

RANDOM CURVES AND SCHRAMM-LOEWNER EVOLUTIONS

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ABSTRACT. This expository paper develops the theory behind Schramm-Loewner Evolutions. While this paper states and proves major results regarding SLEs, its primary goal is to provide an intuitive presentation of the subject. Stochastic theory and complex analysis knowledge is required, but the necessary theory of conformal mapping distortions is given in the appendix.

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1. MOTIVATION AND INTRODUCTION

Consider a drunkard's walk in downtown Chicago. At every intersection, the drunk has an equal chance of veering towards any direction. The twist in this random walk is that whenever the drunkard goes in a loop, he is able to go back in time to when the loop started. He performs this *Loop Erased Random Walk (LERW)* away from the lake until he gets tired. The results of his journey around Chicago are shown in Figure 1, where we scale his journey to include more and more intersections. He keeps doing this scaling up to infinity. Will he ever make it back to the lake? Is there a continuous curve he limits towards? What kind of (mathematical) object is this limit? What are the chances he makes it home?¹ The study of the Schramm-Loewner Evolution (SLE) seeks to answer this drunken Chicagoan.

Much like this drunk, the mathematical narrative to building these SLEs will take us on a loop. There will be two stories in the narrative. First, the pedagogical one. Here, we conveniently start with a curve γ in the upper half plane $\mathbb{H}$. This curve gives rise to a family of sets—denoted by (K_t) —which are attached to the real axis. For each such K_t , the Riemann Mapping Theorem² will assure us of

¹Low.

²See [1] or any complex analysis textbook for the statement and a proof of this classic result.

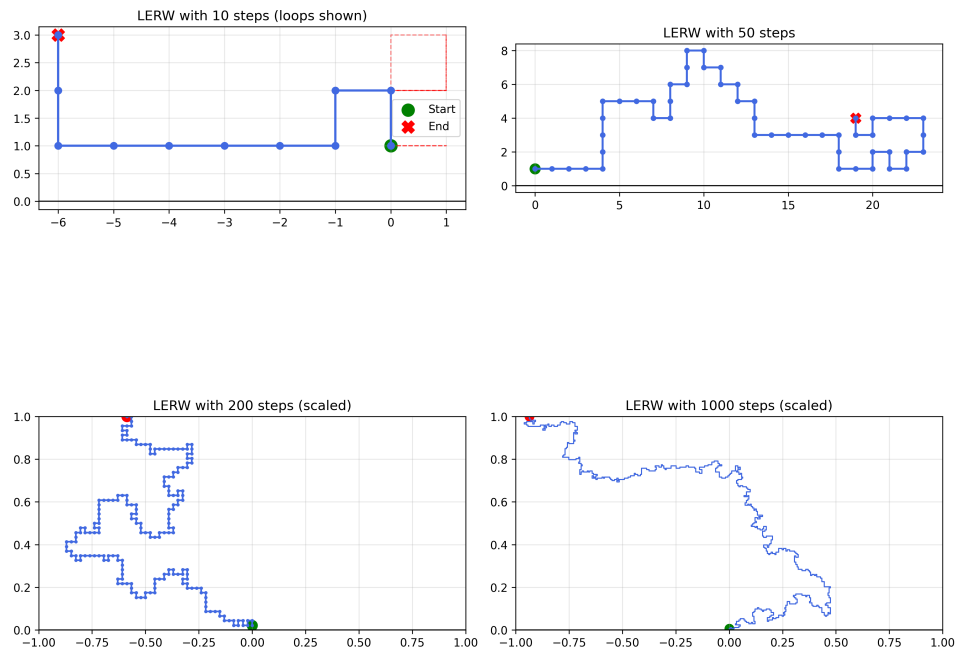


FIGURE 1. Loop Erased Random Walks

the existence of a unique conformal map $g_t : \mathbb{H} \setminus K_t \rightarrow \mathbb{H}$. Placing just a few conditions on the K_t will yield the flow of how the maps g_t change over time, with its evolution governed by the Loewner differential equation, which itself is driven by some continuous function U_t . Placing two more assumptions on this evolution will result in the g_t being directed by $\sqrt{\kappa}B_t$ in place of the U_t , where B_t is a standard Brownian motion and $\kappa > 0$. This story is interesting in its own right, but it ends here, which is why this paper concentrates more on the second story.

The second story begins with a parameter $\kappa > 0$, a standard Brownian motion, and the conformal maps g_t which satisfy the Loewner differential equation. It is these g_t which are called the Schramm-Loewner Evolutions, and we'll see that they give rise to the family (K_t) we will now call the SLE chains. A very deep theorem will see us proving that the evolutions are generated by curves γ . Indeed, it will be these continuous curves we will most be interested in. Among other things, we will see that the κ parameter controls how wild the curve will be, how the curves are the scaling limits of various critical models (the drunkard will see his looped race LERW converging towards an SLE_2 path), and how distortion estimates are important tools in all these results.

The map for both of these stories (the blue arrows being for the first story and the red being for the latter) — as well as for the paper as a whole — is given by Figure 2.

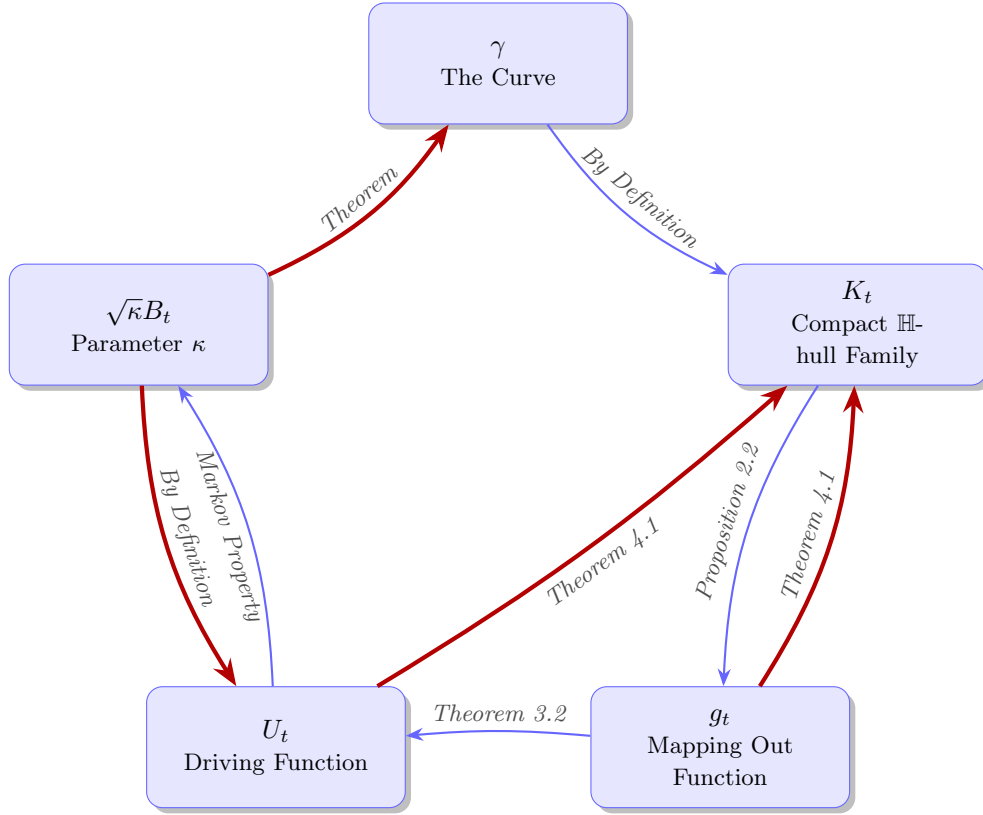


FIGURE 2. The Mathematical Narrative of SLE

2. PRELIMINARIES

For the rest of the paper, we denote $\mathbb{H}$ to be the upper half plane.

Definition 2.1. We define K to be a *compact $\mathbb{H}$ -hull* if $K = \overline{K} \cap \mathbb{H}$ and $\mathbb{H} \setminus K$ is simply connected.

Example 2.2. The recurring example for this section will be the family of growing vertical slits in the upper half plane given by the compact $\mathbb{H}$ -hulls $K_t = [0, 2i\sqrt{t}]$. Note that for each time $t \geq 0$, $\mathbb{H} \setminus [0, 2i\sqrt{t}]$ is simply connected.

Proposition 2.3. Suppose K is a compact $\mathbb{H}$ -hull. Then there exists a unique conformal map $g_K : \mathbb{H} \setminus K \rightarrow \mathbb{H}$ such that $|g_K(z) - z| \rightarrow 0$ as $z \rightarrow \infty$ which admits the Laurent expansion

$$(2.4) \quad g_K(z) = z + \frac{c_1}{z} + O(z^{-2}),$$

where $c_1, c_2, \dots \in \mathbb{R}$.

Proof. (Existence) Define the set $D := \{-\frac{1}{z} \mid z \in \mathbb{H} \setminus K\}$. By the Riemann mapping theorem, there exists some conformal $\phi : \tilde{D} \rightarrow \mathbb{H}$ with $\phi(0) = 0$ and $\phi'(0) > 0$. By the Schwarz reflection principle, we set $\phi(\bar{z}) = \overline{\phi(z)}$ and find that ϕ extends to a conformal transformation defined on $D \cup \{\bar{z} \mid z \in D\} \cup (\mathbb{R} \cap \partial D)$. Noting that for

$z \in \mathbb{R} \setminus \overline{K}$, we have that $z = \bar{z}$ and thus $\phi(z) = \phi(\bar{z}) = \overline{\phi(z)}$, we see that the Taylor expansion of ϕ about zero yields

$$\sum_{i=1}^{\infty} a_i z^i = \overline{\sum_{i=1}^{\infty} a_i z^i} = \sum_{i=1}^{\infty} \overline{a_i} z^i.$$

Hence, all the coefficients in the Taylor expansion of g about 0 are real valued and are such that

$$\frac{1}{\phi(-\frac{1}{z})} = \frac{1}{\phi(-1/z)} = -\frac{z}{a_1} + \frac{a_2}{a_1^2} - \frac{a_2^2 - a_1 a_3}{a_1^3} \cdot \frac{1}{z} + \dots$$

Define $g_K : \mathbb{H} \setminus K \rightarrow \mathbb{H}$ by

$$g_K(z) := -a_1 \frac{1}{\phi(-\frac{1}{z})} + \frac{a_2}{a_1}.$$

Clearly, g_K is conformal and

$$g_K(z) = z + \frac{a_2^2 - a_1 a_3}{a_1^2} \frac{1}{z} + O(z^{-2})$$

and hence $|g_K(z) \rightarrow \infty|$ as $|z| \rightarrow \infty$.

(Uniqueness) Suppose g_K^* is another such conformal map. It is known that since $h := g_K^* \circ g_K^{-1} : \mathbb{H} \rightarrow \mathbb{H}$ is a conformal automorphism, then h must be of the form $h(z) = \frac{az+b}{cz+d}$ with $a, b, c, d > 0$ and $ad - bc = 1$. Thus, $|h(z) - z| \rightarrow 0$ with $|z| \rightarrow \infty$ implies $a = c = 1$, and hence $b = d = 0$. Finally, $h \equiv \text{id}$ implies $g_K^* = g_K$. $\square$

Definition 2.5. Suppose K is a compact $\mathbb{H}$ -hull with the mapping out function g_K . We define the *half-plane capacity* of K to be

$$\text{hcap}(K) = \lim_{z \rightarrow \infty} z(g_K(z) - z).$$

It is not hard to see that $\text{hcap}(K)$ can be uniquely identified with c_1 in (2.4) In a sense, one can think of $\text{hcap}(K)$ as a measure of the size of the set K . To make this statement clearer, consider the following example and proposition.

Example 2.6. Keeping with Example 2.2, it can be easily shown that the unique mapping out function assigned to $K_t = [0, 2i\sqrt{t}]$ is given by $g_t = g_{K_t} : \mathbb{H} \setminus [0, 2i\sqrt{t}] \rightarrow \mathbb{H}$ such that

$$g_t(z) = \sqrt{z^2 + 4t}.$$

Recalling that the expansion of $z \mapsto \sqrt{1+z}$ is given by $1 + \frac{1}{2}z - \frac{1}{8}z^2 + \dots$,

$$g_t(z) = z \sqrt{1 + \frac{4t}{z^2}} = z \left(1 + \frac{1}{2} \left(\frac{4t}{z^2} \right) - \frac{1}{8} \left(\frac{4t}{z^2} \right)^2 + \dots \right) = z + \frac{2t}{z} + \dots$$

It is now clear that $|g_t(z) - z| \rightarrow 0$ as $z \rightarrow \infty$ and that $\text{hcap}(K_t) = 2t$.

Proposition 2.7. Suppose $K, \tilde{K}$ are compact $\mathbb{H}$ -hulls such that $K \subseteq \tilde{K}$. Then

- (1) for $x \in \mathbb{R}$, $K + x$ is a compact $\mathbb{H}$ -hull with $g_{K+x}(z) = g_K(z - x) + x$ such that $\text{hcap}(K + x) = \text{hcap}(K)$;
- (2) for $r > 0$, rK is a compact $\mathbb{H}$ -hull with $g_{rK}(z) = r g_K(\frac{z}{r})$ such that $\text{hcap}(rK) = r^2 \text{hcap}(K)$;
- (3) if $L = g_K(\tilde{K} \setminus K)$, then L is a compact $\mathbb{H}$ -hull with $g_L(z) = \text{hcap}(K) + \text{hcap}(L) = \text{hcap}(\tilde{K})$. Moreover, $\text{hcap}(K) \leq \text{hcap}(\tilde{K})$.

Proof. We will show (3) to give a taste of the other proofs.

Since g_K is conformal, clearly L is a compact $\mathbb{H}$ -hull, and thus the map g_L is well defined. Since $g_{\tilde{K}} = g_L \circ g_K$, we know by (2.4) that

$$z(g_{\tilde{K}}(z) - z) = z(g_L(g_K(z)) - z) = z(g_L(g_K(z)) - g_K(z)) + z(g_K(z) - z),$$

and so $\text{hcap}(\tilde{K}) = \text{hcap}(L) + \text{hcap}(K)$.

For the second claim, it will suffice to show that $\text{hcap}(K) \geq 0$ for any K compact $\mathbb{H}$ -hull. To see this, let B_t be a standard Brownian motion, let τ be the first exit time of B from $\mathbb{H} \setminus K$. Since $\Im(\cdot)$ is harmonic and $z - g_K(z)$ is holomorphic, we can apply the optional stopping theorem to the martingale $B_t - \Im(g_K(B_t))$. Then since $\Im(g(z)) = 0$ for any $z \in \partial\mathbb{H} \setminus K$,

$$\mathbb{E}_Z[\Im(B_\tau)] = \mathbb{E}_z[\Im(B_\tau - g_K(B_\tau))] = \mathbb{E}_z[\Im(B_0 - g_K(B_0))] = \Im(z - g_K(z)).$$

We are done if we can show that $\text{hcap}(K) = \lim_{y \rightarrow \infty} y \mathbb{E}_{iy}[\Im(B_\tau)]$. Since $\text{hcap}(K) \in \mathbb{R}$, then

$$\text{hcap}(K) = \Re(\text{hcap}(K)) = \Re(\lim_{y \rightarrow \infty} iy(g_K(iy) - iy)) = \Im(\lim_{y \rightarrow \infty} y(iy - g_K(iy))).$$

But from what we just showed above, we are done. $\square$

Example 2.8. Consider the closed unit disk in the upper half plane, $\overline{\mathbb{D}^+}$. Since $g_{\overline{\mathbb{D}^+}} = z + \frac{1}{z}$, then thus $\text{hcap}(\overline{\mathbb{D}^+}) = 1$. If we define $\text{rad}(K) = \sup\{|z| \mid z \in K\}$, then $K \subseteq \text{rad}(K) \cdot \overline{\mathbb{D}^+}$. Using Proposition 2.7,

$$\text{hcap}(K) \leq \text{hcap}(\text{rad}(K)\overline{\mathbb{D}^+}) = \text{rad}(K)^2.$$

Proposition 2.9. *Let K be a compact $\mathbb{H}$ -hull and suppose $|z| \geq 2\text{rad}(K)$. Then there is some $c > 0$ such that*

$$(2.10) \quad \left| g_K(z) - z - \frac{\text{hcap}(K)}{|z|} \right| \leq c \frac{\text{rad}(K)\text{hcap}(K)}{|z|^2}.$$

The proof for this proposition can be found in [2].

3. INTRODUCING SLE_κ : THE PEDAGOGICAL APPROACH

Consider a simple curve in the upper half plane given by $\gamma(t)$ with $\gamma(0) = 0$. If we let H_t be the unbounded complement of $\gamma([0, t])$, then we are interested in the map given by Proposition 2.2, $g_t : \mathbb{H} \setminus K_t \rightarrow \mathbb{H}$, where $K_t = \mathbb{H} \setminus H_t$ is the curve and all the regions the curve has “swallowed.” We developed the theory of g_t with the above results, and now we are interested in the flow of $t \mapsto g_t(z)$. This flow is most interesting at or near γ_t , and it stops for points on the curve when $g_t(z) = g_t(\gamma(t)) \in \mathbb{R}$. This time is so important we denote it by $U_t := g_t(\gamma(t))$. In fact, we will show that if we perform a time change so that $\text{hcap}(\gamma([0, t])) = 2t$ for any time t , then the flow of the curve over time is governed by the Loewner Differential Equation given by

$$(3.1) \quad \partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Imposing a Markov property on this flow will yield the very interesting result that U_t is a transformation of a standard Brownian motion.

There’s a much deeper converse to this narrative. If given a dilated standard Brownian motion, and a family g_t whose evolution satisfies (3.1), we can obtain the continuous curve (and its corresponding compact $\mathbb{H}$ -hulls). Such an evolution

is aptly named a Schramm-Loewner Evolution, and the dilation of the Brownian motion will correspond to how wild the curve is.

Theorem 3.2. *Suppose that (K_t) is a family of compact $\mathbb{H}$ -hulls satisfying the following:*

- (1) *For $0 \leq s \leq t$, $K_s \subseteq K_t$.*
- (2) *As $t \downarrow s$, we have $\text{diam}(g_s(K_t \setminus K_s)) \rightarrow 0$.*
- (3) *For any $t \geq 0$, $\text{hcap}(K_t) = 2t$.*

Then if $K_0 \neq \emptyset$, there exists some continuous $U : [0, \infty) \rightarrow \mathbb{R}$ that satisfies (3.1).

Proof. Define $U_t := \bigcap_{s \geq t} \overline{g_t(K_s \setminus K_t)}$. By the first and second assumptions, U_t is a single point and lies on the real line. To see that it is continuous in t , it suffices to show by (2) that

$$|U_t - U_s| \leq 4\text{diam}(g_s(K_t \setminus K_s)).$$

Let $\epsilon > 0$. Define $g_{s,t} := g_t \circ g_s^{-1} = g_{g_s(K_t \setminus K_s)}$. Letting $K = g_t(K_{t+\epsilon} \setminus K_t)$ in (2.10), we see that for $|z| \geq 2\text{rad}(K)$

$$g_K(z) = g_{K+x}(z-x) + x = z + \frac{\text{hcap}(K+x)}{|z-x|^2} + O\left(c \frac{\text{rad}(K+x)\text{hcap}(K+x)}{|z-x|^2}\right).$$

Letting $x = U_t$ and using Proposition 2.7,

$$g_K(z) = g_{t,t+\epsilon}(z) = z + \frac{2\epsilon}{|z - U_t|^2} + 2\epsilon \cdot \text{rad}(K)O\left(\frac{c}{|z - U_t|^2}\right).$$

Computing the difference quotient,

$$\begin{aligned} \frac{g_{t+\epsilon}(z) - g_t(z)}{\epsilon} &= \frac{(g_{t,t+\epsilon} - g_{t,t}(z)) \circ g_t(z)}{\epsilon} \\ &= \frac{2}{|g_t(z) - U_t|^2} + 2\text{rad}(K)O\left(\frac{c}{|g_t(z) - U_t|^2}\right). \end{aligned}$$

Taking $\epsilon \rightarrow 0$ and using the second assumption to see that $\text{rad}(K) \downarrow 0$, we arrive at the result. $\square$

We call U_t in (3.1) the *Loewner driving function*. While it is true that in the above theorem it is (K_t) that generates U_t , we will mostly be interested in the converse. We will see that U_t actually drives the process because it generates a series of random (K_t) sampled from $\mathbb{H}$ -hulls above. Interestingly, the (K_t) form a continuous curve.

Definition 3.3. Let (K_t) be a random family of conformal $\mathbb{H}$ -hulls satisfying (1)-(3) in Theorem 3.2.

- Let τ be a stopping time for the family (K_t) . We say that (K_t) possesses a *conformal Markov property* if for any $s \geq 0$, $\{g_\tau(K_{\tau+s}) - U_\tau\}$ is equal in distribution to the process (K_s) , and if it is independent of any time $t \in [0, \tau)$
- We say that (K_t) is *scale invariant* if for any $r > 0$, (rK_{t/r^2}) is equal in distribution to (K_t) .

Theorem 3.4. *Suppose (K_t) satisfies the conformal Markov property and is scale invariant. Then there exists some $\kappa \geq 0$ such that $\sqrt{\kappa}B_t = U_t$ where B_t is a standard Brownian motion.*

Proof. Clearly, (K_t) satisfying the conformal Markov property implies that $\{U_{\tau+s} - U_s\}_{s \geq 0}$ is equal in distribution to U_s and is independent of any time before τ . Since the increments are independent, there exist constants $\kappa \geq 0$ and $a \in \mathbb{R}$ such that $U_t = \sqrt{\kappa}B_t + at$, where B_t is a standard Brownian motion. Indeed, since K_t is scale invariant, then so is U_t , and thus by the scale invariance of Brownian motion,

$$rU_{t/r^2} = \sqrt{\kappa}rB_{t/r^2} + \frac{a}{r}t = \sqrt{\kappa}B_t + \frac{a}{r}t = \sqrt{\kappa}\tilde{B}_t + at,$$

where $\tilde{B}_t$ is a standard Brownian motion. Hence, $a = 0$, and we are done. $\square$

Definition 3.5. Let B_t be a standard Brownian motion, $\kappa \geq 0$, and g_t be the solution to

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}, \quad g_0(z) = z.$$

We define g_t to be a *Schramm-Loewner evolution* of parameter κ (SLE_κ).

4. SLE_κ IS GENERATED BY A CURVE: THE SECOND APPROACH

Here, we begin with the converse to Theorem 3.2 mentioned in the previous section.

Theorem 4.1. Suppose $U_t : [0, \infty) \rightarrow \mathbb{R}$ is a continuous function and for each $t \geq 0$, let g_t be the solution to the Loewner differential equation in (3.1). Define K_t as the set for which g_t is not defined:

$$\tau(z) := \inf\{t \geq 0 \mid g_t(z) \in \partial\mathbb{H}\} \quad K_t := \{z \in \mathbb{H} \mid \tau(z) \leq t\}.$$

Then g_t is a conformal map of the domain $\mathbb{H} \setminus K_t$ to $\mathbb{H}$ that can be expressed as in (2.4).

Proof. Since g_t is defined for when $g_t - U_t \neq 0$, we see from (3.1) that g_t is analytic. First, we will show that g_t has nonzero derivative on $\mathbb{H} \setminus K_t$. We differentiate g_t with respect to z and see

$$\frac{\partial}{\partial t} g'_t(z) = \frac{\partial}{\partial z} \frac{2}{g_t(z) - U_t} = -\frac{2}{(g_t(z) - U_t)^2} g'_t(z).$$

Since $\partial_t \log g'_t(z) = \frac{\partial_t g'_t(z)}{g'_t(z)}$, then dividing by $g'_t(z)$ above yields

$$\frac{\partial}{\partial t} \log g'_t(z) = -\frac{2}{(g_t(z) - U_t)^2}.$$

Since $U_t \in \mathbb{R}$, then $|\frac{\partial}{\partial t} \log g'_t(z)| \leq \frac{2}{[\Im(g_t)]^2}$. By considering the imaginary component in (3.1), it is not hard to see that $\Im(g'_t(z))$ decreases over time. Letting $t_0 < \tau(z)$, we see that $\log g'_t(z)$ is uniformly bounded for each $t \leq t_0$. Thus, $\log g'_{t_0}(z)$ is well defined and bounded, and thus $g'_{t_0}(z)$ is nonzero for any $t_0 \leq \tau(z)$. Let $z_1, z_2 \in \mathbb{H} \setminus K_t$ be distinct and such that $g_t(z_1) = w_1$ and $g_t(z_2) = w_2$. Now consider

$$\frac{\partial}{\partial t} \log(g'_t(z_1) - g'_t(z_2)) = -\frac{2}{(g_t(z_1) - U_t)(g_t(z_2) - U_t)},$$

which implies by our previous work that $w_1 \neq w_2$.

To see that $g_t(\mathbb{H} \setminus K_t) = \mathbb{H}$, let $w \in \mathbb{H}$. Define $h_t(w)$ for any $0 \leq t \leq T$, where $T \geq 0$ is a fixed time such that

$$\partial h_t(w) = -\frac{2}{h_t(w) - U_{T-t}}, \quad h_0(w) = w.$$

It takes little work to show that h_t is well defined for all such w and $T \geq 0$. However, if we suppose that for some time $0 \leq t \leq T$, it holds that $g_t(z) = h_{T-t}$, then it is clear that z is determined by $z = g_0(z) = h_T(w)$.

Finally, to show g_t satisfies (2.4), we integrate over the Loewner differential equation over $[0, t]$ to see

$$g_t(z) = g_0(z) + \int_0^t \frac{2}{g_s(z) - U_s} ds = z + \int_0^t \frac{2}{g_s(z) - U_s} ds.$$

In the limit as $z \rightarrow \infty$, we see that $g_t(z) - z \rightarrow 0$, implying that the Laurent expansion contains no terms of higher order than one. $\square$

Definition 4.2. We say that SLE_κ is generated by a curve if for every $t \geq 0$,

$$(4.3) \quad \gamma(t) = \lim_{y \downarrow 0} g_t^{-1}(U_t + iy)$$

exists and is a continuous curve. We call $\gamma(t)$ the SLE_κ path.

The following results will show that SLE_κ is generated by a continuous curve for $\kappa \neq 8$. The result holds for $\kappa = 8$, but is outside the scope of the paper (the proof is given by [7]). We follow the methodology outlined in section 3.8 of [5]. Any proof not provided can be found there. First, we provide the necessary assumptions for SLE_κ to be generated by a curve using Theorem 4.4. Then, we briefly mention how SLE_κ satisfies those assumptions. Note that Theorem 4.4 does not directly imply that SLE_κ is generated by a curve, but the ideas presented in them are similar enough that it is still useful to discuss them.

Theorem 4.4. Suppose $c_0 \leq 4$ and for all s, t such that $0 \leq s < t \leq 1$ we have that

$$(4.5) \quad |U_t - U_s| \leq c_0 |s - t|^{\frac{1}{2}}.$$

Then the limit in (4.3) exists and γ is a continuous simple curve.

In order to prove Theorem 4.4, we need the following lemma:

Lemma 4.6. Define $f_t(w) = g_t^{-1}(w)$ and

$$\Delta(r) = 1 + \max_{0 \leq s < t \leq 1, t-s \geq r} \left\{ \frac{|U_t - U_s|}{\sqrt{s-t}} \right\},$$

$$I(y) = \sup_{0 \leq y \leq 1} \left\{ \int_0^y f'_t(U_t + ir) dr \right\}.$$

If $I(y) < \infty$, the limit in (4.3) exists, and there exist c_1 such that

$$|\gamma(t) - \gamma(s)| \leq c_1 I(\sqrt{t-s}) \Delta(t-s)^4.$$

Proof. Let $y' < y$ so that

$$|f_t(U_t + iy) - f_t(U_t + iy')| \leq \int_{y'}^y |f'_t(U_t + ir)| dr.$$

Since the integral exists by $I(y) < \infty$ and is continuous, then taking $y' \downarrow 0$ yields the existence of the limit in (4.3) along with

$$|f_t(U_t + iy) - \gamma(t)| \leq I(y).$$

Now let $\frac{y}{2} \leq y' \leq 2y$. The distortion principle³ implies there exists some $c_2 > 0$ such that

$$I(y) \geq \int_{\frac{y}{2}}^y |f'_t(U_t + ir)| dr \geq c_2 y |f'_t(U_t + iy')| \geq |f_s(U_t + ir) - f_t(U_t + ir)|,$$

where the last inequality is again a consequence of the distortion theorems. Hence, for $0 \leq s \leq t \leq s + \delta^2 \leq 1 + \delta^2$, we apply the triangle inequality and then Corollary 8.6 in the appendix to conclude

$$\begin{aligned} |\gamma(t) - \gamma(s)| &\leq |\gamma(t) - f_t(U_t + iy)| + |f_t(U_t + iy) - f_s(U_t + iy)| + |f_s(U_t + iy) - \gamma(s)| \\ &\leq 2I(y) + |f_t(U_t + iy) - f_s(U_t + iy)| + |f_s(U_t + iy) - f_s(U_s + iy)| \\ &\leq 3I(y) + c\delta \left[1 + \frac{|U_t - U_s|^4}{\delta^4} \right] |f'_s(U_s + i\delta)| \leq c\Delta(\delta^2)I(\delta). \end{aligned}$$

□

Lemma 4.7. *For each $c_0 < 4$, there is some $\theta < 1$ and $c < \infty$ such that if U_t satisfies (4.5), then for all $0 \leq t \leq 1$, if $y \leq 1$, then*

$$|f'_t(iy)| \leq cy^{-\theta}, \quad I(y) \leq cy^{1-\theta}, \quad \theta = 1 - \frac{c_0^2}{16}.$$

Together, Lemmas 4.6 and 4.7 imply Theorem 4.4. Note that the last part of Lemma 4.6 implies that if $I(r) \downarrow 0$ as $r \rightarrow 0$, then γ is a continuous curve. It suffices to show that for $\kappa \neq 8$, the SLE satisfies these conditions.

Theorem 4.8. *For $\kappa \neq 8$, SLE_κ paths are generated by a curve.*

We provide a sketch of Theorem 4.8. To prove this, we will show that there exists $\delta > 0$ such that for sufficiently small y and finite $t_0 < \infty$ the following holds for SLE_κ :

$$\max_{0 \leq t \leq t_0} |f'_t(U_t + iy)| \leq y^{\delta-1}.$$

By the distortion theorem, we can consider $y = 2^{-n}$ and $t_0 = 1$. Let $k = 0, 1, \dots, 2^{2n}$ and suppose $t = k2^{-2n}$. Applying the distortion estimate (Corollary 7.6) to $t \leq s \leq t + 2^{-2n}$ gives

$$|f'_s(U_s + i2^{-n})| \leq c[2^n|U_t - U_s| + 1]^4 |f'_t(U_t + i2^{-n})|.$$

Using similar techniques to those found in Lemma 4.6, we know $|f'_s(U_t + i2^{-n})| \leq c_1|f'_t(U_t + i2^{-n})|$, and since the modulus of continuity of Brownian motion gives $|U_t - U_s| \leq c_2\sqrt{n}2^{-n}$, we have that

$$\max_{0 \leq t \leq 1} \{f'_t(U_t + i2^{-2n})\} \leq Cn^2 \max_{t=k2^{-2n}} \{f'_t(U_t + i2^{-2n}) : k = 0, 1, \dots, 2^{2n}\}.$$

To conclude, it suffices to show that if n is sufficiently large, then

$$|f'_t(U_t + i2^{-n})| \leq 2^{(1-\delta)n}.$$

It is in this claim that our methodology fails for $\kappa = 8$. The proof of this claim, and the proof for when $\kappa = 8$, is outside the scope of the paper, but can be found in [5].

³Found in the appendix as Theorem 7.4.

5. PHASES OF THE SLE_κ PATH

The main results in this section deal with the behavior of the SLE_κ paths depending on the κ . Indeed, we will see that as κ increases into higher ‘phases’, the curve becomes ‘wilder’, up to the point where it is plane-filling. This result is striking in two ways. The first is that a single parameter governs the evolution and behavior of random curves. The second is that this property hints at scaling limits of discrete critical models. Indeed, it is proved that the LERW converges as a scaling limit to an SLE_2 in [6]. See Figure 3 for reference.

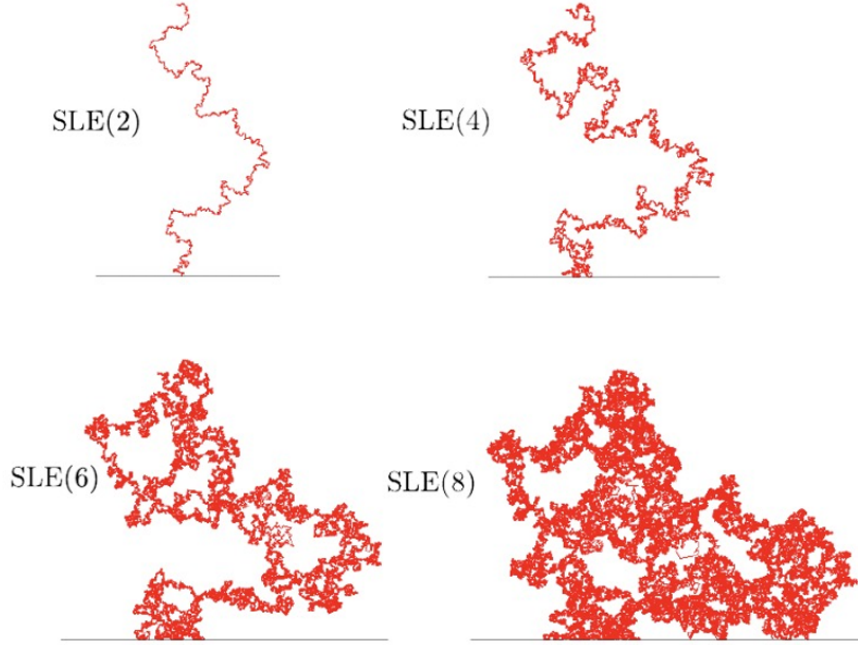


FIGURE 3. Phases of the SLE_κ . Pictured by Hang Du in [8]

Theorem 5.1. *If $\kappa \leq 4$, the SLE_κ path a.s. never intersects itself. For $\kappa > 4$, the SLE_κ path a.s. is self intersecting but not plane-filling. For $\kappa \geq 8$, the SLE_κ path is plane-filling a.s.*

Definition 5.2. X_t is called a *Bessel process* with parameter a if it satisfies the equation

$$(5.3) \quad dX_t = \frac{a}{X_t} dt + dB_t, \quad t < T_0, \quad X_0 = x_0,$$

where $T_0 = \inf\{t \geq 0 \mid X_t = 0\}$, $x_0 > 0$, and B_t is a standard Brownian motion.

Remark 5.4. By setting $a = \frac{d-1}{2}$ in (5.3), X_t becomes a *Bessel process of dimension d* and we write $X_t \sim \text{BES}^d$. When $d \geq 1$ is an integer and B is a d -dimensional Brownian motion then $|B|$ is a d -dimensional Bessel process. To see this calculation as well as more on Bessel processes, see [9].

Theorem 5.5. *Suppose $X_t \sim \text{BES}^d$. If $d < 2$, then X_t hits 0 a.s. If $d \geq 2$, then X_t does not hit 0 a.s.*

Proof. We will show that X_t^{2-d} is a continuous martingale. To see this, consider that by Ito's formula,

$$\begin{aligned} dX_t^{2-d} &= (2-d)X_t^{1-d}dX_t + \frac{1}{2}(1-d)(2-d)X_t^{-d}d\langle X \rangle_t \\ &= (2-d)X_t^{1-d} \left(\left(\frac{d-1}{2} \right) \frac{1}{X_t} dt + dB_t \right) + \frac{1}{2}(2-d)(1-d)X_t^{-d} dt \\ &= \frac{1}{2}(d-1)(d-2)X_t^{1-d} \frac{1}{X_t} dt + (2-d)X_t^{1-d}dB_t + \frac{1}{2}(1-d)(2-d)X_t^{-d} dt \\ &= (2-d)X_t^{1-d}dB_t. \end{aligned}$$

Since X_t^{2-d} is an Ito integral, then it is a continuous martingale. Letting $\tau_z := \inf\{t \geq 0 : X_t = z\}$, we apply the optional stopping theorem to the process $X_{t \wedge \tau_a \wedge \tau_b}^{2-d}$, where $0 < a < b$. Thus, for $d \neq 2$,

$$X_0^{2-d} = \mathbb{E}[X_{\tau_a \wedge \tau_b}^{2-d}] = a^{2-d}\mathbb{P}\{\tau_a < \tau_b\} + b^{2-d}(1 - \mathbb{P}\{\tau_a < \tau_b\}).$$

Hence,

$$\mathbb{P}\{\tau_a < \tau_b\} = \frac{X_0^{2-d} - b^{2-d}}{a^{2-d} - b^{2-d}} \xrightarrow{a \rightarrow 0} \begin{cases} \left(\frac{X_0^{2-d}}{-b^{2-d}} \right) + 1, & d < 2 \\ 0, & d > 2 \end{cases}$$

Thus, as $b \rightarrow \infty$, $\tau_b \rightarrow \infty$, and

$$\mathbb{P}\{\tau_0 < \infty\} = \begin{cases} 1, & d < 2 \\ 0, & d > 2 \end{cases}.$$

For $d = 2$, we can apply the fact that $\log(z)$ is harmonic, and hence can immediately say (as per the optional stopping theorem) that

$$\mathbb{P}\{\tau_a < \tau_b\} = \frac{\log(X_0) - \log(b)}{\log(a) - \log(b)} \xrightarrow{a \rightarrow 0} 0,$$

and thus $\mathbb{P}\{\tau_0 < \infty\} = 0$. $\square$

The most important process in this section will be the *cutting off process* defined by

$$V_t^x := g_t(x) - U_t = g_t(x) - \sqrt{\kappa}B_t,$$

where $x > 0$. If we define $\tau_x = \inf\{t \geq 0 : V_t^x = 0\}$, then we think of τ_x as the first time x which is absorbed into the K_t -hulls and “cut off from infinity.”

Proposition 5.6. *The process $V_t^x/\sqrt{\kappa}$ is a BES^d .*

Proof. Using Ito's formula,

$$dV_t^x = \frac{2}{g_t(x) - \sqrt{\kappa}B_t} dt - \sqrt{\kappa}dB_t = \frac{2}{V_t^x} dt - \sqrt{\kappa}dB_t.$$

Hence,

$$d(V_t^x/\sqrt{\kappa}) = \frac{2}{\kappa} \left(\frac{\sqrt{\kappa}}{V_t^x} \right) - dB_t.$$

Letting $d := 1 + \frac{4}{\kappa}$ so that $\frac{d-1}{2} = \frac{2}{\kappa}$, we see that $V_t^x/\sqrt{\kappa} \sim \text{BES}^d$. $\square$

Theorem 5.7. *If $\kappa \leq 4$, then the SLE_κ path is a simple curve. If $\kappa > 4$, then SLE_κ is self-intersecting.*

Proof. By Theorem 5.5 and Proposition 5.6, it is clear that if $\kappa \leq 4$, then $V_t^x \neq 0$ for any $t \geq 0$ a.s. Letting γ denote the SLE_κ path, we see that γ never hits $\partial\mathbb{H}$. Assume for the sake of contradiction that there exist $0 < t_1 < t_2$ such that $\gamma(t_1) = \gamma(t_2)$. Define

$$\gamma^s(t) = g_t(\gamma(t+s)) - \sqrt{\kappa}B_t.$$

We know by the conformal Markov property that γ^s has the same distribution as γ . In particular, it never hits $\partial\mathbb{H}$. However,

$$\gamma^{t_2-t_1}(t_1) = g_{t_1}(\gamma(t_2)) - \sqrt{\kappa}B_{t_1} = g_{t_1}(\gamma(t_1)) - \sqrt{\kappa}B_{t_1},$$

and by definition, $\gamma(t_1) \in \partial K_{t_1}$, and thus $g_{t_1}(\gamma(t_1)) \in \mathbb{R}$. Hence, $\gamma^{t_2-t_1}(t_1) \in \mathbb{R}$, a contradiction.

If $\kappa > 4$, then $V_t^x = 0$ a.s. and thus γ hits $\mathbb{R}$ a.s. If γ is a simple curve, then $K_t = \gamma([0, t])$, and so for any $s > 0$, $\gamma(t+s) \in \mathbb{H} \setminus K_t$. Since $g_t : \mathbb{H} \setminus K_t \rightarrow \mathbb{H}$, then $\gamma^s(t) \in \mathbb{H} \cup \{0\}$, and never intersects $\mathbb{R}$, a contradiction to the conformal Markov property. $\square$

Lemma 5.8. *Let $r > 1$. Then the events $\{\tau_x = \tau_y\}$ and*

$$E = \sup_{t < \tau_1} \left\{ \frac{V_t^r - V_t^1}{V_t^1} < \infty \right\}$$

are equivalent.

Proof. For ease, we denote $g(x, y) = \mathbb{P}\{\tau_x = \tau_y\}$ to be the probability that x and y are “swallowed” by the SLE_κ path γ at the same time. By scale invariance, $g(x, y) = g(1, \frac{y}{x})$, so it suffices to show that $g(1, r)$ and E are equivalent for any $r > 1$. It will be useful to note that $g(1, r) \xrightarrow{r \rightarrow \infty} 0$.

Suppose E occurs, then there is some $C > 0$ such that for all $t < \tau_1$,

$$V_t^r < V_t^1(C+1)$$

Since $V_{\tau_1} = 0$, then $V_t^r \rightarrow 0$ as $t \uparrow \tau_1$, and thus $\tau_r \leq \tau_1$. It is a fact of the mapping out functions that $g_t(x) < g_t(x')$ if $x < x'$ (see [2]), and thus $V_t^x < V_t^{x'}$ a.s. Hence, we know that $\tau_r \not\leq \tau_1$ since $r > 1$, so then $E \subseteq \{\tau_1 = \tau_r\}$. If we show that $\mathbb{P}\{\tau_1 = \tau_r \cap E^c\} = 0$, we are done. Define

$$\sigma_M := \inf\{t \geq 0 : \frac{V_t^r - V_t^1}{V_t^1} \geq M\},$$

we have by definition that

$$\mathbb{P}\{\tau_1 = \tau_r \mid \sup_{t < \tau_1} \frac{V_t^r - V_t^1}{V_t^1} \geq M\} = \mathbb{P}\{\tau_1 = \tau_r \mid \sigma_M < \tau_1\}.$$

Scaling the curve by the map $z \mapsto \frac{z}{\sqrt{\sigma_M}}$, we use the scale invariance property to show that $g(1, r) = g(1, 1+M)$. Using the conformal Markov property by restarting the SLE_κ path at time σ_M ,

$$\mathbb{P}\{\tau_1 = \tau_r \mid \sigma_M < \tau_1\} = g(1, 1+M) \xrightarrow{M \rightarrow \infty} 0.$$

Thus, $\mathbb{P}\{\tau_1 = \tau_r \cap E^c\} = 0$, and so we have shown that the two events differ by a set of at most measure zero, implying they are equivalent. $\square$

Theorem 5.9. *If $\kappa \in (4, 8)$, then the SLE_κ path almost surely cuts regions off from infinity. If $\kappa \geq 8$, then the SLE_κ path is almost surely plane-filling.*

Proof. Step 1. Consider the process $Z_t = \log \left(\frac{V_t^r - V_t^1}{V_t^1} \right)$. Using Ito's formula, we plug in $d = 1 + \frac{4}{\kappa}$ to show that

$$(5.10) \quad dZ_t = \left(\left(\frac{3}{2} - d \right) \frac{1}{(V_t^1)^2} + \left(\frac{d-1}{2} \right) \left(\frac{V_t^r - V_t^1}{(V_t^1)^2 V_t^r} \right) \right) dt - \frac{1}{V_t^1} dB_t.$$

We will transform Z_t into a Brownian motion using the time change

$$\sigma(t) = \inf\{u \geq 0 \mid \int_0^u \frac{1}{(V_s^1)^2} ds = t\}.$$

We see from this definition that $\sigma(\infty) = \tau_1$ and

$$(5.11) \quad dt = \frac{d\sigma(t)}{(V_{\sigma(t)}^1)^2}.$$

Define the continuous local martingale

$$Y_t := - \int_0^{\sigma(t)} \frac{1}{V_s^1} dB_s.$$

Then using Ito's isometry,

$$\langle Y \rangle_t = \left\langle - \int_0^{\sigma(t)} \frac{1}{V_s^1} dB_s \right\rangle_t = \int_0^{\sigma(t)} \frac{1}{(V_s^1)^2} dt = t,$$

and so Y_t is a standard Brownian motion via Levy's characterization. Defining $Z_t = Z_{\sigma(t)}$, we use (5.10) and (5.11) and compute

$$\begin{aligned} d\tilde{Z}_t &= d(Z_{\sigma(t)}) \\ &= \left(\left(\frac{3}{2} - d \right) \frac{1}{(V_{\sigma(t)}^1)^2} + \left(\frac{d-1}{2} \right) \left(\frac{V_{\sigma(t)}^r - V_{\sigma(t)}^1}{(V_{\sigma(t)}^1)^2 V_{\sigma(t)}^r} \right) \right) d\sigma(t) - \frac{1}{V_{\sigma(t)}^1} dB_{\sigma(t)} \\ &= \left(\left(\frac{3}{2} - d \right) \frac{1}{(V_{\sigma(t)}^1)^2} + \left(\frac{d-1}{2} \right) \left(\frac{V_{\sigma(t)}^r - V_{\sigma(t)}^1}{(V_{\sigma(t)}^1)^2 V_{\sigma(t)}^r} \right) \right) (V_{\sigma(t)}^1)^2 dt + dY_t \\ &= \left(\left(\frac{3}{2} - d \right) + \left(\frac{d-1}{2} \right) \left(\frac{V_{\sigma(t)}^r - V_{\sigma(t)}^1}{V_{\sigma(t)}^r} \right) \right) dt + dY_t. \end{aligned}$$

Integrating,

$$(5.12) \quad \begin{aligned} \tilde{Z}_t &= \tilde{Z}_0 + Y_t + \left(\frac{3}{2} - d \right) t + \left(\frac{d-1}{2} \right) \int_0^t \left(\frac{V_{\sigma(s)}^r - V_{\sigma(s)}^1}{V_{\sigma(s)}^r} \right) ds \\ &\geq \tilde{Z}_0 + Y_t + \left(\frac{3}{2} - d \right) t. \end{aligned}$$

Step 2. In the case when $\kappa \geq 8$, we have $d \leq \frac{3}{2}$ implying that $\tilde{Z}_t \geq \tilde{Z}_0 + Y_t$. Since $\sigma(\infty) = \tau_1$, and Y_t is a standard Brownian motion, then

$$\sup_{t < \tau_1} Z_t = \sup_{t \geq 0} \tilde{Z}_t \geq \tilde{Z}_0 + \sup_{t \geq 0} Y_t = \infty.$$

Since $z \leq e^z$, then

$$\sup_{t < \tau_1} e^{Z_t} = \infty.$$

By Lemma 5.8, this immediately implies that $g(x, y) = 0$.

When $\kappa \in (4, 8)$, then $d \geq \frac{3}{2}$. Let $\epsilon > 0$ and define $\tau = \inf\{t \geq 0 : \log(\tilde{Z}_t) = \epsilon\}$. Letting $r = 1 + \epsilon$, we will show that $g(1, r) = g(1, 1 + \epsilon) > 0$. From (5.12),

$$\begin{aligned} \tilde{Z}_{t \wedge \tau} &= \tilde{Z}_0 + Y_{t \wedge \tau} + \left(\frac{3}{2} - d\right)(t \wedge \tau) + \frac{d-1}{2} \int_0^{t \wedge \tau} e^{\tilde{Z}_s} ds \\ &\leq \tilde{Z}_0 + Y_{t \wedge \tau} + \left(\left(\frac{3}{2} - d\right) + \left(\frac{d-1}{2}\right)\epsilon\right)(t \wedge \tau) \\ &=: \tilde{Z}_0 + Y_{t \wedge \tau} + a(t \wedge \tau), \end{aligned}$$

where we defined $a = \left(\left(\frac{3}{2} - d\right) + \left(\frac{d-1}{2}\right)\epsilon\right)$. For ϵ small we have $a < 0$, and so $Z_t^* = \tilde{Z}_0 + Y_t + at$ is a Brownian motion with negative drift. It is clear from previous work that the events

$$\left\{\sup_{t < \tau_1} \frac{V_t^r - V_t^1}{V_t^1} < \infty\right\}, \quad \left\{\sup_{t < \tau_1} e^{Z_t} < \epsilon\right\}, \quad \left\{\sup_{t \geq 0} \log(\tilde{Z}_t) < \log(\epsilon)\right\}$$

are all equivalent. Since Z_t^* has negative drift, then $\mathbb{P}\{\sup_{t \geq 0} Z_t^* < \log(\epsilon)\} > 0$. Since $Z_t^* \geq \tilde{Z}_t$, then $\mathbb{P}\{\sup_{t \geq 0} \log(\tilde{Z}_t) < \log(\epsilon)\} > 0$ and we have stated this to be equivalent by Lemma 5.8 to $g(x, y) > 0$.

Step 3. Suppose $\kappa \geq 8$. Suppose γ does not fill $\partial\mathbb{H}$. If $x < y$ are on the region not filled by γ , then both x, y are going to be cut off at time t_0 , when the curve closes off the region. Then $\tau_x = \tau_y$, a contradiction to Step 2. It can similarly be shown that if $\kappa \in (4, 8)$ then γ cuts regions off from $\partial\mathbb{H}$.

Step 4. Fix $z \in \mathbb{H}$ such that $z = x + iy$. Call $z_t = x_t + iy_t = g_t(z)$, and let $\rho \in \mathbb{H}$. We can use Ito's formula to calculate that

$$M_t = |g'_t(z)|^{(8-2\kappa+\rho)\rho/(8\kappa)} y_t^{\rho^2/8\kappa} |U_t - z_k|^{\rho/\kappa}$$

is a continuous local martingale. Define $\Upsilon_t := \frac{y_t}{|g'_t(z)|}$. By the Koebe-1/4 Theorem,

$$(5.13) \quad \frac{1}{4} \leq \frac{\Upsilon_t}{\text{dist}(z, \gamma \cup \mathbb{R})} \leq 4$$

Defining $S_t = \sin(\arg(z_t - U_t))$, we see that by plugging in definitions,

$$M_t = |g'_t(z)|^{(8-\kappa+\rho)\rho/(4\kappa)} \Upsilon_t^{\rho(\rho+8)/8\kappa} S_t^{-\rho/\kappa}$$

Letting $\rho = \kappa - 8$, our martingale becomes

$$M_t = \left(\frac{\Upsilon_t^{\frac{1}{8}}}{S_t^{\frac{1}{\kappa}}}\right)^{\kappa-8}.$$

Let $\epsilon > 0$. Define τ_z as

$$\tau_z := \inf\{t \geq 0 \mid \Upsilon_t = 0\}.$$

Thus, $M_{t \wedge \tau_z}$ is a continuous non-negative martingale. Using (5.13), we can apply the martingale convergence theorems and let $M_{t \wedge \tau_z} \rightarrow M_\infty$. We apply the Optional Stopping Theorem to $M_{t \wedge \tau_z}$, to see that

$$\mathbb{E}[M_0] = \mathbb{E}[M_{t \wedge \tau_z}] = \mathbb{P}\{\tau_z < \infty\} M_{t \wedge \tau_z} + \mathbb{P}\{\tau_z = \infty\} M_\infty = \mathbb{P}\{\tau_z = \infty\} M_\infty$$

and thus

$$\mathbb{P}\{\tau_z = \infty\} = \frac{M_0}{M_\infty}.$$

We have shown that for $\kappa > 8$, the curve almost surely fills $\partial\mathbb{H}$. We also know that $\lim_{t \rightarrow \infty} \text{Im}(z_t) \rightarrow \mathbb{R}$. Then S_t limits to 0 or to π a.s. as $t \rightarrow \infty$. Hence, $\mathbb{P}\{\tau_z = \infty\} = 0$. Thus, almost surely there exists some time $t \geq 0$ such that $\gamma(t) = z$. $\square$

6. SLE_6 AS A SCALING LIMIT

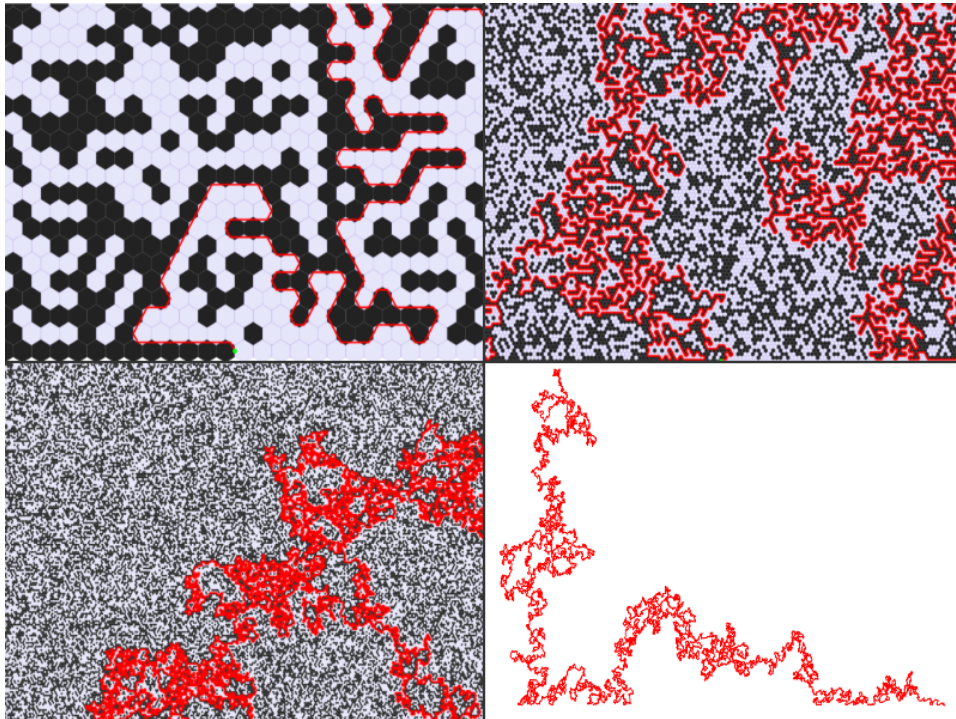


FIGURE 4. Hexagonal Percolation

Consider a hexagonal lattice in the upper half plane. Color all the hexagons bordering the positive real axis lavender, and all the ones bordering the negative real axis black. Independently, color every other hexagon in the half plane either lavender or black with equal probability. From percolation theory, it is known that there is almost surely no infinite cluster of either lavender or black hexagons. Start a path from the origin such that at any step, there must be a lavender block to the right and a black one on its left. Let δ be the size of the hexagons, and define the path to be γ^δ . This percolation exploration satisfies two properties which hint that its limit is an SLE path, the third property will suggest that the limit is indeed SLE_6 . First, we need to define these properties in the context of the critical exploration model. For an illustration of this process, refer to Figure 4. We now introduce some useful terms and later justify that they are well defined.

Definition 6.1. Suppose $D, \tilde{D}$ are simply connected domains, $x, y \in D$, and $\tilde{x}, \tilde{y} \in \tilde{D}$. Let $\varphi : D \rightarrow \tilde{D}$ be a conformal transformation with $\varphi(x) = \tilde{x}$ and $\varphi(y) = \tilde{y}$. For each $\epsilon > 0$, we let γ^δ be the percolation exploration in D from x to y . If

the limit exists, we define $\gamma = \lim_{\delta \rightarrow 0} \gamma^\delta$ and say γ is *conformally invariant* if $\varphi(y) = \lim_{\delta \rightarrow 0} \varphi(\gamma^\delta)$ has the same distribution as the percolation exploration from $\tilde{x}$ to $\tilde{y}$.

Definition 6.2. Let L be a compact $\mathbb{H}$ -hull that is bounded away from the origin. If it exists, let γ be the scaling limit of the percolation exploration in $\mathbb{H}$, and, if it exists, let γ^* be the scaling limit of percolation exploration in $\mathbb{H} \setminus L$. Let T_L (respectively T_L^*) be the hitting time at which γ_t (γ_t^*) intersects L ($\mathbb{H} \setminus L$). We say that γ has the *locality property* if the two processes have the same distribution time up to the hitting time of the set L (up to a time re-parametrization).

While the limits of γ^δ in the previous definitions are not known to exist priori, it is known that they exist. See [10].

- a To study γ^δ , one only needs to consider the hexagons to the left and right of it. Thus, conditioning γ^δ up to time t , the distribution of the remainder of γ^δ in the domain $\gamma^\delta(t)$ to y (for any y) is that of a critical exploration.
- b Smirnov (2001) showed that the scaling limit of the percolation exploration is conformally invariant in [10].
- c By definition, percolation models have the locality property.

Items (a) and (b) together imply that the percolation exploration satisfies the conformal Markov property. These facts imply that the percolation exploration limits towards an SLE_κ path. If we show that SLE_6 satisfies a locality property, then by (c) we have a good candidate for the scaling limit.

Theorem 6.3. *Locality holds if and only if $\kappa = 6$. That is, if γ is an SLE_6 curve, then $\psi(\gamma)$ is an SLE_6 (up until first hitting $\psi(\partial D \setminus \partial \mathbb{H})$ modulo a time parameterization).*

Proposition 6.4. *In the context of Theorem 3.2, let $\varphi : D \rightarrow \varphi(D)$ be a conformal transformation, and define $\tilde{K}_t = \varphi(K_t)$, $\tilde{g}_t = g_{\tilde{K}_t}$, $\tilde{a}_t = \text{hcap}(\tilde{K}_t)$. If we define $\varphi_t = \tilde{g}_t \circ \varphi \circ g_t^{-1}$, then the Loewner driving function is given by $\tilde{U}_t = \varphi_t(U_t)$ such that*

$$\partial_t \tilde{g}_t(z) = \frac{\partial_t \tilde{a}(t)}{\tilde{g}_t(z) - \tilde{U}_t} dt \quad \tilde{g}_0(z) = z,$$

where

$$\tilde{a}(t) = \int_0^t 2(\varphi'_s(U_s))^2 ds.$$

Proof. We assume that $\varphi = \varphi_0$, $\varphi(0) = 0$, and $\varphi'(0) = 1$. We will show this result for $t = 0$ because the argument is the same regardless of t . We claim that if $\text{rad}(K_t) \leq \frac{1}{2}$, then there is some $c < \infty$ so that

$$|\text{hcap}(\varphi(K_t)) - \text{hcap}(K_t)| \leq c\sqrt{\text{rad}(K_t)}\text{hcap}(K_t).$$

A proof for this is found in [5]. Using this, we see that since $\tilde{a}(0) = \text{hcap}(\emptyset) = 0$, then

$$\tilde{a}(t) - \tilde{a}(0) = \text{hcap}(K_t) \left(1 + O(c\sqrt{\text{rad}(K_t)})\right).$$

Hence, in the limit, $\tilde{a}'(0) = 2$, as desired. It is left to the reader to check that after scaling and translating, these assumptions are enough to prove the formula for $\tilde{a}(t)$. By following the proof as in Theorem 3.2, it becomes clear that

$$\partial_t \tilde{g}_t = \frac{\partial_t \tilde{a}(t)}{\tilde{g}_t - \varphi_t(U_t)}.$$

□

Proposition 6.5. *In the context of the previous proposition, the maps (φ_t) satisfy*

$$(6.6) \quad \partial_t \varphi_t(U_t) = \lim_{z \rightarrow U_t} \partial_t \varphi_t(z) = -3\varphi_t''(U_t).$$

Proof. Modulo algebra, we can show via an application of the chain rule that

$$\partial_t \varphi_t(z) = 2 \left(\frac{(\varphi_t'(U_t))^2}{\varphi_t(z) - \varphi_t(U_t)} - \varphi_t'(z) \frac{1}{z - U_t} \right)$$

To see the result, use this expression and expand into Taylor series. □

Proof. (Proof of Theorem 6.3) Define φ_t and $\tilde{U}_t$ as in Propostion 6.4. Using Ito's formula and (6.6),

$$\begin{aligned} d\tilde{U}_t &= d\varphi_t(U_t) \\ &= (\partial_t \varphi_t(U_t) + \frac{\kappa}{2} \varphi_t''(U_t)) dt + \sqrt{\kappa} \varphi_t'(U_t) dB_t \\ &= (-3\varphi_t''(U_t) + \frac{\kappa}{2} \varphi_t''(U_t)) dt + \sqrt{\kappa} \varphi_t'(U_t) dB_t \\ &= (\frac{\kappa - 6}{2}) \varphi_t''(U_t) dt + \sqrt{\kappa} \varphi_t'(U_t) dB_t. \end{aligned}$$

Let

$$\sigma(t) = \inf\{u \geq 0 : \int_0^u (\psi'_s(U_s))^2 ds = t\}$$

so that $dt = (\psi'(U_{\sigma(t)}))^2 d\sigma(t)$. Continuing from the above,

$$d\tilde{U}_t = (\frac{\kappa - 6}{2}) \frac{\psi''_{\sigma(t)}(U_{\sigma(t)})}{(\psi'(U_{\sigma(t)}))^2} dt + \sqrt{\kappa} d\tilde{B}_t,$$

where we defined

$$\tilde{B}_t = \int_0^{\sigma(t)} \psi'_s(U_s) dB_s.$$

By our choice of σ and by Levy's characterization of a Brownian motion, $\tilde{B}_t$ is a standard Brownian motion. For $\kappa = 6$, we see that

$$d\tilde{U}_t^* = \sqrt{6} d\tilde{B}_t.$$

Proposition 5.19 tells us that

$$\partial_t \tilde{g}_{\sigma(t)}(z) = \frac{\partial_t (\tilde{a}(\sigma(t)))}{\tilde{g}_{\sigma(t)} - \tilde{U}_{\sigma(t)}} dt = \frac{2}{\tilde{g}_{\sigma(t)}(z) - \tilde{U}_{\sigma(t)}} dt, \quad g_{\sigma(0)}(z) = z$$

Thus, $(\tilde{K}_{\sigma(t)})$ is equal in distribution to (K_t) . □

This section so far has not shown that the scaling limit of the percolation exploration is SLE_6 . We have, however, shown that SLE_6 is a very good candidate for the limit. The proof that it is indeed the scaling limit is given by Camia and Newman (2007) in [12]. This result has resulted in advances to percolation theory. In particular, Schramm showed left passage probabilities of percolation models. The following Theorem by Schramm is to do with the left passage probability of SLE_κ .

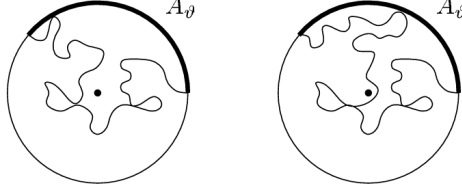


FIGURE 5. From [3]: From the construction of the percolation exploration, left figure represents the event E_θ .

Theorem 6.7. *Let $\kappa \leq 8$. Let E be the event that the SLE_κ path passes to the left of any $z_0 = x_0 + iy_0 \in \mathbb{H}$. Then*

$$\mathbb{P}[E] = \frac{1}{2} + \frac{\Gamma(\frac{4}{\kappa})}{\sqrt{\pi} \Gamma(\frac{8-\kappa}{2\kappa})} {}_2F_1\left(\frac{1}{2}, \frac{4}{\kappa}, \frac{3}{2}; -\frac{x_0^2}{y_0^2}\right) \frac{x_0}{y_0},$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0.$$

and F_1 is Gauss hypergeometric function given by

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (q)_n = q(q+1) \cdots (q+n-1).$$

Theorem 6.8. *Consider a hexagonal lattice in the unit disk of mesh $\delta > 0$. Let $\theta \in [0, 2\pi)$, and let $A_\theta = \{e^{i\nu} : \nu \in [0, \theta]\}$ be the arc of the unit circle towards θ . Let E_θ be the event that there is a cluster of lavender hexagons connected to A_θ such that they also surround the origin. Then*

$$\lim_{\delta \downarrow 0} \mathbf{P}[E_\theta] = \frac{1}{2} - \frac{\Gamma(\frac{2}{3})}{\sqrt{\pi} \Gamma(\frac{1}{6})} {}_2F_1\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{2}; -\cot^2(\frac{\theta}{2})\right) \cot(\frac{\theta}{2}).$$

Proof. We will consider the percolation exploration from $(1, 0)$ to $e^{i\theta}$ as in Figure 5. Let γ^δ be the path of the exploration. From Figure 5, it is clear that E_θ is equal to the event that the origin lies in the component of $\mathbb{D} \setminus \gamma^\delta$ to the right of the path. Translating the setup into the context of Theorem 6.8, we send $(1, 0) \mapsto 0$ and $e^{i\theta} \mapsto \infty$. Generalizing, we can consider any point $z_0 \in \mathbb{H}$ and use the inverse of the map

$$\varphi(z) = e^{i\theta} \frac{z + \cot \frac{\theta}{2} - i}{z + \cot \frac{\theta}{2} + i}$$

whose inverse sends $z_0 = i - \cot(\frac{\theta}{2})$ to the origin. We know that $\lim \delta \downarrow 0 \gamma^\delta = \gamma^{\text{SLE}_6}$, and so we can apply Theorem 6.8 using $x_0 = \cot \frac{\theta}{2}$ and $y_0 = 1$. $\square$

7. APPENDIX

Distortion estimates are one of the most useful tools in the study of complex analysis. In fact, the proofs found in section 4 hinge on them. For the following, we let D be a proper simply connected domain containing the origin, and define

$$S = \{f : \mathbb{D} \rightarrow \overline{D} \mid f \text{ is conformal, } f(0) = 0, f'(0) = 1\}$$

so that if $f \in S$, then clearly the power series of f is given by $f(z) = z + a_2 z^2 + \dots$.

Proposition 7.1. *If $f \in S$, then*

- i $\text{area}(f(\mathbb{D})) = \pi \sum_{n=1}^{\infty} n |a_n|^2$.
- ii *there exists some odd function $h \in S$ such that $h(z)^2 = f(z^2)$*
- iii *For each $n \geq 2$, $|a_n| \leq n$.*

Proof. We prove (i) and a version of (iii) but leave (ii) to the reader. To show (i), we apply Green's theorem to $\gamma = C$ where C is the circle of radius r to see

$$\frac{1}{2i} \int_{\gamma} \bar{z} dz = \frac{1}{2i} \int_{\gamma} (x - iy)(dx + i dy) = \frac{1}{2i} \int \int_{f(r\mathbb{D})} 2i dx dy = \text{area}(f(r\mathbb{D})).$$

Alternatively,

$$\begin{aligned} \frac{1}{2i} \int_{\gamma} \bar{z} dz &= \frac{1}{2i} \int_0^{2\pi} \overline{f(re^{i\theta})} f'(re^{i\theta}) r i d\theta \\ &= \frac{1}{2i} \int_0^{2\pi} \sum_{n=1}^{\infty} \overline{a_n} r^n e^{-in\theta} \sum_{n=1}^{\infty} n a_n r^{n-1} e^{i\theta(n-1)} i r^{i\theta} d\theta = \pi \sum_{n=1}^{\infty} r^{2n} |a_n|^2 n \end{aligned}$$

Sending $r \rightarrow 1$ proves the result.

For (iii), we will only show the case for when $n = 2$. For a general n , this was known as the Bierbach conjecture and was proved by de Branges in 1987 [13]. Consider the power expansion of $h \in S$ from (ii). Since h is odd, its expansion about 0 has no even powers of z . Thus, we can express

$$h(z) = z + c_3 z^3 + c_5 z^5 + \dots$$

Since $f(z^2) = h(z)^2$, we have that

$$z^2 + a_2 z^4 + \dots = (z + c_3 z^3 + c_5 z^5 + \dots)^2 = z^2 + 2c_3 z^4 + \dots$$

implying that $c_3 = \frac{a_2}{2}$. Defining $g(z) = \frac{1}{h(\frac{1}{z})}$ so that

$$g(z) = \frac{1}{z^{-1} + (\frac{a_2}{2})z^{-3} + \dots} = z \left(\frac{1}{1 + \frac{a_2}{2}z^{-2} + \dots} \right)$$

Letting $r = \frac{1}{1 + \frac{a_2}{2}z^{-2} + \dots}$, we recognize the expression in the parenthesis above to be a geometric sum

$$\frac{1}{1+r} = 1 - r + r^2 - \dots = 1 - \frac{a_2}{2}z^{-2} + O\left(\frac{1}{z^4}\right).$$

Hence $g(z) = z - \frac{a_2}{2}z^{-1} + O(z^{-3})$. But then (i) implies that $\frac{|a_2|}{2} \leq 1$, and so we are done. $\square$

Theorem 7.2. (*Koebe 1/4*) *If $f \in S$ and $r \in (0, 1]$, then $B_{\frac{r}{4}}(0) \subset f(r\mathbb{D})$.*

Proof. By replacing f with $g_r(z) = \frac{f(rz)}{r}$, it suffices to consider the $r = 1$ case. Fixing $r = 1$ and $z_0 \notin \mathbb{H}$, if we show that $|z_0| \geq \frac{1}{4}$, we are done. Define

$$\tilde{f}(z) := \frac{z_0 f(z)}{z_0 - f(z)}.$$

Clearly, $f(0) = 0$, and it is not hard to see with the quotient rule that $f'(0) = 1$. Since $\tilde{f} : \mathbb{D} \rightarrow \mathbb{H}$, then $\tilde{f} \in S$, and we can express $\tilde{f}$ using the power series of f

$$\tilde{f}(z) = \frac{z_0 f(z)}{z_0 - f(z)} = \frac{z_0 (\sum_{n=1}^{\infty} a_n z^n)}{z_0 - \sum_{n=1}^{\infty} a_n z^n} = \left(\sum_{n=1}^{\infty} a_n z^n \right) \frac{1}{1 - \sum_{n=1}^{\infty} \frac{a_n}{z_0} z^n}$$

Recognizing the last expression as geometric gives

$$f(\tilde{z}) = (z + a_2 z^2 + O(z^3)) \left(1 + \left(\frac{z}{z_0} + O(z^2) \right) \right) + O(z^2) = z + (a_2 + \frac{1}{z_0}) z^2 + O(z^3)$$

Using (iii) of Lemma 5.14 gives that $|a_2 + \frac{1}{z_0}| \leq 2$. But we know that $|a_2| \leq 2$, so then

$$\left| \frac{1}{|z_0|} \right| \leq \left| \frac{1}{z_0} + a_2 \right| \leq 4$$

and we are done. $\square$

Corollary 7.3. *Suppose $f : D \rightarrow \tilde{D}$ where $D, \tilde{D} \subseteq \mathbb{C}$ are proper domains with $z \in D$. Then*

$$\frac{\text{dist}(f(z), \partial \tilde{D})}{4} \leq |f'(z)| \text{dist}(z, \partial D) \leq 4 \text{dist}(f(z), \partial \tilde{D})$$

Proof. For convenience, we define $d = \text{dist}(z, \partial D)$ and $\tilde{d} = \text{dist}(f(z), \partial \tilde{D})$. After translations and post-composing, we can take without loss of generality $z = f(z) = 0$.

Create a function

$$\tilde{f}(w) = \frac{f(dw)}{df'(0)}.$$

Indeed, we see that $\tilde{f}(0) = 0$ and an application of the product rule yields $f'(0) = 1$. Since $\tilde{f} \in S$, we can apply Theorem 5.15. Let $\epsilon > 0$. There is some δ such that if $1 - \delta$ then for $w \in \mathbb{D} \setminus (1 - \delta)\mathbb{D}$, we apply Koebe's 1/4 Theorem with $r = 1$ to see

$$|\tilde{f}(w)| \geq \frac{1}{4} - \epsilon$$

Hence,

$$\frac{\tilde{d}}{d|f'(0)|} \geq \inf_{w \in \mathbb{D} \setminus (1-\delta)\mathbb{D}} |f(w)| \geq \frac{1}{4} - \epsilon.$$

Rearranging we see that $4\tilde{d} \geq f'(0)d$. The lower bound is given with similar logic using f^{-1} instead of f . $\square$

The following three results are all consequences of Koebe's 1/4 theorem and Proposition 5.14 and are used in the paper.

Theorem 7.4. (*Distortion Theorem*) *Suppose $f \in S$ and $|z| < 1$. Then if $r = |z|$,*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}$$

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}$$

Corollary 7.5. (*Distortion Principle*) *Suppose $V \subset D$ is compact, and let $z, w \in V$. If $f : D \rightarrow f(D)$ is conformal, then there exists some $c = c(D, V) < \infty$ such that*

$$|f'(z)| \leq c|f'(w)|$$

Corollary 7.6. *There exists some $c_0 > 0$ so that if $f : \mathbb{H} \rightarrow f(\mathbb{H})$ and $x \in \mathbb{R}$, then for any $r > 0$,*

$$\frac{|f'(ri)|}{c_0(x^4 + 1)} \leq |f'(rx + ri)| \leq c_0(x^4 + 1)|f'(ri)|,$$

$$|f(rx + ri) - f(ri)| \leq c_0 r(|x|^4 + 1)|f'(ri)|.$$

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