# SCHRAMM-LOEWNER EVOLUTIONS AND PERCOLATION INTERFACES

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Abstract. The Schramm-Loewner evolutions (SLE) are a random family of curves defined using a Brownian motion and a non-negative parameter $\kappa$. After constructing SLE and proving its basic properties, we explore the relationship between critical site percolation on the triangular lattice and SLE with parameter $\kappa = 6$. To do so, we prove Cardy's formula for crossing probabilities and use this to prove that the scaling limit of critical percolation on the triangular lattice's percolation exploration is SLE(6).

## Contents



## 1. Background Information

Schramm-Loewner evolutions are a random family of curves lying in domains of the complex plane $\mathbb{C}$ defined through a non-negative parameter $\kappa$. In this paper, we will mainly consider SLEs defined in the upper half-plane $\mathbb{H} = \{x + iy \in \mathbb{C} : y > 0\}$. Consequently, we assume familiarity with the key theorems in complex analysis, as in e.g. [11] or [20]. We will begin by introducing relevant definitions and results that link complex analysis to our study of SLEs; more information and proofs can be found in, for example, [2, 12].

**Definition 1.1.** Let $K$ be a subset of $\mathbb{H}$. $K$ is a **compact $\mathbb{H}$ hull** if $K$ is bounded and $H = \mathbb{H} \setminus K$ is a simply connected domain.

The Riemann mapping Theorem and the Schwarz reflection principle give us a way to encode the geometry of compact $\mathbb{H}$ hulls into functions; a proof may be found in Section 3.2 of [2].

**Theorem 1.2.** *Let $K$ be a compact $\mathbb{H}$ hull and $H = \mathbb{H}\setminus K$. Then, there is a unique conformal isomorphism $g_K : H \to \mathbb{H}$ such that $g_K - z$ goes to $0$ as $|z|$ goes to $\infty$ in addition to being bounded uniformly in $z \in H$. Also, for some $a_K \in \mathbb{R}$,*

$$g_K(z) = z + \frac{a_K}{z} + O(|z|^{-2}), |z| \to \infty.$$

This function $g_K$ is known as the **mapping-out function**. The condition that $g_K(z) - z \to 0$ at $\infty$ is sometimes referred to as **hydrodynamic normalization**. In particular, Theorem 1.2 gives us a way to parameterize these hulls.

**Definition 1.3.** Let $K$ be a compact $\mathbb{H}$ hull. Then, its **half-plane capacity** is

$$\text{hcap}(K) = \lim_{z\to\infty} z(g_k(z) - z) = a_K.$$

From this definition and from the expansion provided in Theorem 1.2, if $K$ is a compact $\mathbb{H}$ hull, $r > 0$, and we consider $rK = \{rz : z \in K\}$ then $\text{hcap}(rK) = r^2\text{hcap}(K)$. In addition, we may see that translation of a hull by an element of $\mathbb{R}$ has no impact on half-plane capacity. Another way in which we can get a sense of the size of a compact $\mathbb{H}$ hull is by using the following definition.

**Definition 1.4.** Let $K$ be a compact $\mathbb{H}$ hull. The **radius** of $K$ is defined as

$$\text{rad}(K) = \inf\{r \geq 0 : \exists x \in \mathbb{R} \text{ such that } K \subseteq r\overline{\mathbb{D}} + x\}$$

where $\mathbb{D}$ is the open unit disk so $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Here and throughout the paper, we will use this definition of $\mathbb{D}$. We will now summarize some key properties related to half-plane capacity and mapping-out functions. The proofs for all of these statements can be found in Chapter 3 of [2].

**Proposition 1.5.** *Let $K$ be a compact $\mathbb{H}$ hull. Then, the following hold.*

*(1)* $\mathrm{hcap}(K) \geq 0$*;* $\mathrm{hcap}(K) = 0$ *if and only if* $K = \emptyset$*.*
*(2) Let $K_0$ and $K_1$ be compact $\mathbb{H}$ hulls. If $K' = K_0 \cup g_{K_0}^{-1}(K_1)$ then $K'$ is a compact $\mathbb{H}$ hull containing $K_0$, $g_{K'} = g_{K_1} \circ g_{K_0}$, and* $\mathrm{hcap}(K') = \mathrm{hcap}(K_0) + \mathrm{hcap}(K_1)$*. If $K$ contains $K_0$, then it can be expressed in such a form.*
*(3) If $z \in H$,* $|g_K(z) - z| \leq 3\mathrm{rad}(K)$*.*
*(4) There is a finite constant $C$ such that if $r \in (0, \infty)$, $\xi \in \mathbb{R}$, and $K$ is contained in $r\overline{\mathbb{D}} + \xi$, then $|g_K(z) - z - \frac{a_K}{z-\xi}| \leq \frac{Cra_K}{|z-\xi|^2}$ for $2r \leq |z - \xi|$.*

We will use the first property as a characterization of when the half-plane capacity is 0, the second property for when we consider nested sequences of compact $\mathbb{H}$ hulls, the third property as a continuity estimate, and the fourth property as a differentiability estimate.

## 2. Schramm-Loewner Evolutions

In this section, we will introduce Schramm-Loewner evolutions and some key properties using the background information in the previous section.

**Definition 2.1.** Let $(K_t)_{t\geq 0}$ be a family of compact $\mathbb{H}$ hulls. It is **increasing** if $K_s$ is strictly contained in $K_t$ when $s < t$. For such an increasing family, let $K_{t+} = \cap_{s>t} K_s$ and for $s < t$, let $K_{s,t} = g_{K_s}(K_t \setminus K_s)$. Our family of compact $\mathbb{H}$ hulls has the **local growth property** if $\mathrm{rad}(K_{t,t+h}) \to 0$ as $h$ decreases to 0 uniformly on compacts in $t$.

The following proposition will justify our later definition of SLE.

**Proposition 2.2.** *Let $(K_t)_{t\geq 0}$ be an increasing family of compact $\mathbb{H}$ hulls with the local growth property. Then, $K_{t+} = K_t$ for all $t$ and $t \to \mathrm{hcap}(K_t)$ is continuous and strictly increasing on $[0, \infty)$. Finally, if $t \geq 0$, we may find a unique $\xi_t \in \mathbb{R}$ such that $\xi_t \in \overline{K_{t,t+h}}$ for all $h > 0$ and $(\xi_t)_{t\geq 0}$ is continuous. The process $(\xi_t)_{t\geq 0}$ is the* ***Loewner transform*** *of $(K_t)_{t\geq 0}$.*

The proposition tells us how we might encode a growing hull process into a continuous real-valued function. In particular, it gives us a form of right-continuity in hull growth, a continuous parameterization of the hulls, and a continuous well-defined function, the Loewner transform, which describes where new growth is happening at microscopic levels.

*Proof.* Let $K_{t,t+} = g_{K_t}(K_{t_+} \setminus K_t)$. By Proposition 1.5 (2), we know that $\mathrm{hcap}(K_{t+h}) = \mathrm{hcap}(K_t) + \mathrm{hcap}(K_{t,t+h})$. This gives us that $\mathrm{hcap}(K_{t,t+}) \leq \mathrm{hcap}(K_{t,t+h}) \leq \mathrm{rad}(K_{t,t+h})^2$ by the definitions of half-plane capacity and the radius of a compact $\mathbb{H}$ hull. It follows from the local growth property that $t \to \mathrm{hcap}(K_t)$ is continuous. We also have that $\mathrm{hcap}(K_{t,t_+}) = 0$, implying by Proposition 1.5 (1) that $K_{t,t+} = \emptyset$ and $K_{t+} = K_t$.

Now, fix some $t \geq 0$. Note that for all $h > 0$, $\overline{K_{t,t+h}}$ is compact, and if $0 < h' < h$, then $\overline{K_{t,t+h'}}$ is contained in $\overline{K_{t,t+h}}$. By the local growth property, there is a unique $\xi_t \in \mathbb{R} \cap (\bigcap_{h>0} \overline{K_{t,t+h}})$. Now, for $h > 0$, let $z \in K_{t+2h} \setminus K_{t+h}$, $w = g_{K_t}(z)$, and $w' = g_{K_{t+h}}(z)$, so we have $w \in K_{t,t+2h}$ and $w' = g_{K_{t,t+h}}(w) \in K_{t+h,t+2h}$. By definition of the radius and Proposition 1.5 (3), $|\xi_t - w| \leq 2\mathrm{rad}(K_{t,t+2h})$, $|\xi_{t+h} - w'| \leq 2\mathrm{rad}(K_{t+h,t+2h})$, and $|w - w'| \leq 3\mathrm{rad}(K_{t,t+h})$. This ensures by the triangle inequality that

$$|\xi_{t+h} - \xi_t| \leq 2\mathrm{rad}(K_{t+h,t+2h}) + 3\mathrm{rad}(K_{t,t+h}) + 2\mathrm{rad}(K_{t,t+h})$$

which goes to 0 as $h$ goes to 0 uniformly on compacts in $t$, proving continuity. □

The parameterization we worked with in this proposition was parameterization on $[0, \infty)$. But if $T$ is in $(0, \infty)$, we may also consider parameterization on $[0, T)$. The proposition 2.2 implies that the function sending $t$ to $\frac{\mathrm{hcap}(K_t)}{2}$ is a homeomorphism on $[0, T)$. If $\tau$ is the associated inverse homeomorphism, we can find a family $K'_t = K_{\tau(t)}$ of compact $\mathbb{H}$ hulls such that $\mathrm{hcap}(K'_t) = 2t$. We will call such a family **parameterized by half-plane capacity**.

We will now prove that hulls grow according to a differential equation controlled by the Loewner transform, which converts our problem from growing random hulls into analyzing real-valued stochastic processes.

**Proposition 2.3.** *Let $(K_t)_{t\geq 0}$ be a family of compact $\mathbb{H}$ hulls which is increasing, satisfies the local growth property, and which is parameterized by half-plane capacity. Consider its Loewner transform. Let $g_t = g_{K_t}$ for $t \geq 0$, and for $z \in \mathbb{H}$, let $\zeta(z) = \inf\{t \geq 0 : z \in K_t\}$. In the case where this set is empty, we take $\zeta(z) = \infty$. Now, fix some $z \in \mathbb{H}$. The function $g_t(z) : [0, \zeta(z)) \to \mathbb{H}$ is a differentiable function of $t$. Moreover, this function satisfies* ***Loewner's differential equation:***

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \xi_t}.$$

*Finally, if $\zeta(z) < \infty$, then as $t \to \zeta(z)$, then $g_t(z) - \xi_t \to 0$.*

*Proof.* Let $0 \leq s < t < \zeta(z)$. By Proposition 1.5 (2), we get that $\mathrm{hcap}(K_s) + \mathrm{hcap}(K_{s,t})$, so because the family is parameterized by half-plane capacity, we get that $\mathrm{hcap}(K_{s,t}) = 2(t-s)$. Let $z_t = g_t(z)$. We then have that $g_{K_{s,t}}(z_s) = z_t$ and $K_{s,t}$ is contained in $2\mathrm{rad}(K_{s,t})\overline{\mathbb{D}} + \xi_s$. We then apply Proposition 1.5 (3) to the compact $\mathbb{H}$ hull $K_{s,t}$ to get that $|z_t - z_s| \leq 3\mathrm{rad}(K_{s,t})$. The local growth property then directly implies the continuity of $(z_t)_{0\leq t\leq \zeta(z)}$. Now, let $z \in \mathbb{H}$ and $s \leq t < \zeta$. It follows by continuity that $\delta = \inf\{|z_u - \zeta_u| : u \in [0,t]\}$ is positive. So, if $s$ and $t$ are sufficiently close such that $\mathrm{rad}(K_{s,t}) \leq \frac{\delta}{8}$, then we have that $4\mathrm{rad}(K_{s,t}) \leq |z_s - \zeta_s|$. We then apply Proposition 1.5 (4) with compact $\mathbb{H}$ hull $K_{s,t}$ to get that

$$\left| z_t - z_s - \frac{2(t-s)}{z_s - \zeta_s} \right| \leq \frac{4C\mathrm{rad}(K_{s,t})(t-s)}{|z_s - \zeta_s|^2}$$

for some finite constant $C$. Noting that $z_t - z_s = g_t(z) - g_s(z)$, dividing both sides by $t - s$ gives us differentiability and that Loewner's differential equation is satisfied by the local growth property. Finally, let $s < \zeta(z) < t < \infty$. Then, $z \in K_t \setminus K_s$, so $z_s \in K_{s,t}$ and $|z_t - \zeta_s| \leq 2\mathrm{rad}(K_{s,t})$. The local growth property then implies that as $s \to \zeta(z)$, $|z_s - \zeta_s| \to 0$. □

So far, we have shown that some families of hulls satisfying certain properties grow according to the Loewner differential equation controlled by its Loewner transform. It is therefore natural to consider what continuous functions can serve as Lowener transforms for a family of hulls. We now prove in the next two propositions that any continuous real-valued function $(\xi_t)_{t\geq 0}$, which we shall henceforth refer to as the **driving function**, gives rise to a family of compact $\mathbb{H}$ hulls that satisfies Proposition 2.3 with $(\xi_t)_{t\geq 0}$ being the Loewner transform of the family.

**Proposition 2.4.** *Consider a driving function $(\xi_t)_{t\geq 0}$. If $z \in \mathbb{C}\setminus\{\xi_0\}$, then there is a unique $\zeta(z) \in (0, \infty]$ and a unique continuous map $t \to g_t(z) : [0, \zeta(z)) \to \mathbb{C}$ such that if $t$ is in $[0, \zeta(z))$, then $g_t(z) \neq \xi_t$, $|g_t(z) - \xi_t| \to 0$ as $t \to \zeta(z)$ when $\zeta(z)$ is finite, and*

$$g_t(z) = z + \int_0^t \frac{2}{g_s(z) - \xi_s} ds.$$

*The initial value $g_0(z) = z$ is satisfied. Furthermore, letting $\zeta(\xi_0) = 0$ and $C_t = \{z \in \mathbb{C} : t < \zeta(z)\}$, then $C_t$ is open for all $t \geq 0$ and $g_t : C_t \to \mathbb{C}$ is holomorphic.*

The process $g_t(z)$ for $t$ in $[0, \zeta(z))$ is the **maximal solution** starting from $z$ and $\zeta(z)$ is the **lifetime**. The family of maps $(g_t)_{t\geq 0}$ is the **Loewner flow** in $\mathbb{H}$ that corresponds to the driving function. Later, we will consider the complements of the $C_t$ sets in $\mathbb{H}$ and prove that they are compact $\mathbb{H}$ hulls that satisfy the conditions of Proposition 2.3. Our proof utilizes basic properties of the theory of ordinary differential equations; see e.g. [21].

*Proof.* We define the vector field $b(t,z) = \frac{2}{z - \xi_t}$. Aside from the singularity at $\xi_t$, this is holomorphic in $z$ and continuous in $t$. This gives us the integral form of the ODE:

$$g_t(z) = z + \int_0^t b(s, g_s(z)) ds.$$

Furthermore, if we fix $z \neq \xi_0$, then $b(t,z)$ is Lipschitz on compact sets away from $\xi_t$ because $|b(t,z) - b(t,z')| \leq 2n^2|z - z'|$ for $|z - \xi_t| \geq \frac{1}{n}$. The Picard-Lindelöf Theorem implies the existence and uniqueness of a solution on a unique maximal interval $[0, \zeta(z))$ where $g_t(z) \to \xi_t$ with $g_t(z) \neq \xi_t$.

Now, fix some $t \geq 0$. Let $z_0 \in C_t$. We note that this ODE satisfies the conditions for continuous dependence on the initial condition. So, there is a neighborhood $U$ of $z_0$ such that $\zeta(z) > t$ for all $z \in U$ and $g_s(z)$ is well defined on $[0, t]$ for all $z \in U$. This shows that $C_t$ is open. Finally, we know from complex ODE theory that if the vector field $b(t,z)$ is holomorphic in $z$ and continuous in $t$, then the solution $g_t(z)$ is holomorphic in $z$ for fixed $t$, provided that the solution exists. It follows that $g_t$ is holomorphic on the set $C_t$. □

For all $t \geq 0$, define $K_t = \{z \in \mathbb{H} : \zeta(z) \leq t\}$ and $H_t = \mathbb{H} \setminus K_t$. We may also call the family $(K_t)_{t\geq 0}$ a **Loewner chain**. From now on, we will restrict the domains of $\zeta$ and $g_t$ from $\mathbb{C}$ to $\mathbb{H}$, and $C_t$ to $H_t$, respectively. Next, we will summarize the previously discussed correspondence that hull growth is captured by the driving function, which will justify our definition of SLE.

**Proposition 2.5.** *The family $(K_t)_{t\geq 0}$ is an increasing family of compact $\mathbb{H}$ hulls with the local growth property. Additionally, for all $t \geq 0$, $\operatorname{hcap}(K_t) = 2t$ and $g_{K_t} = g_t$. Finally, the driving function $(\xi_t)_{t\geq 0}$ is the Loewner transform of $(K_t)_{t\geq 0}$.*

*Proof.* Let $0 \leq t \leq T$ and let $\hat{\xi}_t = \xi_{T-t}$. We will consider what is sometimes known as the backward Loewner equation

$$w_t^* = -\frac{2}{w_t - \hat{\xi}_t}$$

for $w_0 = w \in \mathbb{H}$ and $0 \leq t \leq T$. We may solve this differential equation for $[0, T]$ in considering the vector field $\hat{b}(t, z) = -\frac{2}{(z-\hat{\xi}_t)}$ which satisfies that the imaginary part is nonnegative. Let $z = w_T$ and $z_t = w_{T-t}$. Then,

$$w_t = w - \int_0^t \frac{2}{w_s - \hat{\xi}_s} ds.$$

By making the change of variables $u = T - s$, it follows that

$$z_t = w_{T-t} = z + \int_{T-t}^T \frac{2}{w_s - \hat{\xi}_s} ds = z + \int_0^T \frac{2}{z_u - \xi_u} du.$$

By the uniqueness of the Cauchy problem on $\hat{b}$ and by our prior expansion, $\zeta(z) > T$, $g_T(z) = w$, and $z$ is the only point in $\mathbb{H}$ such that these properties hold. This implies that $g_T$ is a holomorphic bijection by Proposition 2.4 and therefore a conformal isomorphism. So, $H_T$ is simply connected. Now, fix some $T \geq 0$ and let $r = \max(\sqrt{T}, \sup_{t\leq T} |\xi_t - \xi_0|)$. Let $R \geq 4r$, $z \in \mathbb{H}$ such that $R \leq |z - \xi_0|$, and $\tau = \inf\{t \text{ in } [0, \zeta(z)) : r \leq |g_t(z) - z|\}$. In the case where this set is empty, let $\tau = \zeta(z)$. Note that $0 < \tau \leq \zeta(z)$ and if $t \leq \tau$ and $t < \zeta(z)$ then $|g_t(z) - z| \leq r$. In particular, $\zeta(z) > \tau$ because

$$R - 2r \leq |g_t(z) - \xi_t| = |(g_t(z) - z) + (z - \xi_0) + (\xi_0 - \xi_t)|.$$

Note as well that $|g_t(z) - z| \leq \frac{2t}{R-2r} \leq \frac{t}{r}$ because

$$g_t(z) - z = \int_0^t \frac{2}{g_s(z) - \xi_s} ds.$$

This directly implies that $T \leq \tau$ because otherwise, then $|g_\tau(z) - z| \leq \frac{\tau}{r} < \frac{T}{r} \leq r$, a contradiction. So, $T < \zeta(z)$ and $z \in H_T$. Choosing $R = 4r$, we get that if $z \in K_T$, then $|z - \xi_0| \leq 4r$. Therefore, $K_T$ is bounded and a compact $\mathbb{H}$ hull. Next, we get that

$$z(g_t(z) - z) - 2t = 2\int_0^t \frac{z - g_s(z) + \xi_s}{g_s(z) - \xi_s} ds$$

so that

$$|z(g_t(z) - z) - 2t| \leq \frac{(4r + 2|\xi_0|)t}{R - 2r}.$$

Letting $R \to \infty$, as $|z| \to \infty$, then $z(g_t(z) - z) \to 2t$, which implies that as $|z| \to \infty$, $g_t(z) - z \to 0$. Hence, $g_t = g_{K_t}$ and $\operatorname{hcap}(K_t) = 2t$ for all $t \geq 0$. To finish the proof, let $s \geq 0$, and for fixed $t \geq 0$, let $\xi'_t = \xi_{s+t}$, $H'_t = g_s(H_{s+t})$, $K'_t = \mathbb{H} \setminus H'_t$, and $g'_t = g_{s+t} \circ g_s^{-1}$. Differentiating with respect to $t$, we get that $(g'_t)_{t\geq 0}$ is the Loewner flow with driving function $(\xi'_t)_{t\geq 0}$ with domain $H'_t$, and $K'_t = g_s(K_{s+t} \setminus K_s) = K_{s,s+t}$. Early in the proof, we showed that if $z \in K_T$, then $|z - \xi_0| \leq 4r$, so if $z \in K_{s,s+t}$, then

$$|z - \xi_s| \leq 4\left(\max\{\sup_{s\leq u\leq s+t} |\xi_u - \xi_s|, \sqrt{t}\}\right)$$

which implies that $(K_t)_{t\geq 0}$ has the local growth property with Loewner transform $(\xi_t)_{t\geq 0}$. □

We now arrive at our definition of **Schramm-Loewner evolutions**, which were introduced by Oded Schramm in [15].

**Definition 2.6.** Let $\kappa \geq 0$. For $t \geq 0$, let $\xi_t = \sqrt{\kappa} B_t$ where $(B_t)_{t\geq 0}$ is a standard one dimensional Brownian motion. Note that $(\xi_t)_{t\geq 0}$ then defines a continuous function, so Proposition 2.4 applies. The random family $(K_t)_{t\geq 0}$ as described in Proposition 2.5 is then said to be (chordal) **SLE($\kappa$)**.

There is another version of SLE known as radial SLE. We will only work with chordal SLEs in this paper, but for an overview of radial SLEs and other SLE variants, readers may look at e.g. [7].

Since the definition of SLE relies on Brownian motion, we may therefore translate some SLE problems into problems concerning Brownian motion. For instance, the scale invariance and Markov properties of Brownian motion (see e.g. [13]) provide us with a convenient characterization of SLE.

**Theorem 2.7.** *If $(K_t)_{t\geq 0}$ is an increasing family of compact $\mathbb{H}$ hulls with the local growth property and such that* $\text{hcap}(K_t) = 2t$ *for all $t \geq 0$, then it is an SLE if and only if the below two conditions hold.*

*(1) $(K_t)_{t\geq 0}$ is **scale invariant**. This means that if $\lambda \in (0,\infty)$, then $(\lambda K_{\lambda^{-2}t})_{t\geq 0}$ has the same distribution as $(K_t)_{t\geq 0}$.*

*(2) $(K_t)_{t\geq 0}$ has the **domain Markov property**. This means that if $s \geq 0$, then $(g_{K_s}(K_{s+t} \setminus K_s) - \xi_s)_{t\geq 0}$ has the same distribution as $(K_t)_{t\geq 0}$ and is independent of $\mathcal{F}_s = \sigma(\xi_r : r \leq s)$. In other words, after time $s$, the recentered future hulls look like an independent new copy of the process.*

*Proof.* ($\Longrightarrow$) We know that $\xi_t = \sqrt{\kappa}B_t$ where $B_t$ is a standard Brownian motion. The scaling property of Brownian motion, applied to this driving function, then directly implies that $(K_t)_{t\geq 0}$ is scale invariant. On the other hand, we may show that the Loewner transform for fixed $s \geq 0$ of $(g_{K_s}(K_{s+t} \setminus K_s) - \xi_s)_{t\geq 0}$ is $(\xi_{s+t} - \xi_s)_{t\geq 0}$, so by the Markov properties of Brownian motion, $(K_t)_{t\geq 0}$ has the domain Markov property as well.

($\Longleftarrow$) Properties (1) and (2) imply, respectively, that $(\xi_t)_{t\geq 0}$ is scale-invariant and has stationary independent increments. Since $(\xi_t)_{t\geq 0}$ is continuous by Proposition 2.2, by the Lévy-Khinchin representation, $\xi_t = \sigma B_t + \mu t$, for all $t \geq 0$, where $B_t$ is a standard Brownian motion, $\sigma \geq 0$, and $\mu \in \mathbb{R}$. However, scale invariance forces $\mu = 0$. So, for some $\kappa \geq 0$, $(K_t)_{t\geq 0}$ is an SLE($\kappa$). □

Until now, we have considered SLE in $\mathbb{H}$ with boundary points 0 and $\infty$. However, it may be convenient to translate our study of SLE into an arbitrary proper simply connected planar domain $D$ with arbitrary boundary points $z_0$ and $z_\infty$. We will call these triples **two-pointed domains**. To do so, we will use conformal isomorphisms from two-pointed domains back to the triple $(\mathbb{H}, 0, \infty)$, known as **scales**. More formally, a scale for $D$ is a conformal isomorphism $\phi : D \to \mathbb{H}$ such that $\phi(z_0) = 0$ and $\phi(z_\infty) = \infty$.

Our terminology may also be extended to this new framework. A subset of $D$ is a $D$-hull if $D \setminus K$ is a simply connected neighborhood of $z_\infty$. Considering the associated scale isomorphism $\sigma$, we can therefore get families of $D$-hulls that have the local growth property and such that $\text{hcap}(\sigma(K_t)) = 2t$ for all $t \geq 0$. Such a random variable family is an SLE($\kappa$) in $D$ of scale $\sigma$ if the Loewner transform of $\sigma(K_t))_{t\geq 0}$ is $\sqrt{\kappa}B_t$ where $B_t$ is a standard Brownian motion. Finally, our new equivalent of our mapping-out function and Loewner flow is $g_t = g_{\sigma(K_t)} \circ \sigma$. The next two results prove that regardless of domain and boundary points, conformal invariance and the domain Markov property of SLE still hold.

**Proposition 2.8.** *Consider the two-pointed domains $(D, z_0, z_\infty)$ and $(D', z_0', z_\infty')$, and let $\phi : D \to D'$ be a conformal isomorphism such that $\phi(z_0) = z_0'$ and $\phi(z_\infty) = z_\infty'$. Let $\sigma$ and $\sigma'$ be scales for $D$ and $D'$, respectively, and $\lambda = \sigma' \circ \phi \circ \sigma^{-1} : \mathbb{H} \to \mathbb{H}$. Let $(K_t)_{t\geq 0}$ be an SLE($\kappa$) in $D$ of scale $\sigma$. If $K_t' = \phi(K_{\lambda^{-2}t})$ then $(K_t')_{t\geq 0}$ is an SLE($\kappa$) in $D'$ of scale $\sigma'$.*

*Proof.* First, by definition, $(\sigma(K_t))_{t\geq 0}$ is a (standard) SLE in $\mathbb{H}$ with driving function $\sqrt{\kappa}B_t$ for $t \geq 0$ where $(B_t)_{t\geq 0}$ is a standard Brownian motion. Now, we apply $\sigma'$ to $K_t'$ for $t \geq 0$ to get that $\sigma'(K_t') = \lambda(\sigma(K_t))$. But, note that $\lambda$ is a conformal isomorphism of $\mathbb{H}$ fixing 0 and $\infty$, meaning that for some $a > 0$, $\lambda(z) = az$. Now, because SLE in $\mathbb{H}$ is scale-invariant, this implies that $(a\sigma(K_t))_{t\geq 0}$ is SLE($\kappa$). Because of Brownian scaling, the $(\sqrt{\kappa}B_t)_{t\geq 0}$ has the same distribution as $(a\sqrt{\kappa}B_{a^{-2}t})_{t\geq 0}$. By making a change of variables, this implies the proposition. □

**Proposition 2.9.** *Let $(K_t)_{t\geq 0}$ be an SLE($\kappa$) in the two-pointed domain $(D, z_0, z_\infty)$ of scale $\sigma$ and let $T$ be a finite stopping time. Let $K_t' = K_{T+t} \setminus K_t$ and let $\sigma_T : D \setminus K_T \to \mathbb{H}$ be defined by $\sigma_T(z) = g_T(z) - \xi_T$. Let $z_T = g_T^{-1}(\xi_T)$. Then, $(D_T, z_t, z_\infty)$ is a two-pointed domain with a scale $\sigma_T$. Additionally, conditional on $\mathcal{F}_T$, $(K_t')_{t\geq 0}$ is an SLE($\kappa$) in $(D_T, z_t, z_\infty)$ of scale $\sigma_T$.*

*Proof.* By definition, $(\sigma(K_t))_{t\geq 0}$ is an SLE($\kappa$) in $\mathbb{H}$ with driving function $(\sqrt{\kappa}B_t)_{t\geq 0}$. Note that $\sigma(K_t') = \sigma(K_{T+t}) \setminus \sigma(K_T)$. In describing future growth in $\mathbb{H}$, we remove the effect of $\sigma(K_T)$ by using $g_T$ and therefore consider $g_T(\sigma(K_{T+t})\setminus\sigma(K_T))$. We then get that after time $T$, the new driving function becomes $(\xi_{t+T} - \xi_T)_{t\geq 0}$. By the strong Markov property, since $\xi_t = \sqrt{\kappa}B_t$ and $T$ is a stopping time, $(B_{t+T} - B_T)_{t\geq 0}$ conditionally on $\mathcal{F}_T$ is independent of $\mathcal{F}_T$ and has the same distribution as a standard linear Brownian motion. So, conditionally on $\mathcal{F}_T$, $(\xi_{t+T} - \xi_T)_{t\geq 0}$ has the same distribution of $(\xi_t)_{t\geq 0}$ and is independent of $\mathcal{F}_T$. This implies that $(g_T(\sigma(K_{T+t}) \setminus \sigma(K_T)))_{t\geq 0}$ is an independent SLE($\kappa$) in $\mathbb{H}$ started at 0. Returning to the domain $D$, $\sigma_T$ has the properties that $\sigma_T(z_T) = 0$ with $z_T = \sigma_T^{-1}(0) = \sigma_T^{-1}(\xi_T)$ and $\sigma_T(z_\infty) = \infty$, so $(D_T, z_T, z_\infty)$ is a two-pointed domain with scale $\sigma_T$. Finally, note that $\sigma_T(K'_t) = g_T(\sigma(K_{T+t}) \setminus \sigma(K_T))$ for all $t \geq 0$ by definition. So, by definition, conditionally on $\mathcal{F}_T$, $(K'_t)_{t\geq 0}$ is an SLE($\kappa$) in $(D_T, z_t, z_\infty)$ of scale $\sigma_T$. □

## 3. Site Percolation

A technical and detailed result tells us that for all $\kappa \geq 0$ and if $(K_t)_{t\geq 0}$ is an SLE($\kappa$), there is a continuous path $\gamma : [0, \infty) \to \overline{\mathbb{H}}$, which we'll call the **trace** or **path** of the SLE, such that for all $t \geq 0$, $H_t = \mathbb{H} \setminus K_t$ is equal to the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$. Sometimes referred to as the **Rohde-Schramm Theorem**, this was proven in [14] for the case where $\kappa \neq 8$ and was proven in [9] for when $\kappa = 8$. Sometimes, for simplicity, this curve is also referred to as SLE when it is clear that we are not referring to the hulls.

It turns out that SLE(6) has a property known as locality (see e.g. [10]). This means that if $\gamma$ is an SLE curve in $\mathbb{H}$ from 0 that is stopped after hitting $\partial D \setminus \partial\mathbb{H}$ where $D$ is a simply connected domain in $\mathbb{H}$ such that the origin is in its boundary, then $\gamma$ has the same law as an SLE in $D$ stopped at the corresponding time. As it turns out, percolation under certain conditions also has an analogous locality property. The combination of percolation and SLE(6) having this property therefore suggests that a deeper relationship might be present between the two. This is, in fact, the case, as we will prove in subsequent sections.

Before we make this relationship between SLE(6) and percolation more rigorous, we will first introduce the basics of site percolation which will be relevant in later sections. The basic model of site percolation on a lattice is as follows. For some fixed $p \in [0, 1]$, each lattice site is declared open with probability $p$ and closed with probability $1 - p$, obeying the distribution of a Bernoulli variable, independently of all other sites. In figures, open sites are often pictured in black, and closed sites are often pictured in white. We may then consider the random sublattice with the same vertex set but only with open lattice sites.

Sometimes, it is beneficial to focus on a specific point, for instance, the lattice site containing the origin, and consider the open cluster that contains this point, if it exists. By an open cluster, we mean a connected subset of the random sublattice with only open lattice sets. Let

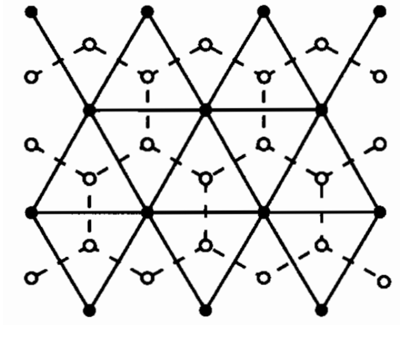
$$\theta(p) = \mathbb{P}_p[\text{ there is an open cluster of infinite cardinality containing the origin }]$$

where $\mathbb{P}_p$ denotes the probability measure where each site has probability $p$ of being open. Then we have $\theta(0) = 0$ and $\theta(1) = 1$. Intuitively, it makes sense that as $p$ increases, so does $\theta(p)$. Hence, a key question in percolation theory is at what value $p_c \in (0, 1)$ does $\theta(p)$ change from being 0 to being positive. More specifically, $p_c = \sup\{p : \theta(p) = 0\}$.

**Definition 3.1.** The **triangular lattice** $\mathbb{T}$ with **mesh** $\delta > 0$ is the set of points $\{\delta(m + ne^{i\pi/3}) : m, n \in \mathbb{Z}\}$.

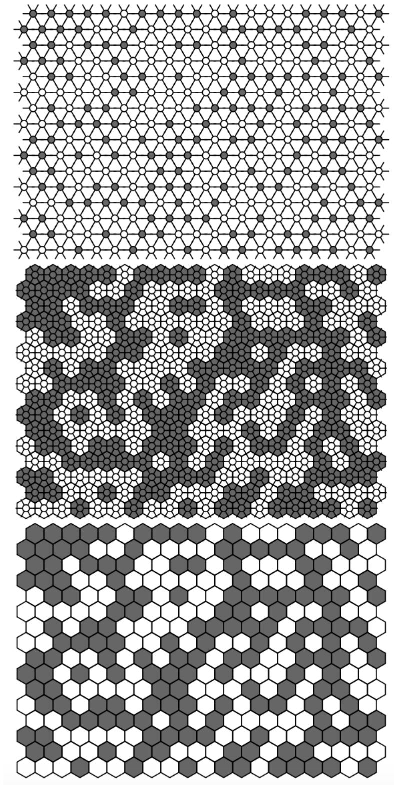
In site percolation, the sites are the triangle vertices, so we may also call the sites vertices. However, for visualization purposes, it may be easier to color cells rather than sites. We may do so by considering the honeycomb lattice, as percolation on $\mathbb{T}$ is equivalent to considering hexagons on a honeycomb lattice. The reason being is that we may view each site of $\mathbb{T}$ as the center of a hexagonal cell on this lattice and this preserves the connections on $\mathbb{T}$. This is pictured in Figures 1 and 2 on this page and in the following.

Figure 1. Overlaying the hexagonal lattice on the triangular lattice. Figure taken from [6].



The critical value for site percolation for $\mathbb{T}$ was proven to be $\frac{1}{2}$ in [8], regardless of our choice of $\delta$. We therefore will restrict ourselves to considering this value, $p = \frac{1}{2}$, in the remainder of this paper. The

FIGURE 2. Transitioning from the triangular lattice to the hexagonal lattice. Figure taken from [19].



value $p = \frac{1}{2}$ has a special property that comes from the definition of the Bernoulli random variable with the parameter $\frac{1}{2}$.

**Definition 3.2.** In percolation, each vertex is either black (state 1) or white (state 0). A **configuration** is an assignment of 0 or 1 to each site. The space of configurations is the product space (equipped with the product measure) $\{0,1\}^V$ where $V$ is the set of vertices.

If we were to define a percolation measure on configurations using the product measure, then by swapping which vertices are black and which vertices are white everywhere, the measure is unchanged due to the symmetry of the Bernoulli($\frac{1}{2}$) random variable. We will primarily be concerned with the following phenomenon.

**Definition 3.3.** A **monochrome crossing** of a domain is a path of sites all of the same color, black or white, from one specified boundary arc to another, and lying inside the domain.

We explicitly define a monochrome crossing due to dual events. For example, if our domain is a rectangle, if there is no black horizontal crossing, there must be a white vertical crossing.

**Definition 3.4.** If $\omega$ and $\omega'$ are two possible configurations, then $\omega \leq \omega'$ if $\omega'$ has at least as many black sites as $\omega$. An event $A$ is **increasing** if whenever $\omega \in A$ and $\omega \leq \omega'$, then $\omega' \in A$.

From this definition, we may see that the event that there is a monochrome crossing is an increasing event. This next inequality, the **FKG inequality**, gives us a lower bound on the probability of two increasing events occurring simultaneously.

**Proposition 3.5.** *If $A$ and $B$ are two increasing events,* $\mathbb{P}[A \cap B] \geq \mathbb{P}[A]\mathbb{P}[B]$.

*Proof.* We will prove this by showing that any two increasing functions on the product space have non-negative covariance under the product measure. In other words, we want to prove that if $f$ and $g$ are increasing, then $\mathbb{E}[fg] - \mathbb{E}[f]\mathbb{E}[g] \geq 0$. Once this is shown, applying this to the indicator functions of $A$ and $B$ give us the desired inequality.

We begin by proving the case where $f$ and $g$ depend only on finitely many sites. We will prove this by induction. In the case where $f$ and $g$ depend only on one site, we get

$$\mathbb{E}[fg] - \mathbb{E}[f]\mathbb{E}[g] = \frac{f(0)g(0) + f(1)g(1) - (f(0)+f(1))(g(0)+g(1))}{4}.$$

Since both $f$ and $g$ are increasing, $f(1)-f(0)$ and $g(0)-g(1)$ are both nonnegative, meaning that $(f(1)-f(0))(g(0)-g(1)) \geq 0$. Expanding our previous expression implies the conclusion in this case. Now, assume that the claim is true when $f$ and $g$ depend on $n-1$ sites. Then, the law of total expectation implies that

$$\mathbb{E}[fg] = \mathbb{E}[\mathbb{E}[fg|\text{1st } (n-1) \text{ sites}]] \geq \mathbb{E}[\mathbb{E}[f|\text{1st } (n-1) \text{ sites}]\mathbb{E}[g|\text{1st } (n-1) \text{ sites}]]$$

because $f$ and $g$ are increasing in the single variable depending on the $n$th site. Now, since we have written increasing functions of the first $(n-1)$ sites, the induction hypothesis and the law of total expectation imply that $\mathbb{E}[fg] \geq \mathbb{E}[f]\mathbb{E}[g]$.

We will now consider the case where $f$ and $g$ depend on infinitely many states in the lattice. We may order the sites and define $f_n = \mathbb{E}[f|\text{1st } n \text{ sites}]$ with an analogous definition for $g_n$ where $n \in \mathbb{N}$. As these functions increase with respect to $n$ variables, it follows that $\mathbb{E}[f_n g_n] \geq \mathbb{E}[f]\mathbb{E}[g]$. Also, by the martingale convergence theorem, $f_n \to f$ almost surely in $\mathcal{L}^2$, so $\mathbb{E}[f_n] \to \mathbb{E}[f]$. The same statements hold for $g$ and our $g_n$ functions. By the triangle inequality, for any $n \in \mathbb{N}$, $|f_n g_n - fg| \leq |f_n - f||g_n| + |f||g_n - g|$. By the Cauchy-Schwarz inequality and $\mathcal{L}^2$ convergence, these errors go to 0. This implies $\mathbb{E}[f_n g_n] \to \mathbb{E}[fg]$. So, taking $n \to \infty$ gives us the desired conclusion. □

We now proceed with the **Russo-Seymour-Welsh estimates**. A fundamental tool in percolation theory, these give us bounds away from 0 and 1 in which we can observe monochrome crossings, independently of our chosen mesh size.

**Lemma 3.6.** *For $a \in \mathbb{N}$ and $b > 0$, define $R(a,b) = \{x+iy \in \mathbb{T} : 0 \leq x \leq a, 0 \leq y \leq b\}$ and $H(a,b)$ to be the event there is a horizontal black crossing of $R(a,b)$. Then, $\mathbb{P}[H[2a,b]] \geq \frac{\mathbb{P}[H[a,b]]^2}{4}$.*

*Proof.* Let $g$ be a deterministic horizontal crossing of black sites and let $g'$ be the reflection of $g$ across the vertical line $I_a = \{x+iy \in \mathbb{C} : x = a\}$. Reflecting will help us to construct a symmetric set-up. We will let $O$ be the connected component of $R(2a,b) \setminus (g \cup g')$ that contains the center point $a$. If $a \in g$ and therefore $O = \emptyset$, then a similar argument as the one we present below holds. Note that the region $O$ lies between the paths $g$ and $g'$ and is symmetric with respect to $I_a$.

If $g$ does not touch $\mathbb{R}$, then the boundary of $O$ is $g$ (the original horizontal crossing), $g'$ (its symmetric image), the portion of the boundary of $R(a,b)$ that is left and bottom, which we will call $J$, and the symmetric image of $J$ lying on the right and top, which we will call $J'$. Coloring $g$ and $J'$ in black, $g'$ and $J$ in white and running critical percolation in $O$, let $A(g)$ be the event that there exists a black crossing that joins $g$ to $J'$ in $O$. Due to the symmetry across $I_a$ and the fact that critical percolation is symmetric with respect to color, we may see that $\mathbb{P}[A(g)] = \frac{1}{2}$. Now, let $\gamma$ be the highest horizontal crossing of $R(a,b)$. Note that the event $\{\gamma = g\}$ depends only on the state of the percolation sites above $g$ and is independent of the percolation below $g$. So, $\{\gamma = g\}$ and $A(g)$ are independent. If both events hold, then there is a black path in $R(2a,b)$ joining the the left boundary of the rectangle to the union of the right boundary with the right half of its bottom boundary. We will call this event $A'$. In the case where $g$ intersects $\mathbb{R}$, then $J'$ is just a part of the lower right boundary, but we may still show that $\mathbb{P}[A(g)] = \frac{1}{2}$ and that $\mathbb{P}[A'|\gamma = g] = \frac{1}{2}$.

Next, note that the event $A'$ is contained in the event $H(a,b) = \bigcup_g \{\gamma = g\}$ so it follows that

$$\mathbb{P}[A'] = \sum_g \mathbb{P}[A' \cap \{\gamma = g\}] \geq \sum_g \frac{\mathbb{P}[A(g)]}{2} = \frac{\mathbb{P}[H(a,b)]}{2}.$$

Let $A''$ be the event that there exists a black crossing of $R(2a,b)$ from the right boundary to the union of the left half of the bottom boundary and the left boundary. We note that by symmetry, $\mathbb{P}[A''] = \mathbb{P}[A']$. Both are increasing events and we note that $A'' \cap A'$ gives us the event $H(2a,b)$, so the FKG inequality implies that $\mathbb{P}[H(2a,b)] \geq \mathbb{P}[A' \cap A''] \geq \frac{\mathbb{P}[H(a,b)]^2}{4}$. □

As a corollary, we obtain the following.

**Corollary 3.7.** *For any $k \in \mathbb{N}$, there is some constant $a_k > 0$ such that $\mathbb{P}[H(kn,n)] \geq a_k$ for $n \in \mathbb{N}$ with $n > 2$, where we use the notation from Lemma 3.6.*

*Proof.* We will switch to our interpretation of percolation using hexagons. We consider a rhombus-shaped domain made out of hexagons as in Figure 4 on the next page.

In this proof, we will use the coordinate system defined with the vectors 1 and $e^{\frac{i\pi}{3}}$. So, the point $(u,v)$ denotes the point $u + ve^{\frac{i\pi}{3}}$ and $[a,b] \times [a',b']$ denotes a parallelogram with sides of length $b-a$ and $b'-a'$ that are parallel in the direction $e^{\frac{i\pi}{3}}$. Since we are considering a rhombus-shaped domain, we are considering domains of the form $[a,b] \times [a',b']$ such that $b-a = b'-a'$.

FIGURE 3. A rough sketch of the proof of Lemma 3.6. Figure taken from [22].

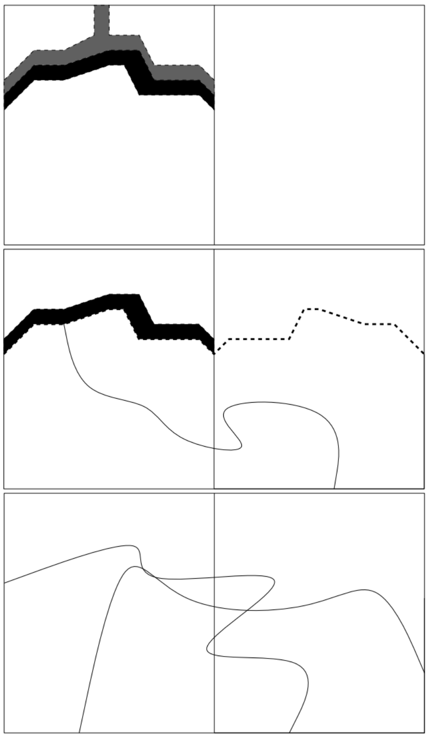
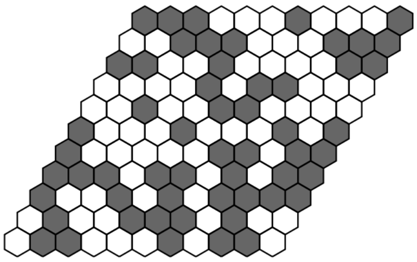


FIGURE 4. Dual events on a rhombus domain. Figure taken from [22].



We can see that a black horizontal crossing occurs if and only if there is no white vertical crossing. Therefore, one and only one of these events occurs. However, since we are considering critical percolation (meaning $p = \frac{1}{2}$), because of symmetry, we may see that these events have the same probability. Therefore, the probability of each of these events is equal to $\frac{1}{2}$. Let $n \in \mathbb{N}$. If we consider a rhombus $[-n, n] \times [0, 2n]$, we note that if there is a horizontal black crossing of it, then it also contains a horizontal black crossing of $R(n, n\sqrt{3})$. Thus, $\mathbb{P}[H(n, n\sqrt{3})]$ is bounded below by $\frac{1}{2}$. By induction and by Lemma 3.6, we may see that for any $k > 0$, there is some constant $a_k > 0$ such that $\mathbb{P}[H(kn, n)] \geq a_k$ regardless of our choice of $n > 2$. ☐

We will now continue to use our hexagonal interpretation of $\mathbb{T}$ to consider some basic consequences of the RSW estimates. In particular, we will consider nested hexagons

$$\Lambda_n = \{ue^{\frac{ik\pi}{3}} + ve^{\frac{i(k+1)\pi}{3}} : u, v \geq 0, u + v \leq n, k \in \{0, 1, 2, 3, 4, 5\}\}.$$

Using these, we may create concentric disjoint sets $A_j = \lambda_{2^{j+1}} \setminus \lambda_{2^j}$ that form a discrete analog to nested annuli with the same modulus.

**Lemma 3.8.** *Let $C_j$ denote the event that there is a white loop in $A_j$ disconnecting the origin from infinity. Then, for $j \geq 2$, then $\mathbb{P}[C_j] \geq c$ for some non-negative constant $c$.*

*Proof.* We cover the hexagonal ring with six rotated rectangles $R_1, ..., R_6$ in a hexagonal loop such that each one lies within $A_j$, has a fixed aspect ratio, and such that if each is crossed in the long direction by a white path, a closed loop is formed encircling the origin. For all $i \in \{1,2,3,4,5,6\}$, let $E_i$ be the event of a white crossing in the long direction. Each one is an increasing event. Furthermore, we have that $A_1 \cap ... \cap A_6$ is contained in the event $C_j$. By Lemmas 3.5 and 3.7, is follows that for some $p < 1$, $\mathbb{P}[C_j] \geq \mathbb{P}[A_1 \cap ... \cap A_6] \geq p^6$. □

**Corollary 3.9.** *Let $j, \ell \geq 2$. Then, the probability of a black path between $\partial\Lambda_{2^j}$ and $\partial\Lambda_{2^{j+l}}$ is at most $(1-c)^\ell$ for some positive constant $c$. In particular, the probability of a monochrome crossing separating the inner and outer boundaries of a fixed shape annulus is bounded above by a constant $q$ only depending on the shape of the annulus, uniformly in $\delta$.*

*Proof.* The full annulus between these two has $\ell$ nested disjoint annuli, $A_j, A_{j+1}, ..., A_{j+\ell-1}$. If a black path exists, then the path must avoid a white crossing in each of these annuli. So, the event we're considering is contained in the event $\bigcap_{k=j}^{j+\ell-1} C_k^c$, where we use the notation from Lemma 3.8. By disjointness of the annuli, these events are independent. So, using the constant $c$ from Lemma 3.8, we get the conclusion. □

In fact, if we repeat the same arguments from Lemma 3.6, Corollary 3.7, Lemma 3.8, and Corollary 3.9 and duality arguments, we get both upper and lower bounds for these domains. We omit the proof here because of the similarity to these previous arguments.

**Lemma 3.10.** *Let $\Omega$ be a planar domain, either a rectangle of a fixed aspect ratio or an annulus of a fixed modulus. Then, there are constants $0 < p < q < 1$ depending only on the shape of $\Omega$, such that for critical site percolation on the triangular lattice with mesh $\delta$, the probability of a monochrome crossing between specified opposite sides in $\Omega$ is between $p$ and $q$, regardless of $\delta$. In the case of a rectangle, opposite sides take on the standard definition. In the case of annuli, the crossing event is the existence of a monochrome path separating the inner and outer boundaries.*

## 4. Cardy's Formula

In this section, we will consider Cardy's crossing probability prediction for critical percolation from conformal field theory, or the probability of finding an unbroken path of open edges connecting one part of the boundary to another. The prediction is as follows. Let $D$ be a bounded simply connected domain containing the origin such that $\partial D$ is a continuous curve. Let $\phi : \mathbb{D} \to D$ be the conformal map from the unit disk to $D$ such that $\phi(0) = 0$ and $\phi'(0) > 0$. Let $z_1, z_2, z_3, z_4 \in \partial D$ be in counterclockwise order so that $z_i = \phi(w_i)$ and $w_1, w_2, w_3, w_4$ are in counterclockwise order. Then, the crossing probability in $D$ from arc $z_1z_2$ to arc $z_3z_4$ is

$$\frac{\Gamma(2/3)}{\Gamma(4/3)\Gamma(1/3)}\nu^{1/3}F_{2,1}(1/3, 2/3, 4/3, \nu)$$

where $\nu = \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_3)(w_2-w_4)}$, $\Gamma$ is the gamma function and $F_{2,1}$ is the hypergeometric function. See [5] for more details.

We will now prove Cardy's formula in the case of percolation on $\mathbb{T}$. This is one of the key steps to proving the correspondence between critical site percolation on $\mathbb{T}$ and SLE(6) as previously discussed. We will follow the arguments of Stanislav Smirnov in [16]. Much of the proof is given by using discrete versions of complex analysis ideas adapted to $\mathbb{T}$, and then translating them into the continuum by taking the limit of the mesh size to 0.

**Notations 4.1.** We will fix $\tau = \exp(\frac{2\pi i}{3})$ (rotation in $\mathbb{C}$ by 120 degrees). We will also fix $\Omega$ to be a simply connected domain with a smooth boundary, and $a(1), a(\tau), a(\tau^2)$ be three unique boundary points of $\Omega$ in counterclockwise order. We will denote arcs between points via concatenation. For example, the (counterclockwise) arc between $a(1)$ and $a(\tau)$ will be denoted as $a(1)a(\tau)$. We will let $\nu$ be the counterclockwise-pointing unit tangent to $\partial\Omega$.

We then have that there are harmonic functions $h_\alpha(z)$ for $\alpha \in \{1, \tau, \tau^2\}$ that are the unique solutions of a mixed **Dirichlet-Neumann problem** along the boundary:

$$\begin{cases} h_\alpha(a(\alpha)) = 1, h_\alpha = 0 \text{ on arc } a(\tau\alpha)a(\tau^2\alpha) \\ \frac{\partial h_\alpha}{\partial(\tau\nu)} = 0 \text{ on arc } a(\alpha)a(\tau\alpha) \\ \frac{\partial h_\alpha}{\partial(-\tau^2\nu)} = 0 \text{ on arc } a(\tau^2\alpha)a(\alpha). \end{cases}$$

By mapping to an equilateral triangle, we may see that the solution of this Dirichlet-Neumann problem for a half-plane is a hypergeometric function. Therefore, we may begin to see the connection to Cardy's formula.

**Definition 4.2.** A **black simple path** is a sequence $\{b_i\}_{i\in\mathbb{N}}$ of black sites such that for all $i \in \mathbb{N}$, $b_i$ is adjacent to $b_{i+1}$.

**Notations 4.3.** Fix some mesh size $\delta > 0$ and let $\alpha \in \{1, \tau, \tau^2\}$. For fixed $z \in \Omega$, we define the event $Q^\delta_\alpha(z)$ as the event that there is a black simple path from arc $a(\alpha)a(\tau\alpha)$ to arc $a(\tau^2\alpha)a(\alpha)$ which separates $z$ from arc $a(\tau\alpha)a(\tau^2\alpha)$. Let $H^\delta_\alpha$ be the function denoting this probability, meaning $H_\alpha(z)^\delta = \mathbb{P}[Q^\delta_\alpha(z)]$. Moreover, if $z$ is the center of a triangle and $z+\eta$ is the center of an adjacent triangle, then we will call $P_\beta(z,\eta)^\delta$ the probability of event $Q^\delta_\beta(z+\eta) \setminus Q^\delta_\beta(z)$. In the event it is clear which mesh size we are taking, we may omit the use of $\delta$ (for example using $H_\alpha$ instead of $H^\delta_\alpha$).

We have now arrived at our first key lemma, known as the $\frac{2\pi}{3}$ **Cauchy-Riemann equations**. The standard Cauchy-Riemann equations describe a sort of invariance under rotations of $\mathbb{C}$ by $\frac{\pi}{2}$, and here we form a similar property for the triangular lattice, but with rotation $\frac{2\pi}{3}$.

**Lemma 4.4.** *Let $z$ be a center of a triangle and $\eta$ be a vector from $z$ to an adjacent triangle. Then, if $\beta \in \{1, \tau, \tau^2\}$, $P_\beta(z, \eta) = P_{\tau\beta}(z, \tau\nu)$, regardless of our mesh size $\delta > 0$.*

This means that by rotating the direction $\eta$ and $\beta$ by 120 degrees, the probability is invariant. So, if we rotate the entire percolation configuration by 120 degrees, the law of a path is unchanged up to rotation. We will later use this to consider increments in rotated directions similarly to how we would view a holomorphic function's derivatives.

*Proof.* We will analyze the event $Q' = Q_\beta(z+\eta) \setminus Q_\beta(z)$. Consider the triangle centered at $z$ with vertices $X$, $Y$, and $Z$, such that $X$ is opposite to the direction $z+\eta$ and the others are labeled in counterclockwise order. For the exploration path to satisfy the $\beta$ turn condition at $z+\eta$ but not at $z$, there must exist a black separating path $\gamma$ between $z$ and $z+\eta$. Let $\gamma$ be the simple black path from the arc $a(\beta)a(\tau\beta)$ to $a(\tau^2\beta)a(\beta)$ closest to the arc $a(\tau\beta)a(\tau^2\beta)$, ensuring the canonical separation of $z$ and $z+\eta$. Therefore, there exist two disjoint black paths from $Y$ and $Z$ to arcs $a(\tau^2\beta)a(\beta)$ and $a(\beta)a(\tau\beta)$, respectively, and $X$ is a white vertex connected by a simple white path to arc $a(\tau\beta)a(\tau^2\beta)$.

We now proceed with a color-switching argument. We fix $\Omega'$ by choosing, for each possible configuration, the counterclockwise most white path from $X$ to arc $a(\tau\beta)a(\tau^2\beta)$ and the clockwise most black path from $Y$ to arc $a(\tau^2\beta)a(\beta)$. Our choice uniquely determines $\Omega'$ as the union of these paths and the region of $\Omega$ between them, which contains $a(\tau^2\beta)$. Now, the existence of a black path from $Z$ to arc $a(\beta)a(\tau\beta)$ only depends on the coloring of $\Omega \setminus \Omega'$. At criticality $p = \frac{1}{2}$, the color inversion in $\Omega \setminus \Omega'$ is measure-preserving and swaps black and white paths. This means that the probability of a black path from $Z$ to its arc is equal to that of a white path with the same endpoints. Conditioning on the choice of $\Omega'$ and averaging over all possible configurations, we find that the probability of $Q'$ is the probability that there are three disjoint simple paths from $X$, $Y$, and $Z$ to arcs $a(\tau\beta)a(\tau^2\beta)$, $a(\beta)a(\tau^2\beta)$, and $a(\beta)a(\tau\beta)$, with colors white, black, and white, respectively. This follows since the color inversion in $\Omega \setminus \Omega'$ at $p = \frac{1}{2}$ preserves the measure and swaps the black and white paths. A global color inversion throughout $\Omega$ maps this event to $Q_{\tau\beta}(z+\tau\eta) \setminus Q_{\tau\beta}(z)$, which completes the proof. ☐

We will now proceed with beginning to bound crossing probabilities by using the Russo-Seymour-Welsh estimates.

**Lemma 4.5.** *For $\beta \in \{1, \tau, \tau^2\}$, there is an exponent $\epsilon > 0$ and constant $C$, both depending only on the domain $\Omega$, such that $H_\beta$ is uniformly $\epsilon$ Hölder continuous with norm at most $C$, regardless of the mesh size. Moreover, $H_\beta$ has boundary values of $0$ on arc $a(\tau\beta)a(\tau^2\beta)$ and as $\delta \to 0$, $H_\beta$ tends to $1$ at $a(\beta)$.*

This lemma gives uniform regularity control over $\{H_\beta\}$ as $\delta \to 0$. It will turn out that this is essential to prove convergence of the discrete observable to its scaling limit. Because this lemma proves that the Hölder norms of the functions $H^\delta_\alpha$ for $\alpha \in \{1, \tau, \tau^2\}$ are uniformly bounded, any sequence of these functions with $\delta$ going to 0 has a uniformly converging subsequence. Later, we will prove that the limit functions are exactly the functions satisfying the mixed Dirichlet-Neumann problem previously discussed.

*Proof.* As a probability, $H_\beta$ is between 0 and 1 everywhere. In order to prove it is Hölder continuous, we will show that for all $z, z'$ in our domain but away from singular boundary points on the marked boundary arcs, the Hölder condition is satisfied. By definition, we note that

$$H_\beta(z) - H_\beta(z') = \mathbb{P}[Q_\beta(z) \setminus Q_\beta(z')] - \mathbb{P}[Q_\beta(z') \setminus Q_\beta(z)]$$

because each set difference can be analyzed using percolation events separating $z$ and $z'$. For either event to happen, the exploration path should turn at $z$ but not at $z'$ or vice versa, which only happens if there is a separating cluster distinguishing $z$ and $z'$. In particular, there must be a monochrome path connecting the segment $[z, z']$ to the boundary arcs involved in defining $Q_\beta$ by similar logic to our proof of Lemma 4.4. So, the probability difference is controlled by the probability of having such a monochrome cluster which connects $[z, z']$ to the boundary arcs. Between $[z, z']$ and arc $a(\alpha)a(\tau\alpha)$, we may find approximately $|\log|z - z'|| - c$ disjoint annuli, where $c$ is a constant. This is because in two dimensions, the number of annuli scales with the logarithm of the distance ratio. For each annulus, and independent of how small each annulus is, the probability that a monochrome path crosses it is at most some $q < 1$ by Lemma 3.10, so the chance of separating $z$ from $z'$ shrinks exponentially in the number of annuli. Since each annulus crossing is independent of the others and has probability of at most $q$, the total separation probability is at most approximately $q^{-c}|z - z'|^{\log q}$. Because $q < 1$, $\log q$ is negative, implying that $|H_\beta(z) - H_\beta(z')| \leq 2q^{-c}|z - z'|^{-\log q}$.

We now proceed to check the boundary conditions for $H_\beta$. In arc $a(\tau\beta)a(\tau^2\beta)$, by construction, paths cannot separate that arc from itself with black paths. So, $H_\beta$ is zero here. We now consider the point $a(\beta)$. Around this point, there are $|\log\delta| - c$ disjoint discrete annuli of fixed shape. By Lemma 3.10, each has a chance of at least $p > 0$ of having a black circuit that separates $a(\beta)$ from other arcs. The probability of $a(\beta)$ being separated from the arc $a(\tau\beta)a(\tau^2\beta)$ is therefore bounded below by $1 - (1 - p)^{|\log\delta| - c}$, which tends to 1 as $\delta \to 0$. □

We will now proceed by considering the discrete analogue of contour integrals.

**Definition 4.6.** Let $\Gamma$ be an equilateral triangular contour with vertices in the centers of lattice triangles with mesh $\delta > 0$ and the bottom side parallel to $\mathbb{R}$. Let $\Gamma$ have vertices $x(1), x(\tau)$, and $x(\tau^2)$, in counterclockwise order, such that $x(1)$ is on top. Then, the discrete contour integral of a function $H(z)$ is

$$\oint_\Gamma^\delta H(z)dz = \delta \sum_{z \in x(\tau)x(\tau^2)} H(z) + \delta\tau \sum_{z \in x(\tau^2)x(1)} H(z) + \delta\tau^2 \sum_{z \in x(1)x(\tau)} H(z).$$

The sums are taken over the centers of the lattice triangles in the corresponding intervals.

**Lemma 4.7.** *Let $\Gamma$ be an equilateral triangle contour, derived from the triangular lattice of mesh $\delta > 0$, with sides of length $l$ such that its vertices are in the center of lattice triangles and its bottom side is parallel to $\mathbb{R}$. Then, if $\beta \in \{1, \tau, \tau^2\}$, then*

$$\oint_\Gamma^\delta H_\beta^\delta(z)dz = \oint_\Gamma^\delta \frac{1}{\tau} H_{\tau\beta}^\delta(z)dz + O(l\delta^\epsilon).$$

This lemma is the discrete analytic condition that allows us to show that the scaling limit is conformally invariant. In the continuum, the contour integral of a holomorphic function is 0. Here, we prove that on small discrete triangular contours, the integral behaves almost as if the function is holomorphic, eventually with a small error. This discrete vanishing of contour integrals allows us to prove convergence to a holomorphic limit as $\delta \to 0$.

*Proof.* We define a bipartition of the triangular grid's triangular centers, coloring faces in a checkerboard fashion. This is so that every edge of the dual lattice connects a black face center to a white face center. Let $\mathcal{B}$ be the set of centers of black triangles inside or on $\Gamma$ and let $\mathcal{W}$ be the set of centers of white triangles strictly inside $\Gamma$. Now, fix $\alpha \in \{1, \tau, \tau^2\}$, and let $\eta$ be a step in one lattice direction scaled by $\delta$ (length $\frac{\delta}{\sqrt{3}}$) collinear with $e^{\frac{\pi i}{6}}(x(\tau^2\alpha) - x(\tau\alpha))$. Let $\eta' = e^{\frac{\pi i}{3}}\eta$, the rotated version of $\eta$ by 60 degrees, corresponding to rotation in the lattice. These vectors allow us to connect shifts in $H_\beta$ to shifts in $H_{\tau\beta}$. Next, we note by definition that the discrete derivative of $H_\beta$ is a difference in probabilities, with

$$\frac{\partial H_\beta(z)}{\partial\theta} = H_\beta(z + \theta) - H_\beta(z) = P_\beta(z, \theta) - P_\beta(z + \theta, -\theta).$$

By this and Lemma 4.4, we may then sum discrete differences of $H_\beta$ over black triangle centers in the region to get

$$\begin{aligned}
\sum_{z \in \mathcal{B} \setminus x(\alpha)x(\tau^2\alpha)} (H_\beta(z + \eta) - H_\beta(z)) &= \sum_{z \in \mathcal{B} \setminus x(\alpha)x(\tau^2\alpha)} (P_\beta(z, \eta) - P_\beta(z + \eta, -\eta)) \\
&= \sum_{z \in \mathcal{B} \setminus x(\alpha)x(\tau^2\alpha)} (P_{\tau\beta}(z, \tau\eta) - P_{\tau\beta}(z + \eta, -\tau\eta)) \\
&= \sum_{z \in \mathcal{B} \setminus x(\alpha)x(\tau\alpha)} (P_{\tau\beta}(z, \tau\eta) - P_{\tau\beta}(z + \eta, -\tau\eta)) + O(l\delta^{\epsilon - 1}) \\
&= \sum_{z \in \mathcal{B} \setminus x(\alpha)x(\tau\alpha)} (H_{\tau\beta}(z + \tau\eta) - H_{\tau\beta}(z)) + O(l\delta^{\epsilon - 1}).
\end{aligned}$$

The error term comes from the number of vertices in the sum (proportional to $\frac{l}{\delta}$) and the local error per term. The latter comes from applying Lemma 4.5 in the case $z' = z + \eta$ to get $P_\beta(z, \eta) \leq C\delta^\epsilon$. So, we get that the sums of discrete differences of $H_\beta$ over black faces turn into sums of rotated differences of $H_{\tau\beta}$ up to small error. By similar logic (shifting by $\eta'$ and the same discrete Cauchy-Riemann relation), we get that

$$\sum_{z\in\mathcal{W}} (H_\beta(z+\eta') - H_\beta(z)) = \sum_{z\in\mathcal{W}} (H_{\tau\beta}(z+\tau\eta') - H_{\tau\beta}(z)) + O(l\delta^{\epsilon-1}).$$

We now proceed by adding the black and white sums, which sums over all internal faces in the region inside $\Gamma$. The sums of differences over the interior cancel except for the boundary terms via telescoping sums:

$$\begin{aligned}
&\sum_{z\in x(\alpha)x(\tau^2\alpha)} H_\beta(z) - \sum_{z\in x(\tau\alpha)x(\tau^2\alpha)} H_\beta(z) \\
&= \sum_{z\in\mathcal{B}\setminus x(\alpha)x(\tau^2\alpha)} (H_\beta(z+\eta) - H_\beta(z)) + \sum_{z\in\mathcal{W}} (H_\beta(z+\tau\eta') - H_\beta(z)) \\
&= \sum_{z\in\mathcal{B}\setminus x(\alpha)x(\tau\alpha)} (H_{\tau\beta}(z+\tau\eta) - H_{\tau\beta}(z)) + \sum_{z\in\mathcal{W}} (H_{\tau\beta}(z+\tau\eta') - H_{\tau\beta}(z)) + O(l\delta^{\epsilon-1}) \\
&= \sum_{z\in x(\alpha)x(\tau\alpha)} H_{\tau\beta}(z) - \sum_{z\in x(\alpha)x(\tau^2\alpha)} H_{\tau\beta}(z) + O(l\delta^{\epsilon-1}).
\end{aligned}$$

This is the discrete analog of parts of our desired contour integral expression. This gives us three versions of this telescoping identity, one for each rotation $\alpha \in \{1, \tau, \tau^2\}$, corresponding to three directions in the triangular lattice. We then may sum each of these (one for each $\alpha$) with complex coefficients to match the integral directions of the triangular contour. When $\alpha = 1$ we choose the coefficient $-\frac{\delta}{2}$, when $\alpha = \tau$ we choose the coefficient $-i\frac{\delta\sqrt{3}}{2}$, and when $\alpha = \tau^2$ we choose the coefficient $\frac{\delta}{2}$. The rotational symmetry ensures that the sum of the boundary sums is approximately $\oint_\Gamma H^\delta_\beta(z)dz$, and after rotation by $\tau$ is approximately $\oint_\Gamma \frac{1}{\tau} H^\delta_{\tau\beta}(z)dz$. The error stays $O(l\delta^\epsilon)$, implying the lemma. □

We now recall that there are subsequences of the functions $H^\delta_\alpha$ for $\alpha \in \{1, \tau, \tau^2\}$ that are guaranteed to uniformly converge. Using the previous lemma, we may now proceed with proving that the limits of subsequences of $H^\delta_\alpha$ functions are equal to the functions satisfying the Dirichlet-Neumann problem.

**Lemma 4.8.** *Let $\{\delta_j\}_{j\in\mathbb{N}}$ be a sequence of mesh sizes going to 0 such that the discrete observables $H^{\delta_j}_\alpha$ converge uniformly in $\Omega$ to some functions $f_\alpha$. Then, these limits must be exactly the solution $h_\alpha$ of the target boundary value problem.*

So, this lemma both identifies the limit in the context of our Dirichlet-Neumann problem and proves the uniqueness of the scaling limit.

*Proof.* As $\delta_j$ goes to 0, $H^\delta_\beta \to f_\beta$ uniformly in $\Omega$, so the discrete sum along a contour approximates the integral. Therefore, $\oint^\delta H^\delta_\beta \to \oint f_\beta$. The discrete contours are on the lattice. As $\delta \to 0$ (potentially after slightly adjusting the contour to match the lattice grid which would vanish in the limit and therefore not affect the integral limit), the approximation from Lemma 4.7 becomes exact. So,

$$\oint_\Gamma f_\beta = \oint_\Gamma \frac{1}{\tau} f_{\tau\beta}$$

. Any subsequential limit must satisfy this integral identity. Next, we aim to eliminate variables using linear combinations. In the previous integral identity, if we take $\beta = \alpha$ and subtract $(\frac{1}{2} + \frac{i}{2\sqrt{3}})$ copies of $\beta = \tau\alpha$ (from the same identity) we get that

$$\oint_\Gamma (f_\alpha(z) + \frac{i}{\sqrt{3}}(f_{\tau\alpha}(z) - f_{\tau^2\alpha}(z)))dz = 0.$$

The choice of coefficients is designed to rotate the variables and match the triangular lattice symmetry. By Morera's theorem, $f_\alpha + \frac{i}{\sqrt{3}}(f_{\tau\alpha} - f_{\tau^2\alpha})$ is analytic in $\Omega$. Since the $f$ functions are real-valued (coming from probability functions), analyticity implies that they are also harmonic functions satisfying the Cauchy-Riemann type equations. So, for any $\alpha$, $f_\alpha$ is harmonic with harmonic conjugate $\frac{1}{\sqrt{3}}(f_{\tau\alpha} - f_{\tau^2\alpha})$. Note that for any unit vector $\eta$,

$$\frac{\partial f_\alpha}{\partial \eta} = \frac{\partial f_{\tau\alpha}}{\partial(\tau\eta)}.$$

This is rotational covariance of the derivatives and is the continuum analog of the discrete Cauchy-Riemann equations. On the arc $a(\alpha)a(\tau\alpha)$, the normal vector $\nu$ points outward, so this implies the derivative in direction $\tau\nu$ vanishes on this arc. Similarly, on the arc $a(\tau^2\alpha)a(\alpha)$, the direction changes and we get that the derivative in direction $-\tau^2\nu$ vanishes. Combined, these give us the mixed Dirichlet-Neumann boundary conditions satisfied by the limit. Moreover, Lemma 4.5 showed that the discrete observables converge with these Dirichlet conditions on the arcs: on arc $a(\tau\alpha)a(\tau^2\alpha)$, $f_\alpha = 0$, and at $a(\alpha), f_\alpha = 1$. So, the $f$s satisfy the mixed Dirichlet-Neumann problem which has one and only one solution, $f_\alpha = h_\alpha$. □

Therefore, we may first apply Lemma 4.5 to get that there is a subsequence of the $\{H_\beta^\delta\}_{\delta>0}$ sequence as $\delta \to 0$ that uniformly converges. Applying Lemma 4.8, we have a characterization of these limit functions. This characterization is sufficient to prove the following theorem.

**Theorem 4.9.** *As $\delta \to 0$, the functions $H_\alpha^\delta$ converge uniformly in $\Omega$ to the functions $h_\alpha$.*

As a corollary, we get Carleson's version of Cardy's formula, an equivalent reformulation. This comes from solving for the $h_\alpha$ functions directly when the domain is an equilateral triangle.

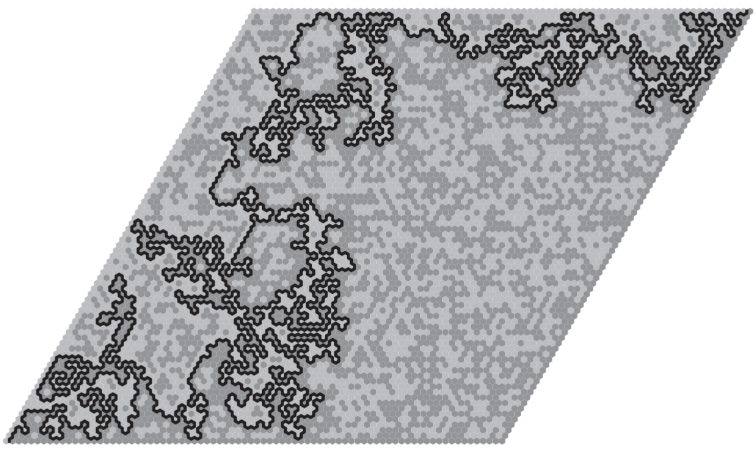
**Theorem 4.10.** *Let $D$ be conformally equivalent to equilateral triangle $ABC$. Let $a, b, c \in \partial D$ that are ordered counter-clockwise so that under a conformal map $\Phi$, $\Phi(a) = A, \Phi(b) = B$, and $\Phi(c) = C$. Let $x$ be an arc $ca$ so that $X = \Phi(x)$ lies on the corresponding image arc $[CA]$. Then, in the scaling limit, the probability there's a crossing in $D$ from arc $ab$ to arc $cx$ (both in the boundary of $D$) tends to $\frac{\mathrm{diam}(CX)}{\mathrm{diam}(CA)}$.*

## 5. Convergence to SLE(6)

Cardy's formula is one of the key steps in proving that percolation explorations of $\mathbb{T}$ converges as $\delta$ goes to 0 to chordal SLE(6). In this section, we will prove this theorem following the outline of [17] and [22]. We note that this approach is different from that of [4], the paper which originally proved this convergence.

Throughout this section, we will fix a bounded simply connected domain $D$ such that $\partial D$ is a continuous curve and with $x, c \in \partial D$ being two distinct boundary points. We will call arc $xc$ the arc between these two points in counterclockwise order and arc $cx$ the arc between these points in clockwise order. For all mesh sizes $\delta$, we will choose an approximation $D_\delta$, $x_\delta$, and $c_\delta$ on the corresponding triangular lattice such that $x_\delta \to x$, $c_\delta \to c$, arc $x_\delta c_\delta \to$ arc $xc$ and arc $c_\delta x_\delta \to$ arc $cx$ as $\delta \to 0$. The convergence of the arcs we view with respect to the metric on the space of curves, which we will define later. We will also assume that our approximations lie within $D$. Fix some $\delta > 0$ and perform critical percolation on $D_\delta$ such that the sites close to arc $c_\delta x_\delta$ are colored black and the sites close to arc $x_\delta c_\delta$ are colored white. We will consider the **exploration process path** $\gamma^\delta$ starting from $x_\delta$ and ending at $c_\delta$ which turns so that on the one side are black sites and on the other side are white sites. The resulting curve is uniquely determined from the exact percolation configuration. We may parameterize the resulting curve $\gamma^\delta$ from $x_\delta$ to $c_\delta$ as a function of $[0, 1]$ to $\mathbb{C}$ with $\gamma^\delta(0) = x_\delta$ and $\gamma^\delta(1) = c_\delta$.

Figure 5. An exploration process on a rhombus. Figure taken from [22].



In proving convergence to SLE(6), we first need to answer the question of why some subsequential limit should exist to begin with, which lies in the notion of **tightness**, which we will later define. The next step is to show that we may parameterize the curves in this subsequence and the limit according to the half-plane capacity, as we previously discussed in relation to SLE. We will then identify a martingale associated with the crossing probabilities using Cardy's formula and use this to conclude that the associated driving function of the limiting curve is exactly that of an SLE(6). We will now begin by proving tightness, but we first need to introduce relevant definitions and notation.

**Definition 5.1.** Let $\mathcal{S}$ be a metric space. A family $\{\gamma^\delta\}_{\delta>0}$ is **tight** in $\mathcal{S}$ if for all $\epsilon > 0$, there exists a compact set $K$ in $\mathcal{S}$ such that $1 - \epsilon < \inf_\delta \mathbb{P}[\gamma^\delta \in K]$.

By Prokhorov's Lemma (see e.g [3]), if a family is tight, then it has a subsequence that converges in distribution. Hence, by proving tightness, we will be proving the existence of a subsequential limit. Moreover, by Skorokhod's representation theorem (see e.g [3]), we can assume that this convergence occurs almost surely.

The metric space we will work with is $\mathcal{S}$, the space of continuous curves from $[0,1]$ into $\overline{D}$ modulo continuous increasing reparameterizations of $[0,1]$. That is, $f_1$ and $f_2$ represent the same curve if and only if there is a continuous, monotonic increasing bijection $\phi : [0,1] \to [0,1]$ such that $f_1 = f_2 \circ \phi$. The metric is

$$d(f_1, f_2) = \inf_{\phi} \sup_{u\in[0,1]} |f_1(u) - f_2 \circ \phi(u)|$$

where the infimum is over all continuous monotonic increasing bijections $\phi$. The metric is discussed in more detail in [1]. With this metric, $\mathcal{S}$ is complete.

**Notations 5.2.** For all $n \in \mathbb{N}$ and for each curve $\gamma$, let $T_0^n = 0$ and for all $j \in \mathbb{N}$, let

$$T_j^n = \inf\{t > T_{j-1}^n(\gamma) : |\gamma(t) - \gamma(T_{j-1}^n(\gamma))| > 16 \cdot 2^{-n}\}.$$

These will mark when our curves move by a large amount. Moreover, we can count the maximum number of such large jumps as $M(n,\gamma)$. In other words, $M(n,\gamma) = \sup\{n \in \mathbb{N} : T_j^n \text{ is well-defined}\}$.

We now begin with proving tightness. We first show a characterization of some compact sets in $\mathcal{S}$ in terms of these jump counts.

**Lemma 5.3.** *Fix a sequence $u(n)$ where for all $n \in \mathbb{N}$, $u(n) > 0$. Consider the set of curves $K = \{\gamma : M(n,\gamma) \le u(n)$ for all $n \ge n_0\}$ where $n_0 \in \mathbb{N}$. Then, $K$ is compact in $\mathcal{S}$.*

*Proof.* Let $(\gamma_k)_{k\in\mathbb{N}}$ be any sequence in $K$. We will show that there exists a convergent subsequence in $K$.

Let $n = n_0$. Each $\gamma_k$ can then be broken up into at most $u(n)$ segments where the curve makes movements of size greater than $16 \cdot 2^{-n}$. In particular, the jump times $T_j^n$ terminate after at most $u(n)$ steps in $[0,1]$. Let $m_k \le u(n)$ be this number. Now, we consider the sequence $\{m_k\}_{k\in\mathbb{N}}$. Since this is a bounded sequence of natural numbers, we may find a subsequence along which it is constant. Without loss of generality, we assume that this is already a constant sequence, so for all $k \in \mathbb{N}$, $m_k = m(n) \le u(n)$ (meaning each curve makes the same number of large jumps). Now, let $j \in \mathbb{N}$ such that $j \le m(n)$. For all $k \in \mathbb{N}$, note that $T_j^n(\gamma_k) \in [0,1]$ and that $\gamma_k(T_j^n(\gamma_k)) \in \overline{D}$. Therefore, there is a convergent subsequence indexed by $k_\ell$ such that as $k_\ell \to \infty$, $T_j^n(\gamma_{k_\ell}) \to T_j^n \in [0,1]$ and $\gamma_{k_\ell}(T_j^n(\gamma_{k_\ell})) \to x_j^n \in \overline{D}$ by compactness. By a similar argument and by diagonalization, we may therefore find a subsequence of our initial sequence, which we'll call $(\gamma_\ell)_{\ell\in\mathbb{N}}$, such that for all $n \in \mathbb{N}$ with $n \ge n_0$, all but finitely many $\gamma_\ell$ curves take the same number of jumps of size at least $16 \cdot 2^{-n}$ and for all appropriate $j$, $T_j^n(\gamma_\ell)$ and $\gamma_\ell(T_j^n(\gamma_\ell))$ converge to some $T_j^n \in [0,1]$ and to some $x_j^n \in \overline{D}$, respectively.

We will now show that this subsequence converges uniformly. We will let $\epsilon > 0$ and let $n \in \mathbb{N}$ be at least $n_0$ and such that $32 \cdot 2^{-n} < \frac{\epsilon}{3}$. Let $k \in \mathbb{N}$. We will define the function $\gamma_k^n : [0,1] \to \mathbb{C}$ such that on each interval $[T_j^n(\gamma_k), T_{j+1}^n(\gamma_k)]$ the function linearly interpolates between $\gamma_k(T_j^n(\gamma_k))$ and $\gamma_k(T_{j+1}^n(\gamma_k))$. When $j = M(n,\gamma_k)$, we will use 1 instead of $T_{j+1}^n(\gamma_k)$. But, we know that for all appropriate $j$, $T_j^n(\gamma_\ell)$ and $\gamma_\ell(T_j^n(\gamma_\ell))$ converge to some $T_j^n \in [0,1]$ and to some $x_j^n \in \overline{D}$, so it follows that the sequence of functions $\gamma_k^n$ converges uniformly. This comes from the fact that polygonal curves, constructed by connecting a fixed number of points, uniformly converge with respect to the supremum norm when the location of the points in the domain and their values in the codomain both converge. Therefore, for $k, k' \in \mathbb{N}$ sufficiently large, $||\gamma_k^n - \gamma_{k'}^n|| < \frac{\epsilon}{3}$. Now, let $t \in [0,1]$. It follows that for any $k$, we have that $|\gamma_k(t) - \gamma_k^n(t)| < 32 \cdot 2^{-n}$ be the triangle inequality. So, it follows that for all $k, k' \in \mathbb{N}$ sufficiently large,

$$d(\gamma_k, \gamma_{k'}) \le ||\gamma_k - \gamma_{k'}|| \le ||\gamma_k - \gamma_k^n|| + ||\gamma_k^n - \gamma_{k'}^n|| + ||\gamma_{k'}^n - \gamma_{k'}|| < 3\left(\frac{\epsilon}{3}\right) = \epsilon.$$

This means that our subsequence is Cauchy in $\mathcal{S}$ and therefore has a limit in $\mathcal{S}$ by completeness to which it converges uniformly to.

Let $\gamma$ be the limit curve of this convergent subsequence $(\gamma_m)_{m\in\mathbb{N}}$. For all $m$, we have that $|\gamma_m(T_j^n(\gamma_m))-\gamma_m(T_{j-1}^n(\gamma_m))|\geq 16\cdot 2^{-n}$ for all appropriate $j$. But, as the jump times and the images of the jump times converge as $m\to\infty$, we have

$$|x_j^n - x_{j-1}^n| = \lim_{m\to\infty}|\gamma_m(T_j^n(\gamma_m)) - \gamma_m(T_{j-1}^n(\gamma_m))| \geq 16\cdot 2^{-n}.$$

So, the limit curve must also move at least this much for each step, meaning that the maximum number of such steps $M(n,\gamma)$ is also at most $u(n)$. As this holds for all $n\in\mathbb{N}$ such that $n\geq n_0$, this proves that $\gamma\in K$ and that $K$ is compact. □

Now, we will examine the random curves themselves. We begin by getting exponential probability bounds for annulus crossings.

**Lemma 5.4.** *For all $k,\delta,r$, and $x$, there are two positive constants $\alpha$ and $C$ such that the probability there exist $k$ disjoint open crossings of the annulus $A(x,r)=\{z : r<|z-x|<4r\}$ is bounded by $C2^{-k\alpha}$. In particular, the probability $\gamma^\delta$ crosses a fixed annulus $A(x,r)$ more than $2k$ times also has the same bound.*

*Proof.* This follows from Corollary 3.9. □

The next lemma links jumps to annulus crossings.

**Lemma 5.5.** *Let $N,n\in\mathbb{N}$ such that $N\leq C'4^n$ where $C'$ is a constant depending on the domain in which the curve lies. Then, we may find a collection of points $x_1,...,x_N$ such that for every segment of the curve $\gamma^\delta$ between the times $T_{j-1}^n$ and $T_j^n$ the curve must cross at least one of the annuli $A(x_i,2^{-n})$. Moreover, if $K$ is sufficiently large, then the probability there exists an $i\leq N$ such that $A(x_i,2^{-n})$ is crossed at least $Kn$ times is at most $C''2^{-n}$ where $C''$ is a positive constant. So, $\mathbb{P}[nKC'4^n\leq M(n)]\leq C''2^{-n}$.*

The exact value of $C'$ depends on the specific domain, but we give a broad proof here.

*Proof.* Let $S$ be a bounded domain where the curve lies and cover it with annuli of radius $2^{-n}$ centered at $x_1,...,x_N$. By definition of the stopping times, $16\cdot 2^{-n}\leq|\gamma(T_j^n)-\gamma(T_{j-1}^n)|$. So, the $j$th segment of the curve moves at least $16\cdot 2^{-n}$, which is much bigger than the radii of our annuli. Therefore, there must be at least one annulus such that the curve crosses the annulus. The second part of the lemma follows from Lemma 5.4 and choosing $K>\frac{4}{\alpha}$. Since there are at most $C'4^n$ total annuli, the probability that one is crossed at least $Kn$ times is therefore $C'4^nC2^{-\alpha\cdot\frac{Kn}{2}}$. Finally, we note that if $M(n)\geq nKC'4^n$, then we must have that some annulus is crossed $Kn$ times, proving the final part of the lemma. □

Now we have the necessary tools to prove tightness.

**Lemma 5.6.** *$\{\gamma^\delta\}_{\delta>0}$ is tight in $\mathcal{S}$.*

*Proof.* We will use the notation from Lemma 5.5. Let $\epsilon>0$. For all $n\in\mathbb{N}$, let $u(n)=nKC'4^n$. Then, $\mathbb{P}[M(n,\gamma^\delta)\leq u(n)]\geq 1-C''2^{-n}$ by Lemma 5.5. Therefore, if $A$ is the event that $\gamma^\delta$ satisfies $M(n,\gamma^\delta)\leq u(n)$ for all $n\geq n_0$ where $n_0\in\mathbb{N}$ is sufficiently large, then $\mathbb{P}[A]\geq 1-\sum_{n\geq n_0}C''2^{-n}\geq 1-\epsilon$ due to the exponential decay and the convergence of the series $\sum_{n\in\mathbb{N}}C''2^{-n}$. Let $K_\epsilon=\{\gamma^\delta : M(n,\gamma)\leq u(n)$ for all $n\geq n_0\}$. By Lemma 5.3, $K_\epsilon$ is compact. Therefore, by the definition of tightness, we have that our family is tight. □

We are now in a position to justify our study of convergence. From our proof of tightness, we have a subsequence $\gamma^{\delta_n}$ and the limiting curve $\gamma$, which we get almost sure convergence to, and which we assume is not constant on any interval. In the remainder of this paper, we will work with this subsequence and its limit. It remains to show that $\gamma$ is an SLE(6). First, we will need to prove that almost surely $\gamma$ can be constructed via Loewner chains.

Our next step comes from relating the domain $D$ to $\mathbb{H}$ where we defined SLEs. We will consider the conformal map $\Phi:\overline{D}\to\overline{\mathbb{H}}$ such that $\Phi(x)=0$ and $\Phi(c)=\infty$. For all $u\in[0,1]$, let $D_u$ be the connected component of $D\setminus\gamma[0,u]$ that has $c$ on the boundary, and the hulls $K_u=D\setminus D_u$. We also assume that $\gamma$ is not constant on any interval. Then $\Phi(\gamma)$ is a continuous curve, so the half-plane capacity of $\Phi(K_u)$ is continuous with respect to $u$. To say that this family of $\mathbb{H}$ hulls is a Loewner chain, we will need to use the fact that there is a parameterization so that $\Phi(\gamma)$ is a Loewner chain, which is proven in the next lemma, and then justify strictly increasing half-plane capacity. From here, we would obtain that $\Phi(\gamma)$ is uniformly continuous on compact intervals of the form $[0,t_0+s]$, which would imply the local growth property.

**Lemma 5.7.** *Let $\overline{\gamma} : [0,1] \to \overline{\mathbb{H}}$ be a simple, continuous, non-constant curve such that $\overline{\gamma}(0) = 0$. Then, there exists a reparameterization $\gamma$ such that $\gamma[0,t]$ has half-plane capacity $2t$ and the associated conformal maps $g_t$ solve Loewner's differential equation with driving function $g_t(\gamma(t))$.*

*Proof.* For all $s \geq 0$, we define the hull $K_s = \overline{\gamma}[0,s]$. Since the curve is simple, these sets are compact and strictly increasing. Now, let $h(s) = \frac{1}{2}\text{hcap}(K_s)$, which is therefore continuous and strictly increasing. Let $s(t)$ be the inverse of $h(s)$ and we define the reparameterized curve $\gamma(t) = \overline{\gamma}(s(t))$. The hulls $K_t = \gamma[0,t]$ then satisfy $\text{hcap}(K_t) = 2t$. Now, for all $t \geq 0$, we consider the mapping-out function $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$ and the driving function $\xi_t = g_t(\gamma(t))$. Note that $\xi_t$ is a continuous, real function. It follows that the Loewner differential equation is satisfied. □

We will now prove that almost surely the half-plane capacity of $K_u$ from $c$ is a strictly increasing function of $u$. To do so, we will prove that this is a strictly increasing family.

**Notation 5.8.** If $z \in D$, let $\sigma_z$ to be the first time $u$ such that $\gamma[0,u]$ disconnects $z$ from $c$ in $D$. Analogous notation holds when we're considering $\gamma^\delta$ (and $c_\delta$).

**Lemma 5.9.** *Let $z \in D$ with rational coordinates. Then, almost surely, $\sigma_z^{\delta_n} \to \sigma_z$ as $n \to \infty$.*

In words, this lemma says that the disconnection times of $z$ by the discrete curve $\gamma^\delta$ converge to the disconnection time by the limit curve $\gamma$, almost surely.

*Proof.* Because $\gamma^{\delta_n}$ converges to $\gamma$, we may directly obtain that $\sigma_z \geq \limsup \sigma_z^{\delta_n}$. We will now prove that $\sigma_z \leq \liminf \sigma_z^{\delta_n}$ which implies the conclusion. Assume for the sake of contradiction that the probability that $\sigma_z^{\delta_n} < \sigma_z - \epsilon$ infinitely often is positive where $\epsilon > 0$. So, for infinitely many $n$, the discrete path $\gamma^{\delta_n}$ disconnects $z$ before time $\sigma_z - \epsilon$. Due to convergence, the discrete path is close to $\gamma$ for large $n$ which has not disconnected $z$ by time $\sigma_z - \epsilon$, meaning that the discrete path completes a disconnection before the limit path in spite of being near it. To disconnect $z$ early, the discrete path traverses a thin annulus around $z$ such that the limit path does not yet separate. Let $A$ be such an annulus around $z$ in the domain $D \setminus \gamma[0, \sigma_z - \frac{\epsilon}{2}]$. As a corollary of Lemma 3.10, we get that the probability that the discrete path traverses $A$ and forms a disconnection is at most $c < 1$. Placing $k$ disjoint annuli between $z$ and the boundary gives us the probability that the discrete path completes all crossings by $\sigma_z - \epsilon$ is at most $c^k$ (the exponential decay rate comes from Lemma 3.10 and a similar argument to Corollary 3.9). However, by considering annuli of the same modulus and just of a smaller radii, which would increase the value of $k$, this would contradict our earlier assumption. So, $\sigma_z \leq \liminf \sigma_z^{\delta_n}$. □

**Lemma 5.10.** *Let $0 < u < u'$. Then, almost surely, there is some $v \in (u, u')$ such that $\gamma(v) \notin \gamma[0,u] \cap \partial D$.*

In this lemma and its proof, we recall from our setup the assumption that $\gamma$ is not constant on any interval.

*Proof.* We consider a countable dense set of points $(x_j)_{j \in \mathbb{N}}$ on the union of the boundaries of the connected components of $D \setminus \gamma[0,u]$. Let $j \in \mathbb{N}$. As a consequence of Corollary 3.9, $x_j$ is almost surely not hit by $\gamma$. So, almost surely, none of our countable dense set of points are hit by $\gamma$. So, since a dense subset is avoided, $\gamma$ cannot remain entirely in $\gamma[0,u] \cap \partial D$ over any time interval, proving the lemma. □

**Lemma 5.11.** *The map $u \to K_u$ is strictly increasing (in the sense of strict inclusion).*

*Proof.* Let $0 < u < u'$ such that $u, u' \in \mathbb{Q}$. By Lemma 5.10, we may find some $v \in (u, u')$ such that $\gamma(v) \notin \gamma[0,u] \cap \partial D$. So, $\gamma(v) \in D \setminus \gamma[0,u]$ and lies in one of its connected components. Assume for the sake of contradiction that the connected component it lies in does not contain the target boundary point $c$. Let $z \in \mathbb{Q}$ be in this component as well. At the same time $u$ or soon after this component is disconnected, the curves $\gamma^{\delta_n}$ disconnect $z$ as well almost surely by Lemma 5.9. Because the discrete curves $\gamma^{\delta_n}$ are simple and end at $c$, they may not re-enter a component that has been disconnected from $c$. This means that after time $u$, they may no longer reach $z$. However, then $\gamma^{\delta_n}(v)$ is not in the same component at $z$, contradicting that $\gamma(v)$ and $z$ lie in the same component and our uniform convergence. Hence, $\gamma(v) \in K_{u'} \setminus K_u$. But, we already know that the map $u \to K_u$ is increasing, so since this holds for all rational $u < u'$, the map is almost surely strictly increasing. □

Therefore, we can reparameterize via a time change to make $\Phi(K_u)$ a family of $\mathbb{H}$ hulls. It remains to identify the process' driving function. Also, we note that if we parameterize each curve $\gamma^{\delta_n}$ by its capacity, then $\sup_t |\gamma^{\delta_n}(t) - \gamma(t)| \to 0$ almost surely, because $\gamma$ is almost surely uniformly continuous.

The next step in our proof is the identification with martingales. First, we will return to a lemma involving Cardy's formula. In doing so, we will add two points and their approximations to our setup to align with the setup of Cardy's formula in the previous section. We will explicitly consider more precisely how the discrete domains $D_\delta$ converge to $D$. In our proof of Theorem 4.10, we implicitly used the fact that subsequential limits for the $H_j^\delta$ functions exist when the interior points of $D$ eventually are contained in $D_\delta$ as $\delta \to 0$. In particular, this depended on Russo-Seymour-Welsh estimates, where this condition was used. We also used Russo-Seymour-Welsh in handling boundary conditions. The next lemma proves that under two conditions, our use of the Russo-Seymour-Welsh estimates was justified, in other words, that the interior points of $D$ are eventually contained in the $D_\delta$ process.

**Lemma 5.12.** *Assume that $(a_\delta, b_\delta, c_\delta) \to (a, b, c)$. Furthermore, assume that for any $z \in \partial_j$, the distance of $z$ to $\partial_j^\delta$ goes to $0$ when $\delta \to 0$ uniformly in $z$. Here, $\partial_j^\delta$ denotes one of the boundary arcs of $\partial D_\delta$ and $\partial_j$ denotes a boundary arc of $\partial D$ between marked points. Then, the interior points of $D$ are eventually contained in $D_\delta$ as $\delta \to 0$.*

The first assumption in this lemma is purely to orient ourselves with the setup of Cardy's formula from the previous section but is not actually used in the proof.

*Proof.* Let $K \subset D$ be compact. The second condition implies that $\partial D_\delta \to \partial D$ uniformly. We also recall that $D$ is a bounded, simply-connected domain such that $\partial D$ is continuous. By the compactness of $K$, we therefore have that for any $z \in K$, there is some $\epsilon > 0$ such that $\text{dist}(z, \partial D) > \epsilon$. We also have that for all sufficiently small $\delta$, $\sup_{w \in \partial D_\delta} \text{dist}(w, \partial D) < \frac{\epsilon}{2}$. This follows from uniform convergence of the boundary arcs. Hence, because $D_\delta$ is simply connected, $K$ is contained in $D_\delta$ by compactness. A singleton set is compact, so we can conclude. $\square$

We now assume that our setup obeys the conditions in Lemma 5.12 so we may apply the results in the previous section. Our next step is to identify a martingale observable using Cardy's formula.

**Lemma 5.13.** *Let $a, b \in \partial D$ be distinct from $x$ and $c$ such that $a, x, b$, and $c$ lie in counterclockwise order. Let their approximations are $a_{\delta_n}$ and $b_{\delta_n}$ such that $a_{\delta_n} \to a$ and $b_{\delta_n} \to b$ as $n \to \infty$. Let $\mathcal{A}^{\delta_n}$ denote the event that $\gamma^{\delta_n}$ hits arc $a_{\delta_n} c_{\delta_n}$ before arc $c_{\delta_n} b_{\delta_n}$. Then, for all $t \geq 0$, we have that*

$$\mathbb{P}[\mathcal{A}^{\delta_n} | \gamma^{\delta_n}[0, t]] \to X_t$$

*almost surely, where $X_t$ is the image of $\gamma_t$ under the conformal map from $D_t = D \backslash \gamma[0, t]$ to the equilateral triangle $ABC$ sending $a, b, c \to A, B, C$.*

*Proof.* Let $t \geq 0$. The connected component of $D \backslash \gamma^{\delta_n}[0, t]$ with $c_{\delta_n}$ on its boundary converges to $D \backslash \gamma[0, t]$ as $n \to \infty$ in the sense of Lemma 5.12, which implies Carathéodory convergence of the domains. The boundary points all converge as well to their respective counterparts. By Theorem 4.10, the crossing probability for critical site percolation in $D_t^{\delta_n}$ between the boundary arcs $a_{\delta_n} c_{\delta_n}$ and $c_{\delta_n} b_{\delta_n}$ converges to Cardy's formula in the limit domain $D_t$. Note that this crossing probability is precisely the conditional probability $\mathbb{P}[\mathcal{A}^{\delta_n} | \gamma^{\delta_n}[0, t]]$. So, the limit of this conditional probability is a deterministic function of the limiting domain and the marked points. In other words, it is the image of the tip $\gamma_t$ under the conformal map $\Phi_t : D_t \to ABC$ normalized so that $\Phi_t(a) = A$, $\Phi_t(b) = B$, and $\Phi_t(c) = C$. If we define $X_t = \Phi_t(\gamma_t)$, then the conclusion follows almost surely by definition. $\square$

We will now prove a continuous version of this lemma.

**Lemma 5.14.** *Let $\mathcal{A}$ be the event that $\gamma$ hits arc $ac$ prior to hitting arc $bc$. Let $T$ be the hitting time of $bc \cup ca$ by $\gamma$ and $X_t = 1_{\mathcal{A}}$ for $t \geq T$. Then, for all $t \geq 0$, $X_t = \mathbb{P}[\mathcal{A} | \gamma[0, t]]$.*

We note a slight abuse of notation in our use of $X_t$. Initially, $X_t$ was defined as the image of the tip $\gamma_t$ under the conformal map of the slit domain $D_t$ to the equilateral triangle $ABC$. However, by Theorem 4.10 and Cardy's formula, the crossing probability in $D_t$ from arc $ac$ to arc $bc$ is the image of $\gamma_t$ under this conformal map. Consequently, we may, equivalently, view $X_t$ as the proposed conditional probability.

*Proof.* Let $f$ be a continuous bounded function on the space of curves. Our goal is to show that $\mathbb{E}[1_{\mathcal{A}} f(\gamma[0, t])] = \mathbb{E}[X_t f(\gamma[0, t])]$. By the dominated convergence theorem and the almost sure convergence of the exploration paths, we get that

$$\begin{aligned}
\mathbb{E}[1_{\mathcal{A}} f(\gamma[0, t])] &= \lim_{n \to \infty} \mathbb{E}[1_{\mathcal{A}^{\delta_n}} f(\gamma^{\delta_n}[0, t])] \\
&= \lim_{n \to \infty} \mathbb{E}[\mathbb{P}[\mathcal{A}^{\delta_n} | \gamma^{\delta_n}[0, t]] f(\gamma^{\delta_n}[0, t])] \\
&= \lim_{n \to \infty} \mathbb{E}[X_t f(\gamma^{\delta_n}[0, t])] \\
&= \mathbb{E}[X_t f(\gamma[0, t])].
\end{aligned}$$

$\square$

This lemma proves that $X_t$ is a continuous function with respect to $t$ and therefore implies that it is a continuous martingale. We are now ready to conclude via a stochastic calculus argument.

**Theorem 5.15.** *$\gamma$ is a (chordal) SLE(6) from $x$ to $c$ in domain $D$.*

*Proof.* We will consider the compact $\mathbb{H}$ hulls $\Phi(K_t)$ defined by $\Phi(\gamma(t))$ as previously discussed. Let $g_t$ be the associated mapping-out functions and let the associated driving function be $\xi_t$. Our objective is to prove that $\xi_t = \sqrt{6}B_t$ where $B_t$ is a standard Brownian motion. Moreover, we'll let $a' = \Phi(a)$ and $b' = \Phi(b)$. Let $\Psi$ be the Schwarz-Christoffel conformal map from $\mathbb{H}$ to $ABC$ such that $\Psi(0) = A$, $\Psi(1) = B$, and $\Psi(\infty) = C$. The restriction of $\Psi$ to $[0, 1]$ is given by

$$\Psi(z) = K \int_0^z \frac{dy}{y^{\frac{2}{3}}(1-y)^{\frac{2}{3}}}$$

for some positive constant $K$. Note that $\Psi$ satisfies the ODE

$$3\Psi''(z) + 2\left(\frac{1}{z} + \frac{1}{z-1}\right)\Psi'(z) = 0.$$

Our first step is to express $X_t$ in terms of the driving function. Let $t \leq T$, where $T$ is defined as in Lemma 5.14. We will consider the transformation that sends $z$ to $\frac{z-g_t(a')}{g_t(b')-g_t(a')}$. This map sends $g_t(a')$ to 0, $g_t(b')$ to 1 and $\infty$ to $\infty$. As these are our boundary points, note that $\xi_t$ is mapped to a quantity in $\mathbb{H}$. Applying $\Psi$ to normalize position, we get the location of $X_t$ in triangle coordinates:

$$X_t = \Psi\left(\frac{\xi_t - g_t(a')}{g_t(b') - g_t(a')}\right).$$

Inverting, it follows that

$$\xi_t = g_t(a') + (g_t(b') - g_t(a'))\Psi^{-1}(X_t).$$

Next, recall Loewner's differential equation: $\partial_t g_t(z) = \frac{2}{g_t(z)-\xi_t}$ for fixed $z$. For $t \leq T$, the functions $g_t(a')$ and $g_t(b')$ are determined from $X_t$ by solving the ODE and replacing $\xi_t$ with our previous expression. So, the functions $g_t(a')$, $g_t(b')$, and $\xi_t$ are therefore measurable with respect to this filtration of $(X_t)_{t\geq 0}$.

Note that $g_t(a')$ and $g_t(b')$ are $C^1$ functions of $t$ and $\Psi^{-1}$ is a $C^2$ function. Therefore, $\xi_t$ is a semi-martingale. Now, let $Z_t = \frac{\xi_t - g_t(a')}{g_t(b')-g_t(a')}$ which lies in $(0, 1)$ while $\xi_t$ is between $g_t(a')$ and $g_t(b')$. The processes $\Psi(Z_t)$ are therefore local martingales. Because $\xi_t$ is a semi-martingale, it say be written as $\xi_t = M_t + V_t$ on $[0, T]$ where $M_t$ is a local martingale and $V_t$ is a finite variation process.

By definition, we have that

$$dZ_t = \frac{dM_t + dV_t}{D_t} - \frac{2dt}{D_t(g_t(a') - \xi_t)} - \frac{\xi_t - g_t(a')}{D_t^2}\left(\frac{2}{g_t(b') - \xi_t} - \frac{2}{g_t(a') - \xi_t}\right)dt$$

where $D_t = g_t(b') - g_t(a')$. But, noting that $Z_t D_t = \xi_t - g_t(a')$ and $(1 - Z_t)D_t = g_t(b') - \xi_t$, this simplifies to

$$\frac{dM_t + dV_t}{D_t} + \left[\frac{2}{D_t^2 Z_t} - \frac{2Z_t}{D_t^2}\left(\frac{1}{1 - Z_t} + \frac{1}{Z_t}\right)\right]dt.$$

We may also see that $d\langle Z\rangle_t = \frac{d\langle M\rangle_t}{D_t^2}$ where $\langle\cdot\rangle$ denotes quadratic variation. By Itô's formula applied to $\Psi(Z_t)$, the drift is

$$\frac{\Psi'(Z_t)dV_t}{D_t} + \frac{d\langle M\rangle_t}{2D_t^2}\Psi''(Z_t) + \Psi'(Z_t)\left[\frac{2}{D_t^2 Z_t} - \frac{2Z_t}{D_t^2}\left(\frac{1}{1 - Z_t} + \frac{1}{Z_t}\right)\right]dt.$$

We will now simplify the $dt$ term. First, we note that the $dt$ term is equal to $\frac{2\Psi'(Z_t)}{D_t^2}\left[\frac{2Z_t-1}{Z_t(Z_t-1)}\right]dt$. But, by definition of the ODE that $\Psi$ satisfies, we have that $\Psi''(y) = -\frac{2}{3}\left(\frac{2y-1}{y(y-1)}\right)\Psi'(y)$. Therefore, the $dt$ term is equal to $-\frac{3\Psi''(Z_t)}{D_t^2}dt = -\frac{6\Psi''(Z_t)}{2D_t^2}dt$.

Because the $X_t$ process is a martingale as previously discussed, the drift terms must vanish. In other words,

$$\frac{dV_t}{g_t(b') - g_t(a')}\Psi'(Z_t) + \frac{\Psi''(Z_t)}{2(g_t(b') - g_t(a'))^2}(d\langle M\rangle_t - 6dt) = 0.$$

Now, we will select a sequence $\{(a_n', b_n')\}_{n\in\mathbb{N}}$ with $a_n' \to -\infty$ and $b_n' \to \infty$ as $n \to \infty$. For all $n \in \mathbb{N}$ and for all $t \geq 0$, let $Z_t^n = \frac{\xi_t - g_t(a_n')}{g_t(b_n')-g_t(a_n')}$. By the Schwarz reflection principle, it follows that $|\Psi'(Z_t^n)|$ and $|\Psi''(Z_t^n)|$ are uniformly bounded away from 0 and away from $\infty$. Therefore, by replacing $a'$ and $b'$ with $a_n'$ and $b_n'$ in the drift expression and taking $n \to \infty$, we must have that $dV_t = 0$. This implies that $\langle M\rangle_t = 6t$ for all $t \geq 0$. It follows that $\xi(\frac{t}{6})$ is a standard Brownian motion. So, $\sqrt{6}B_t = \xi_t$ where

$(B_t)_{t\geq 0}$ is a standard Brownian motion. This implies that $\gamma$ is an SLE(6) in $D$ by definition of SLE in two-pointed domains. □

While this is the final proof in our paper, we end by briefly discussing some ways this result has been used in probability theory. For example, convergence to SLE(6) has been used to compute several arm exponents in percolation, which describe the probability of finding disjoint paths on different parts of a lattice and to characterize the behavior of physical systems by critical points, as in [18]. We also immediately get an alternative proof for the Rohde-Schramm Theorem only for SLE(6). We also directly find that SLE(6) is reversible, which means that the law of the time reversal of an SLE(6) from $c$ to $x$ is the law of an SLE(6) from $c$ to $x$ after reparameterization and in the same domain.

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