MORPHISMS OF FINITE C_p-TAMBARA FIELDS ARE FREE, ÉTALE, GALOIS

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Abstract. Tambara fields are equivariant analogues to fields in classical theory. They appear in representation theory and equivariant homotopy theory. This paper explicitly characterizes all finite C_p-Tamabara fields and all morphisms between them. Each belongs to one of three families: $\underline{\mathbb{F}_{q^n}}, \mathrm{FP}(\mathbb{F}_{q^n})$, or $\mathrm{Coind}_e^{C_p}(\mathbb{F}_{q^n})$. All nontrivial morphisms $k \to \ell$ define ℓ as a free module over k and are étale and Galois.

Contents

Introduction.

The key object of study is Tambara functors, which are the equivariant equivalent to rings. They generalize rings with G-action for G a finite group. Tambara functors were introducted by Tambara in [12] under the name TNR functors to organize structures from group cohomology and representation rings. They are Green functors with aditional structure, which are themselves Mackey functors with aditional structure. More recently, Brun [7] has observed that the homotopy groups of G-$\mathbb{E}_\infty$ ring spectra inherit the structure of a Tambara functor, and the norm structures of Tambara functors are closely related to the Hill-Hopkins-Ravenel norms used in the solution of the Kervaire invariant one problem [8].

For finite group G, a G-Tambara functor roughly consists of the following data:

(1) Rings $k(G/H)$ for each transitive G-set G/H. We say $k(G/H)$ is in level G/H and refer to $k(G/e)$ (resp. $k(G/G)$) as the bottom (resp. top) level.

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(2) Ring maps $k(G/H) \to k(G/K)$ for every morphism of G-sets $G/K \to G/H$, called restriction maps. At each level there is a G-action which is preserved by the restriction maps.
(3) Multiplicative norm and additive transfer maps $k(G/H) \to k(G/K)$ for every morphism of G-sets $G/H \to G/K$.

Example A. Let R be a ring with G action. We call $\mathrm{FP}(R)$ the fixed point functor of R such that for each subgroup $H \leq G$ the corresponding level is

$$\mathrm{FP}(R)(G/H) = R^H,$$

the fixed field of the subgroup. The restriction maps are the natural inclusions of fixed fields, and the norm and transfer maps are calculated using the double coset formula.

For a detailed treatment of the G-Tambara functor construction, [1] and [11] are excellent resources.

When at each level a G-Tambara functor k satisfies $|k(G/H)| < \infty$, we say that k is *level-wise finite*. A morphism $\ell \to k$ of G-Tambara functors consists of level-wise ring-maps $\ell(G/H) \to k(G/H)$ which commute with restriction, norm, transfer, and conjugation. A non-zero G-Tambara functor k is called an *G-Tambara field* if every morphism $k \to \ell$ with $\ell \neq 0$ is monic, defined by Nakaoka in [5] and [6]. He shows in [5, Theorem 4.32] that all restriction maps in k are injective.

This paper will predominantly discuss level-wise finite C_p-Tambara fields and maps between them. To that end, for fixed p, $\mathcal{C}$ will denote the category of level-wise finite C_p Tambara fields. The paper is concerned with classifying the objects and morphisms in the category.

Theorem B (cf. Theorem 1.5)**.** *If k is a level-wise finite G-Tambara field then it is exactly either coinduced or clarified.*

In particular, if k is a level-wise finite C_p Tambara field then it takes exactly one of the following three forms $\underline{\mathbb{F}_{q^n}}, \mathrm{FP}(\mathbb{F}_{q^{n \cdot p}})$, or $\mathrm{Coind}_e^{C_p}(\mathbb{F}_{q^n})$, with the former two being examples of clarified C_p-Tambara fields and the lattermos a coinduced C_p-Tambara field.

The morphisms between objects can then be calculated exactly, since all types of objects are known.

Theorem C (cf. Proposition 2.2, Proposition 2.3, Theorem 2.4, Proposition 2.5, Proposition 2.6)**.** *We explicitly determine all morphisms in $\mathcal{C}$.*

There are certain properties which all morphisms in this category posess.

Theorem D (cf. Theorem 3.3, Theorem 3.6, Corollary 3.13)**.** *All morphisms $k \to \ell$ of level-wise finite C_p Tambara fields define ℓ as a free module over k. They are also étale and Galois.*

Not all modules over Tambara fields are free, so the first part of Theorem D isn't immediate.

1. Objects.

We begin by introducing a basic classification for level-wise finite G-Tambara functors, as either clarified or coinduced.

Definition 1.1. (1) A field-like G-Tambara functor k is clarified if its bottom level $k(G/e)$, is a field. (per p.20 of [1])
(2) A field-like G-Tambara functor k is coinduced if it is of the form $k \cong \mathrm{Coind}_H^G \ell$ where H is a proper subgroup of G and Coind_H^G is the coinduction functor from H-rings to G-rings, right adjoint to the restriction morphism.

These classifications are mutually exclusive, and describe all finite G-Tambara fields.

Lemma 1.2. *Let G be a finite group and k be a field like G-Tambara functor. Then $k \cong \mathrm{Coind}_H^G \ell$ for some ℓ a field like H-Tambara functor ℓ such that $\ell(H/e)$ is a field.*

Proof. This is Corollary B in [1]. □

Lemma 1.3. *If k is as before with the additional requirement that it is not coinduced from G, then k is clarified.*

Proof. From the previous lemma $k \cong \mathrm{Coind}_H^G \ell$ for some ℓ a field like H-Tambara functor ℓ such that $\ell(H/e)$. However, the condition that k is not coinduced requires that $H = G$ since H cannot be a proper subgroup. We conclude

$$k \cong \mathrm{Coind}_G^G \ell = \ell.$$

Since $\ell(G/e)$ is a field so too is $k(G/e)$. □

Lemma 1.4. *Moreover, if k is coinduced then k is not clarified.*

Proof. Writing $k \cong \mathrm{Coind}_H^G \ell$, the condition that k is coinduced means that H is a proper subset of G.

It is the definition of coinduction (as in [1] Definition 2.40) that

$$k(G/e) \cong \mathrm{Coind}_H^G \ell(G/e) \cong \mathrm{Fun}(G/H, \ell(G/e)) \cong \underbrace{\ell(G/H) \times \cdots \times \ell(G/H)}_{|G/H| \text{ times}}.$$

Since H is proper then $|G/H| > 1$ and as the product of more than one field $k(G/e)$ is not a field. □

Theorem 1.5. *If k is a finite G-Tambara field then it is exactly either coinduced or clarified.*

Proof. Immediate from Lemma 1.2, Lemma 1.3, and Lemma 1.4. □

One basic property of $\mathcal{C}$ is that all non-zero morphisms in the category are monic, which is required by the definition of field-like G-Tambara functors.

Now the first key result was stated in [1] Proposition 4.9.

Theorem 1.6. *So long as G is non-trivial, there are no Tambara functor morphisms from a coinduced Tambara functor to one which is clarified.*

Proof. [1] Proposition 4.9. □

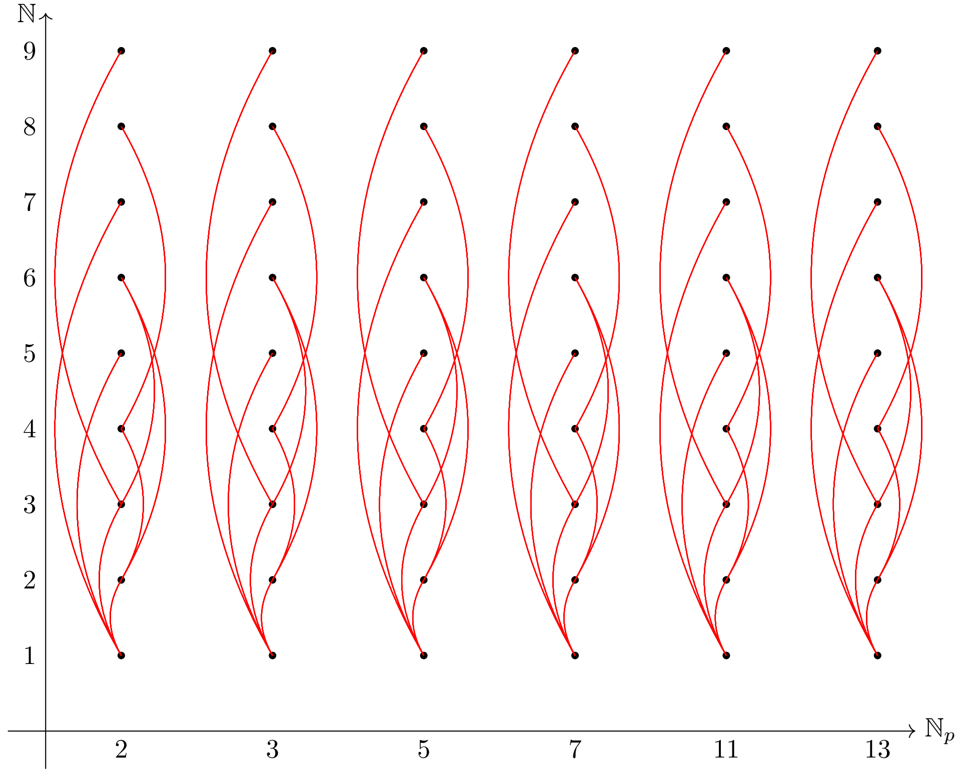

FIGURE 1. Finite fields $\mathbb{F}_{q^n}$ parameterized by $n \in \mathbb{N}$ and $q \in N_p$ with morphisms $\mathbb{F}_{q^n} \to \mathbb{F}_{q^m}$ when $n \mid m$. Lines are drawn when $n \mid m$ and there does not exist k such that $n \mid k$ and $k \mid m$.

1.1. **Finite fields.** The finite fields are precisely the fields $\mathbb{F}_{q^n}$ where q is prime. They are captured diagrammatically in Fig. 1, and are indexed by pairs in $\mathbb{N} \times \mathbb{N}_p$. Field homomorphisms are only between finite fields of the same characteristic. There is a morphism $\mathbb{F}_{q^n} \to \mathbb{F}_{q^m}$ if and only if $n \mid m$. In fact, there are exactly n many such maps, the same as the number of automorphisms of $\mathbb{F}_{q^n}$.

1.2. **Explicit Computation of Objects in $\mathcal{C}$.** Level-wise finite C_p-Tambara fields may exactly either be coinduced or clarified per Theorem 1.5. However, a clarified C_p-Tambara field k is determined precisely by the field $k(C_p/e) = \mathbb{F}$ at its bottom level, and the (possibly trivial) action of C_p on that field.

Definition 1.7. $\underline{R}$ where R is a ring with G-action is and is defined level-wise by $\underline{R(G/H)} = R^H = R$. We often write $FP(R)$ instead when R is a ring with faithful G-action to distinguish fixed point G-Tambara functors with trivial action and non-trivial action.

Theorem 1.8. *Let k be a C_p-Tambara field and let C_p act non-trivially on $k(C_p/e)$. Then the canonical map $k \to \underline{k(C_p/e)}$ is an isomorphism.*

Proof. Due to [2] Corollary 4.8. $\square$

Recall that when k is a clarified level-wise finite C_p-Tambara functor it consists of a finite field with C_p action at the bottom level. However in Theorem 1.8 we show that the C_p-Tambara field is determined entirely by field on the bottom level and the C_p action on that field, since higher level are computed by finding the corresponding fixed fields. In particular, it is precisely the fixed points C_p-Tambara field. By distinguishing fixed point C_p-Tambara fields with trivial action and non-trivial action, there are exactly three possibilities for any level-wise finite C_p-Tambara functor.

1.2.1. *Coinduced.* In the first case, k is a coinduced finite Tambara field. In particular, there is an isomorphism $k \cong \mathrm{Coind}_e^{C_p}\mathbb{F}_{q^n}$, where $\mathrm{Coind}_e^{C_p}\mathbb{F}_{q^n}(C_p/e) = \mathbb{F}_{q^n}^{\times p}$ at the bottom level and $\mathrm{Coind}_e^{C_p}\mathbb{F}_{q^n}(C_p/C_p) = \mathbb{F}_{q^n}$.

(i) The restriction map $\mathrm{Res}_e^{C_p} : k(C_p/C_p) \to k(C_p/e)$ is defined by
$$\mathrm{Res}_e^{C_p}(x) = (x, \ldots, x)$$
where $x \in k(C_p/C_p) = \mathbb{F}_{q^n}$.

(ii) Additive transfer $\mathrm{Tr}_e^{C_p}$ defined by
$$\mathrm{Tr}_e^{C_p}(x_1, \ldots, x_p) = x_1 + \cdots + x_p$$

(iii) and multiplicative norm defined by
$$\mathrm{Nm}_e^{C_p}(x_1, \ldots, x_p) = x_1 \cdots x_p.$$

(iv) Choosing some particular indexing of the factors of C_p so that $g \in C_p$ ranges from 0 to $p-1$, we get
$$c_g(x_1, x_2, \ldots, x_p) = (x_{1+g}, \ldots, x_p, x_1, \ldots x_g)$$
on $k(C_p/e) = \mathbb{F}_{q^n}^{\times p}$ and the action on $k(C_p/C_p) = \mathbb{F}_{q^n}$ is trivial.

1.2.2. *Trivial.* In the second case, k is a clarified finite C_p-Tambara field with trivial action, with $k(C_p/e) = \mathbb{F}_{q^n}$. There is an isomorphism $k \cong \underline{\mathbb{F}_{q^n}}$ where $\underline{\mathbb{F}_{q^n}}(C_p/e) = \mathbb{F}_{q^n}$ and $\underline{\mathbb{F}_{q^n}}(C_p/C_p) = \mathbb{F}_{q^n}$ on the top level which is the fixed field since C_p has the trivial action.

(i) The restriction map $\mathrm{Res}_e^{C_p}$ is the identity map since it is given by the inclusion of fixed points.

(ii) The transfer map $\mathrm{Tr}_e^{C_p}$ is defined by $\mathrm{Tr}_e^{C_p}(x) = p \cdot x$ which can e calculated using the double coset formula,
$$\mathrm{Res}_e^{C_p}\mathrm{Tr}_e^{C_p} = \sum_{g \in C_p} c_g.$$

(iii) The norm map $\mathrm{Nm}_e^{C_p}$ is the identity map as well which can also be calculated by the double coset formula
$$\mathrm{Res}_e^{C_p}\mathrm{Nm}_e^{C_p} = \prod_{g \in C_p/e} c_g.$$

(iv) and the conjugation maps c_g are the identity, since hte C_p action on F_q^n is trivial.

1.2.3. *Fixed Point.* In the third case, k is a clarified C_p-Tambara field with faithful action, with $k(C_p/e) = \mathbb{F}_{q^n}$. In particular, C_p corresponds to a nontrivial subgroup of $\mathrm{Aut}(\mathbb{F}_{q^n}) \cong \mathbb{Z}/n\mathbb{Z}$. Then $p \mid n$ and by writing $n = m{\cdot}p$ there is an isomorphism $k \cong FP(\mathbb{F}_{q^{p\cdot m}})$ which is defined by $FP(\mathbb{F}_{q^{p\cdot m}})(C_p/e) = \mathbb{F}_{q^{p\cdot m}}$ and $FP(\mathbb{F}_{q^{p\cdot m}})(C_p/C_p) = \mathbb{F}_{q^{p\cdot m}}^{C_p} \cong \mathbb{F}_{q^m}$.

(i) The restriction $\mathrm{Res}_e^{C_p} : \mathbb{F}_{q^{p\cdot m}}^{C_p} \to \mathbb{F}_{q^{p\cdot m}}$ is just the inclusion of the fixed field.

(ii) The transfer map $\mathrm{Tr}_e^{C_p} : \mathbb{F}_{q^{p\cdot m}} \to \mathbb{F}_{q^{p\cdot m}}^{C_p}$ is precisely the Galois trace $\mathrm{Tr}_{\mathbb{F}_{q^{p\cdot m}}^{C_p}}^{\mathbb{F}_{q^{p\cdot m}}}$.

Recalling that $C_p \cong \mathrm{Gal}(\mathbb{F}_{q^{p\cdot m}}/\mathbb{F}_{q^{p\cdot m}}^{C_p})$ so that

$$\mathrm{Tr}_e^{C_p}(x) = \sum_{g\in C_p} c_g(x).$$

(iii) The norm map $\mathrm{Nm}_e^{C_p}$ is the Galois norm $\mathrm{Nm}_{\mathbb{F}_{q^{p\cdot m}}^{C_p}}^{\mathbb{F}_{q^{p\cdot m}}}$ calculated by

$$\mathrm{Nm}_e^{C_p}(x) = \prod_{g\in C_p} c_g(x).$$

(iv) On the bottom level, the conjugation c_g is the same as the action of C_p on $\mathbb{F}_{q^{p\cdot m}}$ which is the same as the Galois action. So that in particular $c_g(x) = g \cdot x$.

On the top level, c_g is the original action on the fixed subfield, $\mathbb{F}_{q^{p\cdot m}}^{C_p}$ which is trivial.

Like with the finite fields $\mathbb{F}_{q^n}$, each of $\underline{\mathbb{F}_{q^n}}$, $\mathrm{FP}(\mathbb{F}_{q^{n\cdot p}})$, and $\mathrm{Coind}_e^{C_p}(\mathbb{F}_{q^n})$ are indexed by pairs in $\mathbb{N}\times\mathbb{N}_p$, however we will see that the morphisms may be different from in Fig. 1.

2. Morphisms.

2.1. **Internal Morphisms.** I will begin by describing the morphisms internal to each of these classifications.

For finite fields, a morphism $\mathbb{F}_{q^n} \to \mathbb{F}_{\ell^m}$ exists precisely when $q = \ell$ and $n \mid m$. There are n many morphisms of this form, corresponding to the n automorphisms of $\mathbb{F}_{q^n}$. This is the picture to have in mind as we proceed.

Definition 2.1. a morphism $k \to \ell$ of G-Tambara functors consists of level-wise morphisms $k(G/H) \to \ell(G/H)$ for every $H \leq G$ which commute with restriction, norms, transfers, and conjugation

Proposition 2.2. *Morphisms of the form* $\mathrm{Coind}_e^{C_p}\mathbb{F}_{q^n} \to \mathrm{Coind}_e^{C_p}\mathbb{F}_{\ell^m}$ *exist when* $q = \ell$ *and* $n \mid m$. *When morphisms exist, the* Hom*-set takes the following form.*

$$\mathrm{Hom}(\mathrm{Coind}_e^{C_p}\mathbb{F}_{q^n}, \mathrm{Coind}_e^{C_p}\mathbb{F}_{q^m}) \cong \mathbb{Z}/n \times C_p$$

Proof. Suppose that $\phi : \mathrm{Coind}_e^{C_p}\mathbb{F}_{q^n} \to \mathrm{Coind}_e^{C_p}\mathbb{F}_{\ell^m}$ exists and is not trivial. Then it must be injective, since the codomain is a Tambara field. There are corresponding ring maps

$$\phi : \mathbb{F}_{q^n}^{\times p} \to \mathbb{F}_{\ell^m}^{\times p}$$

and

$$\phi : \mathbb{F}_{q^n} \to \mathbb{F}_{\ell^m}$$

which are both injective. In particular, $\mathbb{F}_{q^n}$ embeds into $\mathbb{F}_{\ell^m}$. We deduce that $q = \ell$ and $n \mid m$.

Write $e_i = (0, \ldots, 1, \ldots, 0)$ as the element in the domain with 1 in the i-th position and 0 elsewhere. Likewise, $f_i = (0, \ldots, 1, \ldots, 0)$ is the similarly defined element in the codomain. We will see that there is some permutation $\sigma \in S_p$ so that $\phi(e_i) = f_{\sigma(i)}$.

In particular, the only idempotent elements in $\mathbb{F}_{q^n}^{\times p}$ and $\mathbb{F}_{\ell^m}^{\times p}$ are those with 0 or 1 in each entry. Then e_i and f_j are idempotents in the sets to which they belong. Moreover, $\phi(e_i) \cdot \phi(e_i) = \phi(e_i \cdot e_i) = \phi(e_i)$ so that it is idempotent. However, the morphism commutes with additive transfer map so that

$$1 = \phi(0 + \cdots + 1 + \cdots + 0) = \phi \circ \mathrm{Tr}_e^{C_p}(e_i) = \mathrm{Tr}_e^{C_p} \circ \phi(e_i)$$

So that in particular, there is exactly one 1 in the entries of $\phi(e_i)$ and the rest of the entries are zeros. As desired, $\phi(e_i) = f_{\sigma(i)}$ for some choice of σ.

However, ϕ also commutes with conjugation maps. We may view $C_p \leq S_p$ as the subgroup generated by cycle permutations. For every choice of $g \in C_p$ we get

$$f_{\sigma g(i)} = \phi \circ c_g(e_i) = c_g \circ \phi(e_i) = f_{g\sigma(i)}$$

Hence σ must belong to the centralizer of C_p, and so is an element of C_p. In particular, on the bottom level ϕ looks like the direct product of p copies of a field map $\mathbb{F}_{q^n} \to \mathbb{F}_{\ell^n}$ up to permutation.

Moreover, all morphisms of finite C_p-Tambara fields of this form are well-defined, so that the fintie C_p-Tambara field morphisms are precisely these.

Now to check all the necessary properties.

There are level-wise morphisms:

$$\phi : \mathbb{F}_{q^n}^{\times p} \to \mathbb{F}_{\ell^m}^{\times p}$$

at the bottom level and

$$\phi : \mathbb{F}_{q^n} \to \mathbb{F}_{\ell^m}$$

at the top level. Let's call $\sigma_\phi \in C_p$ the permutation of indices induced by the map at the bottom level. It commutes with restriction maps.

$$\phi \circ \mathrm{Res}_e^{C_p}(x) = \phi(x, \ldots, x) = (\phi(x), \ldots, \phi(x)) = \mathrm{Res}_e^{C_p} \circ \phi(x).$$

Also additive transfers

$$\phi \circ \mathrm{Tr}_e^{C_p}(x_1, \ldots, x_p) = \phi(x_1) + \cdots + \phi(x_p) = \mathrm{Tr}_e^{C_p} \circ \phi(x_1, \ldots, x_p)$$

And multiplicative norm

$$\phi \circ \mathrm{Nm}_e^{C_p}(x_1, \ldots, x_p) = \phi(x_1) \cdots \phi(x_p) = \mathrm{Nm}_e^{C_p} \circ \phi(x_1, \ldots, x_p).$$

And conjugation.

$$\begin{aligned}\phi \circ c_g(x_1, \ldots, x_p) &= (\phi(x_{\sigma_\phi \sigma_g(1)}), \ldots, \phi(x_{\sigma_\phi \sigma_g(p)})) \\ &= (\phi(x_{\sigma_g \sigma_\phi(1)}), \ldots, \phi(x_{\sigma_g \sigma_\phi(p)})) \\ &= c_g \circ \phi(x_1, \ldots, x_p)\end{aligned}$$

Any morphism of the form $\mathrm{Coind}_e^{C_p}\mathbb{F}_{q^n} \to \mathrm{Coind}_e^{C_p}\mathbb{F}_{q^m}$ consists of a field inclusion $\mathbb{F}_{q^n} \to \mathbb{F}_{q^m}$ at the top level. There are n many maps, corresponding to the automorphisms of $\mathbb{F}_{q^n}$, and they form a group isomorphic to $\mathbb{Z}/n$. Because of the definition of the restriction map, and transfer maps, on the bottom level the field morphism from the top level is applied entry-wise, up to the permutation of entries. However, in order to be compatible with the conjugation maps c_g, the only permissible permutation of entries is to rotate them. The p entry-rotations form a group which is isomorphic to C_p.

Hence each morphism can be described by a field inclusion at the top level, and a possible permutation of entries at the bottom level. The group of automorphisms is then $\mathbb{Z}/n \times C_p$. □

Proposition 2.3. *Morphisms of the form $\underline{\mathbb{F}_{q^n}} \to \underline{\mathbb{F}_{\ell^m}}$ exist when $q = \ell$ and $n \mid m$. Moreover,*

$$\mathrm{Hom}(\underline{\mathbb{F}_{q^n}}, \underline{\mathbb{F}_{q^m}}) \cong Aut(\mathbb{F}_{q^n}) \cong \mathbb{Z}/n.$$

Proof. A morphism $\phi : \underline{\mathbb{F}_{q^n}} \to \underline{\mathbb{F}_{\ell^m}}$ consists of embeddings

$$\phi_{bot} : \mathbb{F}_{q^n} \to \mathbb{F}_{\ell^m}, \qquad \phi_{top} : \mathbb{F}_{q^n} \to \mathbb{F}_{\ell^m}$$

on the bottom and top level respectively. In fact, these are exactly the same embedding of finite fields, since the maps commute with $\mathrm{Res}_e^{C_p}$ which is the identity map.

Any embedding of finite fields, giving rise to a map of trivial action clarified C_p-Tambara fields of the above form commutes straightforwardly with restriction, transfer, norm, and conjugation maps. □

Theorem 2.4. *There exists a morphism $\mathrm{FP}(\mathbb{F}_{q^{p \cdot n}}) \to \mathrm{FP}(\mathbb{F}_{\ell^{p \cdot m}})$ if and only if $q = \ell$, $n \mid m$, and $np \nmid m$. There are exactly np-many morphisms, when one exists. In particular,*

$$\mathrm{Hom}(\mathrm{FP}(\mathbb{F}_{q^{np}}), \mathrm{FP}(\mathbb{F}_{q^{mp}})) \cong Aut(\mathbb{F}_{q^{np}}) \cong \mathbb{Z}/np$$

Proof. For a nontrivial morphism $\phi : \mathrm{FP}(\mathbb{F}_{q^{p \cdot n}}) \to \mathrm{FP}(\mathbb{F}_{\ell^{p \cdot m}})$ there are corresponding level-wise maps

$$\phi : \mathbb{F}_{q^{p \cdot n}} \to \mathbb{F}_{\ell^{p \cdot m}}$$

on the bottom level and

$$\phi : \mathbb{F}_{q^{p \cdot n}}^{C_p} \to \mathbb{F}_{\ell^{p \cdot m}}^{C_p}$$

on the top level. The second map induces a field inclusion $\mathbb{F}_{q^n} \to \mathbb{F}_{\ell^m}$ so that $q = \ell$ and $n \mid m$.

Let $\sigma(x) = x^q$ be the Frobenius map. The C_p action on $\mathbb{F}_{q^{pn}}$ is the action of the group generated by σ^n. Likewise σ^m for $\mathbb{F}_{q^m}$ and its action. Indeed, if $\mathbb{F}_{q^{pn}}$ embeds into $\mathbb{F}_{q^{pm}}$ then it is the same as the fixed field of σ^{pn} over $\mathbb{F}_{q^{mp}}$. We can ask when such an embedding exists which preserves the C_p action in embedding $\mathbb{F}_{q^{pn}}$ into $\mathbb{F}_{q^{pm}}$. This is precisely when $pn \nmid m$. Notice first:

$$\sigma^m \mid_{\mathbb{F}_{q^{pn}}} = \sigma^{(m/n)n} \mid_{\mathbb{F}_{q^{pn}}} = (\sigma^n \mid_{\mathbb{F}_{q^{pn}}})^{(m/n)}.$$

When $p \nmid (m/n)$ (i.e. $pn \nmid m$), the group generated by σ^m over $\mathbb{F}_{q^{np}}$ is of order p. In particular, the action induced will be the same as the original C_p action on $\mathbb{F}_{q^{np}}$ (except with possibly a different generator, the embedding can be composed with an $\mathbb{F}_{q^{np}}$ automorphism so that the actions coincide exactly). Such an embedding will also behave in the expected way when composed with the restriction, transfer, and norm maps. The number is pn since that is the number of ways that $\mathbb{F}_{q^{np}}$ embeds into $\mathbb{F}_{q^{mp}}$. □

Similarly to Proposition 2.2, in Proposition 2.3 and Theorem 2.4 there is an action on the Hom-set by automorphisms of the domain, so these proofs show that the Hom-set described is the unique transitive free Aut-set in each case.

2.2. **External Morphisms.** From here, we may characterize all maps between finite C_p Tambara fields. What remains is computing maps between objects of different classifications.

Proposition 2.5. *(1) There exists a map $\underline{\mathbb{F}_{q^n}} \to \mathrm{FP}(\mathbb{F}_{\ell^{mp}})$ if and only if $q = \ell$ and $n \mid m$. When such a map exists, there are n many.*

(2) There exists a map $\underline{\mathbb{F}_{q^n}} \to \mathrm{Coind}_e^{C_p}\mathbb{F}_{\ell^m}$ if and only if $\ell = q$ and $n \mid m$. If one such map exists there are exactly n many.

(3) There exists a map $\mathrm{FP}(\mathbb{F}_{q^{n\cdot p}}) \to \mathrm{Coind}_e^{C_p}\mathbb{F}_{\ell^m}$ if and only if $\ell = q$, $np \mid m$. If one such map exists there are exactly n many.

For any choice of $n \mid m$ there will either be a map $\mathrm{FP}(\mathbb{F}_{q^{np}}) \to \mathrm{FP}(\mathbb{F}_{q^{mp}})$ or a map $\mathrm{FP}(\mathbb{F}_{q^{np}}) \to \mathrm{Coind}_e^{C_p}(\mathbb{F}_{q^m})$, since exactly either $pn \mid m$ or not.

Proof. Whenever there is a map of one of the following forms $\underline{\mathbb{F}_{q^n}} \to \mathrm{FP}(\mathbb{F}_{\ell^{mp}})$, $\underline{\mathbb{F}_{q^n}} \to \mathrm{Coind}_e^{C_p}\mathbb{F}_{\ell^m}$, or $\mathrm{FP}(\mathbb{F}_{q^{np}}) \to \mathrm{Coind}_e^{C_p}(\mathbb{F}_{\ell^m})$, at the top level this is a map $\underline{\mathbb{F}_{q^n}} \to \mathbb{F}_{\ell^m}$. It follows that $q = \ell$ and $n \mid m$.

(1) For a map $\phi : \underline{\mathbb{F}_{q^n}} \to \mathrm{FP}(\mathbb{F}_{q^{m\cdot p}})$, the map at the bottom level is a field inclusion $\mathbb{F}_{q^n} \to \overline{\mathbb{F}_{q^{mp}}}$.

There are exactly n such maps, corresponding to the automorphisms of $\mathbb{F}_{q^n}$. Since the restriction map is the inclusion of the fixed field, this also describes how $\mathbb{F}_{q^n}$ embeds into $\mathbb{F}_{q^{p\cdot m}}$ at the top level conjugation is consistent, and the transfer and norm maps are calculated by the double coset formula so that everything is consistent.

(2) Suppose $\phi : \underline{\mathbb{F}_{q^n}} \to \mathrm{Coind}_e^{C_p}(\mathbb{F}_{q^m})$. At the bottom, this is a map $\mathbb{F}_{q^n} \to \mathbb{F}_{q^m}^{\times p}$ at the bottom level.

There are n many such morphisms, corresponding to the embeddings $\iota : \mathbb{F}_{q^n} \to \mathbb{F}_{q^m}$. In particular, by composing with the restriction map we see it must take the form $x \mapsto (\iota(x), \dots, \iota(x))$.

(3) In the last case, $\phi : \mathrm{FP}(\mathbb{F}_{q^{np}}) \to \mathrm{Coind}_e^{C_p}(\mathbb{F}_{q^m})$. At the bottom level there is a ring inclusion $\mathbb{F}_{q^{n\cdot p}} \to \mathbb{F}_{\ell^m}^{\times p}$. This means that $np \mid m$. Choose c_g corresponding to an element $g \in C_p$ which generates the cyclic group. On the bottom level, the ring inclusion must take the form

$$\phi(x) = (\phi(c_g^k x), \phi(c_g^{k+1}(x)), \phi(c_g^{k+2}(x)), \ldots, \phi(c_g^{k+p-1}(x)))$$

for $0 \leq k < p$ where the power of c_g is interpreted modulo p. It is necessary that each element in the orbit of an element $x \in \mathbb{F}_{q^{np}}$ be represented as an entry in the corresponding element $\phi(x) \in \mathbb{F}_{\ell^m}^{\oplus p}$ so that it has the expected behavior when composed with the trace and norm maps.

□

Proposition 2.6. *There is no map of C_p-Tambara fields of the form* $\mathrm{FP}(\mathbb{F}_{q^n}) \to \underline{\mathbb{F}_{\ell^m}}$.

Proof. if such a map $\phi : \mathrm{FP}(\mathbb{F}_{q^n}) \to \underline{\mathbb{F}_{\ell^m}}$ did exist it would consist of a field inclusion $\mathbb{F}_{q^{n\cdot p}} \to \mathbb{F}_{\ell^m}$ at the bottom level, and $\mathbb{F}_{q^n} \to \mathbb{F}_{\ell^m}$ at the top level. However, the conjugation is incompatible with such a morphism. Choose $x \in \mathbb{F}_{q^{n\cdot p}}$ so that $c_g(x) \neq x$. Because ϕ is injective then

$$\phi(c_g(x)) \neq \phi(x) = c_g(\phi(x))$$

which is a contradiction. □

This characterizes all maps between level-wise finite C_p-Tambara fields. The picture can be simplified further.

Proposition 2.7. *(1) Maps* $\underline{\mathbb{F}_{q^n}} \to \mathrm{FP}(\mathbb{F}_{q^{mp}})$ *filter through* $\mathrm{FP}(\mathbb{F}_{q^{n\cdot p^{v_p(m/n)+1}}})$ *where $v_p(m/n)$ is the p-adic valuation function on m/n. Alternatively, such maps may be written as a composition* $\underline{\mathbb{F}_{q^n}} \to \underline{\mathbb{F}_{q^m}} \to \mathrm{FP}(\mathbb{F}_{q^{m\cdot p}})$.

(2) Maps $\underline{\mathbb{F}_{q^n}} \to \mathrm{Coind}_e^{C_p}(\mathbb{F}_{q^m})$ *filter through* $\mathrm{Coind}_e^{C_p}(\mathbb{F}_{q^n})$.

(3) Maps $\mathrm{FP}(\mathbb{F}_{q^{np}}) \to \mathrm{Coind}_e^{C_p}(\mathbb{F}_{q^m})$ *filter through* $\mathrm{Coind}_e^{C_p}(\mathbb{F}_{q^{np}})$.

Proof. (1) $\underline{\mathbb{F}_q^n}$ embeds into $\mathrm{FP}(\mathbb{F}_{q^{n\cdot p^{v_p(m/n)+1}}})$ since $n \mid n \cdot p^{v_p(m/n)+1}$. The embeddings $\mathrm{FP}(\mathbb{F}_{q^{n\cdot p^{v_p(m/n)+1}}}) \to \mathrm{FP}(\mathbb{F}_{q^{mp}})$, which exist since $n \cdot p^{v_p(m/n)+1} \nmid m$, capture all possible embeddings of $\mathbb{F}_{q^n} \to \mathbb{F}_{q^m}$ and $\mathbb{F}_{q^n} \to \mathbb{F}_{q^{mp}}$. Choose the morphisms so that $\mathrm{FP}(\mathbb{F}_{q^{n\cdot p^{v_p(m/n)+1}}}) \to \mathrm{FP}(\mathbb{F}_{q^m})$ is consistent with the original embedding.

To see the other fact, notice that $\underline{\mathbb{F}_{q^n}}$ embeds into the fixed field at each level of $\mathrm{FP}(\mathbb{F}_{q^{m\cdot p}})$, which is always isomorphic to $\mathbb{F}_{q^m}$ and is preserved between levels. So the embedding $\underline{\mathbb{F}_{q^n}} \to \mathrm{FP}(\mathbb{F}_{q^m})$ is determined entirely by how $\mathbb{F}_{q^n}$ embeds into $\mathbb{F}_{q^m}$ at the top level. Choose maps $\underline{\mathbb{F}_{q^n}} \to \underline{\mathbb{F}_{q^m}}$ and $\underline{\mathbb{F}_{q^m}} \to \mathrm{FP}(\mathbb{F}_{q^{m\cdot p}})$ which preserve the embedding of $\mathbb{F}_{q^n}$ into $\mathbb{F}_{q^m}$ at the top level. This is precisely what is necessary. Such maps exist since $\underline{\mathbb{F}_q^n} \to \underline{\mathbb{F}_{q^m}}$ corresponds to all embeddings of $\mathbb{F}_{q^n}$ into $\mathbb{F}_{q^m}$ and likewise $\underline{\mathbb{F}_{q^m}} \to \mathrm{FP}(\mathbb{F}_{q^{m\cdot p}})$ corresponds to all embeddings of $\mathbb{F}_{q^m}$ into itself at the top level.

(2) $\underline{\mathbb{F}_{q^n}}$ is a sub-Tambara field to $\mathrm{Coind}_e^{C_p}(\mathbb{F}_{q^n})$. $\mathrm{Coind}_e^{C_p}(\mathbb{F}_{q^n})$ captures all possible embeddings of $\mathbb{F}_{q^n}$ into $\mathbb{F}_{q^m}$, and so by choosing the appropriate map, $\underline{\mathbb{F}_{q^n}} \to \mathrm{Coind}_e^{C_p}\mathbb{F}_{q^m}$ filters through $\mathrm{Coind}_e^{C_p}\mathbb{F}_{q^n}$ as desired.

(3) The last case has similar logic.

□

Fig. 2 and Fig. 3 summarize the morphism structure.

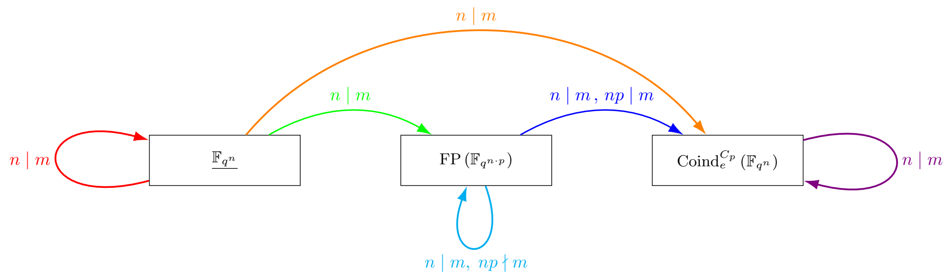

FIGURE 2. Summarizing morphisms between finite C_p Tambara fields for fixed q.

3. PROPERTIES.

3.0.1. *Free Modules.* We show that every map $k \to \ell$ of level-wise-finite C_p-Tambara fields gives ℓ as a free module over k with respects to the map.

Proposition 3.1. *Let k be a Green functor. The free k-module on a single generator in level G/H is isomorphic to the k-module underlying the k-algebra given by the adjunction unit*

$$k \to \mathrm{Coind}_H^G Res_H^G k.$$

Proof. [3] Theorem 4.7. □

With this result in mind, we can make the picture clearer for the C_p Tambara functor case in particular.

Proposition 3.2. *Let, $k \to \ell$ be a map of C_p-Tambara functors. ℓ is a free module over k if and only if it takes one of the following forms:*

(1) $\ell \cong k^{\oplus n}$
(2) $\ell \cong (\mathrm{Coind}_e^{C_p} Res_e^{C_p} k)^{\oplus m}$
(3) $\ell \cong k^{\oplus n} \oplus (\mathrm{Coind}_e^{C_p} Res_e^{C_p} k)^{\oplus m}$

Proof. Notice that ℓ must be a direct sum of free k-modules of a single generator, which take a form according to Proposition 3.1. There are exactly two possible identities for the free module in a single generator over k. These are:

$$\mathrm{Coind}_{C_p}^{C_p} \mathrm{Res}_{C_p}^{C_p} k \cong k$$

and

$$\mathrm{Coind}_e^{C_p} \mathrm{Res}_e^{C_p} k.$$

The three possibilities are direct sums of modules of these forms. □

Now the main result is a matter of checking cases.

Theorem 3.3. *Every nontrivial map $k \to \ell$ of level-wise finite C_p-Tambara fields defines ℓ as a free module over k.*

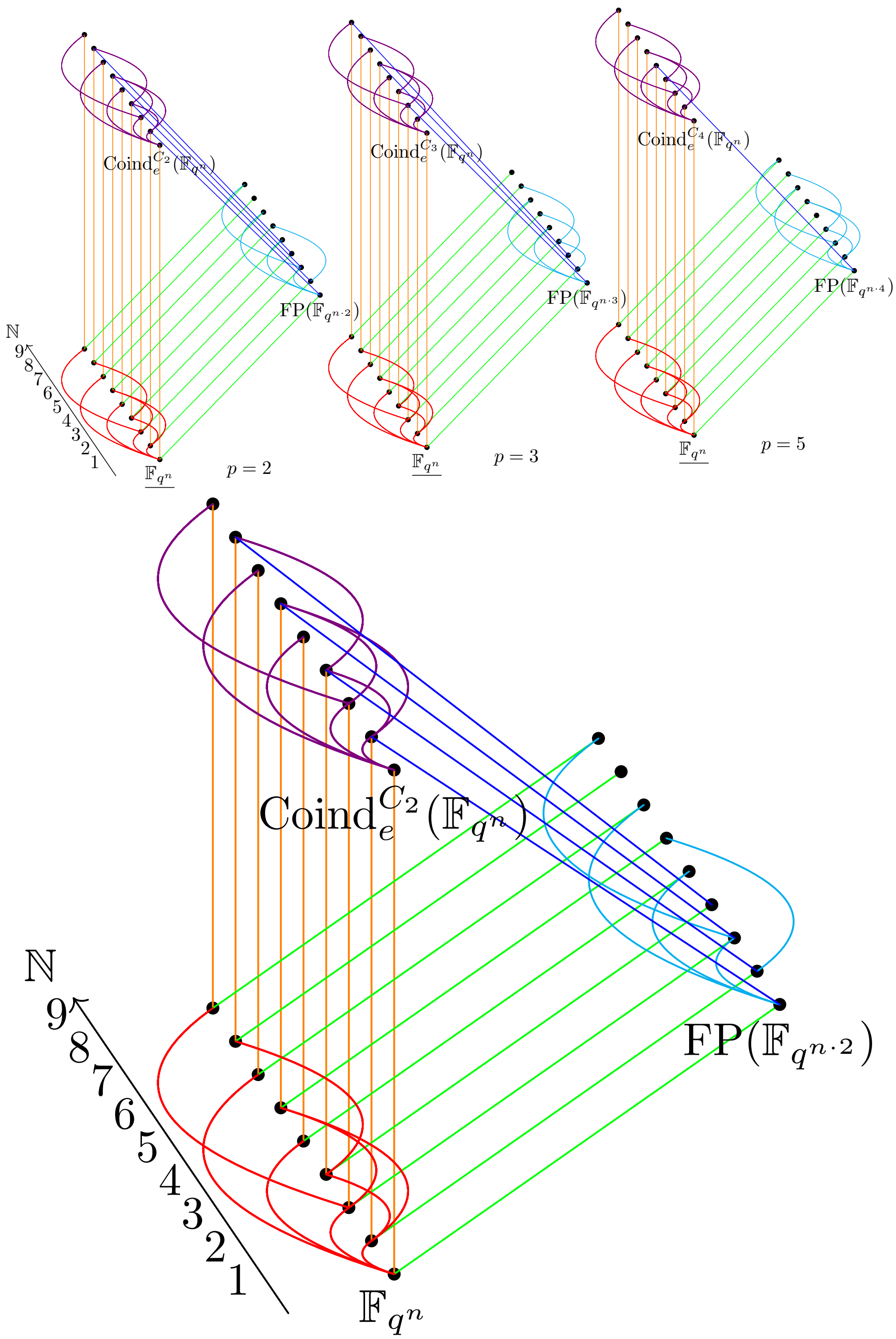

FIGURE 3. Above: morphim structure of finite C_p-Tambara fields with fixed q at $p = 2, 3, 5$. Below: detail for $p = 2$.

Proof. First of all, we compute $\mathrm{Coind}_e^{C_p}\mathrm{Res}_e^{C_p}k$ for the three possibilities for k. These are:

$$\mathrm{Coind}_e^{C_p}\mathrm{Res}_e^{C_p}(\mathrm{Coind}_e^{C_p}\mathbb{F}_q^n) \cong \mathrm{Coind}_e^{C_p}\mathbb{F}_{q^n}$$
$$\mathrm{Coind}_e^{C_p}\mathrm{Res}_e^{C_p}(\underline{\mathbb{F}_{q^n}}) \cong \mathrm{Coind}_e^{C_p}\mathbb{F}_{q^{n\cdot p}}$$
$$\mathrm{Coind}_e^{C_p}\mathrm{Res}_e^{C_p}(\mathrm{FP}(\mathbb{F}_{q^n})) \cong \mathrm{Coind}_e^{C_p}(\mathbb{F}_{q^n}^{\times p}) \cong (\mathrm{Coind}_e^{C_p}\mathbb{F}_{q^n})^{\times p}$$

From this point, the proof requires checking cases. All maps take the form $k \to \ell$. In each case we discuss modules over k, which is why the isomorphisms simplify nicely like they do.

Case 1: $\underline{\mathbb{F}_{q^n}} \to \underline{\mathbb{F}_{q^m}}$ when $n \mid m$. In this case, $\ell \cong k^{\oplus(m/n)}$ since level-wise $\ell(C_p/e) \cong \mathbb{F}_{q^m} \cong (\mathbb{F}_{q^n})^{m/n}$ and the same for $\ell(C_p/C_p)$.

Case 2: $\mathrm{FP}(\mathbb{F}_{q^{n\cdot p}}) \to \mathrm{FP}(\mathbb{F}_{q^{m\cdot p}})$ when $n \mid m$ and $pn \nmid m$. In this case $\ell \cong k^{\oplus(m/n)} \cong \mathrm{FP}(\mathbb{F}_{q^{n\cdot p}})^{\oplus(m/n)}$.

Case 3: $\mathrm{Coind}_e^{C_p}\mathbb{F}_{q^n} \to \mathrm{Coind}_e^{C_p}\mathbb{F}_{q^m}$ where $n \mid m$. Once more, $\ell \cong k^{\oplus m/n} \cong (\mathrm{Coind}_e^{C_p}\mathbb{F}_{q^n})^{\oplus m/n} \cong \mathrm{Coind}_e^{C_p}\mathbb{F}_{q^n}^{\oplus m/n} \cong \mathrm{Coind}_e^{C_p}\mathbb{F}_{q^m}$.

Case 4: $\underline{\mathbb{F}_{q^n}} \to \mathrm{FP}(\mathbb{F}_{q^m})$ where $n \mid m$. As a module over k, we have $\ell \cong (\mathrm{Coind}_e^{C_p}\mathrm{Res}_e^{C_p}k)^{\oplus(m/n)} \cong (\mathrm{Coind}_e^{C_p}\mathbb{F}_{q^n})^{\oplus(m/n)}$. To see this, notice that $(\mathrm{Coind}_e^{C_p}\mathbb{F}_{q^n})^{\oplus(m/n)}(C_p/e) \cong (\mathbb{F}_{q^n}^{\oplus p})^{\oplus(m/n)} \cong \mathbb{F}_{q^{mp}}$ as a $\mathbb{F}_{q^n}$-module. It is similar at the top level where $(\mathrm{Coind}_e^{C_p}\mathbb{F}_{q^n})^{\oplus(m/n)}(C_p/C_p) \cong \mathbb{F}_{q^n}^{\oplus(m/n)} \cong \mathbb{F}_{q^n}$ as an $\mathbb{F}_{q^n}$ module.

Case 5: For $\underline{\mathbb{F}_{q^n}} \to \mathrm{Coind}_e^{C_p}(\mathbb{F}_{q^m})$ we similarly have $\ell \cong (\mathrm{Coind}_e^{C_p}\mathbb{F}_{q^n})^{\oplus(m/n)}$

Case 6: For $\mathrm{FP}(\mathbb{F}_{q^{n\cdot p}}) \to \mathrm{Coind}_e^{C_p}\mathbb{F}_{q^m}$ we similarly calculate $\ell \cong (\mathrm{Coind}_e^{C_p}\mathrm{Res}_e^{C_p}k)^{\oplus(m/np)} \cong ((\mathrm{Coind}_e^{C_p}\mathbb{F}_{q^n})^{\oplus p})^{\oplus(m/np)} \cong \mathrm{Coind}_e^{C_p}\mathbb{F}_{q^m}$ as a module over k.

□

3.0.2. *Étaleness.* All maps between level-wise finite C_p-Tambara fields are étale.

Definition 3.4. ([4])A morphism $R \to S$ of Tambara functors is *formally étale* if $\Omega^1_{S/R} = 0$ and S is flat as a module over (the underlying green functor) R.

Definition 3.5. A morphism of Tambara functors is étale if it is formally étale and finitely presented

We showed in Theorem 3.3 that any morphism $R \to S$ of level-wise finite C_p-Tambara fields has S as a free module over R. Since free implies flat, and level-wise finite implies finitely presented, the remaining work ammounts to showing that the Kähler differentials vanish. The proof will actually not require calculating the Kähler differentials outright since we can scaffold off of results shown in [3] and [10].

Theorem 3.6. *All morphisms $k \to \ell$ of level-wise finite C_p-Tambara fields are étale.*

Proof. The proof consists of checking cases.

(1) $\underline{\mathbb{F}_{q^n}} \to \underline{\mathbb{F}_{q^m}}$. Since every extension $\mathbb{F}_{q^n} \subset \mathbb{F}_{q^m}$ of finite fields is étale then so is $\underline{\mathbb{F}_{q^n}} \to \underline{\mathbb{F}_{q^m}}$ by [10] Corollary 5.2.

(2) In [3] Theorem 5.4 Wisdom proved that to show a map $k \to \ell$ of C_p-Tambara functors is étale, it is sufficient to show all transfers in ℓ are surjective and $k(G/e) \to \ell(G/e)$ is étale. In the remaining cases maps $k \to \ell$ have either $\ell \cong \mathrm{FP}(\mathbb{F}_{q^{np}})$ or $\ell \cong \mathrm{Coind}_e^{C_p}(\mathbb{F}_{q^n})$. In both cases the transfer map is

surjective, in the first case since the Galois trace is surjective onto the fixed field, and in the second because $(x_1, \dots, x_p) \mapsto x_1 + \cdots + x_p$ is also. Moreover, we can check that $k(C_p/e) \to \ell(C_p/e)$ is étale in each case since every finite product of finite separable field extensions is étale. Hence by ̧refetale at bottom each map is étale.

□

3.0.3. *Galois.* Classically, Galois theory allows us to derive a correspondence between subgroups of the automorphism group of certain nicely behaved (i.e. 'Galois') field extensions, and the intermediate fields fixed by these subgroups. We can develop a similar notion of Tambara-Galois field extensions in the context of Tambara fields. We will see that inclusions of level-wise finite C_p-Tambara fields are Galois, and indeed we can exhibit a correspondence analogous to the fundamental theorem of Galois theory from the classical study of Galois field extension.

Definition 3.7. For an inclusion $k \to \ell$ of G-Tambara fields, define

$$G(\ell/k) := \{\sigma \in \mathrm{Aut}(\ell) : \sigma \mid_k = \mathrm{id}\}$$

.

Definition 3.8. An inclusion $k \to \ell$ of G-Tambara fields is Tambara-Galois, or ℓ is Tambara-Galois over k, when the fixed subfield of $G(\ell/k)$ is k.

Recall that in Galois theory L/K is Galois if and only if $G(L/K) = [L : K]$. Now we begin showing that the level-wise finite C_p-Tambara fields are Galois in the above sense.

Lemma 3.9 (Goursat's Lemma)**.** *There is a bijective correspondence between subgroups G of $A \times B$ and quintuples $(\overline{G_1}, G_1, \overline{G_2}, G_2, \theta)$ where $G_1 \trianglelefteq \overline{G_1} \leq A$ and $G_2 \trianglelefteq \overline{G_2} \leq B$ and $\theta : \overline{G_1}/G_1 \to \overline{G_2}/G_2$ is an isomorphism.*

Proof. See [9] for a neat proof.

Going from a quintuple $(\overline{G_1}, G_1, \overline{G_2}, G_2, \theta)$ to a subgroup of $A \times B$ takes the following procedure:

(1) take the natural surjection $\overline{G_1} \times \overline{G_2} \to \overline{G_1}/G_1 \times \overline{G_2}/G_2$
(2) consider the graph $G_\theta = \{([a], [b]) \in \overline{G_1}/G_1 \times \overline{G_2}/G_2 \mid [b] = \theta([a])\}$
(3) then $p^{-1}(G_\theta)$ is our subgroup.

□

Lemma 3.10. *Let $s \mid n$ and σ generate $\mathbb{Z}/n$ and ρ generate C_p. The subgroups of $\mathbb{Z}/n \times C_p$ take exactly the following forms:*

(1) $\mathbb{Z}/s \times C_p$,
(2) $\mathbb{Z}/s \times e$,
(3) and when $sp \mid n$, the diagonal subgroups, $\langle(\sigma^{np/s}, \rho)\rangle$, which are distinct from the above forms.

Proof. According to Lemma 3.9 the subgroups of $\mathbb{Z}/n \times C_p$ correspond to quintuples $(\overline{G_1}, G_1, \overline{G_2}, G_2, \theta)$ where $G_1 \trianglelefteq \overline{G_1} \leq \mathbb{Z}/n$ and $G_2 \trianglelefteq \overline{G_2} \leq C_p$ and $\theta : \overline{G_1}/G_1 \to \overline{G_2}/G_2$ is an isomorphism. Notice that $\overline{G_2}/G_2$ is exactly either C_p or e.

(1) When $\overline{G_2}/G_2 = e$, then $\overline{G_1} = G_1, \overline{G_2} = G_2$, θ is trivial, and $p^{-1}(G_\theta) = \overline{G_1} \times \overline{G_2}$. In particular, $\overline{G_1}$ may be any subgroup of $\mathbb{Z}/n$, and $\overline{G_2}$ any subgroup of C_p. This covers cases (1) and (2) from the statement.
(2) When $p \mid n$, for any $sp \mid n$ we can write $\overline{G_1} = \mathbb{Z}/sp, G_1 = \mathbb{Z}/s, \overline{G_2} = C_p, G_2 = e$ so that $\theta : \overline{G_1}/G_1 \to \overline{G_2}/G_2$ is a map into C_p. The elements of $\mathbb{Z}/sp$ are generated by $\sigma^{n/sp}$ and so $\theta[\sigma^{n/sp}] = [\rho]$, when we choose the appropriate generators. Since $[\sigma^{n/sp}] = \sigma^{n/sp} + \langle\sigma^{n/s}\rangle$, it follows that

$$G = p^{-1}(G_\theta) = \{(a, b) \in \mathbb{Z}/sp \times C_p : \theta([a]) = [b]\} = \langle(\sigma^{np/s}, \rho), (\sigma^{n/s}, e)\rangle.$$

Since $\langle(\sigma^{n/sp}, \rho)\rangle$ contains $\langle(\sigma^{n/s}, e)\rangle$, because we have $(\sigma^{n/sp}, \rho)^p = (\sigma^{n/s}, e)$, then $G = \langle(\sigma^{np/s}, \rho)\rangle$ which has order sp.

$\square$

Theorem 3.11. *(1) The subgroups $\mathbb{Z}/s \leq Aut(\underline{\mathbb{F}_{q^n}}) \cong \mathbb{Z}/n$ for each $s \mid n$ fix $\underline{\mathbb{F}_{q^{n/s}}}$, which is embedded uniquely into $\underline{\mathbb{F}_{q^n}}$ up to automorphisms.*
(2) For $s \mid np$, the subgroups $\mathbb{Z}/s \leq Aut(\mathrm{FP}(\mathbb{F}_{q^{np}})) \cong \mathbb{Z}/np$ fix $\underline{\mathbb{F}_{q^{np/s}}}$ if $(np/s) \mid n$ and $\mathrm{FP}(\mathbb{F}_{q^{np/s}})$ if $(np/s) \nmid n$.
(3) Subgroups of the form $\mathbb{Z}/s \times C_p \leq Aut(\mathrm{Coind}_e^{C_p}(\mathbb{F}_{q^n})) \cong \mathbb{Z}/n \times C_p$ fix $\underline{\mathbb{F}_{q^{n/s}}}$. Subgroups of the form $\mathbb{Z}/s \times e$ fix $\mathrm{Coind}_e^{C_p}(\mathbb{F}_{q^{n/s}})$. When $p \mid n$, for every $sp \mid n$, the diagonal subgroups fix $\mathrm{FP}(\mathbb{F}_{q^{n/s}})$.

Proof. (1) $\mathrm{Aut}(\underline{\mathbb{F}_{q^n}})$ corresponds to the automorphisms of $\mathbb{F}_{q^n}$ per Proposition 2.3. Hence the fixed field of $\mathbb{Z}/s \leq \mathbb{Z}/n \cong \mathrm{Aut}(\underline{\mathbb{F}_{q^n}})$ corresponds to the fixed field in the classical Galois setting on both levels, which is $\underline{\mathbb{F}_{q^{n/s}}}$.
(2) consider $s \mid np$ and the corresponding subgroup $\mathbb{Z}/s \leq \mathbb{Z}/np \cong \mathrm{Aut}(\mathrm{FP}(\mathbb{F}_{q^{np}}))$. Letting $\sigma(x) = x^q$ be the standard frobenius map,
  (a) At the bottom level $\mathrm{FP}(\mathbb{F}_{q^{np}})(C_p/e) = \mathbb{F}_{q^{np}}$ the automorphisms of $\mathbb{Z}/s$ are generated by $\sigma^{np/s}$. The fixed field is $\mathbb{F}_{q^{np/s}}$.
  (b) At the top level, restricting $\sigma^{np/s}$ to $\mathrm{FP}(\mathbb{F}_{q^{np}})(C_p/C_p) = \mathbb{F}_{q^n}$ the fixed field is $\mathbb{F}_{q^{\gcd(n,np/s)}}$.

When $(np/s) \mid n$, then the fixed field is $\mathbb{F}_{q^{np/s}}$ at the top level. Moreover, at the bottom level the fixed field is $\mathbb{F}_{q^{(np/s)}} \leq \mathbb{F}_{q^n}$ so that the C_p action is trivial on the bottom and top levels, consistent with the structure of $\underline{\mathbb{F}_{q^{np/s}}}$.

When $(np/s) \nmid n$, then $\gcd(n, np/s) = n/s$ and so the fixed field is $\mathbb{F}_{q^{n/s}}$ at the top level it is $\mathbb{F}_{q^{np/s}}$ for which there is a non-trivial C_p action. This is consistent with the structure of $\mathrm{FP}(\mathbb{F}_{q^{np/s}})$.
(3) To understand the structure of $\mathrm{Aut}(\mathrm{Coind}_e^{C_p}(\mathbb{F}_{q^n})) \cong \mathbb{Z}/n \times C_p$ recall the $\mathbb{Z}/n = \langle\sigma\rangle$ component acts as σ^k on the top level $\mathbb{F}_{q^n}$ and in each entry on the bottom level, $\mathbb{F}_{q^n}^{\times p}$. The C_p component, ρ acts by cyclically permuting the entries on the bottom level, and acts trivially on the top level.

Per Lemma 3.10 there are three possible subgroups of $\mathrm{Aut}(\mathrm{Coind}_e^{C_p}(\mathbb{F}_{q^n}))$.
  (a) If the subgroup is $\mathbb{Z}/s \times C_p$, then the action is generated by $(\sigma^{n/s}, \rho)$. On the top level, the fixed field is $\mathbb{F}_{q^{n/s}}$. Because ρ permutes the entries on the bottom level, elements $(x_1, \ldots, x_p)$ are only fixed when

$$x_1 = x_2 = \cdots = x_p.$$

Since $\sigma^{n/s}$ also acts on the entries, the fixed field on the bottom level is $\mathbb{F}_{q^{n/s}}$. In paritcular, the fixed functor is embedded into$\mathrm{Coind}_e^{C_p}(\mathbb{F}_{q^n})$ the same way $\underline{\mathbb{F}_{q^{n/s}}}$ is, and in fact is precisely that functor.

(b) If the subgroup is $\mathbb{Z}/s \times e$ then it is generated by $(\sigma^{n/s}, e)$. On the top level, the fixed field is $\mathbb{F}_{q^{n/s}}$, and on the bottom level, since the action is applied entry-by-entry, the fixed field is $\mathbb{F}_{q^{n/s}}^{\times p}$. This is the same structure as $\mathrm{Coind}_e^{C_p}\mathbb{F}_{q^{n/s}}$.

(c) if $sp \mid n$, then the subgroup could also be $\langle(\sigma^{n/sp}, \rho)\rangle$. On the top level, where the action of ρ is trivial, $\mathbb{F}_{q^{n/sp}}$ is fixed. The behavior is interesting on the bottom level. Notice that since $(\sigma^{n/sp}, \rho)^p = (\sigma^{n/s}, e)$, then the fixed field on the bottom level is a sub-field of $\mathbb{F}_{q^{n/s}}$. However, notice also that $\mathbb{F}_{q^{n/s}}$ has a unique subfield of order p. Indeed, the action of $\sigma^{n/sp}$ is to permute these elements cyclically. Letting $x_0, \ldots, x_{p-1}$ be these p elements so that $\sigma^{n/sp}(x_i) = x_{i-1}$ with the index interpreted modulo p. However, ρ also has the effect of permuting entries cyclically so that

$$\rho(\sigma^{n/sp}(x_0), \ldots, \sigma^{n/sp}(x_{p-1}))) = \rho(x_{p-1}, x_0, \ldots, x_{p-2}) = (x_0, \ldots, x_{p-1})$$

and these p cyclic rotations are fixed. Notice also that when $x \in \mathbb{F}_{q^{n/sp}} \leq \mathbb{F}_{q^n}$ the fixed field from $\sigma^{n/sp}$ then $(x, \ldots, x)$ is fixed by $(\sigma^{n/sp}, \rho)$.

In particular, we know that the fixed field contains the following group.

$$G = \langle(x_0, \ldots, x_{p-1}), \{(x, \ldots, x) : x \in \mathbb{F}_{q^{n/sp}}\}\rangle.$$

This consists of two subgroups $\langle(x_0, \ldots, x_{p-1})\rangle$ which has order p and $\{(x, \ldots, x) : x \in \mathbb{F}_{q^{n/sp}}\}$ of order n/sp. They intersect only trivially at $(e, \ldots, e)$, and so hence $|G| = p \cdot (n/sp) = n/s$. We now that the fixed field is contained in $\mathbb{F}_{q^{n/s}}$ and contains G. By a counting argument, the fixed field on the bottom level is precisely $\mathbb{F}_{q^{n/s}}$, embedded in the same manner as in $\mathrm{FP}(\mathbb{F}_{q^{n/s}}) \to \mathrm{Coind}_e^{C_p}(\mathbb{F}_{q^n})$.

$\square$

Moreover all level-wise finite C_p Tambara-fields share the same *prime subfield* – the minimal subfield fixed by the full automorphism group – to borrow terminology from classical Galois theory.

Remark 3.12. (1) The automorphisms $\mathrm{Aut}(\underline{\mathbb{F}_{q^n}}) \cong \mathbb{Z}/n$ of $\underline{F_{q^n}}$ fix $\underline{\mathbb{F}_q}$.

(2) $\mathrm{Aut}(\mathrm{FP}(\mathbb{F}_{q^{np}})) \cong \mathbb{Z}/np$ fixes $\underline{\mathbb{F}_q}$.

(3) $\mathrm{Aut}(\mathrm{Coind}_e^{C_p}\mathbb{F}_{q^n}) \cong \mathbb{Z}/n \times C_p$ fixes $\underline{\mathbb{F}_q}$.

These prime subfields are common choices from Bredon cohomology coefficients, especially $\underline{\mathbb{F}_2}$.

Corollary 3.13. *Every inclusion $k \to \ell$ of level-wise finite C_p-Tambara fields is Galois.*

Proof. Immediate from Theorem 3.11. $\square$

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