BRAID GROUPS IN CONFIGURATION SPACES AND MAPPING CLASS GROUPS

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ABSTRACT. This expository paper is an introduction to the braid group from three different perspectives. We start with the geometric definition of the braid group. Then, we provide some basic topology background to define the braid group as the fundamental group of the configuration space of the complex plane. With this definition, we work on proving that the braid group is torsion-free. Lastly, we define the braid group through the mapping class group and prove that the center of the braid group is infinite cyclic and generated by the Dehn twist about the boundary.

Contents

1. Introduction

This paper introduces the concept of the braid group, which was first used by Emil Artin in 1925. The braid group is named for its geometric counterpart: a braid with strands that cross over each other. One important feature of the braid group is that it has many equivalent definitions in many different mathematical areas, including in the study of configuration spaces, mapping class groups, polynomials, and hyperplane complements. Having this large variety of definitions in different areas allows the braid group to transfer its properties from one field to another. For example, we can prove that the braid group is torsion-free via its representation in the configuration spaces and transfer it to its representation in the field of mapping class groups, and conclude that the mapping class group of an n-punctured disk is torsion-free.

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In this paper, we will start with the geometric definition and Artin presentation of the braid group to gain an intuition of the braid group. Then we will move on to the definition of the braid group via the configuration space and prove that the braid group is torsion-free. Lastly, we will dive into the definition of the braid group via the mapping class group and prove that the center of the braid group is infinite cyclic and generated by the Dehn twist around the boundary. The reader can find more expository information about the braid group in Wilson's paper "The Geometry and Topology of Braid Groups." [6].

2. Geometric Definition of Braid Groups and Artin Presentation

In this section, we will provide the geometric definition of the braid group, along with some diagrams to help readers better understand the concept. First, we will define the braid group. We will use the definition provided by Farb and Margalit in their book *A Primer on Mapping Class Groups* [2].

Definition 2.1. Let $p_1, \dots, p_n$ be distinguished points in the complex plane $\mathbb{C}$. A *braid* is a collection of n paths

$$f_i : [0,1] \longrightarrow \mathbb{C} \times [0,1], \qquad 1 \leq i \leq n,$$

called *strands*, together with a permutation $\bar{s}$ of $\{1, \dots, n\}$ such that:

(i) the strands $f_i([0,1])$ are disjoint,
(ii) $f_i(0) = p_i$,
(iii) $f_i(1) = p_{\bar{s}(i)}$,
(iv) $f_i(t) \in \mathbb{C} \times \{t\}$.

The *braid group on n strands*, denoted B_n, is the group of isotopy classes of braids. For now, readers can consider two braids to be equivalent (isotopic) if they can be continuously deformed from one to the other by only bending the strands. We will define isotopy in Section 3 when we introduce homotopy.

We now show that the braid group is a group.

Proposition 2.2. *The braid group is, in fact, a group.*

Proof. In order to show this statement, we first define the operation for the group. Let $f, g \in B_n$, where f and g are two representative braids for their isotopy classes. Note that f_i, g_i, for every $1 \leq i \leq n$, represent the ith strand in f and g, respectively. We define $f \cdot g$ as:

$$(f_i \cdot g_i)(t) = \begin{cases} f_i(2t), & 0 \leq t \leq \frac{1}{2}, \\ g_{\bar{s}(i)}(2t-1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Here, $\bar{s}$ is a permutation of $\{1, \dots, n\}$. This operation scales f and g by $\frac{1}{2}$, and then concatenates the strands of f on top of the corresponding strands of g, giving a new strand in B_n. The associativity of the group can be verified by reparameterizing t. We define the inverse braid, f^{-1}, by taking the reflection of the original braid through the plane $\mathbb{C} \times \{1\}$, meaning for each f_i, $f_i^{-1}(t) = f_i(1-t)$. The identity braid, id, will be the braid with $\mathrm{id}_i(t) = p_i$ for every $t \in [0,1]$. Geometrically, the identity braid is n straight strands with no crossings at all. Notice that $f \cdot f^{-1}$ is isotopic to the trivial braid, thus showing that the two braids are indeed inverses. $\square$

For the braid group, there is a special subgroup called *the pure braid group*, denoted PB_n. The pure braid group can be considered as the braid group with $f_i(0) = f_i(1) = p_i$.

Next, we will look at the presentations of the braid group. We define the half-twist of the ith strand and the $(i+1)$th strand as σ_i shown below in Figure 1. Every braid in B_n can be written as a word in the half-twists $\sigma_1, \dots, \sigma_{n-1}$.

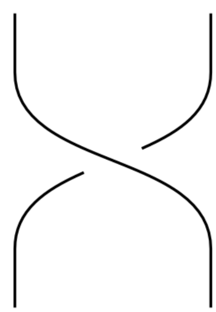

FIGURE 1. Diagram of σ_i

Emil Artin described the braid group presentations in 1947 as follows:

Definition 2.3. The *Artin braid group* on n strands, B_n, is the finitely generated group with generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ that satisfy

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{whenever } |i-j| \geq 2,$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{for } i = 1, \dots, n-2.$$

We provide the diagrams of the two relations (Figure 2 and Figure 3) below for readers to have a better understanding of the Artin presentation.

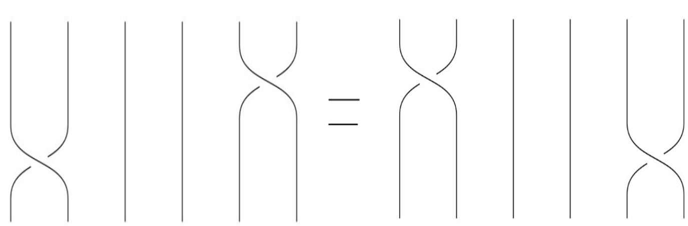

FIGURE 2. Diagram of $\sigma_i\sigma_j = \sigma_j\sigma_i$

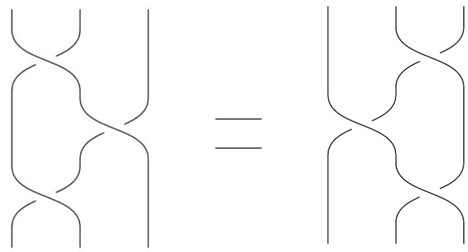

FIGURE 3. Diagram of $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

For the pure braid group with n strands, Artin showed that it is generated by $T_{i,j}$, where $T_{i,j}$ is defined as $(\sigma_{j-1}\,\cdots\,\sigma_{i+1})\,\sigma_i^2\,(\sigma_{j-1}\,\cdots\,\sigma_{i+1})^{-1}$. The geometric representation of $T_{i,j}$ is shown in Figure 4.

Definition 2.4. The *pure braid group* on n strands, PB_n, is the finitely generated group with generators $T_{i,j}$ for $1 \leq i < j \leq n$ that satisfy

$$\begin{aligned}
[T_{p,q}, T_{r,s}] &= 1 \quad \text{for } p < q < r < s, \\
[T_{p,s}, T_{q,r}] &= 1 \quad \text{for } p < q < r < s, \\
T_{p,r}\,T_{q,r}\,T_{p,q} &= T_{q,r}\,T_{p,q}\,T_{p,r} = T_{p,q}\,T_{p,r}\,T_{q,r} \quad \text{for } p < q < r, \\
[T_{r,s}\,T_{p,r}\,T_{r,s}^{-1},\,T_{q,s}] &= 1 \quad \text{for } p < q < r < s.
\end{aligned}$$

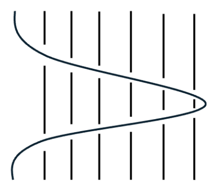

FIGURE 4. Diagram of $T_{i,j}$

We will not verify Artin presentation of the braid group and the pure braid group in this paper. For readers interested in this topic, please refer to Joshua Lieber's "Introduction to Braid Groups"[5].

3. Braid Groups via Configuration Spaces

In this section, we will define the braid group using the fundamental group and the configuration space. Then we will prove the braid group is torsion-free.

3.1. **Braid Groups are the Fundamental Groups of the Configuration Space of the Complex Plane.**

Before discussing the braid group through the lens of the configuration space, we will introduce some algebraic topology concepts related to this topic. The first is the fundamental group, which is built up from homotopy. Note that we will use the definitions and theorems from Hatcher [1] in this subsection.

Let X and Y be topological spaces and let $p, q : X \to Y$ be continuous maps. A *homotopy* from p to q is a continuous map $H : X \times I \to Y$ such that $H(x, 0) = p(x)$ and $H(x, 1) = q(x)$ for all $x \in X$, where $I = [0, 1]$ is the unit interval. We say p and q are *homotopic*, written as $p \simeq q$, if there exists such H. *Isotopy* can be considered as a special case of homotopy, where the path from p to q must lie entirely in the space of embeddings. Next, we will need to understand what a *loop* is to understand the fundamental group. A *path* in a space X is a continuous map $f : I \to X$, where I is the unit interval. A *loop* is a path f with the additional condition that $f(0) = f(1)$. X is called *path-connected* if there exists a path for any two points in X. Finally, we are capable of defining the fundamental group.

Definition 3.1. The *fundamental group* of a topological space X with a base point x_0, denoted as $\pi_1(X, x_0)$, is the set of all loops starting and ending at x_0, up to homotopy.

Proposition 3.2. *The fundamental group of a topological space X is a group.*

Proof. The proof is similar to the proof of Proposition 2.2. □

Now we are ready to understand the relation between the braid group and the configuration space.

Definition 3.3. For a topological space M and a positive integer n, the *unordered configuration space of M on n points* is

$$\mathrm{UConf}_n(M) \;=\; \big\{\{m_1, m_2, \ldots, m_n\} \subset M \;\big|\; m_i \neq m_j \text{ for } i \neq j\big\}.$$

Thus $\mathrm{UConf}_n(M)$ is composed of n-element subsets of M.

Similarly, the *ordered configuration space of M on n points* is

$$\mathrm{Conf}_n(M) \;=\; \big\{(m_1, m_2, \ldots, m_n) \in M^n \;\big|\; m_i \neq m_j \text{ for } i \neq j\big\}.$$

From the definitions above, we know that the unordered configuration space is obtained from the corresponding ordered configuration space by quotienting out the free action of the symmetric group, S_n, that permutes the n coordinates.

Definition 3.4. The n-strand *braid group* is the fundamental group of the unordered configuration space of the complex plane:

$$B_n \;=\; \pi_1\big(\mathrm{UConf}_n(\mathbb{C})\big).$$

The n-strand *pure braid group* is the fundamental group of the ordered configuration space of the complex plane:

$$PB_n \;=\; \pi_1\big(\mathrm{Conf}_n(\mathbb{C})\big).$$

Remark 3.5. Connecting this definition of the braid group and the pure braid group back to the definition in Section 2, we can see that since each strand of a braid is a map $f_i : [0, 1] \to \mathbb{C} \times [0, 1]$, the intersection of any slice $\mathbb{C} \times \{t\}$ with any strand is a point in $\mathrm{UConf}_n(\mathbb{C})$. Thus, the full collection of slices gives an element of $\pi_1(\mathrm{UConf}_n(\mathbb{C}))$. Intuitively, a braid is tracing out a loop in the unordered (respectively, the ordered) configuration space of $\mathbb{C}$. The isotopy of the braids corresponds exactly to the homotopy of loops in $\mathbb{C}$.

3.2. **Braid Groups are Torsion-Free.**

One important property of the braid group is that it is torsion-free. Recall that a *torsion* element $g \in G$, where G is a group with identity element e, is an element for which there exists a positive integer n such that $g^n = e$. The group G is called torsion-free if the only torsion element is the identity element e.

To prove this property, we will show that the configuration space is an Eilenberg-MacLane space, denoted as $K(G, 1)$. Below is the definition of the $K(G, 1)$ space.

Definition 3.6. A path-connected space X whose fundamental group is isomorphic to a given group G and which has a contractible universal covering space is called a $K(G, 1)$ space.

It is known that for a finite $K(G, 1)$, $\pi_1(K(G, 1))$ is torsion-free. The readers can refer to [4] for the detailed proof of this claim. Since, by Definition 3.6, $\pi_1(K(G, 1)) \cong G$, G is torsion-free. Thus, our eventual goal of proving that the braid group is torsion-free is to show that $\mathrm{UConf}_n(\mathbb{C})$ is a $K(B_n, 1)$ space, and $\mathrm{Conf}_n(\mathbb{C})$ is a $K(PB_n, 1)$ space. However, before we delve into the formal proof, we will provide some background definitions on higher homotopy groups, exact sequences, and fiber bundles. The readers can read through Hatcher [1, chap. 4] for more information on these definitions.

A homotopy group, denoted $\pi_n(X, x_0)$, for a space X with a base point $x_0 \in X$ is the set of homotopy classes of maps $f : (I^n, \partial I^n) \longrightarrow (X, x_0)$, where homotopies are required to satisfy $f_t(\partial I^n) = x_0$ for all t. We can see that the previous definition of the fundamental group corresponds to this definition when $n = 1$, since $(I, \partial I) \longrightarrow (X, X_0)$ is mapping the unit interval onto X, while fixing the two endpoints at x_0, forming a loop.

Remark 3.7. Note that for a finite CW complex X, the second condition of a $K(G, 1)$ space is equivalent to saying that all the higher homotopy groups are trivial(i.e., $\pi_k(X)$ are trivial for every $k \geq 2$). This is because a covering space projection $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ induces isomorphism $p_* : \pi_k(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_k(X, x_0)$ for every $k \geq 2$, and a universal cover is simply-connected (path-connected and trivial fundamental group).

Next, we will introduce exact sequences and a proposition that will be frequently used in the actual proof that braid groups are torsion-free.

Definition 3.8. A sequence of homomorphisms of groups

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \cdots$$

is *exact at* A_n if $\ker \alpha_n = \operatorname{Im} \alpha_{n+1}$, which means $\alpha_n \circ \alpha_{n+1}$ is trivial. It is an *exact sequence* if it is exact at every term. An exact sequence of the form

$$1 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 1$$

is called a *short exact sequence*. Notice that $\operatorname{Im} f = \ker g$.

Proposition 3.9. *Let*

$$1 \xrightarrow{i} A \xrightarrow{j} B \xrightarrow{k} 1$$

be a short exact sequence of groups. Then j is an isomorphism, so $A \cong B$.

Proof. To show j is an isomorphism, we only need to show that it is bijective because j is a homomorphism given by the definition of the exact sequence. Since i is a homomorphism that maps the identity to the identity element in A, its image is just the identity element in A. We know that $\mathrm{Im}(i) = \ker(j)$, so $\ker(j)$ is only the identity element in A, proving j is injective. Since k maps the entire B to the identity element, the kernel of k is B. From the exactness of B, $\mathrm{Im}(j) = \ker(k) = B$, it follows that j is also surjective. Thus, j is bijective and is an isomorphism. $\square$

We now define the concepts of fibration and fiber bundle, which are the last two concepts related to our proof.

Definition 3.10. A map $p : E \longrightarrow B$ is said to have the *homotopy lifting property* with respect to a space X if, given a homotopy $g_t : X \longrightarrow B$ and a map $\tilde{g_0} : X \longrightarrow E$ lifting g_0, so $p \cdot \tilde{g_0} = g_0$, then there exists a homotopy $\tilde{g_t} : X \longrightarrow E$ lifting g_t.

Definition 3.11. A *fibration* is a map $p : E \longrightarrow B$ having the homotopy lifting property with respect to all spaces X.

Theorem 3.12. [1, p. 376] *Suppose $p\colon E \to B$ has the homotopy lifting property with respect to disks D^k for all $k \geq 0$. Choose basepoints $b_0 \in B$ and $x_0 \in F := p^{-1}(b_0)$. Then the induced map $p_*\colon \pi_n(E, F, x_0) \longrightarrow \pi_n(B, b_0)$ is an isomorphism for every $n \geq 1$. Consequently, if B is path-connected, there is a long exact sequence*

$$\cdots \longrightarrow \pi_n(F, x_0) \longrightarrow \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \longrightarrow \pi_{n-1}(F, x_0) \longrightarrow \cdots$$

We will not discuss the proof of this theorem in this paper, since our primary focus is on the braid group. The reader should refer to Hatcher [1] for the details of this proof.

Definition 3.13. A *fiber bundle* structure on a space E, with fiber F, consists of a projection map $p : E \longrightarrow B$ such that each point $b \in B$ has an open neighborhood U for which there exists a homeomorphism $h : p^{-1}(U) \longrightarrow U \times F$ making the diagram below commute, where proj_1 is the natural projection onto the first factor.

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "h"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\mathrm{proj}_1"] \\ & U & \end{tikzcd}$$

Note that every fiber bundle induces the long exact sequence in Theorem 3.12, because for a fiber bundle, the open neighborhood U gives the map p the homotopy lifting property required.

We now shift to discuss two propositions about the braid group that will be used in our proof.

Proposition 3.14. *There is a short exact sequence*

$$1 \longrightarrow PB_n \longrightarrow B_n \xrightarrow{\eta} S_n \longrightarrow 1.$$

Proof. For each braid in B_n, we can associate the permutation it induces on its strands and an element of the symmetric group S_n, giving a well-defined group homomorphism η from B_n to S_n. Note that $\eta(\sigma_i) = (i, i+1)$ for $i = 1, ..., n-1$. The kernel of η is the subgroup of B_n that induces the trivial permutation, which is precisely PB_n. Since the map from PB_n to B_n is just the inclusion map, the image of the map is PB_n itself, which is equal to the kernel of η, proving the exactness of B_n. The exactness of the remaining terms is all given by the corresponding definitions. $\square$

Proposition 3.15. *For $n \geq 1$ there is a short exact sequence*

$$1 \longrightarrow F_n \xrightarrow{\iota} PB_{n+1} \xrightarrow{\rho} PB_n \longrightarrow 1.$$

Proof. Given a pure braid $\beta \in PB_{n+1}$, we define the map $\rho : PB_{n+1} \longrightarrow PB_n$ to remove the $(n+1)th$ strand, so that $\rho(\beta) \in PB_n$. The map ρ is a well-defined homomorphism and clearly surjective, proving the exactness of PB_n.

We note that $\pi_1(\bigvee_{i=1}^n S^1) \cong F_n$. Since $\mathbb{C}_n$ deformation retracts to $\bigvee_{i=1}^n S^1$, $F_n \cong \pi_1(\mathbb{C}_n)$. Thus, the map ι can be viewed as mapping $\pi_1(\mathbb{C}_n)$ into PB_{n+1}. The kernel of ρ consists of the pure braids in PB_{n+1} that make the first n strands the trivial braid, meaning that we can think of the first n strands as n vertical strands up to isotopy. The image of ι is letting the (n+1)st strand follow the trace of the loops in $\pi_1(\mathbb{C}_n)$, while fixing the first n strands up to isotopy, which is exactly the kernel of ρ, so $\mathrm{Im}(\iota) = \ker(\rho)$. The map ι is injective because distinct words in the free group give non-isotopic motions of the last strand, proving exactness of F_n. $\square$

Theorem 3.16. [4]*(Fadell Neuwirth Fibration) For $n \geq 1$, the map*

$$\begin{aligned} p : \mathrm{Conf}_{n+1}(\mathbb{C}) &\longrightarrow \mathrm{Conf}_n(\mathbb{C}) \\ (z_1, \ldots, z_{n+1}) &\longmapsto (z_1, \ldots, z_n) \end{aligned}$$

is a locally trivial fiber bundle.

Remark 3.17. We will make a remark on this theorem, but not prove it in this paper. First, the map p can be viewed as the "forgetful" map that forgets the (n+1)st point in the far right. Thus, the fiber of a given base point $(z_1, \ldots, z_n) \in \mathrm{Conf}_n(\mathbb{C})$ is just the n-punctured complex plane, $\mathbb{C} \setminus (z_1, \ldots, z_n)$.

Another fact about this fiber bundle is that it admits a section $s : \mathrm{Conf}_n \longrightarrow \mathrm{Conf}_{n+1}$, defined as $s((z_1, \ldots, z_n)) = (z_1, \ldots, z_n, |z_1| + \cdots + |z_n| + 1)$.

Now, we are ready to walk through the formal proof that the braid group is torsion-free. For simplicity, we will write $\mathrm{Conf}_n(\mathbb{C})$ as M_n and $\mathrm{UConf}_n(\mathbb{C})$ as N_n.

Proposition 3.18. *The braid group is torsion-free.*

Proof. First, by the Fadell-Neuwirth fibration, the map $p : M_{n+1}(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ is a locally trivial fiber bundle, and each fiber F is isomorphic to the complex plane $\mathbb{C}$ with n points removed, $\mathbb{C}_n$. Since we know that $\mathbb{C}_n$ can be deformation retracted to $\bigvee_{i=1}^n S^1$. Thus,

$$\begin{aligned} \pi_1(F) &\cong \pi_1(\mathbb{C}_n) \cong F_n \\ \pi_2(F) &\cong \pi_2(\mathbb{C}_n) \cong 1 \end{aligned}$$

Because $\bigvee_{i=1}^n S^1$ is a 1-dimensional CW complex, by cellular approximation, $\pi_2(\bigvee_{i=1}^n S^1) \cong 1$.

From the long exact sequence of homotopy groups of the Fadell-Neuwirth fibration, we have:

$$\cdots \longrightarrow \pi_2(\mathbb{C}_n) \longrightarrow \pi_2(M_{n+1}) \xrightarrow{p_*} \pi_2(M_n) \xrightarrow{\partial} \pi_1(\mathbb{C}_n) \longrightarrow \pi_1(M_{n+1}) \longrightarrow \pi_1(M_n) \longrightarrow 1$$

However, since we know that $\pi_2(\mathbb{C}_n)$ is trivial, $\pi_1(\mathbb{C}_n) \cong F_n$, and $PB_n \cong \pi_1(M_n)$, the long exact sequence can be written as:

$$\cdots \longrightarrow 1 \longrightarrow \pi_2(M_{n+1}) \xrightarrow{p_*} \pi_2(M_n) \xrightarrow{\partial} F_n \longrightarrow PB_{n+1} \longrightarrow PB_n \longrightarrow 1$$

Note that p_* is the induced homomorphism from the map p in the Fadell-Neuwirth fibration. From Remark 3.17, we know that there is a section $s : M_n \longrightarrow M_{n+1}$, which is obtained by adding a point on the far right of M_n. Since s is continuous, there exists an induced homomorphism $s_* : \pi_2(M_n) \longrightarrow \pi_2(M_{n+1})$. Note that the composite map $p \circ s$ is just adding an (n+1)st point in the far right of the complex plane, then forgetting that point, which is exactly the identity map in M_n. Thus,

$$p_* \circ s_* = (p \circ s)_* = (\mathrm{id}_{M_n})_* = \mathrm{id}_{\pi_2(M_n)}$$

Since p_* is right-invertible, p_* is surjective. Thus, $\mathrm{Im}(p_*) = \pi_2(M_n)$. But, by exactness, $\mathrm{Im}(p_*) = \ker(\partial)$. Thus, we have the short exact sequence from this long exact sequence that:

$$1 \longrightarrow \pi_2(M_{n+1}) \xrightarrow{p_*} \pi_2(M_n) \xrightarrow{\partial} 1$$

Applying Proposition 3.9 to this short exact sequence, we get

$$\pi_2(M_{n+1}) \cong \pi_2(M_n)$$

Now we will prove inductively that $\pi_2(M_n) \cong 1$ for every $n \geq 1$. For the base case $n = 1$, we know that $M_1 \cong \mathbb{C}$, and $\mathbb{C}$ is contractible, so $\pi_2(\mathbb{C}) \cong 1$.

For the inductive step, we assume that $\pi_2(M_n) \cong 1$, then by the isomorphism we get above, we have $\pi_2(M_{n+1}) \cong \pi_2(M_n) \cong 1$. Thus, we finish the proof that $\pi_2(M_n) \cong 1$.

Last, we will prove inductively that all the higher homotopy groups of M_n are trivial (i.e. $\pi_k(M_n) \cong 1$ for every $k \geq 2$). We have already shown the base case in the previous inductive proof. The key step of the proof is to notice that for all $k \geq 2$, $\pi_k(F) \cong \pi_k(\mathbb{C}_n) \cong 1$, which then by a similar argument for the case of $\pi_2(M_n)$, we can force an isomorphism from the long exact sequence and get $\pi_k(M_{n+1}) \cong \pi_k(M_n)$. Then by the same argument used in the previous induction for the base case $k = 2$, we conclude that for every $k \geq 2$

$$\pi_k(M_n) \cong 1$$

Thus, by the definition of a $K(G, 1)$ space, we can see that M_n is a $K(PB_n, 1)$. Thus, the n-strand pure braid group is torsion-free.

Since N_n is the space obtained by quotienting out the symmetric group S_n from M_n, there exists a natural projection $h : M_n \longrightarrow N_n$. This map is the regular covering map with deck group S_n. Thus, M_n is a covering space of N_n. Because the universal cover of M_n is contractible, it follows that the universal cover of N_n is also contractible. Thus, N_n is also an Eilenberg-MacLane space, making it torsion-free. □

4. Braid Groups via Mapping Class Groups

In this section, we will look at the braid group through the lens of the mapping class group. We will first introduce the definition of the mapping class group. The definitions, theorems, and proofs for this section will be based on *A Primer on*

Mapping Class Groups [2]. Then we will give an example of the mapping class group, specifically the mapping class group of an annulus, which is isomorphic to $\mathbb{Z}$. Lastly, we will show that the center of the braid group is infinite cyclic and generated by the Dehn twist around the boundary.

4.1. **Braid Groups are the Mapping Class Groups of the n-punctured Disk.**

We start by defining the mapping class group.

Definition 4.1. Let $S = S_{g,n}$ be the oriented surface of genus g with n punctures. The *mapping class group* of S, denoted as Mod(S), is the group

$$\mathrm{Mod(S)} = \pi_0\big(Homeo^+(S, \partial S)\big).$$

This definition means that the mapping class group of an oriented surface is the group of all orientation-preserving homeomorphisms that fix the boundary of the surface pointwise, up to isotopy. To help the readers understand, we will show why $\mathrm{Mod}(\mathbb{D}^2)$ is trivial. This result also comes out to be an important lemma in the theory of mapping class groups, known as the Alexander Lemma.

Proposition 4.2. *(Alexander Lemma)* $\mathrm{Mod}(\mathbb{D}^2)$ *is trivial*

Proof. We first define $p : \mathbb{D}^2 \longrightarrow \mathbb{D}^2$ to be a homeomorphism that restricts to the identity on $\partial\mathbb{D}^2$. We define

$$H(x,t) = \begin{cases} (1-t)p(\frac{x}{1-t}), & 0 \leq |x| < 1-t, \\ x, & 1-t \leq |x| \leq 1. \end{cases}$$

for $0 \leq t \leq 1$. We can see that $H(x,t)$ is a well-defined isotopy for any orientation-preserving homeomorphism in $\mathbb{D}^2$ that fixes the boundary to the identity. Figure 5 provides a visual representation of the isotopy $H(x,t)$. □

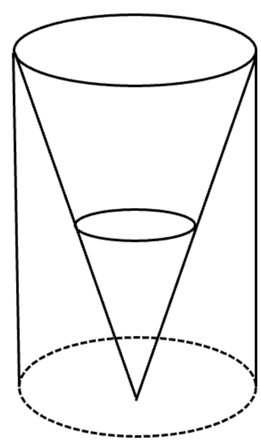

FIGURE 5. Diagram of $H(x,t)$

From the Alexander Lemma, we can immediately conclude that $\mathrm{Mod}(\mathbb{D}^2_1)$ (i.e. the mapping class group of a closed disk with one puncture point) is also trivial.

Next, we will move on to another example: the mapping class group of an annulus is $\mathbb{Z}$. This example will help clarify the concept of Dehn twists and will be directly applied in the later proof relating to the center of the braid group. The proof of the example will be based on the proof in *A Primer on Mapping Class Groups* [2, p. 51], but we will provide more detail in the proof.

Proposition 4.3. $\mathrm{Mod}(A) \cong \mathbb{Z}$

Proof. Note that the universal cover of an annulus, $\tilde{A}$, is the infinite strip $\mathbb{R} \times [0, 1]$. There exists a map $p : \tilde{A} \longrightarrow A$ defined as $p(x, y) = (e^{2\pi i x}, y)$. Observe that shifting x by an integer n does not change $p(x, y)$, so the deck transformation group for $\tilde{A}$ is generated by $T : (x, y) \mapsto (x + 1, y)$. Now let $h \in \mathrm{Mod}(A)$ and let γ be a representative homeomorphism of h. We define $\tilde{\gamma}$ to be the preferred lift of γ that fixes the origin, so that $p \cdot \tilde{\gamma} = \gamma \cdot p$. Next, we define $\tilde{\gamma}_1 : \mathbb{R} \longrightarrow \mathbb{R}$ to be the restriction of $\tilde{\gamma}$ to $\mathbb{R} \times \{1\}$. Since homeomorphisms in the mapping class group fix the boundary, $\tilde{\gamma}_1$ is the lift to $\mathbb{R}$ of the identity on one of the boundaries of A, so $p \cdot \tilde{\gamma}_1 = \mathrm{id} \cdot p = p$. By definition, $\tilde{\gamma}_1$ is a deck transformation and is, thus, in the deck transformation group with the form $\tilde{\gamma}_1(x) = x + n, x \in \mathbb{R}$ for a unique $n \in \mathbb{Z}$. Note that $\tilde{\gamma}_1(0) = n \in \mathbb{Z}$. By defining $\rho(h) = \tilde{\gamma}_1(0)$, we get a well-defined map $\rho : \mathrm{Mod}(A) \longrightarrow \mathbb{Z}$.

Take two homeomorphisms g and f representing two different classes in $\mathrm{Mod}(A)$. Note that $\rho(g \cdot f) = m + n = \rho(g) + \rho(f)$ since the composition of the two maps is sent to the composition of two integer translations. Thus, the map $\rho : \mathrm{Mod}(A) \longrightarrow \mathbb{Z}$ is a homomorphism. To prove the proposition, we only need to show ρ is bijective.

We will first show that ρ is surjective. We define M to be the linear transformation of $\mathbb{R}^2$ given by

$$M = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

Note that M preserves $\mathbb{R} \times [0, 1]$ setwise, making it a homeomorphism in $\tilde{A}$. Thus, we define $\tilde{\phi}$ to be $M\big|_{\mathbb{R}\times[0,1]}$, and $\tilde{\phi}$ is a well-defined oriented homeomorphism in $\tilde{A}$. The homeomorphism $\tilde{\phi}$ is equivariant with respect to the group of deck transformations as shown below:

$$\tilde{\phi}(T(x, y)) = \tilde{\phi}(x + n, y) = ((x + n) + ny, y) = (x + ny, y) + (n, 0) = T(\tilde{\phi}(x, y))$$

for some T in the deck transformation group. Thus, $\tilde{\phi}$ descends to an oriented homeomorphism that fixes the boundary components in A. We then define $\tilde{\phi}_1$ to be $\tilde{\phi}\big|_{\mathbb{R}\times\{1\}}$. Thus, we have $\tilde{\phi}_1(0) = n$ and $\rho([\phi]) = \tilde{\phi}_1(0) = n$, proving ρ is surjective.

It remains to show ρ is injective. Let $s \in \mathrm{Mod}(A)$ be an element in the kernel of ρ, and ψ be the representative homeomorphism of s. To show that ρ is injective, it is sufficient to show that ψ is isotopic to the identity. Since $\rho(\psi) = 0$, the preferred lift of ψ, $\tilde{\psi}$, fixes the boundary components of $\tilde{A}$. Thus, we can define a straight-line homotopy $H_t(x, y) = (1 - t)\mathrm{id} + t\tilde{\psi}(x, y)$ for $t \in [0, 1]$. Thus, we need to show that $H_t(x, y)$ is equivariant with respect to the group of deck transformations to push it down to A so that ψ is isotopic to the identity. To show $H_t(x, y)$ is equivariant with respect to the group of deck transformations, it is sufficient to show that $\tilde{\psi}(T) = T(\tilde{\psi})$ for any T in the deck transformation group. By the covering space theory, we have

$$\tilde{\psi}(T) = \psi_*(T) \cdot \tilde{\psi}$$

where ψ_* is the induced automorphism of the fundamental groups of the annulus. However, since ψ fixes the boundary component of A and $\pi_1(A) \cong \mathbb{Z}$, $\psi_*(T) = T$. Thus, $\tilde{\psi}(T) = T(\tilde{\psi})$, and ρ is injective. $\square$

The mapping class group of an annulus A is directly related to an important concept called Dehn twist. Let $T : A \longrightarrow A$ be a twist map of A. We define $T(\theta, t) = (\theta + 2\pi t, t)$. The map T takes a proper arc that has one endpoint on the inner bound and the other on the outer bound and twists it so that the arc winds around the annulus once, while the endpoints are fixed. From the definition, we observe that T is an orientation-preserving homeomorphism that fixes ∂A pointwise. Thus, $[T] \in \mathrm{Mod}(A)$. Note that the map T here is the same map that was used to show that ρ is surjective in Proposition 4.3. The mapping class group of A is isomorphic to $\mathbb{Z}$, which can be interpreted as fixing the boundary components pointwise, how many times the homeomorphism winds around the annulus up to isotopy. Hence, the mapping class group of the annulus can be seen as generated by $[T]$. Now we are ready to define the Dehn twist.

Definition 4.4. Let S be an oriented surface and let α be a simple closed curve. Choose a regular neighborhood N of α in S and an orientation-preserving homeomorphism $\varphi : A \longrightarrow N$. We obtain a homeomorphism $T_\alpha : S \longrightarrow S$, called a *Dehn twist about* α as follows:

$$T_\alpha(x) = \begin{cases} \varphi \circ T \circ \varphi^{-1}(x), & \text{if } x \in N, \\ x, & \text{if } x \in S \setminus N. \end{cases}$$

From the definition of the Dehn twist, we can think about it as taking a small neighborhood of a simple closed curve, so the small neighborhood is homeomorphic to an annulus since it has an outer bound and an inner bound, with the inner bound being the simple closed curve. Then, we perform the twist map T in that small neighborhood while fixing every point outside that neighborhood.

Now we will define the braid group as a mapping class group. If we have a closed disk $\mathbb{D}^2$ with the set of n-marked points, then the mapping class group of this n-marked points disk $\mathbb{D}_n$ is isomorphic to the braid group with n strands.

Theorem 4.5. $B_n \cong \mathrm{Mod}(\mathbb{D}_n)$

Remark 4.6. Here, we will not prove this theorem, but will offer an intuitive understanding of it. The reader can find the proof of the theorem in [2, p. 243]. Since the n-punctured points are unlabeled, the isotopy classes of orientation-preserving homeomorphisms fix $\partial\mathbb{D}_n$ and permute the n-punctured points. If we trace the path of the homeomorphism for the n-punctured points, then we get precisely the crossing between the n strands for the braid group.

4.2. **The Center of the Braid Group is Infinite Cyclic.**

Next, we will discuss the center of the braid group. Recall that the definition of the *center* of a group G, denoted $Z(G)$, is the set of elements that commute with all elements in G. For the center of the braid group, we have the following Theorem. The proof for this theorem will be based on the proof in [7, p. 9].

Theorem 4.7. *For $n \geq 2$, the center of the braid group B_n and the center of the pure braid group PB_n are infinite cyclic. They are generated by the Dehn twist* *about the boundary,*

$$\Delta_{n-1}^2 \;=\; \big(\sigma_1\,(\sigma_2\sigma_1)\,\cdots\,(\sigma_{n-1}\cdots\sigma_1)\big)^2.$$

Proof. We first define $\delta_j = \sigma_j\cdots\sigma_1$ and $\Delta_j = \sigma_1(\sigma_2\sigma_1)\cdots(\sigma_j\cdots\sigma_1)$ for every $j = 1,2,...,n-1$. Then, from Artin presentation (Definition 2.3), we have the following three relationships:

(1) $\sigma_i = \sigma_{i-1}^{\delta_j}$ when $j \geq i \geq 2$, and $\sigma_1 = \sigma_j^{\delta_j^{-1}\delta_j}$ for all $j \geq 1$;
(2) $\sigma_i = \sigma_{j-i+1}^{\Delta_j}$ when $j \geq i \geq 2$, and $\sigma_1 = \sigma_j^{\Delta_j}$ for all $j \geq 1$;
(3) $[\,\sigma_i,\ \Delta_j^2\,] = 1$ for all $j \geq i$.

Here, the representation $\sigma_{i-1}^{\delta_j}$ means $\delta_j^{-1}\sigma_{i-1}\delta_j$. The readers can verify these relations using Artin's presentation. From relation (3), we observe that Δ_{n-1}^2 commutes with every σ_i, making $\Delta_{n-1}^2 \in Z(B_n)$.

Now, we will prove why Δ_{n-1}^2 generates $Z(B_n)$. Let ϕ_i be a simple closed curve in $\mathbb{D}_n$ around only p_i and p_{i+1} (the ith and $(i+1)$th puncture points). Let $\beta_i \in B_n$ be such that $\mathrm{Fix}(\beta_i) = \{\phi_i\}$, where we define $\mathrm{Fix}(\beta_i)$ as the set of ϕ_i that β_i preserves up to isotopy. Take any $z \in Z(B_n)$, since $\beta_i = z\beta_i z^{-1}$ we observe that:

$$\{\phi_i\} = \mathrm{Fix}(\beta_i) = \mathrm{Fix}(z\beta_i z^{-1}) = z\mathrm{Fix}(\beta_i) = \ \{z(\phi_i)\}$$

Here we will explain the middle part of the equation above, $\mathrm{Fix}(z\beta_i z^{-1}) = z\mathrm{Fix}(\beta_i)$. Take $[\phi_i] \in \mathrm{Fix}(\beta_i)$, then $z\beta_i z^{-1}(z[\phi_i]) = z\beta_i([\phi_i]) = z[\phi_i]$. Thus, $z\mathrm{Fix}(\beta_i) = \mathrm{Fix}(z\beta_i z^{-1})$.

Now, because $\{\phi_i\} = \{z(\phi_i)\}$, we may assume without loss of generality that z fixes each ϕ_i pointwise. Let G_i be the region enclosed by ϕ_i. We define $N = \{G_1 \setminus G_2, G_1 \cap G_2, \ldots, G_{n-2} \cap G_{n-1}, G_{n-1} \setminus G_{n-2}\}$ and let $H \in N$. Figure 6 shows the set N for a 3-punctured disk. The black dotted line represents the simple closed curve ϕ_1, while the blue dotted line represents the simple closed curve ϕ_2.

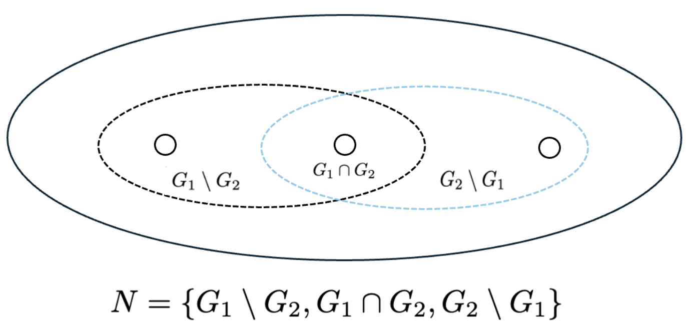

FIGURE 6. N for a 3-punctured Disk

We observe from Figure 6 that for any $H \in N$, H is homeomorphic to an annulus. Since $\mathrm{Mod}(A) \cong \mathbb{Z}$, we may assume that z acts trivially on H. Because the elements in N are disjoint, and the unions of the elements in the set contain all n punctured points, z acts trivially on $G_1 \cup G_2 \cup \ldots \cup G_{n-1} \cup G_{n-2}$. Note that the region $\mathbb{D}_n \setminus N$ is homeomorphic to an annulus. Since z fixes a region enclosing the punctures of $\mathbb{D}_n$ pointwise, it must be a multiple of the Dehn twist about the boundary of $\mathbb{D}_n$. $\square$

5. Conclusion

This paper introduced the geometric definition, the configuration space definition, and the mapping class group definition for the braid group. The paper focused on proving two properties of the braid group: the braid group is torsion-free and the center of the braid group is infinite cyclic. The braid group has some equivalent definitions in mathematical areas such as knot theory, polynomials, and hyperplane complements. Thus, readers can research how the braid group is applied in these fields and can transfer the two properties proved in this paper to these fields. The beauty of the braid group lies in its role as a bridge between different mathematical areas, connecting their various properties and allowing us to move freely between them.

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References

[1] A. Hatcher, *Algebraic Topology.* Cambridge University Press, Cambridge, 2002.
[2] B. Farb and D. Margalit. *A Primer on Mapping Class Groups.* Princeton Mathematical Series 49. Princeton University Press, Princeton and Oxford, 2012.
[3] J. González–Meneses, "Basic results on braid groups," *Ann. Math. Blaise Pascal* **18** (1) (2011), 15–59.
[4] E. Fadell and L. Neuwirth, "Configuration spaces," *Mathematica Scandinavica* **10** (1962), 111–118.
[5] J. Lieber, "Introduction to Braid Groups," VIGRE REU report, University of Chicago (2011), `http://www.math.uchicago.edu/~may/VIGRE/VIGRE2011/REUPapers/Lieber.pdf`.
[6] J. Wilson, "The Geometry and Topology of Braid Groups," `http://www.math.lsa.umich.edu/jchw/RTG-Braids.pdf`.
[7] I. Steinmann, "Braid Groups and Reflection Groups" (2023).