AUTOMORPHISMS OF $\mathbb{R}^n$, STABLE HOMEOMORPHISMS, AND KIRBY'S TORUS TRICK

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Abstract. We investigate the space of all homeomorphisms from $\mathbb{R}^n$ to itself with the goal of understanding its general topological behavior. We begin by proving some interesting initial facts using fairly elementary techniques, but we are quickly led to quite deep rabbit holes. In particular, considerations of path-connectedness bring us to studying the Stable Homeomorphism Theorem, an important result in geometric topology. We dedicate much of the paper to proving it, following Kirby's delightful proof which marked the introduction of his "torus trick".

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1. Introduction

Definition 1.1. We denote by $\mathrm{Top}(n)$ the set of all homeomorphisms from $\mathbb{R}^n$ to itself. We give $\mathrm{Top}(n)$ the compact-open topology, with which it is a topological group under composition.

The goal of this paper is to explore this space. The literature on it is somewhat disorganized, and it is usually considered in a subsidiary role, rather than being in the limelight. This is despite the fact that it is not just a natural object to study, but quite an interesting one, and one which comes up in a number of places. Indeed, when one considers fiber bundles with fiber $\mathbb{R}^n$, but does not wish to preserve the linear structure, $\mathrm{Top}(n)$ is the natural structure group to impose. Thus $\mathrm{BTop}(n)$,

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the classifying space for $\mathrm{Top}(n)$, plays the same role in the topological category that $\mathrm{BPL}(n)$ and $\mathrm{BDiff}(n) \simeq \mathrm{BO}(n)$ play in the PL and differentiable categories.

In fact, in his famous paper [22], Kister showed that the entire theory of microbundles, a generalization of fiber bundles, can be reduced to the study of fiber bundles of this kind, with structure group $\mathrm{Top}_*(n)$ – the origin-fixing subgroup of $\mathrm{Top}(n)$. If one needs convincing that $\mathrm{Top}(n)$ is interesting, consider the following: it is unknown whether there are any finitely generated torsion-free groups that are *not* subgroups of $\mathrm{Top}(n)$ [29]. Further, despite some significant progress in recent years, the homology of $\mathrm{Top}(n)$ remains unknown. Indeed, just to pursue our goals in this paper, we will have to cover a theorem that was a significant open problem for many years. Suffice it to say, this is quite an intriguing and enigmatic object.

We will begin with an elementary discussion of basic topological facts about this space, which we will take care to deal with explicitly and clearly. In particular we will emphasize proving and justifying the definitions and properties that are often stated without explanation in the literature (or not stated at all). This will lead us to discussing the Stable Homeomorphism Theorem, which is equivalent to the fact that $\pi_0(\mathrm{Top}(n)) \cong \mathbb{Z}/2\mathbb{Z}$. We will go into detail on Kirby's well-known "torus trick", which he used to prove that theorem. We will assume basic knowledge of point-set topology and some familiarity with topological and smooth manifolds, including covering spaces, but anything beyond that will be explained. Finally, we take care to provide a wealth of references and suitable sources for further reading, especially when we use results without proof.

Remark 1.2. A note on terminology: Some authors refer to $\mathrm{Top}(n)$ as $\mathrm{TOP}(n)$ or occasionally TOP_n. Further, sometimes this refers to all self-homeomorphisms of $\mathbb{R}^n$ fixing a basepoint (usually 0), we will call that set $\mathrm{Top}_*(n)$. Others use the same term to mean the topological group of all homeomorphisms $S^n \to S^n$ fixing a basepoint (for concreteness, say the north pole N). We'll call this object $\mathrm{Top}_S(n)$. Thankfully, there are very convenient connections between these objects, which we will detail soon (see Propositions 2.4 and 2.5). More generally, $\mathrm{Homeo}(M)$ is often used to mean the homeomorphism group of any manifold M, and some authors just use this convention for $\mathbb{R}^n$. Finally, some sources only call $\mathrm{Top}(n)$ a topological monoid, but it is indeed a topological group [1, Theorem 4]. When confusion is impossible, we will refer to homeomorphisms from a space to itself as automorphisms.

2. General Topological Properties

A prudent place to begin is a discussion about the topology we have elected to put on our space. Any sensible function space should be such that functions whose outputs are near each other in the codomain are close in the function topology. We might try to impose a metric like

$$\rho(f, g) = \sup_{x \in X} d(f(x), g(x))$$

on the function space Y^X (supposing of course that Y is a metric space, which it is in the case we're interested in). This is actually not too naïve, and turns out to be a valid metric on $\mathrm{Top}_S(n)$. The trouble of course is when Y is unbounded, as it is for $\mathrm{Top}(n)$. Even something as innocuous as a rotation in $\mathbb{R}^2$ by some tiny $\varepsilon > 0$ will move points arbitrarily far from the identity. So trying to construct a topology preserving global information about the functions is doomed to failure. The natural next thing to try is a topology which considers functions close if they are close on some compact region. Closeness of course being measured by containment in an open set, we come to the following definition:

Definition 2.1. The **compact-open** topology on a space of functions $Y^X = \{f : X \to Y\}$ is the topology generated by a subbasis of all sets of the form

$$V(K, U) = \{f \in Y^X \mid f(K) \subset U\},$$

where K is compact and U is open. That is, it is the coarsest topology in which every $V(K, U)$ is open.

There are certainly other topologies one might put on function spaces, and $\mathrm{Top}(n)$ has been considered with others, such as in [46] and [31]. However, the following key fact should persuade the reader that compact-open is the "correct" topology to put on $\mathrm{Top}(n)$. Note than an **isotopy** is a homotopy that is a homeomorphism at all times.

Proposition 2.2. *Isotopies of maps from $\mathbb{R}^n$ to itself are the same as paths in $Top(n)$. That is, any isotopy $H : \mathbb{R}^n \times I \to \mathbb{R}^n$ yields a path $\gamma : I \to Top(n)$ with $\gamma(t)(x) = H(x, t)$, and any such path yields an isotopy.*

Proof. If we are given a path or an isotopy, we can always form the other via the recipe $\gamma(t)(x) = H(x, t)$, and the condition that H be a homeomorphism for all t says the same thing as γ mapping into $\mathrm{Top}(n)$. So all we need to check is that continuity of one of these maps implies the other.

First suppose H is continuous. Then

$$\gamma^{-1}(V(K, U)) = \{t \in I \mid \gamma(t)(K) \subset U\} = \{t \in I \mid K \times \{t\} \subset H^{-1}(U)\}.$$

Since $H^{-1}(U)$ is open, every point in $K \times \{t\}$ has a ball of maximal radius around it contained in $H^{-1}(U)$. But K is compact, so the minimum of these radii is achieved at some point and is some nonzero ε. We then have that $K \times (t - \varepsilon, t + \varepsilon) \subset H^{-1}(U)$, hence $(t - \varepsilon, t + \varepsilon) \subset \gamma^{-1}(V(K, U))$ for all $t \in \gamma^{-1}(V(K, U))$. Thus $\gamma^{-1}(V(K, U))$ is open, so γ is continuous.

Now suppose γ is continuous. Then for any open $U \subset \mathbb{R}^n$,

$$H^{-1}(U) = \{(x, t) \in \mathbb{R}^n \times I \mid \gamma(t)(x) \in U\} = \bigcup_{t \in I} \gamma(t)^{-1}(U) \times \{t\}.$$

Since $\gamma(t)$ is a homeomorphism, $\gamma(t)^{-1}(U)$ is open for all t. In particular, this means that around every $x_0 \in \gamma(t_0)^{-1}(U)$, there is some precompact ball O_{x_0} whose closure is contained in $\gamma(t_0)^{-1}(U)$. Thus $\gamma(t_0) \in V(\overline{O_{x_0}}, U)$, and since γ is continuous, there is some $(t_0 - \delta, t_0 + \delta) \subset \gamma^{-1}(V(\overline{O_{x_0}}, U))$. Thus $O_{x_0} \times (t_0 - \delta, t_0 + \delta) \subset H^{-1}(U)$ is a neighborhood around (x_0, t_0), so $H^{-1}(U)$ is open, hence H is continuous. □

In fact, something stronger is true: *compact-open is the finest topology for which the above is true* [1, Theorem 2]. See also [19].

Lemma 2.3. *The evaluation map $\varphi : f \mapsto f(x)$ for some fixed x is continuous.*

Proof. This is standard, just note $\varphi^{-1}(U) = V(\{x\}, U)$. □

We now clear up some of the definitional inconsistencies mentioned above.

Proposition 2.4. *Top*(n) *and Top*$_S(n)$ *are isomorphic as groups.*

Proof. Let $\sigma : S^n \smallsetminus N \to \mathbb{R}^n$ be stereographic projection from the north pole. Define $\Phi : \mathrm{Top}(n) \to \mathrm{Top}_S(n)$ and $\Psi : \mathrm{Top}_S(n) \to \mathrm{Top}(n)$ by

$$\Phi(f) = \sigma^{-1} f \sigma \qquad \text{and} \qquad \Psi(g) = \sigma g \sigma^{-1},$$

where it should be understood that $\Phi(f) : N \mapsto N$. Clearly these are inverses of each other, and since σ is a homeomorphism, $\Psi(g)$ is indeed always a homeomorphism. Similarly, $\Phi(f)$ is evidently a homeomorphism of $S^n \smallsetminus N$ to itself, but it remains to check continuity at N. But this is quickly seen, since any homeomorphism f must map every compact region around 0 to some other compact region, and thus maps neighborhoods of infinity to neighborhoods of infinity. The same is true for $\Phi(f)^{-1} = \sigma^{-1} f^{-1} \sigma$, so $\Phi(f)$ is indeed an element of $\mathrm{Top}_S(n)$.

Finally, Φ is certainly a group homomorphism:

$$\Phi(fg) = \sigma^{-1} f g \sigma = \sigma^{-1} f \sigma \sigma^{-1} g \sigma = \Phi(f)\Phi(g),$$

as is Ψ.

Note that these maps do not provide a homeomorphism between these spaces because stereographic projection identifies every closed subset of $\mathbb{R}^n$ with a compact subset of S^n (just by adding the north pole to the set if it is unbounded in $\mathbb{R}^n$). As such, if these were homeomorphisms, every $V(K, U)$ would be open not just for compact K but for all closed K, which is not the case. $\square$

Proposition 2.5. *Top*(n) *is homeomorphic to Top*$_*(n) \times \mathbb{R}^n$*, but they are not isomorphic as groups.*

Proof. Let $T_x : \mathbb{R}^n \to \mathbb{R}^n$ be translation by x. Define maps $\Gamma : \mathrm{Top}_*(n) \times \mathbb{R}^n \to \mathrm{Top}(n)$ and $\Theta : \mathrm{Top}(n) \to \mathrm{Top}_*(n) \times \mathbb{R}^n$ by

$$\Gamma(f, x) = T_x \circ f \qquad\qquad \Theta(g) = (T_{-g(0)} \circ g, g(0))$$

First, these maps are inverses of each other:

$$\Gamma(\Theta(g)) = \Gamma(T_{-g(0)} \circ g, g(0)) = T_{g(0)} \circ T_{-g(0)} \circ g = g$$

$$\begin{aligned} \Theta(\Gamma(f, x)) &= \Theta(T_x \circ f) = (T_{-T_x(f(0))} \circ T_x \circ f, T_x(f(0))) \\ &= (T_{-T_x(0)} \circ T_x \circ f, T_x(0)) = (T_{-x} \circ T_x \circ f, x) = (f, x) \end{aligned}$$

We need to check continuity of these maps, and since evaluation and composition are continuous (the former by Lemma 2.3 and the latter because $\mathrm{Top}(n)$ is a topological group), we just need to verify that $T : x \mapsto T_x$ is continuous. To that end, consider any $x \in T^{-1}(V(K, U))$, which is some value such that $K + x \subset U$. Every point in $K + x$ is contained in some maximal open ball inside U, and Since K is compact, there is some nonzero minimum of these balls' radii. If ε is this minimum, then any $y \in B_\varepsilon(x)$ is also in $T^{-1}(V(K, U))$, so this set is open, hence T is continuous.

Note however that these maps are not group homomorphisms:

$$\Gamma(f \circ g, x + y) = T_{x+y} \circ f \circ g = T_x \circ T_y \circ f \circ g \neq T_x \circ f \circ T_y \circ g = \Gamma(f, x) \circ \Gamma(g, y).$$

We defer the proof that $\mathrm{Top}(n) \ncong \mathrm{Top}_*(n) \times \mathbb{R}^n$ as groups until after we have discussed orientation-preserving maps. $\square$

2.1. **Metrizability.** Perhaps surprisingly, Top(n) is metrizable. One valid metric is

$$\rho(f,g) = \sup_{r>0} \min(1/r, \sup_{x\in D_r} d(f(x), g(x))),$$

where D_r is the closed ball of radius r centered at the origin. This can equivalently be written as

$$\rho(f,g) = \inf_{r>0} \max(1/r, \sup_{x\in D_r} d(f(x), g(x))).$$

Either expression may be more sensibly interpreted as "$\rho(f,g) = 1/r$, where r is the largest value such that f and g differ by at most $1/r$ on D_r". Hence a smaller distance under ρ corresponds to a larger region on which f and g are closer. These metrics are asserted to be valid in [11] and [37] respectively (although they define it in terms of cubes rather than balls). The standard conditions for a metric are straightforward to check for ρ, and indeed ρ yields the right topology:

Proposition 2.6. *The metric topology induced by ρ on Top(n) is the same as the compact-open topology.*

Proof. Let any $V(K,U)$ be given, and fix any $f \in V(K,U)$. Since K is compact, there is some $\varepsilon > 0$ such that

$$P_\varepsilon = \bigcup_{y\in f(K)} B_\varepsilon(y) \subset U.$$

Next, choose $r > 0$ large enough that $K \subset D_r$ and $1/r < \varepsilon$. Now consider:

$$\begin{aligned} B^\rho_{1/r}(f) &= \{g \mid d(f(x), g(x)) < 1/r \ \text{ for all } \ x \in D_r\} \\ &\subset \{g \mid d(f(x), g(x)) < \varepsilon \ \text{ for all } \ x \in K\} \\ &\subset \{g \mid g(K) \subset P_\varepsilon\} \\ &= V(K, P_\varepsilon) \\ &\subset V(K,U) \end{aligned}$$

So ρ's metric topology is at least as fine as compact-open. For the converse, let any $B^\rho_\ell(f)$ be given. Since $D_{1/\ell}$ is compact, we may cover $f(D_{1/\ell})$ by finitely many balls of radius $\ell/3$, call them $\{B_{\ell/3}(x_i)\}_{i=1}^m$. Let

$$\mathcal{O} = \bigcap_{i=1}^m V\left(\overline{f^{-1}(B_{\ell/3}(x_i))}, B_{\ell/2}(x_i)\right),$$

and observe that $\mathcal{O}$ is open in the compact-open topology and contains only maps that send elements of $D_{1/\ell}$ to within ℓ of where f sends them, so $\mathcal{O} \subset B^\rho_\ell(f)$. □

Largely by virtue of it being a metric space, Top(n) has basically every nice topological property one could ask for. Indeed, we automatically know that Top(n) is first countable, paracompact, and satisfies all the usual separation axioms. The main difficulty in dealing with it is its sheer size, although even that is not too bad:

Proposition 2.7. *Top(n) is second countable.*

Proof. Let $\{U_i\}_{i\in\mathbb{N}}$ be some countable basis of $\mathbb{R}^n$. Let $\mathcal{B}$ be the set of all finite intersections of sets of the form $V(\overline{U}_i, U_j)$. Note that $\mathcal{B}$ is countable, we claim it is the desired basis. Fix any $V(K,U)$ and any $f \in V(K,U)$. We need to find some set in $\mathcal{B}$ within $V(K,U)$ that contains f. Since $f(K)$ is compact, we can cover it with finitely many U_i's, call them $\{U_{i_j}\}_{j=1}^m$, which may each be chosen to be within

U. Each $x \in K$ is in some $f^{-1}(U_{i_j})$, and we can find some U_k containing x such that $\overline{U_k} \subset f^{-1}(U_{i_j})$. By compactness again, we only need finitely many such U_k's to cover K. The intersection of all such $V(\overline{U_k}, U_{i_j})$'s is the desired basis set. $\square$

2.2. **Orientation and Path-connectedness.** Even the simple question of how many path components $\mathrm{Top}(n)$ has turns out to be very difficult, although for now we can develop some general theory. First, we can always split $\mathrm{Top}(n)$ into at least two path-components by considering orientation-preserving and orientation-reversing maps.

More formally[1], we can define a map on $\mathbb{R}^n$ to be orientation-preserving if it has degree 1 as a map on S^n. That is, in the same way as in Proposition 2.4, we lift the map up to the sphere via stereographic projection, mapping the north pole to itself. If this automorphism of the sphere has degree 1, the original map is orientation-preserving, and if it has degree -1, it is orientation-reversing.

Recall that the **degree** of a continuous map $f : S^n \to S^n$ is equal to $f_*(1) \in \mathbb{Z}$, where we are identifying $H_n(S^n) \cong \mathbb{Z}$ and f_* is the map f induces on homology. Since every element of $\mathrm{Top}_S(n)$ is a homeomorphism, they all induce isomorphisms on homology groups, hence the only possible degrees are ± 1. Also recall the standard fact from degree theory that two maps between S^n are homotopic if and only if they have the same degree [32].

So indeed, there cannot be any paths in $\mathrm{Top}(n)$ between orientation-preserving and orientation-reversing maps, as that would correspond to an isotopy on $\mathbb{R}^n$, which could be lifted to an isotopy on S^n.

We will refer to the collection of orientation-preserving maps in $\mathrm{Top}(n)$ (that is, maps corresponding to degree 1 maps on the sphere) as $\mathrm{STop}(n)$. Note that degree theory does not give us that $\mathrm{STop}(n)$ and the complementary set are themselves path-connected, as being of the same degree only guarantees the existence of a homotopy, rather than the isotopy we need. Thanks to the following result, we can largely reduce our study of $\mathrm{Top}(n)$ to focusing on $\mathrm{STop}(n)$:

Proposition 2.8. *The spaces of orientation-preserving and orientation-reversing maps in $Top(n)$ are homeomorphic. That is, $STop(n) \cong Top(n) \setminus STop(n)$.*

Proof. Fix any orientation-reversing $f \in \mathrm{Top}(n)$. Multiplication by f is an automorphism of $\mathrm{Top}(n)$ since it is a topological group. Because degree is multiplicative (i.e. $\deg gh = (\deg g)(\deg h)$), this is a homeomorphism sending orientation-preserving maps to orientation-reversing maps and vice versa. $\square$

Note that this also shows that $\mathrm{STop}(n)$ is an index 2 subgroup of $\mathrm{Top}(n)$.

Proving much more here is almost impossible with elementary techniques; it will be the goal of the next section to prove that $\mathrm{STop}(n)$ is path-connected. Local path-connectedness is also difficult to show in general for $\mathrm{Top}(n)$, but after laying the groundwork for path-connectedness it will be achievable. We can however tie up one loose end:

Continuation of proof of 2.5. To show that there is no group isomorphism between $\mathrm{Top}(n)$ and $\mathrm{Top}_*(n) \times \mathbb{R}^n$, we appeal to the recent fact [2] that every map in

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[1]There are many formulations of orientation, but suffice it to say, they are all equivalent. For some of their definitions and discussions of equivalence, see [15], [25, Chapter 15], and [39].

STop(n) is a commutator. And because $[f,g] = fgf^{-1}g^{-1}$ is orientation-preserving for any $f, g \in \mathrm{Top}(n)$ (since the composition of two orientation-reversing maps is orientation-preserving), we get that STop(n) is in fact the entire commutator subgroup of Top(n). Thus, since STop(n) has index 2 in Top(n), we get that any abelian quotient of Top(n) has at most two elements. But this is manifestly untrue for $\mathrm{Top}_*(n) \times \mathbb{R}^n$: we can take $\mathrm{Top}_*(n) \times \mathbb{R}^n / \mathrm{Top}_*(n) \times 0 \cong \mathbb{R}^n$. □

2.3. **Top(1).** Before continuing further, we provide a fairly thorough description of the one-dimensional situation, which is the only directly tractable case (the case of Top(0) is uninteresting - there is only one map $\mathbb{R}^0 \to \mathbb{R}^0$). The key fact that makes this case accessible is the standard result from elementary analysis that any continuous bijective map from $\mathbb{R}$ to itself is either strictly increasing or strictly decreasing.

Proposition 2.9. *STop*(1) *is contractible and open, hence Top*$(1) \simeq S^0$.

Proof. Any $f \in \mathrm{STop}(1)$ is strictly increasing, so we consider the straight-line isotopy:

$$H(x,t) = tx + (1-t)f(x).$$

This is a homotopy between f and the identity map, and since it is a linear combination of increasing functions with nonnegative coefficients (that are never both zero), it is increasing. So $H_t \in \mathrm{STop}(1)$ for all t, meaning this is an isotopy. It also varies continuously with f, so it is actually a deformation retraction $\mathrm{STop}(1) \simeq \{\mathrm{Id}\}$.

For openness, it suffices to prove that the identity has a neighborhood in STop(1). Indeed it does, namely

$$\bigcup_{n\in\mathbb{N}} V(\{0\}, (-1/n, 1/n)) \cap V(\{1\}, (1/n, \infty)).$$

□

3. The Stable Homeomorphism Theorem

Our goal in this section is to prove the following result (in high dimensions), which turns out to be equivalent to a basic topological property of Top(n):

Theorem 3.1 (Stable Homeomorphism Theorem)**.** *Every orientation-preserving automorphism of* $\mathbb{R}^n$ *is stable.*

An automorphism f is said to be **stable** if $f = h_1 h_2 \cdots h_m$, where each h_i is an automorphism of $\mathbb{R}^n$ and there is some open U_i such that $h_i|_{U_i}$ is the identity. We can also consider stability of a map defined only on an open subset of $\mathbb{R}^n$ by requiring each point in the domain have a neighborhood such that its restriction to that neighborhood can be be extended to a stable automorphism of $\mathbb{R}^n$ as above[2].

For some examples of maps that can be easily shown to be stable, the reader is directed to the upcoming lemmas.

This theorem and the closely related Annulus Theorem were open problems for many years. In low dimensions they were proven case by case, dimension 4 was particularly difficult as usual and took until the 80's to be proved [33]. However, for dimensions 5 and above, an ingenious technique of Kirby (his "torus trick") allows it to be reduced to an easier problem that can be handled with piecewise

[2]This is a reasonable definition to make in light of Lemma 3.5 below.

linear surgery theory [20]. We will discuss his method and prove all the core ideas. Before getting too into the weeds, we justify why we should want this result:

Lemma 3.2. *Stable maps are isotopic to the identity, so the Stable Homeomorphism Theorem implies $STop(n)$ is path-connected.*

Proof. If f is stable, we can write $f = h_1 \cdots h_m$ with $h_i|_{U_i} = \mathrm{Id}|_{U_i}$. We claim each h_i is isotopic to the identity. To prove this, it suffices to consider some h such that $h|_{B_1(0)} = \mathrm{Id}|_{B_1(0)}$. We use a version of Alexander's isotopy:

$$H(x,t) = \begin{cases} \frac{1}{t}h(xt) & \frac{1}{2t} \le |x| \\ x & |x| < \frac{1}{2t} \end{cases}$$

Note that $H(x,0) = \mathrm{Id}(x)$ and $H(x,1) = h(x)$. Also, H is continuous because at $|x| = \frac{1}{2t}$, we get $\frac{1}{t}h(xt) = 2|x|h(\frac{x}{2|x|}) = 2|x|\frac{x}{2|x|} = x$, since h is the identity on $B_1(0)$. Also, $H(x,t_0)$ is a homeomorphism for each fixed t_0 since each of the two forms it takes is a homeomorphism onto its domain:

$$\frac{1}{t_0}h(t_0 B_{1/2t_0}(0)^c) = \frac{1}{t_0}h(B_{1/2}(0)^c) = \frac{1}{t_0}B_{1/2}(0)^c = B_{1/2t_0}(0)^c.$$

So indeed there are isotopies $H_i : h_i \simeq \mathrm{Id}$ for each i. By composing these isotopies we get an isotopy between f and the identity, as desired. □

In fact, we will see that path-connectedness of $\mathrm{STop}(n)$ is actually *equivalent* to the Stable Homeomorphism Theorem. Before we start proving things, we must discuss some necessary machinery. We begin with an introduction to some structures on manifolds that we will need.

3.1. **Piecewise-linear Topology.**

We need to discuss a geometric framework which serves as a sort of middle ground between the topological and smooth categories. We will only introduce the very basics here, just enough to define piecewise-linear manifolds and to see their connection to other kinds of manifolds. We will mainly be summarizing from [34], [14], [35], [26, Lecture 2], and [7, Appendix B]

A **polyhedron** $P \subset \mathbb{R}^n$ is a subset such that every $a \in P$ has a neighborhood around it contained in a cone in P. That is, there is some compact $L \subset P$ such that $aL = \{\lambda a + \mu q \mid q \in L; \lambda, \mu \in [0,1]; \lambda + \mu = 1\}$ is contained in P and contains a neighborhood of a in P. In this situation, L is called a link of a, and aL is called a star. Any open subset of $\mathbb{R}^n$ is a polyhedron, and this is the situation we will generally be concerned with. Indeed, just take a small enough precompact ball around any point and its boundary will be the desired link:

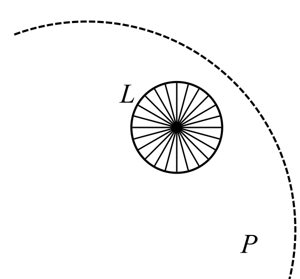

FIGURE 1. Every open set in $\mathbb{R}^n$ is a polyhedron.

A continuous map $f : P \to Q$ between polyhedra is **piecewise-linear** (PL) if there is a triangulation of P such that f is affine when restricted to each simplex[3]. Here by affine, we mean a linear map composed with a translation. If a PL map has a PL inverse, then it is a PL homeomorphism (or equivalence or isomorphism – this is just the isomorphism in the PL category).

A **PL manifold** is a polyhedron such that every point has a neighborhood which is PL homeomorphic to an n-ball [26] [34] [7]. Equivalently, and more conveniently for our purposes, a PL manifold is a topological manifold equipped with a maximal PL atlas [35][6]. That is, a collection of charts such that the transition map between any pair of them is a PL map. This gives a nice abstract perspective on these manifolds and gives a convenient symmetry to the definition of smooth (and stable, as we will see) manifolds. It also makes it sensible to speak of different PL structures on the same topological manifold. We can then consider a PL map on PL manifolds as we have defined it above or as a map whose local representation under PL charts is PL. That is, if φ and ψ are charts in the PL structures for M and N, then $f : M \to N$ is PL if $\psi \circ f \circ \varphi^{-1}$ is PL as a map between polyhedra in $\mathbb{R}^n$. This is equivalent to the above definition if M and N are themselves viewed as polyhedra [18].

It is worth noting that not every topological manifold has a PL structure, and in fact even triangulable manifolds need not be PL manifolds. Further, if a manifold has a PL structure, it need not be unique up to PL homeomorphism.

What is true however is that every smooth manifold has a PL structure, which is unique if we insist upon "Whitehead compatibility". Specifically, given a smooth manifold, there is a PL structure such that the transition maps between the PL charts and smooth charts are piecewise-differentiable (PD). So if ψ is a smooth chart and φ is a PL chart, then $\psi\varphi^{-1}$ restricts to a differentiable map on each simplex of some triangulation, with injective differential at every point [26, Lecture 3] [6]. Further, this PL structure is *unique* up to PL homeomorphism [45]. To see that this is even a sensible condition to impose on the PL structure, note that if the above is true for ψ and φ, then for any other smooth chart χ and PL chart θ, we have

$$\chi\theta^{-1} = \underbrace{\chi\psi^{-1}}_{\text{smooth}}\underbrace{\psi\varphi^{-1}}_{\text{PD}}\underbrace{\varphi\theta^{-1}}_{\text{PL}}.$$

Since linear maps are certainly differentiable[4], we get that $\chi\theta^{-1}$ is piecewise-differentiable. We will see later that the stable structure induced by a smooth structure is the same as that which is induced by the associated PL structure.

Just to get a taste of how PL manifolds work, let's look at a simple example.

Example 3.3. Consider the open unit interval $(0, 1)$, which has a natural topological manifold structure, and consider the map $\varphi : (0, 1) \to (0, 1)$ given by

$$\varphi(x) = \begin{cases} \frac{1}{2}x & x \leq \frac{1}{2} \\ \frac{3}{2}x - \frac{1}{2} & x > \frac{1}{2} \end{cases}$$

[3]One often sees another definition, but this is equivalent [34, Ex. 1.5 and Cor. 2.3] and more convenient for us.

[4]Although C^1 is sufficient for much of what we will be doing, for ease we will always mean C^∞ when we say smooth or differentiable.

This is a PL map between subsets of $\mathbb{R}$, but what about as a map of manifolds? We could take this to be a chart defining a PL structure on $(0, 1)$, but since it's already PL, that structure would be the same as the one defined by the identity map. We could put a more interesting structure on our manifold with a chart like

$$\psi(x) = \begin{cases} x^2 & x \le \frac{1}{2} \\ \frac{3}{2}x - \frac{1}{2} & x > \frac{1}{2} \end{cases}$$

This certainly isn't a PL map, so it couldn't be part of the same PL structure as φ or the identity, since the transition map between the charts wouldn't be PL. It nonetheless determines a perfectly good PL structure, consisting of all charts χ such that $\chi\psi^{-1}$ and $\psi\chi^{-1}$ are PL.

Even though φ and ψ define different PL structures, the resultant PL manifolds are still PL isomorphic. Indeed, if M is $(0, 1)$ with the identity structure and N is the same space with the ψ structure, then ψ itself is a PL homeomorphism $N \to M$. It is PL because $\mathrm{Id}\circ\psi\circ\psi^{-1} = \mathrm{Id}$ is, and its inverse is as well, since $\psi\circ\psi^{-1}\circ\mathrm{Id} = \mathrm{Id}$.

We are interested in PL manifolds now because of one of the key steps in the proof of the Stable Homeomorphism Theorem, and because *all piecewise-linear homeomorphisms are stable*. This enables us to leverage stronger structures to get at stability, but before proving this fact, we need to introduce a more robust framework for stability.

3.2. **Stable Manifolds.** Recall that we can consider whether a map defined only on an open subset of $\mathbb{R}^n$ is stable. Because of this and the fact that compositions of stable maps are stable (Lemma 3.4), we can define a stable structure on a topological manifold. We do so in the same way as we define smooth or PL structures: we take a maximal collection of charts such that for any φ and ψ, the map $\psi\varphi^{-1}$ is stable as a map between subsets of $\mathbb{R}^n$. We thus get the notion of **stable manifolds**, and maps between such manifolds are stable if their local versions are. That is, $f : M \to N$ is stable if $\psi\circ f\circ\varphi^{-1}$ is stable for every pair of charts.

We will see soon (Lemmas 3.12 and 3.10) that all orientation-preserving diffeomorphisms and PL maps on $\mathbb{R}^n$ are stable, so oriented smooth and PL manifolds automatically inherit a stable structure. We can just take the stable charts to be all the oriented smooth/PL charts, as well as all that are stably compatible with them. Unless otherwise stated, any PL or smooth manifold (particularly $\mathbb{T}^n$) will be assumed to have that stable structure.

Something important to note is that the stable structure induced by a smooth structure is the same as that induced by the PL structure one can put on the smooth manifold. Indeed, suppose ψ is a smooth chart on some n-manifold M and φ is a PL chart in the induced PL structure on M. Then $\psi\varphi^{-1}$ is piecewise-differentiable with injective differential (this is the Whitehead compatibility condition). Restricting to the interior of some n-simplex in the triangulation, we have that $\psi\varphi^{-1}$ is a local diffeomorphism. So by Lemma 3.12, this transition map is stable. But then $\varphi\psi^{-1}$, when suitably restricted, is also a diffeomorphism, hence stable. Thus one can take either the smooth or PL charts as defining the stable structure, and the others come along anyways. Because of this, we can freely speak of stable maps between smooth manifolds, and we can switch between viewing them as PL and smooth manifolds without any issues.

3.3. **Initial Lemmas.** We start with some basic facts about stable maps that we will need repeatedly. If the reader is willing to believe that various "nice" homeomorphisms (linear maps, diffeomorphisms, etc.) and maps "near to" stable maps are stable, then this section can be skipped painlessly enough.

Lemma 3.4. *A composition of stable maps is stable (in $\mathbb{R}^n$ or on a manifold).*

Proof. First, in $\mathbb{R}^n$, if f and g are each compositions of homeomorphisms which are each the identity on an open subset, then their composition is as well. Next, suppose they are only defined on open subsets of $\mathbb{R}^n$ such that their restriction around any point can be extended to a stable map on $\mathbb{R}^n$. Then, the composition of those extensions will be stable, and this will be an extension of a restriction of the composition of the original maps.

In general, let M, N, and P be stable manifolds, and suppose $h : M \to N$ and $k : N \to P$ are stable maps. That means that given any stable charts φ, ψ, and χ on M, N, and P respectively, the maps

$$\psi \circ h \circ \varphi^{-1} \qquad \text{and} \qquad \chi \circ k \circ \psi^{-1}$$

are stable as maps on (open subsets of) $\mathbb{R}^n$. We already know the composition of stable maps on $\mathbb{R}^n$ is stable, so

$$(\chi \circ k \circ \psi^{-1}) \circ (\psi \circ h \circ \varphi^{-1}) = \chi \circ k \circ h \circ \varphi^{-1}$$

is stable, hence kh is stable as a map of manifolds. □

Lemma 3.5. *If two automorphisms of $\mathbb{R}^n$ agree on some open subset and one of them is stable, then so is the other. The same is true on connected manifolds.*

Proof. Let f, g be maps on $\mathbb{R}^n$ as described, with g stable and $f|_U = g|_U$. Then $g^{-1}f$ is the identity on U, so $f = gg^{-1}f$ is a composition of stable homeomorphisms, hence stable.

If we treat them now as maps between manifolds, then their local representations on U will be the same, hence both stable there. Let $\mathscr{S}$ be the collection of all stable charts φ such that there is any chart ψ with $\psi \circ f \circ \varphi^{-1}$ stable. Let $\mathscr{N}$ be all remaining stable charts. Now, let S be the union of the domains of the charts in $\mathscr{S}$, and N the same for the charts in $\mathscr{N}$. Both S and N are open, and S is nonempty by above. Crucially, they are disjoint, since if a chart $\theta \in \mathscr{S}$ and a chart $\chi \in \mathscr{N}$ overlap, then on that overlap,

$$\psi \circ f \circ \chi^{-1} = \underbrace{(\psi \circ f \circ \theta^{-1})}_{\text{stable}} \circ \underbrace{(\theta \circ \chi^{-1})}_{\text{stable}},$$

hence $\chi \in \mathscr{S}$. Thus, to avoid a disconnection, we must have that N is empty, hence f is stable with respect to any chart in the domain. Applying the same argument to charts on the codomain, we get that f is stable with respect to any pair of charts, hence stable. □

This is a rather surprising fact. It indicates that stability is somehow locally detectable, which is good for our definition of stable manifolds, but is peculiar given that the definition of stability is seemingly so nonlocal. It's further surprising how simple the proof is, with the key argument being just a sentence long. It might be viewed best as a first hint that stability isn't so strong a condition after all. In any case, it means that often the easiest way of proving some map is stable is to construct a map that is the identity somewhere and that map somewhere else.

This next lemma will be of crucial importance for showing that all automorphisms of the torus are stable. It's worth noting however that the condition it requires is quite strong. Indeed, every neighborhood around a map will contains maps that differ from it at some points by arbitrarily large amounts.

Lemma 3.6. *For maps $f, g \in Top(n)$, if $d(f(x), g(x))$ is bounded for all $x \in \mathbb{R}^n$, and g is stable, then so is f.*

Proof. (After [8, Lemma 5 & Theorem 5], but the proof there is quite scant on details). If $f(x)$ and $g(x)$ are within some distance for all x, then the same is true for $fg^{-1}(x)$ and x. Further, if fg^{-1} is stable, and so is g, then $f = (fg^{-1})g$ is also stable. Thus it suffices to consider the case where one of the maps, say g, is the identity. For the purposes of this proof, B_a is the open ball of radius a around the origin.

Fix some r large enough that $f(B_1) \subset B_r$ and choose any $s > r$. Next, define a map $h : B_s \to \mathbb{R}^n$ that is the identity on B_r and which radially stretches $B_s \smallsetminus B_r$ homeomorphically onto $\mathbb{R}^n \smallsetminus B_r$. Explicitly, we could take

$$h(x) = \begin{cases} x & |x| < r \\ \frac{s-r}{s-|x|}x & r \leq |x| < s \end{cases}$$

Note that $h^{-1}fh$ is a automorphism of B_s, as is $h^{-1}f^{-1}h$. We now define $j : \mathbb{R}^n \to \mathbb{R}^n$ by

$$j(x) = \begin{cases} h^{-1}f^{-1}h(x) & |x| < s \\ x & |x| \geq s \end{cases}$$

Each of the two forms j takes are automorphisms of their domains, so to see j is a homeomorphism, it suffices to check continuity (and continuity of the inverse) at ∂B_s. Indeed, for $x \in B_s$, the distortion caused by f or f^{-1} is applied to an enlarged version of the space, and thus will be lessened overall since the space is scaled back down by h^{-1} afterwards. As $|x| \to s$, the scale factor gets larger, so the distortion becomes less and less. And since the initial distortion is bounded, the distortion tends to zero, meaning the map tends to the identity.

So j is a homeomorphism, and stable because it is the identity outside of B_s. Also, since $f(B_1) \subset B_r$ and h leaves B_r untouched, jf is the identity on B_1. So jf is stable, meaning that $f = j^{-1}(jf)$ is stable. □

We also note a related result:

Lemma 3.7. *For maps $f, g \in Top(n)$, if $d(f(x), g(x))$ is bounded for all $x \in \mathbb{R}^n$, then f is isotopic to g.*

Proof. It suffices to prove this in the case $g = \mathrm{Id}$, since then the general case follows by multiplying the isotopy from Id to $f^{-1}g$ by f. We use a version of Alexander's trick: define $H : \mathbb{R}^n \times I \to \mathbb{R}^n$ by

$$H(x, t) = \begin{cases} tf(x/t) & t \in (0, 1] \\ x & t = 0 \end{cases}.$$

Clearly this goes from Id to f, and at every $t \neq 0$, we have the inverse $tf^{-1}(x/t)$, so we just need to check continuity. And indeed:

$$\lim_{t\to 0} d(H(x,t),x) = \lim_{t\to 0} td(f(x/t),x/t) \leq \lim_{t\to 0} tB = 0.$$

Note that one could also prove this by appealing to the previous lemma for Id and $f^{-1}g$, then applying Lemma 3.2. □

Finally, we will prove that some special cases of the Stable Homeomorphism Theorem are true, where we restrict to various nice subsets of $\mathrm{Top}(n)$. We will need these to prove the general case, and also to show that smooth and PL manifolds do in fact have induced stable structures.

Lemma 3.8. *Any orientation-preserving (i.e. positive determinant) linear transformation of $\mathbb{R}^n$ to itself is stable.*

Proof. Let some positive-determinant $T \in GL(n)$ be given, and suppose first that T maps every sphere centered at the origin to itself homeomorphically (i.e. T is length-preserving). Recall the standard fact that $GL^+(n)$ is path-connected [42, Theorem 3.68], meaning there is an isotopy $H : T \simeq \mathrm{Id}$ that is linear at all times. By normalizing the images of the basis vectors at each time, we can make it length-preserving at all times. Since the vectors never pass through zero, this presents no continuity or invertibility issues. Now consider the following map:

$$g(x) = \begin{cases} T(x) & x \in B_1(0) \\ H(x, |x|-1) & x \in B_2(0) \smallsetminus B_1(0) \\ x & x \in B_2(0)^c \end{cases}$$

This is evidently a stable homeomorphism that agrees with T on some region, so T is stable.

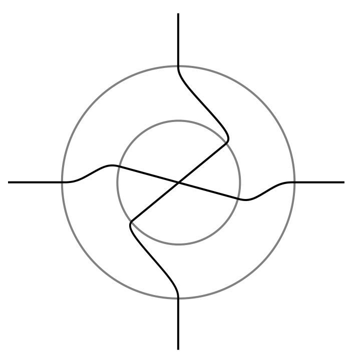

FIGURE 2. The homeomorphism g, bridging between T and Id using the length-preserving isotopy.

For the general case, we just scale the region around the origin along the directions of the images of each basis vector so that the transformation is length-preserving. We've seen that the resultant map is stable, so we need now only show that these scaling maps are stable. Then the original map will be the inverse of the scaling maps composed with the normalized map, hence stable.

We need to prove that for any nonzero $v \in \mathbb{R}^n$ and positive $\alpha \in \mathbb{R}$, we can stretch a region by a factor of α in the direction v with a stable homeomorphism. We first consider this problem in the one-dimensional case, which will look like this:

$$s(x) = \begin{cases} x & x > nK \\ \alpha K + \frac{n-\alpha}{n-1}(x-K) & nK \geq x \geq K \\ \alpha x & K > x \end{cases},$$

where K is some large number and $n > \max(\alpha, 1)$. Notice that $\alpha K + \frac{n-\alpha}{n-1}(x-K)$ is αK at $x = K$ and is nK at $x = nK$. Further, $\frac{n-\alpha}{n-1} > 0$, so this intermediate section is increasing, meaning s is a (stable) homeomorphism. In the n-dimensional case, just take a basis containing v and define a map which is s on the v component and the identity on all the rest. □

Lemma 3.9. *Any translation in $\mathbb{R}^n$ is stable.*

Proof. As in last part of the proof of Lemma 3.8, it suffices to consider the one-dimensional case. So, let $f(x) = x + a$ for some fixed positive $a \in \mathbb{R}$. Then, consider

$$g(x) = \begin{cases} x + a & 1 \leq x \\ x + ax & 0 < x < 1 \\ x & x \leq 0 \end{cases}$$

This is a homeomorphism which equals Id in one region and f in another, hence f is stable. If a is negative, then f will be the inverse of a translation by a positive number, hence also stable. □

Lemma 3.10. *Any orientation-preserving PL map is stable.*

Proof. A PL map certainly agrees with an affine map on some region, and an affine map is just a linear map composed with a translation. □

The next two lemmas are adapted from [9, Chapter 5]. An alternative proof of Lemma 3.12, as well as general discussion of stable maps, can be found in [5].

Lemma 3.11. *Let $h : \mathbb{R}^n \times I \to \mathbb{R}^n$ be a smooth isotopy and fix any $x \in \mathbb{R}^n$. Then there is a smooth isotopy $H : \mathbb{R}^n \times I \to \mathbb{R}^n$ such that $H_t \circ h_0 = h_t$ in a neighborhood of x and H_t is the identity outside of some compact set.*

The proof depends on the notion of flows on manifolds, which is too far afield for us to discuss in the proper detail. The idea is to make a smooth vector field on $\mathbb{R}^n \times I$ that is uniform in the t direction in one region and which depends on h in another region. This is done in such a way that integrating the vector field (that is, "following where the points go as if pushed by the vector field") yields h in one region and the identity isotopy elsewhere. This fact will only be used to prove the following:

Lemma 3.12. *Any orientation-preserving self-diffeomorphism of $\mathbb{R}^n$ is stable.*

Proof. Let f be such a map. We first want to construct an isotopy from f to the identity. By translating first (which is certainly isotopic to the identity), we may assume $f(0) = 0$. The following clever isotopy can be found in [32, §6]:

$$J(x,t) = \begin{cases} \frac{1}{t} f(tx) & t \in (0,1] \\ Df_0(x) & t = 0 \end{cases}$$

where Df_0 is the differential of f at the origin. This is evidently a homeomorphism (in fact a diffeomorphism) for each t, and it's continuous by the definition of the derivative. Thus this is an isotopy from f to a linear map. Since isotopies can only connect orientation-preserving maps to other orientation-preserving maps, we know that $Df_0 \in GL^+(n)$. Since $GL(n)$ is path-connected smooth manifold, we can take a smooth path from Df_0 to Id. We thus get a smooth isotopy $h : f \simeq \mathrm{Id}$.

By the previous lemma, there is an isotopy H such that $H_t \circ f = h_t$ in a region around 0, and such that H_t is the identity outside some compact set. So then H_1 is f^{-1} in one region and the identity in another. By Lemma 3.5, we have that f^{-1} is stable, hence so is f. □

Remark 3.13. We briefly digress here to discuss $\mathrm{Diff}(n)$, the space of all self-diffeomorphisms of $\mathbb{R}^n$. Just like $\mathrm{Top}(n)$, it is fruitful to split it into orientation-preserving and orientation-reversing maps, between which there are no paths. In fact, the isotopy J in the previous proof actually serves as a deformation retract of the orientation-preserving subset of $\mathrm{Diff}(n)$ onto $GL^+(n)$. By Gram-Schmidt orthogonalization, we know that $GL(n) \simeq O(n)$, so in fact $\mathrm{Diff}(n) \simeq O(n)$ for all n. A central result of Kirby and Siebenmann [21] says that for $n \leq 3$ this also coincides with $\mathrm{Top}(n)$. That is, $\mathrm{Top}(n) \simeq O(n) \simeq \mathrm{Diff}(n)$.

3.4. **Main Results.** Up next is the key to the whole game. One of Kirby's big insights is that any homeomorphism of the torus lifts to a homeomorphism of $\mathbb{R}^n$ which is fully determined by its action on $[0,1]^n$. Since the homeomorphism can only move that fundamental domain so far, it is forced to be bounded everywhere (after some correction to realign everything). We will prove this result slightly differently to Kirby to avoid some technical dependencies, but the idea is the same. See his classic paper [20] for the original proofs, which we will be adapting throughout this section. Some other treatments of the argument can be found in [9, Chapters 13 and 14], [7, Appendix B.3.2], and [43, Section 9.3].

Lemma 3.14. *Any orientation-preserving automorphism of $\mathbb{T}^n$ is stable.*

Proof. Fix any such homeomorphism f. Let $\varepsilon : \mathbb{R}^n \to \mathbb{T}^n$ be the standard universal covering map of the torus (just the exponential $e^{2\pi i x}$ in each coordinate). The composition $f\varepsilon$ is also a covering map of $\mathbb{T}^n$, so by a standard fact about covering spaces [24, Theorem 11.40], there is a homeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$ such that the following commutes:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{F} & \mathbb{R}^n \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\varepsilon} \\ \mathbb{T}^n & \xrightarrow[f]{} & \mathbb{T}^n \end{array}$$

Since ε is orientation-preserving, F also has to be orientation-preserving because $\varepsilon F = f\varepsilon$ is.

Note that ε maps two points to the same point on the torus if and only if they differ by an integer in each coordinate. So, we get that

$$\varepsilon F(x) = f\varepsilon(x) = f\varepsilon(\{x_1\}, \ldots, \{x_n\}) = \varepsilon F(\{x_1\}, \ldots, \{x_n\}),$$

where $\{x_i\}$ is the fractional part of x_i. Thus,

$$F(x) = F(\{x_1\}, \ldots, \{x_n\}) + A(x),$$

where A maps into $\mathbb{Z}^n$. By composing with a translation, we can get a map $\tilde{F}$ that fixes the origin and for which the above relation is still true. For $x \in [0,1)^n$, we can see that $A(x) = 0$ since then $x_i = \{x_i\}$. So $F(\{x_1\}, \dots, \{x_n\})$ is continuous on each unit cube, meaning A must be constant on each unit cube for F to be continuous. Thus A depends only on the integer part of each x_i, so it can be viewed as a map on $\mathbb{Z}^n$, which we claim is $\mathbb{Z}$-linear. Indeed, for any $y \in \mathbb{Z}^n$, by continuity we get

$$\begin{aligned}\tilde{F}(y + e_i) &= \lim_{d \to 1^-} \tilde{F}(y + de_i)\\ \tilde{F}(0) + A(y + e_i) &= \lim_{d \to 1^-} \tilde{F}(de_i) + A(y + de_i)\\ A(y + e_i) &= \tilde{F}(e_i) + A(y).\end{aligned}$$

Also,

$$\tilde{F}(y) = \tilde{F}(\{y_1\}, \dots, \{y_n\}) + A(y) = \tilde{F}(0) + A(y) = A(y),$$

so $\tilde{F}|_{\mathbb{Z}^n} = A|_{\mathbb{Z}^n}$. Then $\mathbb{Z}$-linearity of A follows from the above and induction. Extend A to a linear map on all of $\mathbb{R}^n$ by just reusing the same matrix. Notice that A and $\tilde{F}$ are of bounded distance apart on all of space since they differ by $\tilde{F}(\{x_1\}, \dots, \{x_n\})$, so the maximum distortion occurs on the unit cube, which is compact.

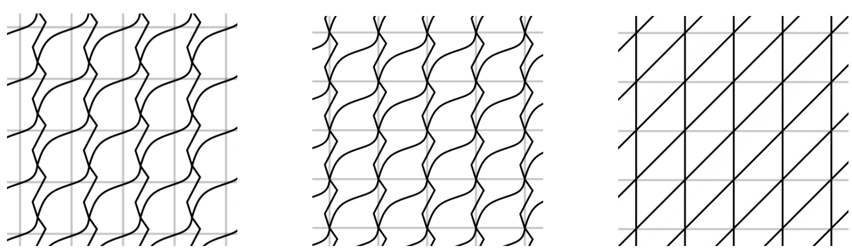

FIGURE 3. The maps F, $\tilde{F}$, and A.

So, by Lemma 3.7, there is an isotopy between A and $\tilde{F}$. Since F was orientation-preserving, so is $\tilde{F}$, and thus so is A, because an orientation-reversing map cannot be isotopic to an orientation-preserving map. Hence, as A is an orientation-preserving linear map, it is stable. Thus $\tilde{F}$ is stable by Lemma 3.6, meaning F is, since all translations are stable.

Since the covering map ε is a smooth covering map (onto $\mathbb{T}^n$ with its *standard* smooth structure), it is a local diffeomorphism. Thus the fact that F is stable says exactly that any local coordinate representation of f is stable, hence f is stable. □

Before the main event, it's worth pausing a moment to consider the key technical machine that makes it possible. To shift the problem to the torus, Kirby immerses a punctured n-torus into $\mathbb{R}^n$. First, a clarification of terminology: what we are looking for is not an embedding, but a smooth map that has invertible differential at every point. So there's no global injectivity requirement, but it needs to be a local diffeomorphism. It is a result of Hirsch [16, Theorem 4.7] that any parallelizable, open, smooth n-manifold can be smoothly immersed in $\mathbb{R}^n$. Many more dimensions are required in general (the Whitney Embedding Theorem requires $2n$), so parallelizability is doing much of the work here. The n-torus is indeed parallelizable (meaning it admits a global frame), just take unit vectors in the counterclockwise direction around each circle. However, the torus is not an open manifold, so we need to consider a punctured torus. We could just remove a single point, but it will be convenient later to have removed a small disc (for concreteness, say the image of $B_\delta(\frac{1}{2}, \dots, \frac{1}{2})$ under the usual covering map for some small $\delta > 0$). So we have a smooth immersion $\alpha : \mathbb{T}^n \setminus D \to \mathbb{R}^n$ (which is also a submersion and local diffeomorphism because they have the same dimension). For explicit constructions of this map, see [21, Essay I, Appendix B], [10], and [36, Lemma 5.6.1].

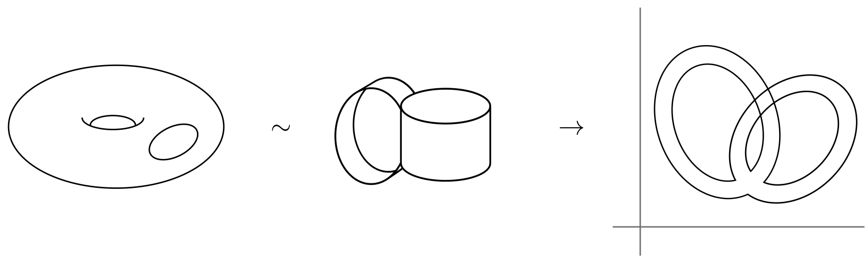

FIGURE 4. Immersing $\mathbb{T}^2 \setminus D$ into $\mathbb{R}^2$.

Before digging into Kirby's fantastic proof of the Stable Homeomorphism Theorem, let's give an outline of the technique. Very broadly, the idea of the torus trick is that instead of solving a difficult problem on $\mathbb{R}^n$, we solve it on the torus, then by immersing the torus into $\mathbb{R}^n$, we somehow translate the problem in $\mathbb{R}^n$ to a problem on the torus. Then we close up the hole on the torus and apply the known solution there. For us, the lemma above takes the role of the known solution on the torus. The argument gets tricky and technical at a few spots, so in somewhat more detail, this is the progression we'll be following:

(1) Immerse a punctured n-torus into $\mathbb{R}^n$.
(2) Apply any (orientation-preserving) homeomorphism of $\mathbb{R}^n$.
(3) Declare this composed map from the punctured torus to $\mathbb{R}^n$ to also be an immersion, and consider the new smooth structure this forces the punctured torus to have.
(4) The homeomorphism of $\mathbb{R}^n$ will be stable iff the identity map between the original punctured torus and the punctured torus with the new smooth structure is stable.
(5) Extend the identity to a homeomorphism of the unpunctured tori (this requires a fair amount of technical fiddling).
(6) Get a PL homeomorphism of the tori.
(7) Use this new stable map together with the fact about all self-homeomorphisms of $\mathbb{T}^n$ being stable to get that the identity map was stable.
(8) Thus the map on $\mathbb{R}^n$ is stable.

Theorem 3.15 (The Stable Homeomorphism Theorem)**.** *Every orientation-preserving automorphism of $\mathbb{R}^n$ is stable for $n \geq 6$.*

Proof. Fix some orientation-preserving homeomorphism f and consider the composition $f\alpha : \mathbb{T}^n \setminus D \to \mathbb{R}^n$, where α is some immersion as discussed above. This generally will not be a smooth immersion, unless f happens to be a diffeomorphism, so we consider a new structure on $\mathbb{T}^n \setminus D$ so that it is. That is, we take $f\alpha$ to define the smooth charts on the punctured torus, which we will call $\tau^n \setminus D$ when it has this smooth structure. We also take the orientation on $\tau^n \setminus D$ to be such that these charts are orientation-preserving (that is, we pull back the orientation by $f\alpha$).

Note that α can be viewed as defining the standard (oriented) smooth structure on $\mathbb{T}^n \setminus D$ in the same way (we can choose α to be orientation-preserving). So, a local representation of the identity map $\mathrm{Id} : \mathbb{T}^n \setminus D \to \tau^n \setminus D$ is $(f\alpha) \circ \mathrm{Id} \circ \sigma$, where σ is a local section of α (which exists since α is a submersion). But $(f\alpha) \circ \mathrm{Id} \circ \sigma = f$, so we can see that f is stable if and only if Id is stable as a map of stable manifolds. As usual, the stable structures are inherited from the (oriented) smooth structures. Note that since f is orientation-preserving, we have that Id is orientation-preserving as well. This is summarized in the following:

$$\begin{tikzcd}
\mathbb{T}^n \setminus D \arrow[r, "\mathrm{Id}"] \arrow[d, "\alpha"'] & \tau^n \setminus D \arrow[d, "f\alpha"] \\
\mathbb{R}^n \arrow[r, "f"'] & \mathbb{R}^n
\end{tikzcd}$$

At this point, we would like to apply what we know about stable homeomorphisms on the torus, but before we can do that, we need to patch up the hole we made. That is, we want to extend the identity to a homeomorphism $\mathbb{T}^n \to \tau^n$, where τ^n is some manifold extending the punctured torus with the new smooth structure. This turns out to be geometrically nontrivial, so we'll go through it step by step, but thankfully, some well-established theorems in geometric topology will handle it pretty easily. Since PL structures induce stable structures just as well as smooth structures but give us a bit more freedom, this is the stage at which we pivot to viewing the torus as a PL manifold.

First, by Theorem 1 in [3], we know that there is a smooth manifold with boundary whose interior is diffeomorphic to $\tau^n \setminus D$. The boundary is evidently homotopy equivalent to a sphere, and has a smooth structure, hence a PL structure. We want to glue a disc along the boundary, but to ensure that what results is a PL manifold, we need to know that the PL structure on the boundary is (isomorphic to) the usual PL sphere. The easiest way to see this is to appeal to the generalized Poincaré conjecture[5], which is known to be true in the PL category for dimension 5 and above [38] [34].

So, we are free to glue a disc (with its usual PL structure) along its boundary to the boundary of our torus, and since these boundaries are PL isomorphic, we get a PL manifold. Call this manifold τ^n. We now wish to extend $\mathrm{Id} : \mathbb{T}^n \setminus D \to \tau^n \setminus D$ to a map between the complete tori. To do this, we apply the Schönflies theorem [4] to $\partial 2D$, that is, the image of $\partial B_{2\delta}(\frac{1}{2}, \ldots, \frac{1}{2})$. This tells us that the region in τ^n bounded by $\partial 2D$ is homeomorphic to an n-ball. The same is true for $\mathbb{T}^n$, so we just need to map these balls to each other to extend our map. This is done by "coning": if one has some homeomorphism $k : S^{n-1} \to S^{n-1}$, then it can be extended to $x \mapsto |x|k(x/|x|)$ on the whole ball (with $0 \mapsto 0$). This gives a homeomorphism of the balls bounded by $\partial 2D$ in the two tori, and we thus get a homeomorphism $h : \mathbb{T}^n \to \tau^n$ extending the identity map (on $\mathbb{T}^n \setminus 2D$).

[5]This where we need $n \geq 6$, although a workaround using a result of Wall [40] and the h-cobordism theorem can make it work for $n = 5$.

Now, let $\eta : \mathbb{T}^n \to \mathbb{T}^n$ be the 2^n-fold covering map given by the squaring map in each component:

$$\eta(z_1, \ldots, z_n) = (z_1^2, \ldots, z_n^2),$$

where here we are considering S^1 as a subset of $\mathbb{C}$. On the torus with its standard PL structure, this is a PL covering map (locally it's just stretching by a factor of two). Let $\hat{\tau}^n$ be the PL covering space of τ^n for which η is a PL covering map. That is, $\hat{\tau}^n$ is a torus with the PL structure such that η is PL. In the same way that we lifted the homeomorphism between tori to a homeomorphism of covering spaces in the proof of Lemma 3.14, we can lift h to a map $\hat{h} : \mathbb{T}^n \to \hat{\tau}^n$. It is a result of Hsiang-Shaneson [17] and Wall [41] (quickly searched for after Kirby's initial announcement) that for $n \geq 5$ there is in fact a PL homeomorphism between $\mathbb{T}^n$ and $\hat{\tau}^n$, call it g. This is all shown in the following diagram, which commutes excluding g.

$$\begin{tikzcd}
\mathbb{T}^n \arrow[r, shift left, "g"] \arrow[r, shift right, "\hat{h}"'] \arrow[d, "\eta"] & \hat{\tau}^n \arrow[d, "\eta"] \\
\mathbb{T}^n \arrow[r, "h"'] & \tau^n
\end{tikzcd}$$

Note that since Id is orientation-preserving, we have that h and $\hat{h}$ are too ($\hat{h}$ because η is orientation-preserving as well). The map g can also be chosen to be orientation-preserving – if it is not, just negate one component of the map. So, since g is PL, it is stable. Now consider the orientation-preserving map $g^{-1}\hat{h} : \mathbb{T}^n \to \mathbb{T}^n$. By Lemma 3.14, this composition is stable, so $\hat{h} = gg^{-1}\hat{h}$, being a composition of stable maps, is stable. Since η is a local PL homeomorphism on both sides of the diagram, this implies h is stable. Restricting h to $\mathbb{T}^n \smallsetminus 2D$, we get that the identity map $\mathbb{T}^n \smallsetminus 2D \to \tau^n \smallsetminus 2D$ is stable. Hence, as discussed above, f is stable, as desired. $\square$

The proof we have given works only in high dimensions, but as mentioned above, the Stable Homeomorphism Theorem is true in all dimensions. Up to dimension 3 was well-known by the time of Kirby's paper, and dimension 4 was proved in [33], so from here on we will drop the dimension condition. We are now ready to achieve one of our major goals.

Theorem 3.16. *The space Top(n) has exactly two path components.*

Proof. Since the spaces of orientation-preserving maps and orientation-reversing maps are homeomorphic, and there are no isotopies between them, this follows from the Stable Homeomorphism Theorem and Lemma 3.2. $\square$

Remark 3.17. The mapping class group of a manifold is defined to be its homeomorphism group modulo isotopy. That is, it is the group of isotopy classes of homeomorphisms. What we have just shown amounts to the fact that the mapping class group of $\mathbb{R}^n$ (and S^n) is $\mathbb{Z}/2\mathbb{Z}$ for all $n > 0$.

The reader may have a few objections or questions at this point. Notably, we haven't shown that STop(n) is actually an open set, so this may only be a path-disconnection (i.e. Top(n) may still be connected). Also, a reasonable response to all this is to question whether it was actually necessary. How do we know that we didn't needlessly prove an overpowered tool for our purpose? The next result, another application by Kirby [20, Theorem 2] of his torus trick, should lay both these concerns to rest.

Theorem 3.18. *$Top(n)$ is locally path-connected.*

Proof. Since it is a topological group, it suffices to prove that the identity map has a path-connected neighborhood. Since stable maps are isotopic to the identity, we need only find a neighborhood of stable maps around the identity. For any compact $C \subset \mathbb{R}^n$ and any $\varepsilon > 0$, we define a set $N(C, \varepsilon)$ as follows. Cover C with finitely many $\frac{\varepsilon}{4}$-balls, call them B_i. Let $2B_i$ be the ball centered at the same point as B_i but with radius $\frac{\varepsilon}{2}$. Then let

$$N(C, \varepsilon) = \bigcap_i V(\overline{B_i}, 2B_i).$$

Certainly $\mathrm{Id} \in N(C, \varepsilon)$, and it is a finite intersection of open sets, hence open. Further, note that for any $f \in N(C, \varepsilon)$, every point in C is moved by f by less than ε. That is, $d(x, f(x)) < \varepsilon$ for all $x \in C$. We will prove that every map in $N(C, \varepsilon)$ is stable.

The idea this time is to immerse tori with different sized holes into $\mathbb{R}^n$, then after applying the homeomorphism, consider that as a map from one punctured torus into another. Then, again, we close up the hole and use what we know about stable maps between tori. The geometric technicalities will take some clearing up, but it won't be too bad with the experience we have.

First, we use Hirsch's immersion α to map $\mathbb{T}^n \setminus D$ into $\mathbb{R}^n$. We will be considering $\alpha(\mathbb{T}^n \setminus nD)$, where nD is just the same disc removed from the torus, but scaled up by a factor of n. Choose ε small enough that every point in $\alpha(\mathbb{T}^n \setminus nD)$ has an ε-ball around it contained in $\alpha(\mathbb{T}^n \setminus (n-1)D)$ for $2 \leq n \leq 4$. That is, such that every point in $\alpha(\mathbb{T}^n \setminus nD)$ can be moved by up to ε without leaving $\alpha(\mathbb{T}^n \setminus (n-1)D)$. This is possible because ∂nD is compact, so we can find some constant-width collar in $\alpha(nD \setminus (n-1)D)$.

Now, fix any $f \in N(C, \varepsilon)$, where C is any compact set containing $\alpha(\mathbb{T}^n \setminus D)$. We then have that $f\alpha(\mathbb{T}^n \setminus 2D) \subset \alpha(\mathbb{T}^n \setminus D)$. We would now like to lift f up to a map on the punctured tori.

This is fairly easily done; since α is an immersion, there is a neighborhood around every point on which it is an embedding. Just restrict to a smaller neighborhood inside each of those such that after mapping down by α then perturbing by f, the image is still in the image of the larger neighborhood. Reduce ε if necessary to make this possible, and to make it so that any two points which α maps to the same place have disjoint such neighborhoods. We can then pull back up by the embedding and thus get a locally defined map $\hat{f} : \mathbb{T}^n \setminus 2D \to \mathbb{T}^n \setminus D$. This glues together properly since we are always just applying α, then f, then an inverse of α, which is always unique on any overlaps.

We can see $\hat{f}$ is an embedding because of the disjointness condition we required, which forces any points that might be sent to the same place to be lifted back to different neighborhoods.

By the same arguments as in the proof of 3.15, we can extend the map $\hat{f}$ to a map $\tilde{f} : \mathbb{T}^n \to \mathbb{T}^n$. More specifically, since $\hat{f}(\partial 3D) \subset 4D \setminus 2D$, we can map $3D$ to the ball bounded by $\hat{f}(\partial 3D)$ and get a homeomorphism $\tilde{f}$. By Lemma 3.14, we know that $\tilde{f}$ is stable, and since it agrees with $\hat{f}$ on an open region (namely,

$\mathbb{T}^n \setminus 4D$), that map is also stable. But then, shifting down by α, which can act as charts since it is locally an embedding, we get that f is stable, as desired. □

Corollary 3.19. *The set of stable maps in Top(n) is open and closed.*

Proof. It is open by the proof of the theorem, but note that it is a subgroup of Top(n), since compositions and inverses of stable maps are stable. So, since multiplication is a homeomorphism, we get that every coset of the set of stable maps is also open. Take the union of all of the other cosets and we get that the complement of the set of stable maps is open. □

Corollary 3.20. *The set of stable maps in Top(n) is exactly the component of the identity.*

Proof. From the previous corollary and the fact that it is path-connected (because all stable maps are isotopic to the identity). □

So indeed, the Stable Homeomorphism Theorem was necessary to our understanding the structure of Top(n). Putting it together, we get the following:

Corollary 3.21. *Top(n) has two components, STop(n) and a space homeomorphic to STop(n), which are also its path components.*

4. Further Directions

Although we now have a good grasp on the basic point-set topology of Top(n), there are still many questions to be asked. The answers to some of these questions are known, but many are not. The remarkable work of Kirby and Siebenmann in the 60s and 70s yields some information, notably that Top$(n) \simeq O(n)$ for $n \leq 3$, but their understanding is far from complete. See [21, Essay 5], particularly Section 5. Another good source summarizing some of this is [13, Chapter 7].

It was actually known before Kirby, thanks to some elaborate surgery theory of Černavskiĭ [46], that Top(n) is locally contractible, as is the space of homeomorphisms for any compact manifold. In the same paper where he proved the Stable Homeomorphism Theorem, Kirby reproved the former result very elegantly.

One of the primary reasons people are interested in Top(n) is for its classifying space, BTop(n). But even less is known about that space than about Top(n). Somewhat surprisingly, the colimit of these spaces, that is,

$$\mathrm{BTop} = \bigcup_n \mathrm{BTop}(n),$$

is significantly better understood. Note that the union here makes sense since we can embed Top(n) into Top$(n+1)$ by just not touching the last component, hence we can map BTop(n) into BTop$(n+1)$. The main source here is [27], and [44] has some more modern results. For instance, the homology of BTop is known, unlike BTop(n). Of course, if one knew the homology of Top(n), a standard Serre spectral sequence argument should do the trick, but that too is difficult.

However, very recently there has been some progress here. In [23], BTop(n) is shown to have finitely generated homotopy and homology groups, and [12] computes the bounded cohomology of Top(n). Regardless, deep understanding of these spaces remains elusive.

Finally, one could reasonably wonder about the algebraic properties of Top(n). These are also difficult; as mentioned in the introduction, it is an open question whether every finitely generated torsion-free group is a subgroup of the homeomorphism group of a manifold. One notable fact is that $\mathrm{Top}(n)$ has a normal subgroup made up of all compactly supported maps. That is, all maps which are the identity except on a compact set. This group is actually significantly better understood, and is known to have zero group homology [30]. The equivalent of this on $\mathrm{Top}_S(n)$ is all maps that are the identity on some neighborhood of the north pole. One incredible recent fact is that any homomorphism from $\mathrm{Top}(n)$ to any separable topological group is *automatically continuous* [28]. There also turns out to be an interesting connection between homeomorphism groups and Lie groups which is discussed well in [29].

Suffice it to say, $\mathrm{Top}(n)$ and automorphism groups of manifolds more generally are objects of remarkable wonder and complexity.

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References

As one of the main goals (and challenges) of this paper was to synthesize the very diffuse literature on this subject, the bibliography is necessarily somewhat bloated. To help the reader navigate it, I'd like to indicate some of the most helpful sources. I have tried to do so in each section, but here I'll list the best place to start for each topic we have discussed.

First and foremost, the material on the Stable Homeomorphism Theorem came primarily from Kirby's classic paper [20], in which he discusses and proves the key results. For general discussions of the basic properties of topological and smooth manifolds, I am very fond of Lee's textbooks [24] and [25]. For the PL category, Rourke and Sanderson's introduction [34] is quite good. For background on some of the facts from surgery theory that we used, the recent book by Chang and Weinberger [7] is excellent.

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