corresponding polytabloids. Notice that these relations come from mixing entries across adjacent columns. To capture this uniformly for arbitrary shapes we introduce Garnir elements.

Definition 4.3 (Garnir element)**.** Let t be a λ-tableau and fix an index $i \geq 1$. Choose subsets

$$X \subseteq \{\text{entries in the } i\text{th column of } t\}, \quad Y \subseteq \{\text{entries in the } (i+1)\text{st column of } t\}.$$

Let $S_{X\cup Y} \leq S_n$ be the subgroup permuting the labels in $X \cup Y$ and fixing the others. The *Garnir element* attached to (X,Y) is

$$G_{X,Y} = \sum_{\sigma \in S_{X\cup Y}} \operatorname{sgn}(\sigma)\sigma \in \mathbb{C}[S_n].$$

Proposition 4.4 (Garnir relation)**.** *Let $\lambda^\dagger$ be the conjugate partition, so $\lambda_i^\dagger$ is the height of the ith column. If $|X \cup Y| > \lambda_i^\dagger$, then for every λ-tableau t it holds that*

$$G_{X,Y} \cdot e_t \;=\; 0.$$

Proof. First, we have that $e_t = \kappa_t\{t\}$ with $\kappa_t = \sum_{c\in C_t} \operatorname{sgn}(c)\, c$. Fix $\sigma \in C_t$, and consider the tabloid $\{\sigma t\}$. Since $|X \cup Y| > \lambda_i^\dagger$ but the first i columns of t have only $\lambda_i^\dagger$ rows in column i, by the pigeonhole principle it follows that there exist some distinct $a, b \in X \cup Y$ that are in the same row of σt. Furthermore, for the transpose $\tau = (ab) \in S_{X\cup Y}$ we have that $\tau \cdot \{\sigma t\} = \{\sigma t\}$, as the same row means it is the same tabloid. Furthermore, $\operatorname{sgn}(\tau) = -1$. Therefore the contribution of $\{\sigma t\}$ to $G_{X,Y} \cdot \{\sigma t\}$ cancels and we are left with

$$G_{X,Y} \cdot \{\sigma t\} = \sum_{\rho \in S_{X\cup Y}} \operatorname{sgn}(\rho)\, \rho \cdot \{\sigma t\} = \cdots + \{\sigma t\} - \{\sigma t\} + \cdots = 0.$$

Since e_t is a signed sum of such $\{\sigma t\}$ with $\sigma \in C_t$, it follows that $G_{X,Y} \cdot e_t = 0$ as desired. □

Remark 4.5 (Coset form of the relation)**.** The direct product $S_X \times S_Y$ is contained in C_t. Therefore it acts on e_t by sign. It follows that $G_{X,Y}$ acts on e_t as

$$\big|S_X \times S_Y\big| \cdot \sum_{\rho \in S_{X\cup Y}/(S_X \times S_Y)} \operatorname{sgn}(\rho)\, \rho,$$

a finite signed sum over coset representatives. As such, Proposition 4.4 gives a linear relation of polytabloids that exchanges entries between the ith and $(i+1)$st columns. These Garnir relations are the additional relations beyond row-symmetry and column-antisymmetry, which we need to reduce arbitrary e_t to a linear combination of standard polytabloids. Essentially, Garnir relations allow to exchange entries between adjacent columns, which in turn allows us to standardize the tabloids.

We now consider an example of a Garnir relation, and see how Garnir elements produce linear relations between polytabloids, which in turn allow us to exchange entries between adjacent columns to rewrite e_t in terms of other polytabloids.

Consider the following tableau for $n = 5$:

$$t = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array},$$

and choose a vertical Garnir belt formed from adjacent columns 1 and 2 with $X = \{4\}$ and $Y = \{2, 5\}$. Then for the coset form of the Garnir relation we get

$$S_{X\cup Y}/\big(S_X \times S_Y\big) = \{\mathrm{id},\ (4\ 5),\ (2\ 4\ 5)\}.$$

which applying e yields

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array} \ - \ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 5 & 4 \\ \cline{1-2} \end{array} \ + \ \begin{array}{|c|c|c|}\hline 1 & 4 & 3 \\ \hline 5 & 2 \\ \cline{1-2} \end{array} \ = \ 0$$

As such, we can eliminate any configuration where the "belt" contains too many symbols for the left column height, rewriting e_t as a linear combination of polytabloids in which entries are shifted to the right column. Iterating such relations (together with row-symmetry and column-antisymmetry) reduces arbitrary polytabloids to combinations of polytabloids associated to standard tableaux (which we can now explicitly define).

4.2. **Standard tableaux and basis.**

Definition 4.6. A λ-tableau t is standard if its entries strictly increase from left to right in each row and strictly increase from top to bottom in each column.

We can now formulate and prove the standard basis theorem, the main result we have been working towards.

Theorem 4.7. *For a fixed partition $\lambda \vdash n$, the set $\{e_t \mid t \text{ a standard } \lambda\text{-tableau}\}$ is a $\mathbb{C}$-basis of S^λ.*

Proof. For a tabloid $\{t\}$ and any $1 \le i \le n$, let $r_i(\{t\})$ be the row index of the entry i in (any representative of) $\{t\}$. Define a total order $<$ on λ-tabloids lexicographically, that is $\{t\} < \{t'\}$ if and only if

$$\big(r_n(\{t\}), r_{n-1}(\{t\}), \dots\big) <_{\text{lex}} \big(r_n(\{t'\}), r_{n-1}(\{t'\}), \dots\big)$$

In other words, $\{t\} < \{t'\}$ if and only if

$$r_n(\{t\}) < r_n(\{t'\}), \quad \text{or}$$

$$r_n(\{t\}) = r_n(\{t'\}) \text{ and } r_{n-1}(\{t\}) < r_{n-1}(\{t'\}), \quad \text{or}$$

$$r_n(\{t\}) = r_n(\{t'\}) \text{ and } r_{n-1}(\{t\}) = r_{n-1}(\{t'\}) \text{ and } r_{n-2}(\{t\}) < r_{n-2}(\{t'\}), \quad \text{or}$$

$$\cdots$$

Now we will show that the leading tabloid for standard t is maximal. Fix a standard tableau t and consider C_t, the subgroup permuting entries within each column of t. We claim that among the tabloids appearing in $C_t\{t\}$, the unique maximal one is $\{t\}$ itself. Write $e_t = \kappa_t\{t\} = \sum_{c\in C_t} \mathrm{sgn}(c)\, c\{t\}$. In a standard tableau, each column is strictly increasing, meaning that the row of n in t is the lowest possible within its column. Then, any $c \in C_t$ can only move n up in that column, so $r_n(c\{t\}) \leq r_n(\{t\})$. Note that equality is achieved if and only if c fixes the position of n. Then for r_n, the same argument applies to $n-1$, then $n-2$, and so on. As such, for every $c \in C_t$, it holds that $c\{t\} \leq \{t\}$, with equality only when c acts trivially on all columns, i.e. only only if $c = 1$. As such,

$$e_t = \{t\} + \sum_{\{u\}<\{t\}} \alpha_u \{u\} \qquad (\alpha_u \in \mathbb{C}),$$

so $\{t\}$ is the (unique) leading term of e_t with coefficient 1. (No other $c \in C_t$ yields the same tabloid, since $C_t \cap R_t = \{1\}$ for a rectangular grid with distinct labels.)

Now, we aim to show linear independence of standard e_t. Suppose $\sum_{t \text{ standard}} a_t e_t = 0$. Pick t_0 standard such that $\{t_0\}$ is the biggest among $\{\{t\} \mid a_t \neq 0\}$. Per above, $\{t_0\}$ appears in e_{t_0} with coefficient 1, while in each e_t with $t \neq t_0$ all tabloids are less than $\{t_0\}$. But now if we compare coefficients of $\{t_0\}$ we would get $a_{t_0} = 0$, which is a contradiction. Therefore all the e_t with t standard are linearly independent.

We can now show that every e_t is a $\mathbb{Z}$-linear combination of $e_{t'}$ with t' standard, by induction. Assume t is not standard. By column-antisymmetrizing within each column (if needed), we can ensure columns of t are strictly increasing. Then there exist adjacent columns where a row decreases (with $a_k > b_k$):

$$\begin{array}{|c|c|}
\hline
a_1 & b_1 \\ \hline
a_2 & b_2 \\ \hline
\vdots & \vdots \\ \hline
a_k & b_k \\ \hline
\vdots & \vdots \\ \hline
\vdots & b_s \\ \hline
\vdots & \multicolumn{1}{c}{} \\ \cline{1-1}
\vdots & \multicolumn{1}{c}{} \\ \cline{1-1}
a_r & \multicolumn{1}{c}{} \\ \cline{1-1}
\end{array}$$

We can now form the Garnir belt by taking $X = \{a_k, \dots, a_r\}$ from the left column and $Y = \{b_1, \dots, b_k\}$ from the right column. The Garnir relation (Proposition 4.4) gives us that the signed sum over shuffles of $X \cup Y$ that respect internal orders of X and Y turns e_t to zero:

$$\sum_{\rho \in S_{X \cup Y}/(S_X \times S_Y)} \operatorname{sgn}(\rho)\, e_{\rho \cdot t} \;=\; 0.$$

In each nontrivial term $\rho \cdot t$ moves some a_j into a b-slot, strictly increasing the tuple $\big(r_n, r_{n-1}, \dots\big)$. Then $\{\rho \cdot t\}$ is strictly larger than $\{t\}$ in the lexicographical order. Therefore,

$$e_t \;=\; \sum_{\rho \neq 1} \beta_\rho\, e_{\rho \cdot t},$$

with all $\{\rho \cdot t\} > \{t\}$. By induction on the well-founded order, each $e_{\rho \cdot t}$ is a $\mathbb{Z}$-linear combination of $e_{t'}$ with t' standard. Thus e_t lies in the span of the standard polytabloids.

Combining the above shows the standard polytabloids form a basis of S^λ, as desired. $\square$

Corollary 4.8. $\dim_{\mathbb{C}} S^\lambda = \#\{$*standard* λ*-tableaux*$\}$*, and every* e_t *is a* $\mathbb{Z}$*-linear combination of* $e_{t'}$ *with* t' *standard (hence the matrices of* S_n *in this basis have integer entries)*[2]

5. Transpose and sign twist

We began the construction of Specht modules by considering permutation modules for transposed tableau (the M_λ and N_λ). The following result adds to the intuition about the connection of transposition and sgn.

First, we remind a standard result from representation theory:

Theorem 5.1 (Frobenius reciprocity: left and right adjoints)**.** *Let* $H \leq G$ *be finite groups and work over* $\mathbb{C}$*.*

(Left adjoint) *For any* H*-representation* W *and* G*-representation* V *there is a natural isomorphism*

$$\operatorname{Hom}_G\big(\operatorname{Ind}_H^G W,\ V\big) \;\cong\; \operatorname{Hom}_H\big(W,\ \operatorname{Res}_H^G V\big).$$

(Right adjoint) *Let the* coinduced *representation be*

$$\operatorname{CoInd}_H^G W \;:=\; \operatorname{Hom}_{\mathbb{C}[H]}(\mathbb{C}[G],\, W),$$

with G *acting by* $\big((g\cdot F)(x) = F(g^{-1}x)\big)$ *for* $F \in \operatorname{CoInd}_H^G W$ *and* $x \in \mathbb{C}[G]$*. Then for any* G*-representation* V *there is a natural isomorphism*

$$\operatorname{Hom}_G\big(V,\ \operatorname{CoInd}_H^G W\big) \;\cong\; \operatorname{Hom}_H\big(\operatorname{Res}_H^G V,\ W\big).$$

In particular, for finite groups over $\mathbb{C}$ *one has a natural* G*-isomorphism* $\operatorname{Ind}_H^G \cong \operatorname{CoInd}_H^G$*, so both adjunctions may be stated with* Ind_H^G*.*

Proposition 5.2. *For any partition* $\lambda \vdash n$*, there is a canonical* S_n*-isomorphism*

$$S^\lambda \otimes \eta \;\cong\; S^{\lambda^\dagger}.$$

Proof. Fix a λ-tableau t and let $t^\dagger$ be its transpose (a $\lambda^\dagger$-tableau). As previously, let R_t and C_t be the row and column stabilizers of t, and let

$$\rho_t = \sum_{r\in R_t} r \in \mathbb{C}[S_n], \qquad \kappa_{t^\dagger} = \sum_{c\in C_{t^\dagger}} \operatorname{sgn}(c)$$

Recall S^λ is spanned by polytabloids $e_t = \kappa_t\{t\}$ inside M_λ, and $M_\mu = \mathbb{C}[S_n/S_\mu]$ for any μ. Then by Frobenius reciprocity 5.1,

$$\operatorname{Hom}_{S_n}\left(M_{\lambda^\dagger},\, S^\lambda{\otimes}\eta\right) \;\cong\; \left(S^\lambda{\otimes}\eta\right)^{S_{\lambda^\dagger}},$$

with the isomorphism sending f to $f(\{t^\dagger\})$ (here $S_{\lambda^\dagger} = R_{t^\dagger}$). Since $R_{t^\dagger} = C_t$, and $c\cdot e_t = \operatorname{sgn}(c)\, e_t$ for $c \in C_t$ while η adds another factor $\operatorname{sgn}(c)$ in $S^\lambda \otimes \eta$, so $e_t \otimes 1$ is fixed by $R_{t^\dagger}$. Thus there is a unique S_n-equivariant map

$$f:\ M_{\lambda^\dagger} \longrightarrow S^\lambda \otimes \eta$$

with

$$f(\{t^\dagger\}) = e_t \otimes 1.$$

[2] There exists an explicit expression for the number of standard tableaux, known as the hook length formula. See more in [2].

Now, compute

$$f(e_{t^\dagger}) = f\big(\kappa_{t^\dagger}\{t^\dagger\}\big) = \kappa_{t^\dagger}\cdot(e_t\otimes 1) = \Big(\sum_{r\in R_t}\operatorname{sgn}(r)\,r\Big)\cdot(e_t\otimes 1) =$$

$$= \sum_{r\in R_t}\operatorname{sgn}(r)\,(re_t)\otimes\operatorname{sgn}(r) = \Big(\sum_{r\in R_t} r\Big)e_t\otimes 1 = \rho_t e_t\otimes 1,$$

using $R_{t^\dagger} = C_t$, the invariance $re_t = e_t$ for $r\in R_t$, and that $\operatorname{sgn}(r)^2 = 1$. We want to show that $\rho_t e_t \neq 0$. Since we have an S_n-invariant inner product on M_λ for which $\langle\{u\},\{v\}\rangle = \delta_{\{u\},\{v\}}$, it must then hold that

$$\langle \rho_t e_t, \{t\}\rangle = \langle e_t, \rho_t\{t\}\rangle = |R_t|\cdot\langle e_t,\{t\}\rangle = |R_t| \neq 0,$$

since the coefficient of $\{t\}$ in $e_t = \kappa_t\{t\}$ is 1. As such, $f(e_{t^\dagger})\neq 0$, as desired.

Finally, for the image we notice that the restriction $f|_{S^{\lambda^\dagger}} : S^{\lambda^\dagger}\to S^\lambda\otimes\eta$ is a nonzero S_n-map between irreducible representations, and therefore an isomorphism by Schur's lemma. This proves $S^\lambda\otimes\eta\cong S^{\lambda^\dagger}$. □

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