# THE LUBIN-TATE DEFORMATION THEOREM AND THE MORAVA $E$-THEORY

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Abstract. This paper gives a concise exposition of the Lubin–Tate deformation theorem for one-dimensional commutative formal group laws of finite height. For a perfect field of characteristic $p$ and a height-$n$ law, it verifies Schlessinger's criteria, identifies the universal deformation ring as a power-series ring over the Witt vectors in $(n-1)$ variables, and explains the resulting universal deformation. As an application, it constructs the associated Morava $E$-theory via Landweber exactness. An appendix reformulates the discussion in the language of formal groups, and clarifies the compatibility of height and deformation across the two contexts.

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## 1. Introduction

One of the central concepts discussed in this paper is that of formal group laws over a commutative ring $R$.

**Definition 1.1.** Let $R$ be a commutative ring with unit. A *one–dimensional commutative formal group law* over $R$ is a formal power series

$$F(x, y) \in R[[x, y]]$$

satisfying the following three axioms.

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(1) *Identity:* $F(x,0) = x$ and $F(0,y) = y$.
(2) *Commutativity:* $F(x,y) = F(y,x)$ in $R[[x,y]]$.
(3) *Associativity:* $F(x,F(y,z)) = F(F(x,y),z)$ in $R[[x,y,z]]$.

For simplicity, throughout this discussion we will refer to such power series as *formal group laws* over $R$, and sometimes we write $F(x,y)$ as $x +_F y$. Fixing the ring $R$, a morphism $f : F \to G$, between two formal group laws $F$ and $G$ over $R$ is defined to be a single-variable power series $f \in R[[x]]$ with zero constant term such that

$$f(x +_F y) = f(x) +_G f(y).$$

It is straightforward to verify that formal group laws over $R$ together with their morphisms form a category, denoted $\mathcal{F}\mathrm{gl}\,(R)$, for any commutative ring $R$.

The notion of formal group laws was first introduced in [1] as formal analogues of Lie groups. It provided a framework for encoding the infinitesimal behavior of Lie groups in purely algebraic terms, making it possible to work over general coefficient rings (not just $\mathbb{R}$ or $\mathbb{C}$) and to apply algebraic techniques to problems in differential geometry and number theory.

However, the power of formal group laws is not confined to the realm of Lie groups. Two consecutive papers by J. Lubin and J. Tate established the foundational role of formal group laws in both algebraic number theory and algebraic topology.

The first paper [7] addressed a problem in number theory. Let $K$ be a non-Archimedean local field with ring of integers $\mathcal{O}_K$, maximal ideal $\mathfrak{m}_K$, and uniformizer $\pi$. The paper provided an explicit construction of the totally ramified abelian extensions of $K$ via a formal group law over $\mathcal{O}_K$, now known as the *Lubin–Tate formal group law*.

Consider the family of power series $\mathcal{F}_\pi \subseteq K[[x]]$ defined by

$$\mathcal{F}_\pi = \{f \in \mathcal{O}_K[[x]] : f(x) \equiv \pi x \pmod{\deg 2}, f(x) \equiv x^q \pmod{\pi}\},$$

where $q$ is the cardinality of $\mathcal{O}_K/\mathfrak{m}_K$. The formal group law is constructed from the following key lemma, now known as the *Lubin–Tate lemma*.

**Lemma 1.2** ([7, Lem. 1])**.** *Let $f, g \in \mathcal{F}_\pi$, and let $L(x_1,\ldots,x_k) = \sum_{i=1}^k a_i x_i \in \mathcal{O}_K[x_1,\ldots,x_k]$ be a linear form. Then there exists a unique power series $F(x_1,\ldots,x_k) \in \mathcal{O}_K[[x_1,\ldots,x_k]]$ such that*

*(1) $F(x_1,\ldots,x_k) \equiv L(x_1,\ldots,x_k) \pmod{\deg 2}$, and*
*(2) $f(F(x_1,\ldots,x_k)) = F(g(x_k),\ldots,g(x_k))$.*

For a given $g \in \mathcal{F}_\pi$, the Lubin–Tate formal group law $F = F_g \in \mathcal{O}_K[[x,y]]$ is defined to be the one obtained by applying Lemma 1.2 to the case $f = g$ and $L(x,y) = x + y$.

Using the Lubin–Tate lemma again, for each $a \in \mathcal{O}_K$ we define $[a]_F[x] \in \mathcal{O}_K[x]$ to be the unique power series satisfying

$$[a]_F(x) \equiv ax \pmod{\deg 2}, \quad g([a]_F(x)) = [a]_F(f(x)).$$

These so-called *a-series* gives a useful description of the endomorphism ring of $F$, as stated in the following theorem.

**Theorem 1.3** ([7, Thm. 1])**.** *For $g \in \mathcal{F}_\pi$, the formal group law $F = F_g$ is the unique formal group law over $\mathcal{O}_K$ such that $[\pi]_F(T) = g(T)$. For every $a \in \mathcal{O}_K$, the series*

*$[a]_F(T)$ defines an endomorphism of $F$. Moreover, the assignment $a \mapsto [a]_F$ is an injective ring homomorphism $\mathcal{O}_K \to \mathrm{End}_{\mathcal{O}_K}(F)$.*

Once $F$ is determined, the $\pi^n$–torsion points of $F$ are defined by

$$F[\pi^n] \;:=\; \{\, x \in \mathfrak{m}_{\overline{K}} \mid [\pi^n]_F(x) = 0 \,\},$$

where $\mathfrak{m}_{\overline{K}}$ denotes the maximal ideal of the valuation ring $\mathcal{O}_{\overline{K}}$ of a fixed algebraic closure $\overline{K}$ of $K$. We then set

$$K_\pi^{(n)} := K\big(F[\pi^n]\big),$$

the extension of $K$ obtained by adjoining all $\pi^n$–torsion points of $F$.

By construction, we have a tower of finite extensions:

$$K \subset K_\pi^{(1)} \subset K_\pi^{(2)} \subset \cdots ,$$

and we write

$$K_\pi^{(\infty)} := \bigcup_{n \geq 1} K_\pi^{(n)}.$$

**Theorem 1.4** ([7, Thm. 2])**.** *Let $F$ be the Lubin–Tate formal group law for a fixed uniformizer $\pi$ of $K$. Then:*

*(1) For $n \geq 1$, the extension $K_\pi^{(n)}/K$ is totally ramified of degree $q^{n-1}(q-1)$.*

*(2) The extension $K_\pi^{(\infty)}/K$ is abelian, and its Galois group is canonically isomorphic to $\mathcal{O}_K^\times$ via the* Lubin–Tate reciprocity map*, which sends $a \in \mathcal{O}_K^\times$ to its action on the torsion points:*

$$a \;\mapsto\; \big(x \mapsto [a]_F(x)\big).$$

*(3) In particular, $K_\pi^{(\infty)}$ is the maximal abelian totally ramified extension of $K$.*

We refer readers to Chapter 5 of [12] and Chapter 7 of [2] for a clearer exposition of this result and its further applications.

We now return to the discussion of the Lubin–Tate formal group law $F \in \mathcal{F}\mathrm{gl}\,(\mathcal{O}_K)$. We can examine its reduction to other related base rings. Let $k = \mathcal{O}_K/\mathfrak{m}_K$ be the residue field, with ramification index $e$ and residue degree $f$ (so $k \cong \mathbb{F}_q = \mathbb{F}_{p^f}$ is a perfect field with characteristic $p$). Let $F_0 \in \mathcal{F}\mathrm{gl}\,(k)$ be defined by $F_0 = F \pmod{\pi}$. Then $F$ may be regarded as a lift of $F_0$ to $\mathcal{O}_K$ subject to the condition $[\pi]_F = g$ for some fixed power series $g \in \mathcal{O}_K[[x]]$.

This leads to a natural question: are there other lifts of $F_0$ to $\mathcal{O}_K$ that are not isomorphic to $F$? If so, how many? More generally, let $k$ be a perfect field with char $(k) = p$, and let $R$ be a complete Noetherian local ring whose residue field is isomorphic to $k$. For a fixed $F_0 \in \mathcal{F}\mathrm{gl}\,(k)$, how many lifts to $R$ exist, and how many isomorphism classes do they form? More formally, such lifts are called deformations, and the *Lubin–Tate deformation theorem*, proved in the second paper [8], provides the answer: the number of isomorphism classes depends on an invariant named height of $F_0$. The larger the height of $F_0$, the more lifts it admits. In the case of local fields, $F_0$ has height exactly $ef$, and hence there are $|\mathfrak{m}_K|^{ef-1}$ non-isomorphic liftings of $F_0$.

This paper focuses on the deformation theorem. Section 2 reviews preliminaries, including the notion of height for formal group laws. Section 3 presents a proof of the deformation theorem via Schlessinger's criteria—an important tool for establishing (pro-)representability of functors on Artinian local rings. Section 4 explains how this theorem leads to the construction of Morava $E$-theory, a cornerstone of modern chromatic homotopy theory. Finally, Appendix A introduces formal groups—an algebro-geometric generalization of formal group laws—and shows how their deformations reveal the moduli-theoretic structures.

## 2. Preliminaries

Assume we are given a field $k$, a formal group law $F_0 \in \mathcal{F}\mathrm{gl}\,(k)$, and a local ring $R$ whose residue field is $k$. The Lubin–Tate deformation theorem determines the number of isomorphism classes of lifts of $F_0$ to $R$, but only in the case where $\mathrm{char}(k) = p > 0$. That is because the characteristic-zero case is trivial. Indeed, any local ring whose residue field has characteristic zero is a $\mathbb{Q}$-algebra, and a direct computation yields the following proposition, which shows that there is only one isomorphism class of formal group laws over a fixed $\mathbb{Q}$-algebra.

**Proposition 2.1.** *Let $R$ be a $\mathbb{Q}$-algebra, and let $F \in \mathcal{F}\mathrm{gl}\,(R)$. Then the formal power series*

$$\log_F(x) = \int \frac{dx}{\partial_y F(x,y)|_{y=0}} \in R[[x]]$$

*satisfies*

$$\log_F(F(x,y)) = \log_F(x) + \log_F(y).$$

*Consequently, $\log_F$ is an isomorphism of formal group laws from $F$ to $G(x,y) = x+y$, the additive formal group law.*

In positive characteristic the situation is quite different: one introduces the notion of height to distinguish isomorphism classes.

### 2.1. Heights.

The height of a formal group laws is defined only over rings of positive characteristic. So throughout this section we fix a prime number $p$, a commutative $\mathbb{F}_p$-algebra $R$, and a formal group law $F \in \mathcal{F}\mathrm{gl}\,(R)$. We first introduce the *$n$-series* of $F$, which plays an important role in the definition of height.

**Definition 2.2.** Fix $F \in \mathcal{F}\mathrm{gl}\,(R)$. For each integer $n \geq 0$ define the *$n$-series* $[n]_F(x) \in R[[x]]$ recursively by

$$[0]_F(x) = 0, \quad [n]_F(x) = F([n-1]_F(x), x)\ (n \geq 1).$$

**Remark 2.3.** By induction and the three axioms of formal group laws, one checks immediately that $[n]_F(F(x,y)) = F([n]_F(x), [n]_F(y))$, i.e. $[n]_F$ is an endomorphism of $F$ for every $n$. Moreover, since

$$F(x,y) = x + y + (\text{ terms of total degree } \geq 2),$$

we have $[n]_F(x) = nx + O(x^2)$. Therefore, the $n$-series agree with the $a$-series constructed from Lemma 1.2; this justifies the shared notation.

Since $R$ is a $\mathbb{F}_p$-algebra, we have $p = 0$ in $R$. Thus, the linear term of $[p]_F$ vanishes. This yields a clean description of the $p$-series:

**Proposition 2.4.** *Let $R$ be an $\mathbb{F}_p$-algebra and $F \in \mathcal{F}\mathrm{gl}\,(R)$. Then either $[p]_F(x) = 0$, or else there exists $n \geq 0$ such that $[p]_F(x) = f(x^{p^n})$ with $f(x) = cx + O(x^2)$ for some $c \neq 0$.*

The height of such a formal group law $F$ can therefore be defined as follows.

**Definition 2.5.** Let $F$ be a formal group law over a commutative $\mathbb{F}_p$-algebra $R$. We say that $F$ has height at least $n$ if $[p]_F(x) = f(x^{p^n})$ for some $f \in R[[x]]$. We say that $F$ has height exactly $n$ if, in addition, $f$ is invertible in $R[[x]]$. We say that $F$ has infinity height if $[p]_F(x) = 0$.

Some tools should be introduced before the proof of Proposition 2.4. Let $R[[x]]dx$ be the free rank 1 $R[[x]]$-module on a symbol $dx$, whose elements are called the $R[[x]]$ *differentials*. (Actually, these are differentials over the formal spectrum $\mathrm{Spf}(R[[x]])$, whose definition will be stated in Definition 3.1 and are more thoroughly discussed in Appendix A.) Given an $R[[x]]$ differential $g(x)dx$ and a formal group law $F \in \mathcal{F}\mathrm{gl}\,(R)$, we define

$$F^*(g(x)dx) = g(F(x,y))\left(\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy\right) \in R[[x,y]]\{dx, dy\}.$$

We say $g(x)dx$ is a *translation invariant* differential of $F$ if

$$F^*(g(x)dx) = g(x)dx + g(y)dy.$$

The submodule of $R[[x]]$ translation invariant differentials for a given $F \in \mathcal{F}\mathrm{gl}\,(R)$ is free of rank 1 over $R$, generated by

$$\omega_F := (\partial_y F(x,y)|_{y=0})^{-1}dx = (1 + a_1x + a_2x^2 + \cdots)dx.$$

It is the unique invariant differential whose constant term is 1.

*Proof of Proposition 2.4.* Consider a slightly more general situation. let $F$ and $F'$ be formal group laws over $R$, and let $h \in xR[[x]]$ be a morphism $F \to F'$. It induces a map

$$h^* : R[[x]]dx \to R[[x]]dx, \quad g(x)dx \mapsto (g \circ h(x))dh,$$

where $dh = h'(x)dx$ is the formal derivative of $h$. A direct computation shows that $h^*$ sends translation invariant differentials of $F$ to translation invariant differentials of $F'$. Hence, there exists $c \in R$ such that $h^*\omega_F = c\omega_{F'}$. Writing $\omega_F(x) = u(x)dx$ and $\omega_{F'}(x) = u'(x)dx$ with $u(0) = u'(0) = 1$, we obtain

$$u(h(x))h'(x)dx = cu'(x)dx.$$

Evaluating at $x = 0$ gives $c = h'(0)$, so $h(x) \equiv cx \pmod{x^2}$. Now take $F = F'$ and $h = [p]_F$. Then $c = [p]'_F(0) = p$, which is 0 in $R$. That implies $[p]^*_F\omega_F = 0$. By construction $\omega_F$ is a unit in the $R[[x]]$-module $R[[x]]dx$, so $[p]^*_F = 0$.

Now, write $[p]_F(x) = \sum_i a_ix^i$ for $a_i \in R$. Since $[p]^*_F = 0$, we have $ia_i = 0$ for all $i > 0$. Over an $\mathbb{F}_p$-algebra every integer $i$ not divisible by $p$ is a unit. That implies $a_i = 0$ unless $p \mid i$. Consequently, there exists $f_1 \in R[[x]]$ with $[p]_F(x) = f_1(x^p)$.

Next, let $F^{(p)} \in \mathcal{F}\mathrm{gl}\,(R)$ denote the formal group law obtained from $F$ by raising every coefficient to the $p$-th power. This implies $F^{(p)}(x^p, y^p) = F(x,y)^p$. Therefore, we have

$$\begin{aligned} F(f_1(x^p), f_1(y^p)) &= F([p]_F(x), [p]_F(y)) \\ &= [p]_F(F(x,y)) = f_1(F(x,y)^p) = f_1(F^{(p)}(x^p, y^p)). \end{aligned}$$

Since the variables are formal, this is equivalent to $F(f_1(x), f_1(y)) = f_1(F^{(p)}(x,y))$. We can therefore regard $f_1$ as a morphism $F^{(p)} \to F$. For every $n \geq 1$, as long as the linear term of $f_n$ is zero, iterating this argument produces $f_{n+1} \in R[[x]]$ such that $[p]_F(x) = f_{n+1}(x^{p^{n+1}})$. If the process never terminates, the lowest degree term of $[p]_F(x)$ would have arbitrarily large degree, forcing $[p]_F(x) = 0$. This completes the proof. $\square$

The following result is Lazard's theorem, and it leads to another description of height.

**Theorem 2.6** ([6, Thm. 2])**.** *Let $A := \mathbb{Z}[\{a_{i,j}\}_{i,j\geq 1}]$ and define*

$$F_A(x,y) = x + y + \sum_{i,j\geq 1} a_{i,j}\, x^i y^j \in A[[x,y]].$$

*Let $I \subset A$ be the ideal generated by the coefficients of $F_A(x,y) - F_A(y,x)$ and $F_A(F_A(x,y),z) - F_A(x, F_A(y,z))$. Set $L := A/I$ and let $c_{i,j}$ be the images of $a_{i,j}$. Then*

$$F_{\mathrm{Laz}}(x,y) = x + y + \sum_{i,j\geq 1} c_{i,j}\, x^i y^j \in L[[x,y]]$$

*is a formal group law, and for every commutative ring $R$ it gives a natural bijection*

$$\begin{aligned} &\mathrm{Hom}_{\mathbf{CRing}}(L,R) \to \mathcal{F}\mathrm{gl}\,(R),\\ &\quad (\phi : L \to R) \mapsto \phi_*(F_{\mathrm{Laz}}) = \sum_{i,j} \phi(c_{i,j}) x^i y^j, \end{aligned}$$

*where* **CRing** *denotes the category of commutative rings. Moreover, $L$ carries a natural grading with $L \cong \mathbb{Z}[t_1, t_2, \dots]$, $|t_i| = 2i$.*

We call $L$ the *Lazard ring*, and $F_{\mathrm{Laz}}$ the *universal formal group law*. Let $v_n \in L$ be the coefficient of $x^{p^n}$ in the $p$-series of $F_{\mathrm{Laz}}$. For any $F \in \mathcal{F}\mathrm{gl}\,(R)$, Lazard's theorem associates to $F$ a unique ring map $\phi_F : L \to R$. We set $v_n(F) := \phi_F(v_n) \in R$. By construction, $v_n(F)$ is precisely the coefficient of $x^{p^n}$ in the $p$-series $[p]_F(x)$. Therefore, if $R$ is an $\mathbb{F}_p$-algebra, then $F \in \mathcal{F}\mathrm{gl}\,(R)$ has height at least $n$ if and only if $v_i(F) = 0$ for all $i < n$. If in addition $v_n(F)$ is invertible, then $F$ has height exactly $n$ (and conversely).

Note that every formal group law has height at least 1, since the linear term of $[p]_F(x)$ must vanish. Moreover, if $F$ and $F'$ are isomorphic formal group laws over the same $\mathbb{F}_p$-algebra $R$, then they have the same height. Indeed, if $g \in R[[x]]$ is an isomorphism from $F$ to $F'$, then

$$[p]_{F'}(x) = g \circ [p]_F \circ g^{-1}(x).$$

Since $g$ is invertible, it has the form $g(x) = rx + O(x^2)$ for some $r \in R^\times$. Thus, if $[p]_F(x) = cx^{p^n} + O(x^{p^n+1})$ then $[p]_{F'}(x) = cr^{1-p^n} x^{p^n} + O(x^{p^n+1})$, maintaining the same height.

We end this part of discussion by presenting some examples of formal group laws and computing their heights.

**Example 2.7.** Let $R$ be any $\mathbb{F}_p$-algebra. The additive formal group law $F_a(x,y) = x + y \in \mathcal{F}\mathrm{gl}\,(R)$ has infinite height, because $[p]_{F_a}(x) = px = 0$. The multiplicative formal group law $F_m(x,y) = x + y + xy \in \mathcal{F}\mathrm{gl}\,(R)$ has height 1, because $[p]_{F_m}(x) = (1+x)^p - 1 = x^p$.

**Example 2.8.** Further classical examples arise from the Lubin–Tate lemma (Lemma 1.2). Fix an integer $n \geq 1$. In the context of the lemma, we take $K/\mathbb{Q}_p$ be the unique unramified extension of degree $n$, with ring of integers $\mathcal{O}_K = W(\mathbb{F}_{p^n})$ (here $W(-)$ represents the Witt vectors) and residue field $\mathbb{F}_{p^n}$. We then take

$$f(x) = px + x^{p^n} \in \mathcal{O}_K[[x]],$$

which indeed satisfies $f \in \mathcal{F}_\pi$. Apply the lemma with $k = 2$, $L(x, y) = x + y$, and $g = f$. We obtain a unique power series $F_n(x, y) \in \mathcal{O}_K[[x, y]]$ such that

$$F_n(x, y) \equiv x + y \pmod{\deg \geq 2}, \quad f(F_n(x, y)) = F_n(f(x), f(y)).$$

By uniqueness, $F_n$ satisfies the three axioms of formal group laws, and its $p$-series is $[p]_{F_n} = f$. Reducing the coefficients modulo $p$ yields a formal group law $H_n \in \mathcal{F}\mathrm{gl}\,(\mathbb{F}_{p^n})$ with $[p]_{H_n}(x) = x^{p^n}$. So $H_n$ has height exactly $n$; this is the *Honda formal group law of height $n$.*

2.2. **Infinitesimal Thickenings and Deformations.** We now define deformations of a formal group law: informally, they record how a formal group law over a residue field can be lifted to larger base rings.

Throughout this section, we fix $k$ to be a field of characteristic $p > 0$.

**Definition 2.9.** An *infinitesimal thickening* of $k$ is an Artinian local ring $A$ equipped with a surjective ring map $\alpha : A \twoheadrightarrow k$ whose kernel $\ker(\alpha) = \mathfrak{m}_A$ is the unique maximal ideal of $A$.

Morphisms of infinitesimal thickenings over the same field $k$ are local ring maps $f : A \to A'$ that commute with the quotient maps to $k$. In this way, all infinitesimal thickenings over $k$ and their morphisms form a category, denoted $\mathbf{Art}_k$.

**Definition 2.10.** Let $A \in \mathbf{Art}_k$ be an infinitesimal thickening, and let $F_0$ be a formal group law over $k$. A *deformation* of $F_0$ over $A$ is a formal group law $F \in \mathcal{F}\mathrm{gl}\,(A)$ such that $F \equiv F_0 \pmod{\mathfrak{m}_A}$.

An isomorphism of deformations $F \to F'$ is an isomorphism of formal group laws $f(x) : F \to F' \in A[[x]]$ with $f(x) \equiv x \pmod{\mathfrak{m}_A}$. These form a groupoid $\mathbf{Def}_{F_0}(A)$. Equivalently, $A \mapsto \mathbf{Def}_{F_0}(A)$ defines a groupoid-valued functor on $\mathbf{Art}_k$.

The height of a formal group law over a fixed field has even stronger implications in the context of deformations. Suppose the formal group law $F_0$ over $k$ has height exactly $n$; then $v_i(F_0) = 0$ for all $i < n$. If we lift $F_0$ to a thickening $A \in \mathbf{Art}_k$, we may choose the parameters $v_i(F) \in \mathfrak{m}_A$ arbitrarily for $i < n$. However, we do not have freedom on all higher coefficients $v_j(F)$ for $j \geq n$, because they must coincide with $v_j(F_0)$. Consequently, the larger the height of $F_0$, the more free parameters a lift possesses; in this sense, height measures the rigidity of liftings of a given formal group law.

Before formalizing this statement in the next section, we first make an important observation about the deformation functor: if formal group law $F_0 \in \mathcal{F}\mathrm{gl}\,(k)$ has finite height, then the functor $\mathbf{Def}_{F_0}$ factors through $\mathbf{Set}$.

**Proposition 2.11** (see, e.g., [13, Thm. 17.15])**.** *Let $A \in \mathbf{Art}_k$, and let $F_0 \in \mathcal{F}\mathrm{gl}\,(k)$ have finite height. Then the groupoid $\mathbf{Def}_{F_0}(A)$ is discrete. That is every object has only the identity automorphism.*

Therefore, when $F_0$ has finite height we may (and will) regard $\mathbf{Def}_{F_0} : \mathbf{Art}_k \to \mathbf{Set}$ as the functor sending $A$ to the set of isomorphism classes of $F_0$'s deformations over $A$.

## 3. The Lubin–Tate Theorem

3.1. **Statement of the Main Theorem.** We now turn to our main topic: the Lubin–Tate deformation theorem. This result provides a complete description of the deformation space of a finite-height formal group. Before stating it, we introduce one further piece of notation borrowed from algebraic geometry. A more thorough treatment of this notion can be found in Appendix A.

**Definition 3.1.** Let $A$ be a commutative ring, and $I \subseteq A$ be an ideal. We define a functor $\mathrm{Spf}(A) : \mathbf{CRing} \to \mathbf{Set}$ by

$$\mathrm{Spf}(A)(B) = \varinjlim_n \mathrm{Hom}_{\mathbf{CRing}}(A/I^n, B).$$

That is, $\mathrm{Spf}(A)$ maps a ring $B$ to the collection of all ring maps $A \to B$ that annihilate some power of $I$.

We call $\mathrm{Spf}(A)$ the *formal spectrum* of $A$. A functor $F : \mathbf{CRing} \to \mathbf{Set}$ is said to be *pro-representable* by a ring $A$ if $F$ is naturally isomorphic to $\mathrm{Spf}(A)$.

In fact, if we equip $A$ with the $I$-adic (linear) topology, whose neighborhood basis at 0 is $\{I^n\}_{n\geq 1}$, then there is a canonical identification

$$\varinjlim_n \mathrm{Hom}_{\mathbf{CRing}}(A/I^n, B) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathbf{CRing}}(A, B),$$

where the right-hand side denotes the set of all continuous ring maps $A \to B$ when $B$ is given the discrete topology.

**Theorem 3.2** ([8, Thm. 3.1])**.** *Let $k$ be a perfect field $k$ of characteristic $p > 0$, and let $F_0 \in \mathcal{F}\mathrm{gl}\,(k)$ be a formal group law of height exactly $n < \infty$. Then there exists a complete local Noetherian $W(k)$-algebra $E_0 := E_0(F_0, k)$, non-canonically isomorphic to $W(k)[[u_1, \dots, u_{n-1}]]$, that pro-represents the functor $\mathbf{Def}_{F_0} : \mathbf{Art}_k \to \mathbf{Set}$.*

*More precisely, there exists a universal formal group law $F_{\mathrm{univ}} \in \mathcal{F}\mathrm{gl}\,(E_0)$ with $F_{\mathrm{univ}} \equiv F_0 \pmod{\mathfrak{m}_{E_0}}$, which induces a natural bijection*

$$\mathrm{Spf}(E_0)(A) \cong \mathbf{Def}_{F_0}(A), \quad f \mapsto f_*(F_{\mathrm{univ}})$$

*for every infinitesimal thickening $A \in \mathbf{Art}_k$. Here $f$ is understood to be a local ring map $f : E_0 \to A$.*

The proof of Theorem 3.2 relies heavily on Schlessinger's criteria, first introduced in [17]. These criteria provide sufficient conditions for a functor on the category of local Artinian rings to be pro-representable.

3.2. **Schlessinger's Criteria.** To introduce Schlessinger's criteria, we first set up a relative version of the category $\mathbf{Art}_k$.

**Definition 3.3.** Let $k$ be a field of characteristic $p > 0$, and let $R$ be a local Noetherian ring with residue field $k$. An $R$-algebra $A$ is called an *infinitesimal thickening* of $k$ if $A$ is an Artinian local ring and the structure map $R \to A$ induces an isomorphism

$$k \cong R/\mathfrak{m}_R \cong A/\mathfrak{m}_A.$$

Equivalently, $A$ has the same residue field as $R$. We write $\mathbf{Art}_R$ for the full subcategory of $\mathbf{CRing}_R$ (the category of commutative $R$-algebras) whose objects are infinitesimal $R$-algebra thickenings.

**Proposition 3.4** (Schlessinger's criteria, [17, Thm. 2.11])**.** *Let $F : \mathbf{Art}_R \to \mathbf{Set}$ be a functor. Assume $F$ satisfies the following properties:*

*(1) The functor $F$ assigns a singleton to $k$.*
*(2) (Mayer–Vietoris property): for any pullback diagram $A_0 \to A_{0,1} \leftarrow A_1$ in $\mathbf{Art}_R$ with both maps surjective, the canonical map*
$$F(A_0 \times_{A_{0,1}} A_1) \to F(A_0) \times_{F(A_{0,1})} F(A_1)$$
*is a bijection.*
*(3) (Formal Smoothness) For every surjection $A \to A'$ in $\mathbf{Art}_R$, the induced map $F(A) \to F(A')$ is also surjective.*
*(4) The tangent space $F(k[\epsilon]/\epsilon^2)$ has finite $k$-dimension $n < \infty$.*

*Then $F$ is pro-representable by the formal power series ring $R[[u_1, \dots, u_n]]$. Equivalently, for every thickening $A \in \mathbf{Art}_R$ there is a natural isomorphism*
$$F(A) \cong \mathrm{Spf}(R[[u_1, \dots, u_n]])(A) \cong \mathfrak{m}_A^{\times n}.$$

For brevity, we write
$$k[\epsilon] := k[\epsilon]/\epsilon^2, \quad \text{and } k[\epsilon_1, \dots, \epsilon_n] := k[\epsilon_1, \dots, \epsilon_i]/(\epsilon_1, \dots, \epsilon_i)^2.$$
We observe that if $F$ satisfies the first three properties, then $F(k[\epsilon])$ automatically acquires the structure of a $k$-vector space, which justifies the terminology *tangent space*. To see this, first note that $k[\epsilon]$ is an abelian group object in $\mathbf{CRing}_k$, witnessed by the addition map
$$k[\epsilon] \times_k k[\epsilon] \cong k[\epsilon_1, \epsilon_2] \to k[\epsilon], \quad x + y\epsilon_1 + z\epsilon_2 \mapsto x + (y + z)\epsilon.$$
Assume $F$ satisfies the first three properties. Applying $F$ to the map above yields a set map
$$F(k[\epsilon]) \times F(k[\epsilon]) \cong F(k[\epsilon]) \times_{F(k)} F(k[\epsilon]) \cong F(k[\epsilon] \times_k k[\epsilon]) \to F(k[\epsilon]).$$
Here, the condition $F(k) = \mathrm{pt}$ shows that the fiber product over $F(k)$ is trivial, giving the first isomorphism. The second isomorphism is guaranteed by the Mayer–Vietoris property. Consequently, $F(k[\epsilon])$ becomes an abelian group object. Moreover, for each $a \in k$, we can define a scalar multiplication map
$$k[\epsilon] \to k[\epsilon], \quad x + y\epsilon \to x + ay\epsilon.$$
It is immediate to check that these two maps endow $F(k[\epsilon])$ with the structure of a $k$-vector space.

We are now prepared to prove Schlessinger's criteria.

*Proof.* We begin by constructing a natural transformation $\mathrm{Spf}(R[[u_1, \dots, u_n]]) \to F$. For simplicity, write $R[[u]] := R[[u_1, \dots, u_n]]$, and let $\mathfrak{n} = (\mathfrak{m}, u_1, \dots, u_n)$ denote its unique maximal ideal. By Yoneda's lemma, after passing to inverse limits, giving such a natural transformation is equivalent to choosing a distinguished element $\bar{a} \in \varprojlim_i F(R[[u]]/\mathfrak{n}^i)$. To produce this element, consider the surjective local $R$-algebra map
$$\phi : R[[u]] \to k[\epsilon_1, \dots, \epsilon_n], \quad u_i \mapsto \epsilon_i \ (1 \le i \le n).$$
Since $(\epsilon_1, \dots, \epsilon_n)^2 = 0$, we have $\phi(\mathfrak{n}^2) = 0$, and therefore $\phi(\mathfrak{n}^i) = 0$ for all $i \ge 2$. Hence, $\phi$ is continuous with respect to the $\mathfrak{n}$-adic topology and induces a compatible system of surjections
$$\dots \to R[[u]]/\mathfrak{n}^3 \to R[[u]]/\mathfrak{n}^2 \to k[\epsilon_1, \cdots, \epsilon_n].$$

By the formal smoothness of $F$, applying $F$ to this tower yields another sequence of surjections

$$\cdots \to F(R[[u]]/\mathfrak{n}^3) \to F(R[[u]]/\mathfrak{n}^2) \to F(k[\epsilon_1, \ldots, \epsilon_n]).$$

Now, choose a $k$-basis $\{a_1, \ldots, a_n\}$ of $F(k[\epsilon])$ and set

$$a = (a_1, \ldots, a_n) \in F(k[\epsilon_1, \ldots, \epsilon_n]) \cong F(k[\epsilon] \times_k \cdots \times_k k[\epsilon]) \cong F(k[\epsilon])^{\times n}.$$

We can then lift $a$ inductively to obtain an element $\bar{a} \in \varprojlim_i F(R[[u]]/\mathfrak{n}^i)$. The induced natural transformation is

$$\mathrm{Spf}(R[[u]])(A) \to F(A), \quad (f : R[[u]] \to A) \mapsto F(f)(\bar{a}),$$

for any thickening $A \in \mathbf{Art}_R$.

The second step is to show that the natural transformation $\mathrm{Spf}(R[[u]]) \to F$ just constructed induces an isomorphism on tangent spaces. The tangent space of $\mathrm{Spf}(R[[u]])$ is

$$\begin{aligned} \mathrm{Spf}(R[[u]])(k[\epsilon]) \cong \mathrm{Hom}^{\mathrm{cont}}(R[[u]], k[\epsilon]) \cong (\epsilon)^{\times n}, \text{ via} \\ (f : R[[u]] \to k[\epsilon]) \mapsto (f(u_1), \ldots, f(u_n)). \end{aligned}$$

For each $1 \leq i \leq n$, define local $R$-algebra maps $\sigma_i : R[[u]] \to k[\epsilon]$ by $u_i \mapsto \epsilon$ and $u_j \mapsto 0$ for $j \neq i$. Then $\{\sigma_1, \ldots, \sigma_n\}$ is a $k$-basis of $\mathrm{Spf}(R[[u]])(k[\epsilon])$. Moreover, under the induced map on tangent spaces each $\sigma_i$ is mapped to $F(\sigma_i)(\bar{a}) = a_i \in F(k[\epsilon])$. Thus, a basis of $\mathrm{Spf}(R[[u]])(k[\epsilon])$ maps to a basis of $F(k[\epsilon])$, and the tangent spaces are indeed isomorphic.

For the third step, we consider functors $F$ and $G$, both satisfying the four properties of Schlessinger's criteria. We claim that any natural transformation $\phi : F \to G$ is in fact a natural isomorphism. More precisely, our goal is to show that $\phi_A : F(A) \to G(A)$ is an isomorphism for all $A \in \mathbf{Art}_R$. We establish this claim by induction on $\mathrm{length}(A)$. The base case $\mathrm{length}(A) = 1$ is immediate, since in this case $A = k$, and the map $F(k) \to G(k)$ must be a bijection because $F(k) = \mathrm{pt} = G(k)$.

For the inductive step, take $A \in \mathbf{Art}_R$ with $\mathrm{length}(A) = n > 1$. Because $A$ is Artinian, there exists a nonzero element $x \in A$ that annihilates $\mathfrak{m}_A$. In the commutative square

$$\begin{tikzcd} F(A) \arrow[r, "\phi_A"] \arrow[d] & G(A) \arrow[d] \\ F(A/x) \arrow[r, "\phi_{A/x}"'] & G(A/x) \end{tikzcd}$$

the map $\phi_{A/x}$ is an isomorphism by the induction hypothesis, since $\mathrm{length}(A/x) < n$. As $F$ and $G$ are formally smooth, the two vertical arrows are surjective. Therefore, it suffices to show that the fibers of $F(A) \to F(A/x)$ are carried bijectively to the fibers of $G(A) \to G(A/x)$. To see this, consider the action map

$$\alpha : k[\epsilon] \times_k A \to A, \quad (\tilde{a} + b\epsilon, a) \mapsto a + bx,$$

where $\tilde{a}$ denotes the image of $a$ under the quotient $A \to A/\mathfrak{m}_A \cong k$. Applying $F$ on $\alpha$ yields an action of the tangent space $F(k[\epsilon])$ on $F(A)$:

$$\alpha_F : F(k[\epsilon]) \times F(A) \cong F(k[\epsilon] \times_k A) \to F(A).$$

Composing $\alpha$ with the quotient $A \to A/x$ gives the map $(\tilde{a}+b\epsilon, a) \mapsto a$. Therefore, the action $\alpha_F$ preserves the image in $F(A/x)$; thus $F(k[\epsilon])$ acts only on the fibers of $F(A) \to F(A/x)$. We claim that this action is free and transitive, i.e., $F(A)$ is a $F(k[\epsilon])$-torsor over $F(A/x)$. With the trivial $F(k[\epsilon])$-action on $F(A/x)$, the surjection $F(A) \to F(A/x)$ is $F(k[\epsilon])$-equivariant, hence it suffices to show that $\alpha_F$ induces an isomorphism

$$F(k[\epsilon]) \times F(A) \cong F(A) \times_{F(A/x)} F(A).$$

The Mayer–Vietoris property of $F$ yields a pullback diagram

$$\begin{tikzcd} F(A \times_{A/x} A) \arrow[r] \arrow[d] & F(A) \arrow[d] \\ F(A) \arrow[r] & F(A/x) \end{tikzcd}$$

Since $x$ annihilates $\mathfrak{m}_A$, there is an isomorphism

$$k[\epsilon] \times_k A \cong A \times_{A/x} A, \quad (\tilde{a}+b\epsilon, a) \mapsto (a+bx, a).$$

Applying $F$ to this, we obtain

$$F(A \times_{A/x} A) \cong F(k[\epsilon] \times_k A) \cong F(k[\epsilon]) \times F(A).$$

Putting the right-hand side into the pullback diagram, we finally get the desired isomorphism $F(k[\epsilon]) \times F(A) \cong F(A) \times_{F(A/x)} F(A)$.

An identical argument shows that $G(A) \to G(A/x)$ is a $G(k[\epsilon])$-torsor. Recall that the tangent spaces of $F$ and $G$ are both isomorphic to the tangent space of $\mathrm{Spf}(R[[u]])$; in particular, they are naturally isomorphic to each other. Hence, $\phi_A : F(A) \to G(A)$ is a map of torsors with isomorphic structure groups, and the base map $F(A/x) \to G(A/x)$ is a bijection. It follows that $\phi_A$ is a bijection. This completes the proof. □

3.3. **Applying Schlessinger's Criteria.** To apply Schlessinger's criteria to the functor $\mathbf{Def}_{F_0} : \mathbf{Art}_k \to \mathbf{Set}$, we first choose a complete local Noetherian ring $R$ with residue field $k$, and relate the category $\mathbf{Art}_k$ to $\mathbf{Art}_R$. When $k$ is perfect, there is a canonical choice, namely the ring of Witt vectors $W(k)$.

**Proposition 3.5.** *Let $k$ be a perfect field of characteristic $p > 0$. For any $A \in$ $\mathbf{Art}_k$, there exists a unique ring map $W(k) \to A$ compatible with the natural projections to $k$. Hence, the forgetful functor $\mathbf{Art}_{W(k)} \to \mathbf{Art}_k$ is an equivalence of categories.*

*Proof.* Since $k$ has characteristic $p$, each $A \in \mathbf{Art}_k$ is a $p$-complete local ring. The existence and uniqueness of the map $W(k) \to A$ then follow from Proposition 10 and Theorem 5 in [18, §2.5]. □

With $R = W(k)$, we can verify that the functor $\mathbf{Def}_{F_0} : \mathbf{Art}_{W(k)} \to \mathbf{Set}$ satisfies Schlessinger's four conditions.

*Proof.* To begin, $\mathbf{Def}_{F_0}(k) = \mathrm{pt}$, since the only deformation of $F_0$ over $k$ is $F_0$ itself.

To prove the Mayer–Vietoris property, let $A_0 \to A_{0,1}$ and $A_1 \to A_{0,1}$ be surjections in $\mathbf{Art}_{W(k)}$. Let $L$ be the Lazard ring. By Lazard's theorem, every formal group law $F$ over $A_0 \times_{A_{0,1}} A_1$ is classified by a ring map $\phi_F : L \to A_0 \times_{A_{0,1}} A_1$. If $F$ is a deformation of $F_0$, then composing $\phi_F$ with the projection map to $k$ yields a ring map $L \to k$ corresponding to $F_0$. This condition is equivalent to giving ring maps $L \to A_0$ and $L \to A_1$ that agree on $A_{0,1}$, and that both reduce to $F_0$ after composing with projections to $k$. Equivalently, it amounts to choosing deformations of $F_0$ over $A_0$ and $A_1$ whose restriction to $A_{0,1}$ coincide. By Proposition 2.11, this correspondence passes through isomorphism classes, and hence yields a bijection

$$\mathbf{Def}_{F_0}(A_0 \times_{A_{0,1}} A_1) \cong \mathbf{Def}_{F_0}(A_0) \times_{\mathbf{Def}_{F_0}(A_{0,1})} \mathbf{Def}_{F_0}(A_1).$$

This is exactly what we need.

To prove the formal smoothness condition, let $A \to A'$ be a surjection in $\mathbf{Art}_{W(k)}$. Every deformation of $F_0$ over $A'$ is classified by a ring map $L \to A'$. Because $A \to A'$ is surjective, this map lifts to $L \to A$, which corresponds to a deformation of $F_0$ over $A$. Hence, the set map $\mathbf{Def}_{F_0}(A) \to \mathbf{Def}_{F_0}(A')$ is surjective.

Finally, we show that the tangent space $\mathbf{Def}_{F_0}(k[\epsilon])$ is a $k$-vector space of dimension $n-1$ when $F_0 \in \mathcal{F}\mathrm{gl}\,(k)$ has height exactly $n$. Fix a deformation $F$ of $F_0$ over $k[\epsilon]$, We know $F \equiv F_0 \pmod{\epsilon}$. Since the height of $F_0$ is $n$, we have $v_i(F_0) = 0$ for all $i < n$. It follows that

$$v_i(F) = a_i\epsilon \quad (a_i \in k, i < n).$$

We claim that isomorphic deformations determine the same coefficients $a_i$. Indeed, if $F$ and $F'$ are isomorphic deformations over $k[\epsilon]$, then there exists an isomorphism of formal group laws $\phi : F \to F'$ with $\phi \in k[\epsilon][[x]]$ of the form of $\phi(x) = x + a(x)\epsilon$. As a power series, $\phi$ has an inverse $\psi(x) = x - a(x)\epsilon$ under composition, since for any $\delta \in k[\epsilon][[x]]$ with $\delta^2 = 0$ the first-order Taylor expansion gives

$$a(x+\delta) = a(x) + a'(x)\delta,$$

where $a'$ is the formal derivative of $a$. Taking $\delta = \pm a(x)\epsilon$ shows $\psi \circ \phi = \phi \circ \psi = \mathrm{id}$. Consequently, the $p$-series satisfy

$$[p]_{F'}(x) = \phi \circ [p]_F \circ \psi(x).$$

Moreover, we know $[p]_F(x) \equiv [p]_{F_0}(x) \pmod{\epsilon}$. So if we write $[p]_F(x) = \sum c_i x^{p^i}$, then $c_i$ are divisible by $\epsilon$ for all $i < n$. Since $\epsilon^2 = 0$, we compute

$$\begin{aligned}\phi \circ [p]_F \circ \psi(x) &= \sum_i c_i(x - a(x)\epsilon)^{p^i} + a\bigg(\sum_i c_i(x - a(x)\epsilon)^{p^i}\bigg)\epsilon \\ &= \sum_i c_i x^{p^i} + a\bigg(\sum_i c_i x^{p^i}\bigg)\epsilon.\end{aligned}$$

In the last term on the right-hand side, all contributions with $i < n$ vanish, since $c_i\epsilon \in (\epsilon^2) = 0$. Consequently, $[p]_F \equiv [p]_{F'} \pmod{x^{p^n}}$, and in particular $v_i(F) = v_i(F')$ for all $i < n$. Therefore, we obtain a well-defined map

$$\Phi : \mathbf{Def}_{F_0}(k[\epsilon]) \to k^{n-1}, \quad F \mapsto (a_0 = p, a_1, \dots, a_{n-1}).$$

Observe that the map constructed above is a map of $k$-vector spaces. Indeed, as in Schlessinger's criteria, the sum of two (isomorphism classes of) deformations $F_1$ and $F_2$ over $k[\epsilon]$ is induced by the composite

$$L \to k[\epsilon] \times_k k[\epsilon] \cong k[\epsilon_1, \epsilon_2] \to k[\epsilon],$$

where the last arrow is the addition map $x + y\epsilon_1 + z\epsilon_2 \mapsto x + (y+z)\epsilon$. That implies $v_i(F_1 + F_2) = v_i(F_1) + v_i(F_2)$. So $\Phi$ is a map of abelian groups. The $k$-linearity is similar: for $a \in k$, scalar multiplication of a deformation $F_1$ is induced by

$$L \to k[\epsilon] \to k[\epsilon],$$

with the second arrow given by $x + y\epsilon \mapsto x + ay\epsilon$. Therefore $v_i(aF_1) = av_i(F_1)$. It follows that $\Phi$ is $k$-linear.

Finally, we show that $\Phi$ is a bijection. For surjectivity, recall that the Lazard ring $L$ is graded and is (non-canonically) isomorphic to $L \cong \mathbb{Z}[t_1, t_2, \ldots]$ with $|t_i| = 2i$. Let $I \subseteq L$ be the ideal of positive-degree elements. We may choose generators $t_i \in L$, each homogeneous of degree $2i$, so that each $t_i$ lifts any chosen generator of the graded piece $(I/I^2)_{2i}$. In particular, we may take $t_{p^i-1} = v_i$ for every $i > 0$. Since these classes generate $L$, their images under a ring map $L \to k[\epsilon]$ can be prescribed freely. Thus, for any choice of $a_1, \ldots, a_{n-1} \in k$, define a map $L \to k[\epsilon]$ by sending $v_i \mapsto a_i\epsilon$ for $0 < i < n$ and $v_i \mapsto v_i(F_0)$ for $i \geq n$. This yields a deformation realizing the prescribed coefficients, so $\Phi$ is surjective.

To prove injectivity, let $F \in \ker(\Phi)$. That means $v_i(F) = 0$ for all $i < n$. It suffices to show that $F$ is isomorphic to the trivial deformation of $F_0$ over $k[\epsilon]$, which we again denote by $F_0$. Any isomorphism $\phi : F \to F_0$ is given by a power series with some compatibility conditions. Accordingly, define

$$A_{F,F_0} = \frac{k[\epsilon][b_0, b_1, \ldots]}{\{\text{compatibility conditions}\}}$$

to be the $k[\epsilon]$-algebra representing such isomorphisms: giving an isomorphism $\phi : F \to F_0$ is equivalent to giving a map $\theta : A_{F,F_0} \to k[\epsilon]$, and the coefficients of $\phi$ are the images $\theta(b_i)$. Since both $F$ and $F_0$ have height exactly $n$, a technical result in the theory of formal group laws (see, e.g., [13, Thm. 15.2]) shows that $A_{F,F_0}$ is a filtered colimit of finite étale $k[\epsilon]$-algebras; in particular, $A_{F,F_0}$ is formally étale over $k[\epsilon]$. Therefore, as indicated in the diagram below, every isomorphism $\phi : F \to F_0$ corresponds to a lift $A_{F,F_0} \to k[\epsilon]$ of the map classifying the identity of $F_0$ over $k$.

$$\begin{tikzcd}
A_{F,F_0} \arrow[r, "\tilde{\mathrm{id}}"] \arrow[dr, dashed] & k \\
k[\epsilon] \arrow[u, "\phi"] \arrow[r, "\mathrm{id}"'] & k[\epsilon] \arrow[u]
\end{tikzcd}$$

Such a lift exists because of the formally étale property of $A_{F,F_0}$. Therefore, we know $F \cong F_0$. That completes the whole proof. $\square$

From the proof of Schlessinger's criteria, we can now describe the construction of the universal formal group law $F_{\mathrm{univ}} \in \mathcal{F}\mathrm{gl}\,(E_0)$. Recall that $\mathbf{Def}_{F_0}(k[\epsilon])$ is a $(n-1)$-dimensional $k$-vector space. For each $0 < i < n$, choose $F_i \in \mathbf{Def}_{F_0}(k[\epsilon])$ such that $v_i(F_i) = \epsilon$ and $v_j(F_i) = 0$ for all $j \neq i$, $j < n$. Since each $F_i$ is a deformation, we have $v_j(F_i) = v_j(F_0)$ for all $j \geq n$. The tuple $F = (F_1, \ldots, F_{n-1})$ thereby defines a deformation of $F_0$ over $k[\epsilon]^{\times_k n-1} \cong k[\epsilon_1, \ldots, \epsilon_{n-1}]$. Under the natural map

$$E_0 = W(k)[[u_1, \ldots, u_{n-1}]] \to k[\epsilon_1, \ldots \epsilon_{n-1}], \quad u_i \mapsto \epsilon_i,$$

any formal group law over $E_0$ lifting $F$ can serve as the universal formal group law. Equivalently, we require $v_i(F_{\mathrm{univ}}) = u_i$ for all $0 < i < n$.

## 4. The Morava $E$-Theory

Fix a perfect field $k$ of characteristic $p$ and a formal group law $F_0 \in \mathcal{F}\mathrm{gl}\,(k)$ of height exactly $n$. From the *Lubin–Tate ring* $E_0 = E_0(F_0, k)$ we construct the

Morava $E$-theory, a cohomology theory central to chromatic homotopy theory. This section carries out that construction, beginning with a brief review of the relationship between spectra and formal group laws.

**Definition 4.1.** Let $E$ be a commutative ring spectrum. A *complex orientation* of $E$ is a distinguished class $x_E \in \widetilde{E}^2(\mathbb{C}P^\infty)$ whose restriction to

$$\widetilde{E}(\mathbb{C}P^1) \cong \widetilde{E}(S^2) \cong \pi_0(E)$$

is the unit 1. Here $\mathbb{C}P^n$ denotes the $n$-dimensional complex projective space. We call a spectrum $E$ *complex orientable* if it admits a complex orientation.

**Proposition 4.2** ([15, Lem. 4.1.4])**.** *Let $E$ be a spectrum with complex orientation $x_E$. Then there are isomorphisms*

$$E^*(\mathbb{C}P^\infty) \cong E^*(\mathrm{pt})[[x_E]] \text{ and } E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*(\mathrm{pt})[[x_E \otimes 1, 1 \otimes x_E]].$$

*Moreover, let $t : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ be the map corresponding to the tensor product of complex line bundles. The power series $F_E \in E^*(\mathrm{pt})[[x, y]]$ defined by*

$$t^*(x_E) = F_E(x_E \otimes 1, 1 \otimes x_E)$$

*is a formal group law over $E^*(\mathrm{pt})$.*

This proposition shows that every complex orientation determines a formal group law. As basic example, consider the integral Eilenberg–Mac Lane spectrum $H\mathbb{Z}$. The usual generator of $H\mathbb{Z}^2(\mathbb{C}P^\infty)$ is a complex orientation of $H\mathbb{Z}$, and the associated formal group law is the additive law $F(x, y) = x + y$ over $\mathbb{Z}$.

Another example arises from the complex cobordism spectrum MU. Construct MU from the Thom spaces $\mathrm{MU}(n)$ of complex vector bundles over $\mathrm{BU}(n)$, the classifying space of the unitary group $\mathrm{U}(n)$ (see, e.g., [15, §4.1]). The composite

$$\mathbb{C}P^\infty = \mathrm{BU}(1) \xrightarrow{\simeq} \mathrm{MU}(1) \to \mathrm{MU}$$

determines a complex orientation $x_{\mathrm{MU}} \in \mathrm{MU}^2(\mathbb{C}P^\infty)$. Quillen proved that the corresponding formal group law is the universal formal group law.

**Theorem 4.3** ([14, Thm. 2])**.** *The formal group law $F_{\mathrm{MU}} \in \mathcal{F}\mathrm{gl}\,(\mathrm{MU}^*(\mathrm{pt}))$ induced by $x_{\mathrm{MU}}$ corresponds, by Lazard's theorem, to a ring map $\theta : L \to \mathrm{MU}^*(\mathrm{pt})$. This map $\theta$ is an isomorphism. Under this identification, $F_{\mathrm{MU}}$ is the universal formal group law $F_{\mathrm{Laz}} \in \mathcal{F}\mathrm{gl}\,(L)$.*

By Quillen's theorem, specifying a complex orientation on a spectrum $E$ induces a ring map $\mathrm{MU}^*(\mathrm{pt}) \cong L \to E^*(\mathrm{pt})$, which classifies the associated formal group law. However, the converse does not hold in general: a formal group law over $E^*(\mathrm{pt})$ need not arise from a complex orientation of $E$. The Landweber exact functor theorem gives a precise algebraic criterion for when such a realization exists.

**Definition 4.4.** Let $R$ be a commutative ring and $M$ be an $R$-module. A sequence of elements $r_0, r_1, , r_2, \ldots \in R$ is *regular* for $M$ if $r_0$ is a non-zero-divisor on $M$, and for each $i \geq 1$ the element $r_i$ is a non-zero-divisor on $M/(r_0, \ldots, r_{i-1})M$.

**Theorem 4.5** (Landweber Exact Functor Theorem; see, e.g., [16, Thm. B.6.2])**.** *Let $M$ be a module over the Lazard ring $L$. If, for each prime $p$, the sequence $v_0 = p, v_1, v_2, \ldots$ is regular on $M$, then $M$ determines a homology theory $E_*$ given by*

$$E_*(X) = \mathrm{MU}_*(X) \otimes_L M.$$

If an $L$-module $M$ satisfies the hypothesis of Theorem 4.5, we say that $M$ is *Landweber exact*. By Lazard's theorem, every formal group law $F$ over a commutative ring $R$ induces a ring map $L \to R$. Viewing $R$ as an $L$-module via this map, we say that $F$ is Landweber exact if the $L$-module $R$ is Landweber exact.

For every Landweber exact formal group law $F$ over $E_*$, Theorem 4.5 gives a homology theory $E_*(-)$. By Brown's representability theorem, there exists a spectrum $E$ with $\pi_*(E) = E_*$ and $E_*(X) \cong \pi_*(E \wedge X)$ for all spectra $X$. Therefore, $E$ is a complex orientable spectrum, with an orientation given by the image of $x_{\mathrm{MU}}$ under the canonical map $\mathrm{MU} \to E$.

**Proposition 4.6.** *Let $k$ be a perfect field of characteristic $p > 0$, and let $F_0 \in \mathcal{F}\mathrm{gl}\,(k)$ have height exactly $n$. Form the graded ring*

$$E(n)_* := E_0[\beta^{\pm 1}] \cong W(k)[u_1, \ldots, u_{n-1}][\beta^{\pm 1}],$$

*where $E_0 = E_0(F, k)$ and $|\beta| = 2$. Define a formal group law $F \in \mathcal{F}\mathrm{gl}\,(E(n)_*)$ from the universal one $F_{\mathrm{univ}} \in \mathcal{F}\mathrm{gl}\,(E_0)$ by prescribing*

$$v_i(F) = v_i(F_{\mathrm{univ}})\beta^i$$

*for all $i \geq 0$. Then $F$ is Landweber exact.*

*Proof.* For every prime number $q \neq p$, the Landweber conditions are vacuous. Indeed, $E_0$ is $p$-local, so $q$ is a unit in $E_0$ (and hence in $E(n)_*$), whence $q$ is not a zero divisor and $E(n)_*/q = 0$. The remaining checks over $E(n)_*/q$ are therefore automatic.

It remains to show that the sequence $v_0 = p, v_1, v_2, \ldots$ is regular on $E(n)_*$. Since $W(k)$ is a domain, the polynomial extension $E(n)_*$ over $W(k)$ is also a domain. In particular, $v_0 = p \neq 0$ is not a zero-divisor. By our analysis of $F_{\mathrm{univ}}$ in the end of Section 3, the map $L \to E(n)_*$ determined by $F$ sends

$$v_i \mapsto u_i\beta^i \ (0 < i < n), \quad \text{and } v_n \mapsto v_n(F_0)\beta^n.$$

Therefore, $v_i$ is not a zero-divisor in the domain $E(n)_*/(v_0, \ldots v_{i-1}) \cong k[u_i, \ldots, u_{n-1}][\beta^{\pm 1}]$ for all $0 < i < n$. The element $v_n$ is not a zero divisor in $E(n)_*/(v_0, \ldots, v_{n-1}) \cong k[\beta^{\pm 1}]$, because $F_0$ has height $n$ implies $v_n(F_0) \neq 0$. Finally, $E(n)_*/(v_0, \ldots, v_n) \cong 0$ implies that all the $v_j$'s for $j > n$ are trivially non-zero-divisors. This completes the proof. $\square$

Proposition 4.6 therefore yields a homology theory $E(n)_*(-)$ with coefficient ring $E(n)_*$, and a complex orientable spectrum $E(n)$ representing it. From $E(n)$ we obtain the associated cohomology theory

$$E(n)^*(X) := [X, E(n)]_*.$$

This is the *Morava $E$-theory.*

The Morava $E$-theory and its associated spectrum are important in several respects. First, it is even-periodic and complex-orientable by construction. Another notable feature is that it admits a canonical structure as an $E_\infty$-ring spectrum.

**Theorem 4.7** (Goerss-Hopkins-Miller, [4, Cor. 7.6])**.** *The Lubin-Tate spectrum $E(n)$ admits a unique $E_\infty$ ring structure compatible with the ring structure on its homotopy groups.*

An $E_\infty$-ring spectrum is a spectrum with a multiplication that is commutative and associative up to all higher coherences, providing the strongest form of multiplicative structure available in homotopy theory. This endows $E(n)$ with powerful multiplicative and equivariant properties, making it possible to import methods from algebraic geometry and deformation theory into stable homotopy theory. See [10] for an introduction to $E_\infty$-ring spectra in the original operadic framework. For the recognition and comparison results that identify $E_\infty$-ring spectra with honest commutative ring spectra—i.e. commutative monoids in any symmetric monoidal category of spectra—and for the machine that constructs these from space-level data, see [11]. For an $\infty$-categorical account, see [9, Ch. 7].

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## Appendix A. Formal Groups: An Algebro-Geometric Approach

In modern chromatic homotopy theory, formal group laws are typically studied within an algebro-geometric framework. Many results—such as the Landweber exact functor theorem and the theory of the Morava stabilizer group—admit natural reinterpretations in this setting (see, e.g., [13, §16, 19]). More specifically, every one-dimensional commutative formal group law over a ring $R$ corresponds to a formal group, which is an abelian group object in $\mathbf{Shv}_R$, the category of étale sheaves over $\operatorname{Spec} R$. The collection of all such formal groups assembles into the moduli stack of formal groups, denoted $\mathcal{M}_{\mathrm{FG}}$. This stack plays a central role in chromatic homotopy theory: its geometry governs localization and periodicity phenomena for spectra.

A.1. **Functor of Points.** The language of formal groups is most naturally introduced after adopting the functor-of-points viewpoint on sheaves and schemes. We briefly introduce this perspective, assuming familiarity with the classical definition of schemes as locally ringed spaces; see [5] and [3] for background.

Let **Sch** denotes the category of schemes. Intuitively, the *functor of points* of a scheme $X$ is the functor

$$h_X : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set},$$

which sends a scheme $Y$ to $h_X(Y) = \mathrm{Hom}_{\mathbf{Sch}}(Y, X)$, and a morphism $f : Y \to Z$ to the set map $h_X(Z) \to h_X(Y)$ given by pre-composition with $f$. The assignment

$$h : \mathbf{Sch} \to \mathrm{Fun}(\mathbf{Sch}^{\mathrm{op}}, \mathbf{Set}), \quad X \mapsto h_X$$

is the Yoneda embedding: it is fully faithful and identifies **Sch** with the full subcategory of representable functors. Using the canonical identification of affine schemes $\mathbf{AffSch}^{\mathrm{op}} \cong \mathbf{CRing}$, one obtains an even cleaner description.

**Proposition A.1** (Proposition 6.2, [3])**.** *The restriction of the functor of points to affine schemes determines a scheme. Equivalently, the functor*

$$h' : \mathbf{Sch} \to \mathrm{Fun}(\mathbf{AffSch}^{\mathrm{op}}, \mathbf{Set}) \cong \mathrm{Fun}(\mathbf{CRing}, \mathbf{Set}), \quad X \mapsto h_X|_{\mathbf{AffSch}}$$

*is fully faithful.*

An identical construction applies to schemes over $\mathrm{Spec}(R)$ for a fixed commutative ring $R$, yielding a functor $h' : \mathbf{Sch}_R \to \mathrm{Fun}(\mathbf{CRing}_R, \mathbf{Set})$, where $\mathbf{CRing}_R$ denotes the category of commutative $R$-algebras.

Therefore, we can *redefine* schemes by their functor of points—namely, as certain functors $\mathbf{CRing} \to \mathbf{Set}$. We make this explicit as follows.

**Definition A.2.** An *étale sheaf* is a functor $X : \mathbf{CRing} \to \mathbf{Set}$ that satisfies étale descent. Explicitly, for any finite collection of étale ring maps $\{A \to B_i\}_{i=1}^n$ such that $A \to \prod B_i$ is faithfully flat, the diagram

$$X(A) \to \prod_{i=1}^n X(B_i) \rightrightarrows \prod_{i,j=1}^n X(B_i \otimes_A B_j)$$

is an equalizer.

**Example A.3.** Let $A$ be a commutative ring. The *affine scheme* $\mathrm{Spec}(A)$ is the étale sheaf $\mathrm{Spec}(A) : \mathbf{CRing} \to \mathbf{Set}$ given by

$$\mathrm{Spec}(A)(B) = \mathrm{Hom}_{\mathbf{CRing}}(A, B).$$

In general, an affine scheme is an étale sheaf that is isomorphic (as a functor) to one arising in this way. A *scheme* is an étale sheaf that is Zariski-locally affine (i.e., it admits a Zariski cover by affine schemes).

Let $A$ be a commutative ring and $I \subseteq A$ an ideal. The formal spectrum $\mathrm{Spf}(A)$ with respect to $I$ (Definition 3.1) is also an étale sheaf. An *affine formal scheme* is a sheaf isomorphic to such a formal spectrum.

A final note is that the same constructions work over a fixed base $\mathrm{Spec}(R)$: we define étale sheaves, affine schemes, general schemes, and affine formal schemes over $\mathrm{Spec}(R)$ via functors $\mathbf{CRing}_R \to \mathbf{Set}$.

A.2. **Formal Groups and Their Height.** We are now ready to define the notion of a formal group.

**Definition A.4.** Let $R$ be a commutative ring, and $F \in R[[x, y]]$ a formal group law. The associated *formal group* is the abelian group object $\mathbb{G}_F : \mathbf{CRing}_R \to \mathbf{Ab}$ in $\mathbf{Shv}_R$, defined on an $R$-algebra $B$ by

$$\mathbb{G}_F(B) = \mathrm{Nil}(B),$$

the set of nilpotent elements of $B$. The abelian group structure is given by

$$(b_1, b_2) \mapsto F(b_1, b_2) = b_1 +_F b_2.$$

**Proposition A.5.** *For any formal group law $F \in R[[x, y]]$ and any commutative $R$-algebra $B$, the operation $+_F$ makes $\mathrm{Nil}(B)$ into a well-defined abelian group.*

*Proof.* First, $b_1 +_F b_2$ is always well-defined: since $b_1$, $b_2$ are nilpotent, the power series $F(b_1, b_2)$ involves only finitely many nonzero terms. It remains to show that every $b \in \mathrm{Nil}(B)$ has an inverse.

We argue by induction on the least integer $n$ such that $b^n = 0$. The base case $n = 1$ is trivial, since then $b = 0$ already has an inverse. Suppose $n > 1$. The expansion

$$x +_F y = x + y + \text{ (higher order terms)}$$

implies $b +_F (-b) = b + (-b) + b^2x = b^2x$ for some $x \in B$. By induction hypothesis, $(b^2x)^{n-1} = 0$ ensures the existence of some $a \in \mathrm{Nil}(B)$ with $b^2x +_F a = 0$. So $-b +_F a$ is an inverse of $b$. This completes the proof. $\square$

We also note that, as an étale sheaf, $\mathbb{G}_F : \mathbf{CRing}_R \to \mathbf{Set}$ is an affine formal scheme: $\mathbb{G}_F \cong \mathrm{Spf}(R[[x]])$, where for the right-hand side we take the associated ideal to be the unique maximal ideal $(x) \subseteq R[[x]]$.

**Definition A.6.** Let $R$ be a commutative ring. A *formal group* over $\mathrm{Spec}(R)$ is an abelian group object $\mathbb{G} : \mathbf{CRing}_R \to \mathbf{Ab}$ in the category $\mathbf{Shv}_R$ that is Zariski-locally of the form $\mathbb{G}_F$ for some formal group law $F$.

Concretely, $\mathbb{G}$ is a formal group if there exists elements $f_1, \ldots, f_n \in R$ generating the unit ideal such that, for each $i$, the restriction of $\mathbb{G}$ to $\mathbf{CRing}_{R_{f_i}}$ is isomorphic to $\mathbb{G}_{F_i}$ for some $F_i \in \mathcal{F}\mathrm{gl}\,(R_{f_i})$.

Formal groups also have a notion of height, compatible with the height of formal group laws, as described below. For brevity, we fix a prime number $p$, a commutative $\mathbb{F}_p$-algebra $R$, and a formal group $\mathbb{G} : \mathbf{CRing}_R \to \mathbf{Ab}$ over $\mathrm{Spec}(R)$.

Recall the Frobenius endomorphism $\mathrm{Frob} : R \to R$ given by $\mathrm{Frob}(r) = r^p$. It induces an étale sheaf map $\mathrm{Frob} : \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ by pre-composition. We therefore define the *Frobenius twist* $\mathbb{G}^{(1)}$ of $\mathbb{G}$ to be the pullback of $\mathbb{G}$ along the Frobenius sheaf map, i.e.

$$\begin{tikzcd}
\mathbb{G} \arrow[dr, dashed, "\mathrm{Frob}_{\mathbb{G}/R}"] \arrow[drr, bend left, "\mathrm{Frob}_{\mathbb{G}}"] \arrow[ddr, bend right] & & \\
& \mathbb{G}^{(1)} \arrow[r] \arrow[d] & \mathbb{G} \arrow[d, "\pi"] \\
& \mathrm{Spec}(R) \arrow[r, "\mathrm{Frob}"'] & \mathrm{Spec}(R)
\end{tikzcd}$$

Concretely speaking, for an $R$-algebra structure $\alpha : R \to A$, we have

$$\mathbb{G}^{(1)}(A, \alpha) = \{(g, \beta) : g \in \mathbb{G}(A, \alpha), \pi(g) = \beta \circ \mathrm{Frob}\} = \mathbb{G}(A, \alpha \circ \mathrm{Frob}).$$

Therefore, $\mathbb{G}^{(1)}$ is again an abelian group object; in particular, it is a formal group. If $\mathbb{G}$ is *coordinatizable*, i.e. $\mathbb{G} \cong \mathbb{G}_F$ for some formal group law $F(x, y) = \sum c_{i,j} x^i y^j \in \mathcal{F}\mathrm{gl}\,(R)$, then $\mathbb{G}^{(1)} \cong \mathbb{G}_{F'}$ is also coordinatizable, with $F' \in \mathcal{F}\mathrm{gl}\,(R)$ defined by $F'(x, y) = c_{i,j}^p x^i y^j$.

Since $\mathbb{G}$ is an étale sheaf over $\mathrm{Spec}(R)$, it can be expressed as a filtered colimit of affine schemes $\mathrm{Spec}(A_\alpha)$, where each $A_\alpha$ is an $R$-algebra. Because every $A_\alpha$ carries the Frobenius endomorphism, each $\mathrm{Spec}(A_\alpha)$ admits a Frobenius endomorphism over $\mathrm{Spec}(R)$. Passing to the colimit yields an induced map of étale sheaves $\mathrm{Frob}_{\mathbb{G}} : \mathbb{G} \to \mathbb{G}$ over $\mathrm{Spec}(R)$. By the universal property of pullback, we then obtain a canonical morphism of formal groups $\mathrm{Frob}_{\mathbb{G}/R} : \mathbb{G} \to \mathbb{G}^{(1)}$, as the diagram above shows. We call it the *relative Frobenius map*. If, moreover, the formal group

$\mathbb{G} \cong \mathbb{G}_F$ is coordinatizable, this map corresponds to the morphism of formal group laws $\phi : F \to F'$ given by $\phi(x) = x^p$.

**Definition A.7.** Let $\mathbb{G}$ be a formal group over $\operatorname{Spec}(R)$. We say that $\mathbb{G}$ has *height at least* $n$ if the multiplication by $p$ map $p : \mathbb{G} \to \mathbb{G}$ factors through the $n$-fold relative Frobenius:

$$\begin{tikzcd} \mathbb{G} \arrow[r, "\mathrm{Frob}^n_{\mathbb{G}/R}"] \arrow[dr, "p"'] & \mathbb{G}^{(n)} \arrow[d, dashed] \\ & \mathbb{G} \end{tikzcd}$$

We say that $\mathbb{G}$ has *height exactly* $n$ if it has height at least $n$, and the induced map $\mathbb{G}^{(n)} \to \mathbb{G}$ is an isomorphism. We say that $\mathbb{G}$ has *infinity height* if it has height at least $n$ for every positive integer $n$.

The notion of height admits a more concrete description when the formal group is coordinatizable. Suppose $\mathbb{G} \cong \mathbb{G}_F$ for some formal group law $F \in \mathcal{F}\mathrm{gl}\,(R)$. In this case, the multiplication-by-$p$ map corresponds, in coordinates, to the $p$-series $[p]_F(x) \in R[[x]]$ Using the canonical identification $\mathbb{G}_F \cong \operatorname{Spf}(R[[x]])$ (as sheaves), the diagram in Definition A.7 becomes

$$\begin{tikzcd} R[[x]] & R[[x]] \arrow[l, "x^{p^n} \leftarrow\!\shortmid x"'] \\ & R[[x]] \arrow[ul, "{[p]_F}"] \arrow[u, dashed, "g"'] \end{tikzcd}$$

It follows that $\mathbb{G}_F$ has height at least $n$ if and only if $[p]_F(x) = g(x^{p^n})$ for some power series $g \in R[[x]]$, and it has height exactly $n$ if and only if in addition $g$ is invertible. This agrees with the usual definition of height of a formal group law.

A.3. **Deformations of Formal Groups.** Finally, we define deformations of formal groups, and show that the Lubin-Tate deformation theorem applies to formal groups as well.

Throughout this section, we fix a field $k$ of characteristic $p > 0$.

**Definition A.8.** Let $A \in \mathbf{Art}_k$ be an infinitesimal thickening, and let $\mathbb{G}_0$ be a formal group over $\operatorname{Spec}(k)$. A *deformation* of $\mathbb{G}_0$ over $A$ is a formal group $\mathbb{G}$ over $\operatorname{Spec}(A)$ together with an isomorphism of formal groups over $\operatorname{Spec}(k)$

$$\phi_{\mathbb{G}} : \mathbb{G} \times_{\operatorname{Spec}(A)} \operatorname{Spec}(k) \to \mathbb{G}_0.$$

An isomorphism of deformations $(\mathbb{G}, \phi_{\mathbb{G}}) \to (\mathbb{G}', \phi_{\mathbb{G}'})$ is an isomorphism of formal groups $f : \mathbb{G} \to \mathbb{G}'$ such that the diagram

$$\begin{tikzcd} \mathbb{G} \times_{\operatorname{Spec}(A)} \operatorname{Spec}(k) \arrow[rr, "f_*"] \arrow[dr, "\phi_{\mathbb{G}}"'] & & \mathbb{G}' \times_{\operatorname{Spec}(A)} \operatorname{Spec}(k) \arrow[dl, "\phi_{\mathbb{G}'}"] \\ & \mathbb{G}_0 & \end{tikzcd}$$

commutes. For a fixed thickening $A$, the deformations of $\mathbb{G}_0$ over $A$ and their isomorphisms form a groupoid, which we denote by $\mathbf{Def}_{\mathbb{G}_0}(A)$.

There are two important observations. First, since both the field $k$ and the thickening $A \in \mathbf{Art}_k$ are local, every formal group over either base is coordinatizable. Indeed, let $\mathbb{G}$ be a formal group over $A$. Choosing elements $a_1, \ldots, a_n \in A$ generating the unit ideal such that each restriction $\mathbb{G}|_{\mathrm{Spec}(A_{a_i})}$ is coordinatizable. Because $A$ is local, some $a_i$ is invertible; hence $\mathrm{Spec}(A_i) = \mathrm{Spec}(A)$, and therefore $\mathbb{G}$ itself is coordinatizable. The same argument applies to formal groups over $\mathrm{Spec}(k)$.

Second, the deformation theories defined via coordinates and via formal groups agree.

**Proposition A.9.** *Let $A \in \mathbf{Art}_k$, $F_0 \in \mathcal{F}\mathrm{gl}\,(k)$, and let $\mathbb{G}_0 = \mathbb{G}_{F_0}$ be the corresponding formal group over* $\mathrm{Spec}(k)$. *Then the groupoids $\mathbf{Def}_{F_0}(A)$ and $\mathbf{Def}_{\mathbb{G}_0}(A)$ are equivalent.*

*Proof.* We show that the functor

$$\begin{aligned}\mathbf{Def}_{F_0}(A) &\to \mathbf{Def}_{\mathbb{G}_0}(A)\\ F &\mapsto (\mathbb{G}_F, \mathrm{id} : \mathbb{G}_F \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) = \mathbb{G}_0 \to \mathbb{G}_0)\end{aligned}$$

gives an equivalence of categories.

To show that it is fully faithful, it suffices to exhibit a bijection

$$\mathrm{Hom}_{\mathbf{Def}_{F_0}(A)}(F, F') \cong \mathrm{Hom}_{\ \mathbf{Def}_{\mathbb{G}_0}(A)}((\mathbb{G}_F, \mathrm{id}), (\mathbb{G}_{F'}, \mathrm{id}))$$

for any two deformations $F, F' \in \mathbf{Def}_{F_0}(A)$. This is possible because each isomorphism $f : \mathbb{G}_F \to \mathbb{G}_{F'}$ is associated to an induced map $f_*$ by Definition A.8, and $f_*$ commutes with the identity map $\mathrm{id} : \mathbb{G}_F \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) = \mathbb{G}_0 \to \mathbb{G}_0$. Identifying $\mathbb{G}_0 \cong \mathrm{Spf}(k[[x]])$, we see that as a power series $f_*(x)$ is the identity $\mathrm{id} = x \in k[[x]]$. That happens if and only if $f(x) \equiv x \pmod{\mathfrak{m}_A}$, which is exactly the defining condition for an isomorphism in $\mathbf{Def}_{F_0}(A)$.

It remains to prove essential surjectivity. Given any $(\mathbb{G}, \phi) \in \mathbf{Def}_{\mathbb{G}_0}(A)$, we must produce a formal group law $F \in \mathbf{Def}_{F_0}(A)$ and an isomorphism of deformations $(\mathbb{G}, \phi) \to (\mathbb{G}_F, \mathrm{id})$. Since $A$ is local, $\mathbb{G}$ is coordinatizable: there exists $F' \in \mathcal{F}\mathrm{gl}\,(A)$ with $\mathbb{G} \cong \mathbb{G}_{F'}$. Under this coordinate, $\phi$ is an isomorphism of formal group laws $\phi : F' \mod \mathfrak{m}_A \to F_0$, hence a power series in $k[[x]]$. Choose a lift $\tilde{\phi} \in A[[x]]$ along the surjection $\alpha : A \to k$. Define $F = \tilde{\phi} \circ F' \circ \tilde{\phi}^{-1}$. Then

$$F \mod \mathfrak{m}_A = \phi \circ (F \mod \mathfrak{m}_A) \circ \phi^{-1} = F_0,$$

so $F \in \mathbf{Def}_{F_0}(A)$. Since the isomorphism of formal groups $\tilde{\phi} : \mathbb{G} \to \mathbb{G}_F$ makes the diagram

$$\begin{tikzcd} \mathbb{G} \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) \arrow[rr, "\tilde{\phi}_* = \phi"] \arrow[dr, "\phi"'] & & \mathbb{G}_{F'} \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) \arrow[dl, "\mathrm{id}"] \\ & \mathbb{G}_0 & \end{tikzcd}$$

commute, $\tilde{\phi}$ indeed induces an isomorphism of deformations $(\mathbb{G}, \phi) \to (\mathbb{G}_F, \mathrm{id})$. This completes the proof. ∎

Therefore, working with deformations of formal group laws or with deformations of formal groups themselves is essentially the same problem: results established in one framework immediately translate into the other. So Theorem 3.2 can be interpreted as a description of the geometric structure of infinitesimal formal neighborhoods of the height-$n$ stratification of $\mathcal{M}_{\mathrm{FG}}$.

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