BERS' PROOF OF NIELSON-THURSTON CLASSIFICATION

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ABSTRACT. The Nielson-Thurston Classification of mapping class groups states that we can classify the self-homeomorphisms of a surface, which are abstract and random, into three basic types that are concrete and easier to understand. Instead of presenting Thurston's original approach to prove this classification, which includes a zillion of beautiful and insightful ideas that take up an entire book to elucidate (see [11]), we present Bers' proof, which adopts a completely different approach that utilizes more Teichmüller theory.

CONTENTS

1. OVERVIEW

The aim of this paper is to prove the following theorem:

Theorem 1.1. ***(Nielson-Thurston Classification).*** *Let S be a surface of genus $g \geq 2$. Each $[f] \in Mod(S)$ has a representative f of one of the following three types: Periodic, Reducible, Pseudo-Anosov. In particular, if $[f]$ is not periodic, then it is either reducible or pseudo-Anosov, but not both.*

Date: August 20th, 2025.

This classification was first proposed by Jakob Nielson in 1944 in his paper [13] and was 'completed' by William Thurston in his famous paper [12]. To prove this theorem, Thurston revolutionized the study of Geometry and Topology of manifolds by developing a zillion of new tools and ideas which influenced the development of mathematics in the late 20th century. In 1978, a few years after Thurston proposed his idea, Lipman Bers gave a different proof of the Nielson-Thurston Classification, which used more Teichmüller theory and is analogous to the proof of the classification of elements of $Isom^+(\mathbb{H}^2)$ by their translation lengths.

The structure of this paper is designed as follows. In the first part, we will present some basic facts in the theory of mapping class groups which include a detailed discussion of the three different types of mapping classes we are interested in. In the second part, we will present some results in Teichmüller theory that are used in Bers' proof of Nielson-Thurston Classification. In the third part, we present Bers' proof of that theorem.

2. Definitions and Conventions

2.1. **Conventions and basic hyperbolic geometry.** We will denote by $S_{g,n,b}$ a surface of genus g with n punctures and b boundaries. When g, n, b are not specified, we will just use S to denote an arbitrary surface. Recall that the *Euler characterisstic* of a surface S is

$$\chi(S) = 2 - 2g - (b + n).$$

For simplicity, we assume $\chi(S) < 0$ for all surfaces S in this paper.

We use $i(\alpha, \beta)$ to denote the geometric intersection number between two isotopy classes of simple closed curves in the given surface where the geometric intersection number is defined to be the minimal number of intersection points (no orientation is involved) between a representative of α and a representative of β.

There is a basic fact from hyperbolic geometry which states that if closed surface S admits a hyperbolic metric (the sectional curvature is constantly negative), then there exists a unique geodesic representative for every isotopy class of simple closed curves which minimizes the length with respect to the given hyperbolic metric. For a proof, see Proposition 1.3 in [1]. In this paper, we use $\ell_X(c)$ to denote the length of the geodesic representative of a class of simple closed curves in X with respect to the hyperbolic metric on X. We use $\ell(X)$ to denote $min\{\ell_X(c)\}$ where c is in the collection of closed curves in X.

All hyperbolic metrics considered in this paper are assumed to be complete and finite-area.

Now let's move on to the *mapping class groups.*

2.2. **Mapping class groups.**

Definition 2.1. The mapping class group of a surface S, denoted $\mathrm{Mod}(S)$, is the group of isotopy classes of homeomorphisms fixing the boundary, denoted $Homeo^+(S, \partial S)$.

Elements of $\mathrm{Mod}(S)$ are called *mapping classes.* A simple but essential family of *mapping classes* are the *Dehn twists.*

Example 2.2. (Dehn Twists) Consider the annulus $A = S^1 \times [0, 1]$. Let $T : A \to A$ be the twist of A given by the formula

$$T(\theta, t) = (\theta + 2\pi t, t).$$

Now, let S be an arbitrary surface and let α be a simple closed curve in S. Let N be a regular neighborhood of α and choose an orientation-preserving homeomorphism $\phi : A \to N$. Composing T with ϕ and extending it by the identity outside the regular neighborhood N, we obtain a homeomorphism $T_\alpha : S \to S$, called the *Dehn twist about* α.

We now describe three types of mapping classes in $\mathrm{Mod}(S)$, namely, the periodic elements, the reducible elements, and the pseudo-Anosov elements. Nilson-Thurston theorem states that every element in $\mathrm{Mod}(S)$ is one the three types. In particular, if $[f]$ is not periodic, then it is either reducible or pseudo-Anosov, and it cannot be both.

2.2.1. *periodic elements.* We say that an element $[f] \in \mathrm{Mod}(S)$ is periodic if it is finite-order. In other words, there exists $n \in \mathbb{N}$ such that $[f^n]$ is isotopic to identity.

2.2.2. *reducible elements.* We say an element $[f] \in \mathrm{Mod}(S)$ is reducible if there is a nonempty set $\{c_1, c_2, \cdots, c_n\}$ of isotopy classes of essential simple closed curves in S so that $i(c_i, c_j) = 0$ for all i and j and so that $\{f(c_i)\} = \{c_i\}$. The collection is called a reduction system for $[f]$. Note that it is possible for an element $[f]$ to be both reducible and periodic.

2.2.3. *pseudo-Anosov elements.* Before we give the definition of pseudo-Anosov mapping classes, we first give the definition of another important concept, the measured foliations.

A singular foliation $\mathcal{F}$ is a decomposition of S into a disjoint union of subsets of S, called the *leaves* of $\mathcal{F}$, and a finite set of points of S, called the singular points such that the following condition holds:

(1) For each nonsingular point $p \in S$, there is a smooth chart from an open set in $\mathbb{R}^2$ that takes the leaves to horizontal lines. The transition maps between any two of these charts are smooth maps that take horizontal lines to horizontal lines
(2) For each singular point $p \in S$, there is a smooth chart from a neighborhood of p to $\mathbb{R}^2$ that takes leaves to the level sets of a k-proned saddle, $k \geq 3$.

A measured foliation $(\mathcal{F}, \mu)$ is a singular foliation $\mathcal{F}$ on S with a transverse measure μ , which, intuitively, is a length function that assigns to each arc that is transverse to the foliations and starts and ends on the leaves a positive real number so that the value is invariant under leaf-preserving isotopy of the arc. In other words, we are free to 'move' the arc along the leaves 'isotopically' without changing its 'length' with respect to the measure.

We say that two measured foliations are *transverse* if their leaves are transverse away from the singularities.

There is a natural action of Homeo(S) on the set of measured foliations of S given by

$$\phi \cdot (\mathcal{F}, \mu) = (\phi(\mathcal{F}), \phi_* \mu),$$

where $\phi_*\mu(\gamma)$ is defined as $\mu(\phi^{-1}(\gamma))$ for an arc γ transverse to $\phi(\mathcal{F})$

We say that an element $[f] \in \mathrm{Mod}(S)$ is pseudo-Anosov if there is a pair of transverse measured foliations $(\mathcal{F}^u, \mu_u)$ and $(\mathcal{F}^s, \mu_s)$ on S, a number $\lambda > 1$ and a representative homeomorphism f of $[f]$ so that

$$\phi \cdot (\mathcal{F}^u, \mu_u) = (\mathcal{F}^u, \lambda \mu_u) \text{ and } \phi \cdot (\mathcal{F}^s, \mu_s) = (\mathcal{F}^s, \tfrac{1}{\lambda} \mu_u).$$

Intuitively, one of the two transverse foliations is stretched by f and the other is shrunk by f.

The study of mapping class groups is closely related to Teichmüller Theory. In particular, Bers' proof of Nielson-Thurston Classification utilizes several classical results in Teichmüller Theory which we will discuss in the next section.

3. Teichmüller Theory

3.1. **Teichmüller Space.** To prove the Nielson-Thurston classification for $\mathrm{Mod}(S)$, we consider its action on another space associated to S, the Teichmüller space $\mathrm{Teich}(S)$.

A hyperbolic structure on a surface S is a homeomorphism $\sigma : S \to X$, where X is a surface with a complete finite-area hyperbolic metric with totally geodesic boundary. σ and X endow the surface S a pull-back metric. We record this hyperbolic structure on S by the pair (X, σ). Two hyperbolic structures (X_1, σ_1) (X_2, σ_2) are said to be *homotopic* if there is an isometry $I : X_1 \to X_2$ such that $I \circ \sigma_1 : S \to X_2$ and $\sigma_2 : S \to X_2$ are homotopic. One definition of the Teichmüller space of S is the following.

Definition 3.1. Let $S_{g,n,b}$ be a surface. Then the *Teichmüller space* of $S_{g,n,b}$ is defined as $\mathrm{Teich}(S_{g,n,b})$={hyperbolic structures on S}/homotopy.

Another definition of Teichmüller space is by Riemann surface structures. A Riemann surface structure on S is a homeomorphism $\sigma : S \to X$ where X is a Riemann surface. This homeomorphism gives a Riemann surface structure on S. We say that two Riemann surface structures on S, (X_1, σ_1), (X_1, σ_2) are conformal if there exists a biholomorphism $I : X_1 \to X_2$ such that $I \circ \sigma_1 : S \to X_2$ and $\sigma_2 : S \to X_2$ are homotopic.

Definition 3.2. (Another definition of Teichmüller space). Let $S_{g,n,b}$ be a surface. Then the *Teichmüller space* of $S_{g,n,b}$ is defined as $\mathrm{Teich}(S_{g,n,b})$:={Riemann surface structures on S}/conformal structures.

The equivalence of these two definitions is derived from the uniformization theorem. For details, see [5].

Theorem 3.3. ***(Fricke's theorem).*** *For $g \geq 2$, we have*

$$Teich(S_g) \cong \mathbb{R}^{6g-6}$$

For a detailed proof and discussion of the previous result, see [1] pages 275-284.

3.2. **Quasiconformal maps.** Let $f : X \to Y$ be a homeomorphism between Riemann surfaces that is smooth outside a finite number of points. Assume also that f^{-1} is smooth outside a finite number of points. We consider f's behavior on local charts $f : U \to V$ where U and V are open subsets of $\mathbb{C}$.

Using the usual notation for maps $\mathbb{R}^2 \to \mathbb{R}^2$, we can write $f(x, y) = (a(x, y), b(x, y))$, where $a, b : \mathbb{R}^2 \to \mathbb{R}$. The derivative df is then the real linear map

$$df = \begin{pmatrix} a_x & a_y \\ b_x & b_y \end{pmatrix}.$$

We can also write $df = f_x dx + f_y dy$, where $f_x = (a_x, b_x)$ and $f_y = (a_y, b_y)$.

Switching to the complex notation and writing $z = x + iy$, we have $f_x = a_x + ib_x$, $f_y = a_x + ib_y$. We can rewrite

$$df = f_z dz + f_{\bar{z}} d\bar{z},$$

where

$$f_z = \frac{1}{2}(f_x - if_y)$$
$$f_{\bar{z}} = \frac{1}{2}(f_x + if_y).$$

We define μ_f of f at p to be $\mu_f(p) = f_{\bar{z}}(p)/f_z(p)$.

Notice that the condition that $f_{\bar{z}} \equiv 0$ on U is equivalent to f to be holomorphic on U. Also, since

$$|f_z|^2 - |f_{\bar{z}}|^2 = a_x b_y - a_y b_x,$$

we see that f is orientation-preserving if and only if $|f_z| > |f_{\bar{z}}|$, which is the same as $|\mu_f| < 1$.

We now define the dilatation of f at p to be

$$K_f(p) = \frac{1 + |\mu_f(p)|}{1 - |\mu_f(p)|}.$$

$K_f(p)$ has a geometric interpretation as follows. The map df_p takes the unit circle in TU_p to an ellipse E in $TV_{f(p)}$, and $K_f(p)$ is the ratio of the length of the major axis of E to the length of the minor axis of E. For a detailed explanation, see [1] page 296.

The dilatation of the map f is defined to be the supremum of $K_f(p)$ for all points $p \in X$ where f is differentiable. If $K_f < \infty$, then we say f is K_f-quasiconformal or simply quasiconformal.

We have the following lemma

Lemma 3.4. *Let $f : X \to Y$ be a homeomorphism between Riemann surfaces. Then f is $1-$quasiconformal if and only if f is a conformal map or, equivalently, a biholomorphism.*

For a proof of this lemma, see Lemma 11.1 in [1].

Quasiconformal maps behave well under composition. We have the following fact.

Lemma 3.5. *Let f and g be two quasicomformal maps $\mathbb{C} \to \mathbb{C}$. We have*

$$K_{f \circ g} \leq K_f K_g,$$

with equality if and only if arg(μ_f)=arg(μ_g) or one of μ_f and μ_g is 0.

The proof of the previous lemma can be found in [6] section 1.2. We can generalize the previous lemma to maps between Riemann surfaces by the following proposition:

Proposition 3.6. *Suppose $f,\ g : X \to Y$ are quasiconformal homeomorphisms between Riemann surfaces. We have:*

(1) $f \circ g$ is quasiconformal and

$$K_{f \circ g} \leq K_f K_g.$$

(2) The inverse f^{-1} is quasiconformal and

$$K_{f^{-1}} = K_f.$$

(3) If g is conformal, then

$$K_{f\circ g} = K_f K_g$$

A natural question to ask is what happens if we restrict to a mapping class of homeomorphism rather than a particular homeomorphism. There will be an infimum of K_f among all representatives f of the mapping class $[f]$. Will that infimum be realized? If so, is that representative unique? These problems are sometimes referred to as Teichmüller's extremal problems. The answer to both questions is yes. Before we give a reinterpretation of this result, we first discuss a crucial construction in Teichmüller theory–the quadratic differentials.

3.3. **Quadratic differentials.** Let $\{z_\alpha : U \to \mathbb{C}\}$ be an atlas for a Riemann surface X. A holomorphic quadratic differential q on X is specified by a collection of expressions $\{\phi_\alpha(z_\alpha)dz_\alpha^2\}$ with the following properties:

(1) Each $\phi_\alpha : z_\alpha(U_\alpha) \to \mathbb{C}$ is a holomorphic function with a finite set of zeros
(2) For any two coordinate charts z_α and z_β, we have

$$\phi_\beta(z_\beta)(\frac{dz_\beta}{dz_\alpha})^2 = \phi_\alpha(z_\alpha)$$

The second condition can be rephrased as the collection $\{\phi_\alpha(z_\alpha)dz_\alpha^2\}$ is invariant under change of local coordinates. In particular, the order of zeros of ϕ_α and ϕ_β agrees at each point where they are defined.

Notice that q at each point z is in fact a map from T_zX to $\mathbb{C}$. Suppose q is given by $\phi_\alpha(z)dz^2$. Let $v \in T_zX$ and $dz(v) = \alpha$ for some $\alpha \in \mathbb{C}$. Then we have $q(v) = \phi_\alpha(z)\alpha^2$. The condition (2) above in fact ensures that there is a well-defined $q(v)$ in $\mathbb{C}$ that is independent of the choice of local charts for q.

Let q be a holomorphic quadratic differential on a compact Riemann surface X. It is a nontrivial fact that we can find local coordinates called natural coordinates around each point $p \in X$ so that in these local coordinates we have $q(z) = z^k dz^2$ for some $k \geq 0$ where k is the order of zero of q at p. A detailed discussion of this fact can be found in [1] pg 311. We will just assume it without proving it.

Given a holomorphic quadratic differential q on a Riemann surface X, we can obtain a foliation by taking the set of smooth paths in X whose tangent vectors at each point evaluate to positive real numbers under q. This foliation is called the horizontal foliation for q. If we take the paths in X whose tangent vectors evaluate to negative real numbers under q, the resulting foliation is called the vertical foliation for q.

The measure associated with the horizontal foliation obtained by q is defined as follows. Suppose that q is given by $\phi(z)dz^2$ within a given chart. Let α be a smooth arc. Then the transverse measure for the horizontal foliation is given by

$$\mu(\alpha) = \int_\alpha |Im(\sqrt{\phi(z)}dz)|.$$

By taking the real parts instead of the imaginary parts, we obtain a transverse measure on the vertical foliation. These definitions make sense since for the following reason. If in local coordinates the quadratic differential is given by dz^2, then

$$q(v) = \alpha^2.$$

If α^2 is a positive real number, then α is a positive real number, which implies that the direction of the tangent vector v is precisely the direction of the horizontal lines in $\mathbb{R}^2$. Similarly, if α^2 is a negative real number, then α is pure imaginary. This implies that v is in the vertical direction. Moreover, if $q = dz^2$, then

$$\mu(\alpha) = \int_\alpha |Im(\sqrt{\phi(z)}dz)|$$

is in fact measuring the total variation of α in the vertical direction. In other words, in local charts, it coincides with the measure $|dy|$.

If z_0 is a zero of the quadratic differential q, then $q(v) = z_0^k\alpha^2$ where $dz(v) = \alpha$. The tangential vectors v for the paths in the horizontal foliation is given by the $\alpha^{k+2} \in \mathbb{R}$. Scaling by a factor in $\mathbb{R}$, we have $\alpha^{k+2} = 1$, which shows that the horizontal foliation has the form of a $(k+2)-$pronged singular point.

3.4. **Teichmüller Theorem and Teichmüller metric on Teich(S).** At the end of section 3.3, we asked the following Teichmüller's extremal problem: Fix a homeomorphism $f : X \to Y$ of Riemann surfaces and consinder the set of dilatations of quasiconformal homeomorphisms $X \to Y$ in the homotopy class of f. Is the infimum of this set realized? If so, is the minimizing map unique? The answer to both questions is yes. A reformulation of this nontrivial result is the uniqueness and existence of Teichmüller mappings. We say a homeomorphism $f : X \to Y$ is a Teichmüller mapping if there are holomorphic quadratic differentials q_X and q_Y on X and Y and a positive real number K so that the following two conditions hold:

(1) The homeomorphism f takes the zeros of q_X to the zeros of q_Y
(2) If $p \in X$ is not a zero of q_X, then with respect to a set of natural coordinates for q_X and q_Y based at p and $f(p)$, the homeomorphism f can be written as
$$f(x+iy) = \sqrt{K}x + i\frac{1}{\sqrt{K}}y.$$

Since $f_z = \frac{K+1}{2\sqrt{K}}$ and $f_{\bar{z}} = \frac{K-1}{2\sqrt{K}}$, the dilatation of f is

$$K_f = \begin{cases} K & \text{if } K \geq 1, \\ \frac{1}{K} & \text{if } K \leq 1. \end{cases}$$

We describe f by saying that it has q_X as its initial differential and q_Y as its terminal differential. Both of these differentials give rise to measured foliations on X and Y respectively as described in section 3.3. Intuitively, the Teichmüller mapping f maps the horizontal (resp. vertical) foliation of q_X to the horizontal foliation (resp. vertical) and 'stretches' the horizontal and 'shrinks' the vertical foliations by a factor of $\sqrt{K}$.

Teichmüller's extremal problem can be answered by the following two theorems whose proofs will be omitted.

Theorem 3.7. ***(Teichmüller's existence theorem)****. Let X and Y be closed Riemann surfaces of genus $g \geq 2$ and let $f : X \to Y$ be a homeomorphism. Then, there exists a Teichmüller mapping $h : X \to Y$ homotopic to f*

Theorem 3.8. ***(Teichmüller's uniqueness theorem)****. Let X and Y be closed Riemann surfaces of genus $g \geq 2$ and let $h : X \to Y$ be a Teichmüller mapping. Then, for every quasiconformal homeomorphism $f : X \to Y$ homotopic to h, we have*

$$K_f \geq K_h.$$

Equality holds if and only if $h = f$.

The proof of these two theorems was first given in [8] and [9]; see also [3].

With these two theorems, we can describe a metric on the Teichmüller space of S. Let (X, σ), (Y, τ) be two points in $Teich(S)$. Then there is a map $f = \tau \circ \sigma^{-1}$. Let h be the Teichmüller mapping in the mapping class $[f]$ whose existence is guaranteed by Teichmüller's existence theorem and whose uniqueness is guaranteed by Teichmüller's uniqueness theorem. We define the Teichmüller distance between (X, σ) and (Y, τ) to be

$$d((X, \sigma), (Y, \tau)) = \frac{1}{2} log(K_h).$$

3.5. **Moduli space as the quotient of Teichmüller space.** In this subsection, we introduce the moduli space of a surface S as the quotient of its Teichmüller space by the action of its mapping class group. We will present two classical results: Proper discontinuity of the action of Mod(S) on Teich(S), and Mumford's compactness criterion.

Let (X, σ) be a point in Teich(S). The mapping class group Mod(S) acts on Teich(S) as follows. Suppose $[f] \in$Mod(S). Choose a representative f of $[f]$ and set

$$f \cdot (X, \sigma) = (X, \sigma \circ f).$$

Notice that $f \cdot (X, \sigma)$ is well-defined since for another representative g of $[f]$, we have that $(X, \sigma \circ f)$ and $(X, \sigma \circ g)$ correspond to the same point in Teich(S) by the equivalence relation in the definition of Teichmüller spaces.

The moduli space of hyperbolic surfaces that are homeomorphic to S is defined to be the quotient space

$$\mathcal{M}(S) = Teich(S)/Mod(S).$$

Intuitively, by modding out the action of mapping class group on Teich(S), we consider only the hyperbolic structures of a surface without considering the marking.

Definition 3.9. Let $[f] \in$Mod(S). We define $\tau(f)$=$inf\{d((X, \sigma), f \cdot (X, \sigma))\}$ for $(X, \sigma) \in$Teich(S).

Recall that in hyperbolic geometry, we classify the group of isometries of $\mathbb{H}^2$ by considering whether the 'translation length' is obtained. Similarly, we classify Mod(S) with the same strategy.

Definition 3.10. Let $[f] \in$ Mod(S). We have the following classification

(1) $[f]$ is hyperbolic if $\tau(f) > 0$ and $\exists (X, \sigma) \in$Teich(S) such that $d((X, \sigma), f \cdot (X, \sigma)) = \tau(f)$
(2) $[f]$ is elliptic if $\tau(f) = 0$ and $\exists (X, \sigma) \in$Teich(S) such that $d((X, \sigma), f \cdot (X, \sigma)) = 0$
(3) $[f]$ is parabolic if $\tau(f)$ is not obtained.

It is an important result that the mapping class group acts properly discontinuously on Teich(S). Here being properly discontinuous is defined as follows. Let G be a group that acts on a topological space X by homeomorphisms. Then we say that the action is properly discontinuous if, for any compact $B \subset X$, the set

$$\{g \in G : g \cdot B \cap B \neq \emptyset\}$$

is finite. To prove the proper discontinuity of the action of Mod(S), we prove the following two lemmas.

Given a hyperbolic surface X homeomorphic to S, we can consider the spectrum of lengths of geodesic representatives of isotopy class of simple closed curves which is defined to be the set

$$rls(X) = \{\ell_X(c) \subset \mathbb{R}_+\}$$

where c are geodesic representatives of isotopy class of simple closed curves in X (they exist and are unique by discussion in 2.1).

Lemma 3.11. ***(Discreteness of the length spectrum)****. Let X be any closed hyperbolic surface. The set rls(X) is a closed discrete subset of $\mathbb{R}$. Furthermore, for each $L \in \mathbb{R}$, the set $\{c$: c an isotopy class of simple closed curves in X with $\ell_X(c) \leq L\}$ is finite*

Proof. see Lemma 12.4 in [1] □

Lemma 3.12. ***(Wolpert's lemma)****. Let X_1, X_2 be hyperbolic surfaces and let $f : X_1 \to X_2$ be a K-quasiconformal map. For any isotopy class c of simple closed curves in X_1, the following inequality holds:*

$$\frac{\ell_{X_1}(c)}{K} \leq \ell_{X_2}(f(c)) \leq K\ell_{X_1}(c)$$

Proof. see Lemma 12.5 in [1] □

Theorem 3.13. *Let $g \geq 1$. The action of Mod(S_g) on Teich(S_g) is properly discontinous.*

Proof. Let K be a compact set in Teich(S_g). We need to show that the set of $f \in$ Mod(S_g) such that $((f \cdot K) \cap K \neq \emptyset)$ is finite. Let (X, σ) be an arbitrary point of K and D a real number such that the ball centered on (X, σ) with radius D contains K. Choose c_1 c_2 to be two isotopy classes of simple closed curves in S_g that fill S (this means the complement of these two curves is a disjoint union of disks or punctured disks). For a proof of the existence of c_1, c_2, see Proposition 3.5 in [1]. Let $L = max\{\ell_X(c_1), \ell_X(c_2)\}$. Suppose $f \in$Mod(S) and $(f \cdot K) \cap K \neq \emptyset$. It follows that $d((X, \sigma), f \cdot (X, \sigma)) \leq 2D$. By Lemma 3.12, $\ell_{f \cdot X}(c_i) \leq e^{4D}L$. Since $\ell_{f \cdot X}(c_i) = \ell_X(f^{-1}(c_i))$, $\ell_X(f^{-1}(c_i)) \leq e^{4D}L$. By Lemma 3.11, there are finitely many isotopy classes of simple closed curves b in S_g so that $\ell_X(b) \leq e^{4D}L$. Thus, there are only finitely many possibilities for $f^{-1}(c_1)$ and $f^{-1}(c_2)$. Since we know that c_i fills S, then by Alexandar's method (see Lemma 4.8 and Corollary 4.9 in this paper), there are only finitely many choices for f^{-1} once $f^{-1}(c_i)$ are determined. Thus there are finitely many possibilities for f that satisfy $(f \cdot K) \cap K \neq \emptyset$. □

Theorem 3.13 extends to the case of $S = S_{g,n}$ where $\chi(S) \leq 0$. An immediate corollary of Theorem 3.11 is that the Teichmüller metric on Teich(S) induces a metric on $\mathcal{M}(S)$. In other words, the induced pseudometric on $\mathcal{M}(S)$ is in fact a metric. This is because if we set the induced metric on $\mathcal{M}(S)$ to be the infimum of distance between any two representatives, then by the proper discontinuity of the action, we have the additional property that two orbits have distance 0 if and only if they are equal.

The remaining part of this section is about another important fact in the theory of moduli spaces–Mumford's compactness criterion.

A natural question arises after we define the moduli space—Is $\mathcal{M}(S)$ compact? The answer to the previous question is no. However, there exists an exhaustion of $\mathcal{M}(S)$ by compact sets.

Let $\mathcal{X} \in \mathcal{M}(S)$. Recall that we defined $\ell(X) = min_{c \in X}\{\ell_X(c)\}$. The $\epsilon-$thick part of $\mathcal{M}(S)$ is the set

$$\mathcal{M}_\epsilon(S) = \{\mathcal{X} \in \mathcal{M}(S) : \ell(X) \geq \epsilon\}.$$

Mumford proves the following theorem:

Theorem 3.14. ***(Mumford's compactness criterion)****. Let $g \geq 1$. For each $\epsilon > 0$, the space $\mathcal{M}_\epsilon(S_g)$ is compact.*

Proof. See [10]. □

A generalization for $S = S_{g,n}$ is given by Bers in [4].
We can finally move on to Bers' proof.

4. Bers' proof

In this section, we will explain in detail Bers' proof of the Nielson-Thurston classification. The main reference for this section is Bers' original paper [2] and Chapter 13 of [1]. The surface S treated in this section is assumed to be $S_{g,n}$ with no boundary.

4.1. Step 1: Elliptic and periodic are the same.

In this section, we prove the following theorem:

Theorem 4.1. *Let $[f] \in Mod(S)$. $[f]$ is periodic if and only if it induces an elliptic action on Teich(S_g), or equivalently, $\exists(X, \sigma)$ such that $d((X,\sigma), f \cdot (X,\sigma)) = 0$.*

Proof. ($\Rightarrow$) Let f_0 be a periodic mapping. We have f_0^n is isotopic to identity for some n. Let ds be a Riemanninan metric on S. Consider the new metric

$$ds_0 = ds + f_0^\star(ds) + \cdots + (f_0^\star)^{n-1}(ds).$$

We have f induces a conformal map on S with this metric. Let the point (X, σ) denote the point in Teich(S) corresponding to this hyperbolic metric. Since f is conformal, $d((X,\sigma), f \cdot (X,\sigma)) = 0$.

($\Leftarrow$) Let $\sigma \circ f \circ \sigma^{-1} : X \to X$ be the Teichmüller map from X onto itself. If $d((X,\sigma), f \cdot (X,\sigma)) = 0$, then $\sigma \circ f \circ \sigma^{-1}$ is conformal. We still use f to denote $\sigma \circ f \circ \sigma^{-1}$ for simplicity. If f is conformal, then for any isotopy class of simple closed curve $[c]$, we have $\ell_X(c) = \ell_X(f(c))$. By the discreteness of the length spectrum of X, there is only a finite number of isotopy classes of simple closed curves $[\gamma]$ in X with $\ell_X(\gamma) = \ell_X(c)$. Therefore, $f^n([c]) = [c]$ for some n. We can choose a finite set of simple closed curves $\{c_i\}$ such that

(1) The c_i are pairwise in minimal position.
(2) The c_i are pairwise nonisotopic.
(3) For distinct i, j, k, at least one of $c_i \cap c_j$, $c_i \cap c_k$, $c_k \cap c_j$ is empty.
(4) $\{[c_1]\} = \{[f(c_1)]\}$
(5) The complement of $\cup c_i$ in X is disjoint union of punctured disks or disks

Then by Alexander method (see Lemma 4.8, Corollary 4.9), f^n is isotopic to identity. □

4.2. **Step 2: The irreducible case.** In this section, we will prove the following result.

Theorem 4.2. *If $[f] \in Mod(S)$ is an irreducible element, then its action on Teich(S) can only be elliptic or hyperbolic.*

To prove this theorem, we need the following lemma.

Lemma 4.3. ***(Collar lemma)*** *Let γ be a simple closed geodesic on a hyperbolic surface X. Then $N_\gamma = \{x \in X : d(x,\gamma) \leq w\}$ is an embedded annulus, where w is given by*

$$w = \sinh^{-1}\left(\frac{1}{\sinh(\frac{1}{2}\ell(\gamma))}\right).$$

Proof. see [1] Lemma 13.6. □

Intuitively, this lemma says that the shorter a simple closed geodesic curve is, the longer a new simple closed geodesic needs to be to intersect it. This is because as $\ell(\gamma)$ decreases, $\sinh(\ell(\gamma)))$ decreases, $\frac{1}{\sinh(\frac{1}{2}\ell(\gamma))}$ increases, and $\sinh^{-1}(\frac{1}{\sinh(\frac{1}{2}\ell(\gamma))})$ increases. Thus, the embedded annulus has a greater height. If a curve c intersects γ, it has to pass through the two boundaries. If not, then there are two situations. First, it lives in the annulus, which means that it is in the same homotopy class with γ for there is only one homotopy class of simple closed curve in the embedded annulus. Since there is only one geodesic representative for any homotopy class of simple closed curve, $\gamma = c$. Second, it intersects one boundary at least twice and does not intersect the other boundary. In this case, it forms a 'bigon' with the boundary it intersects, which contradicts the fact c is a geodesic since a geodesic is distance minimizing. Therefore, c has to be long enough to intersect γ.

What we need from collar lemma is much weaker than the statement.

Corollary 4.4. *Let S be a topological surface, There is a constant $\delta(S)$ associated to S with the following property: Let g be any hyperbolic metric on S and c_1, c_2 be two closed geodesics in arbitrary two distinct homotopy classes with lengths smaller than $\delta(S)$; lemma 4.3 implies that c_1 and c_2 are disjoint.*

Proof. Any sufficiently small $\delta(S)$ would work. □

The following lemma combines the previous corollary and Wolpert's inequality (Lemma 3.12) together and tells us that given a Riemann surface X an irreducible self-mapping $[f]$, the length of every isotopy class of simple closed curve is bounded below by $(\frac{1}{K_X(f)})^{3g-3+n}\delta(S)$.

Lemma 4.5. *Let (X, σ) be a point in Teich(S), f be an irreducible self-mapping of X. Then given any simple closed geodesic γ in X, we have*

$$\ell_X(\gamma) \geq \left(\frac{1}{K_X(f)}\right)^{3g-3+n}\delta(S).$$

Proof. We prove this lemma by contradiction. Suppose there exists geodesic γ in X such that $\ell_X(\gamma) < (\frac{1}{K_X(f)})^{3g-3+n}\delta(S)$. Then we consider the sequence of closed geodesics $\{\gamma_i = f^i(\gamma)\}$ in X. Since $\ell_X(\gamma) < (\frac{1}{K_X(f)})^{3g-3+n}\delta(S)$, by Lemma 3.12, $\ell_X(\gamma_i) = \ell_X(f(\gamma_{i-1})) < K_X(f)\ell_X(\gamma_{i-1})$. Inductively,

$$\ell_X(\gamma_i) < \left(\frac{1}{K_X(f)}\right)^{3g-3+n-i}\delta(S). \tag{4.6}$$

Since $K_X(f) \geq 1$, from (4.6), we have for $0 \leq i \leq 3g-3+n$,

$$\ell_X(\gamma_i) < \delta(S).$$

By Corollary 4.4, we have γ_i are pairwise disjoint for $0 \leq i \leq 3g-3+n$. However, there are at most $3g-3+n$ isotopy classes of pairwise disjoint essential simple closed curves in X (cf. Section 8.3 [1]). Thus, two of the γ_i are in the same homotopy class. It follows that f permutes γ_i which implies that it is reducible. Contradict! □

We now can prove Theorem 4.2.

Proof. Suppose $\{(X_i, \sigma_i)\}$ is a sequence of points in Teich(S) where

$$\lim_{i\to\infty} d((X_i,\sigma_i), f\cdot(X_i,\sigma_i)) = \tau(f).$$

Recall that

$$d((X_i,\sigma_i), f\cdot(X_i,\sigma_i)) = \frac{1}{2}log(K_X(\sigma_i \circ f \circ \sigma_i^{-1}))$$

where $\sigma_i \circ f \circ \sigma_i$ are chosen to be the Teichmüller maps. Abuse the notation, we can choose a subsequence of (X_i, σ_i) such that $d((X_i,\sigma_i), f\cdot(X_i,\sigma_i))$ is bounded above by some constant A and still denote them by (X_i, σ_i). Thus, by Lemma 4.5, we have $\ell(X_i) \geq A'$ for some constant A'.

Projecting (X_i, σ_i) down to the Moduli space $\mathcal{M}(S)$, we obtain a sequence $\{\mathcal{X}_i\}$ in $\mathcal{M}(S)$. Notice that since $\ell(X_i) \geq A'$, we have $\ell(\mathcal{X}_i) \geq A'$. By Mumford's Compactness Criterion 3.14, this sequence lies in a compact set K. Thus, a subsequence of it converges. We still use $\{\mathcal{X}_i\}$ to denote this converging subsequence.

Take the one componenet $\bar{K}$ of the preimage of the compact set in Teich(S) and consider the preimage of $\{\mathcal{X}_i\}$ in $\bar{K}$. We denote this sequence by (X_i, τ_i). Notice that each (X_i, τ_i) differs from (X_i, σ_i) by an element in Mod(S). In other words, there exists χ_i such that $\tau_i = \sigma_i \circ \chi_i$. Moreover, since χ_i acts by isometries on Teich(S),

$$d((X_i,\sigma_i), f\cdot(X_i,\sigma_i)) = d((X_i,\tau_i), \chi_i \circ f\chi_i^{-1}\cdot(X_i,\tau_i))$$

We know $\{(X_i, \tau_i)\}$ converges since it lies in a compact set. Let

$$\lim_{i\to\infty}(X_i,\tau_i) = (X,\tau).$$

We claim that there is an $N > 0$ such that for $i \geq N$

$$d((X,\tau), \chi_i \circ f \circ \chi_i^{-1}\cdot(X,\tau)) = \tau(f).$$

Fixing i and applying the triangle inequality to the points (X,τ), (X_i,τ_i), $\chi_i \circ f \circ \chi_i^{-1}\cdot(X_i,\tau_i)$, $\chi_i \circ f \circ \chi_i^{-1}\cdot(X,\tau)$, we obtain

$$\begin{aligned} d((X,\tau), \chi_i \circ f \circ \chi_i^{-1}\cdot(X,\tau)) \leq d((X,\tau),(X_i,\tau_i)) \\ +d((X_i,\tau_i), \chi_i \circ f \circ \chi_i^{-1}\cdot(X_i,\tau_i)) \\ +d(\chi_i \circ f \circ \chi_i^{-1}\cdot(X_i,\tau_i), \chi_i \circ f \circ \chi_i^{-1}\cdot(X,\tau)) \end{aligned}$$

Since (X_i,τ_i) converges to (X,τ) and $\chi_i \circ f \circ \chi_i^{-1}$ are isometry on Teich(S), the first and the third term in the above inequality converge to zero as $i \to \infty$. Therefore,

$$\lim_{i\to\infty} d((X,\tau), \chi_i \circ f \circ \chi_i^{-1}\cdot(X,\tau)) \leq \lim_{i\to\infty} d((X_i,\tau_i), \chi_i \circ f \circ \chi_i^{-1}\cdot(X_i,\tau_i)).$$

We know

$$\lim_{i\to\infty} d((X_i,\tau_i),\chi_i\circ f\circ\chi_i^{-1}\cdot(X_i,\tau_i))=\tau(f),$$

and $d((X,\tau),\chi_i\circ f\circ\chi_i^{-1}\cdot(X,\tau))\geq\tau(f)$. Therefore,

$$\lim_{i\to\infty} d((X,\tau),\chi_i\circ f\circ\chi_i^{-1}\cdot(X,\tau))=\tau(f).$$

However, since $\mathrm{Mod}(S)$ acts properly discontinuously on $\mathrm{Teich}(S)$, $\chi_i\circ f\circ\chi_i^{-1}$ is eventually constant. Thus, for some $N>>0$ and $i\geq N$, we have

$$d((X,\tau),\chi_i\circ f\circ\chi_i^{-1}\cdot(X,\tau))=\tau(f).$$

It follows that the action of $[f]$ on $\mathrm{Teich}(S)$ can only be elliptic or hyperbolic. □

4.3. **Step 3: The reducible case.** In this section, we will prove the following result.

Theorem 4.7. *If $[f]\in$Mod(S) is a reducible element, then its action on Teich(S) is parabolic.*

Before proving theorem 4.7, we prove the following lemma 4.8 and its corollary 4.9. Let $\{c_1,\cdots,c_n\}$ be a reduction system for a mapping class $[f]\in\mathrm{Mod}(S)$. The following lemma tells us that for some $n\in\mathbb{N}$, we have f^n fixes each component of $S-\{c_1,\cdots,c_n\}$. Notice that f is parabolic if and only if f^n is parabolic. Thus, it suffices to show the action of f^n on $\mathrm{Teich}(S)$ is parabolic.

Lemma 4.8. *(Alexander method) Let S be a compact surface, possibly with marked points, and let $[f]\in$ Mod(S). Let $c_1,\cdots,c_n$ be a collection of simple closed curves with the following properties.*

1. *The c_i are pairwise in minimal position.*
2. *The c_i are pairwise nonisotopic.*
3. *For distinct i,j,k, at least one of $c_i\cap c_j$, $c_i\cap c_k$, $c_k\cap c_j$ is empty.*
4. *$\{[c_1],\cdots,[c_n]\}=\{[f(c_1)],\cdots,[f(c_n)]\}$*

Then, $f(\cup c_i)$ is isotopic to $\cup c_i$. Moreover, if we regard $\cup c_i$ as a graph Γ in S with vertices at the intersection points, then f gives an automorphism of Γ.

This is a classical result in the theory of Mapping class groups. A complete proof of this result can be found in Chapter 2 of [1]. Notice that since f is an automorphism of a finite graph, and the automorphism group of a finite graph is necessarily finite, we may choose $n\in\mathbb{N}$ so that f^n is the identity automorphism of the graph Γ, which means that it fixes each vertex and fixes each edge with orientation. Since f is orientation-preserving (it is in the mapping class group), it follows that f preserves the orientation of the graph, which means that it, after possibly modifying it by an isotopy, fixes Γ pointwise and send each complementary region into itself, which is precisely what we want. In particular, if the system of curves $\{c_i\}$ fills the surface S, which means that the surface obtained by cutting along $\{c_i\}$ is a disjoint union of disks and once-punctured disks, then f has a power that is isotopic to the identity. We have proved the following corollary.

Corollary 4.9. *Let S and $\{c_i\}$ and $[f]\in$Mod(S) satisfy the condition in Lemma 4.8. Suppose further that $\{c_i\}$ fills S. Then f has a nontrivial power that is isotopic to the identity.*

Denote the components of $S - \{c_1, \cdots, c_n\}$ by $\{S_i\}$. Each S_i is a punctured Riemann surface whose punctures either come from the reduction system or the punctures of the original surface S. Without loss of generality, we assume $f|_{S_i}$ are irreducible since if they are not, then we can add more curves in the reduction system of f until f restricts to an irreducible mapping class on each component.

Notice that we have proved that if f is an irreducible element, then f induces a hyperbolic or elliptic action on Teich(S) in step 2. Therefore, by the property of irreducible action, there exists (X_i, σ_i) on each S_i such that $d((X_i,\sigma_i), f|_{S_i} \cdot (X_i,\sigma_i)) = \tau(f|_{S_i})$. Let $A_i = \tau(f|_{S_i})$ and $A = max_i\{\tau(f|_{S_i})\}$. We will prove Theorem 4.7 by proving the following two claims.

(1) $\forall (X,\sigma) \in$Teich(S), $d((X,\sigma), f \cdot (X,\sigma)) > A$.
(2) $\forall \epsilon > 0$, $\exists (X,\sigma) \in$ Teich(S) such that $d((X,\sigma), f \cdot (X,\sigma)) < A + \epsilon$.

Proof. (1) We assume $A > 0$ since the inequality in (1) certainly holds when $A = 0$. Without loss of generality and for simplicity, we assume that each $\sigma(c_i)$ is a geodesic in X. Let (X,σ) denote an arbitrary point in Teich(S). It restricts to a hyperbolic structure $(\bar{X}_i, \bar{\sigma}_i)$ on each $\bar{S}_i$ where $\bar{S}_i$ are the surfaces obtained by cutting S along the reduction system $\{c_i\}$ with the boundary curves $\{c_i\}$ retained. In other words, each S_i is obtained by 'capping' $\bar{S}_i$ with punctured disks on each of their boundaries.

We claim that $d((\bar{X}_i,\bar{\sigma}_i), f|_{\bar{X}_i} \cdot (\bar{X}_i,\bar{\sigma}_i)) > A_i$, which would then imply $d((X,\sigma), f \cdot (X,\sigma)) > A$.

Let $\bar{f}_i : \bar{X}_i \to \bar{X}_i$ be the Teichmüller map in the isotopy class of $f|_{\bar{X}_i}$. (It exists and is unique by Teichmüller Theorem). We use $K_{\bar{X}_i}(\bar{f}_i)$ to denote the dilatation of $\bar{f}_i$. We also have an initial quadratic differential $\bar{\Phi}_i$ on each $\bar{X}_i$. Locally, this quadratic differential is written as $\bar{\phi}_i(z)dz^2$ which gives $\bar{X}_i$ a finite-area Euclidean metric which is locally given by

$$|\bar{\phi}_i(z)|dx \wedge dy.$$

It has the property

$$\int_{\bar{X}_i} |\bar{\phi}_i(z)|dx \wedge dy < \infty.$$

We now see $\bar{X}_i$ as embedded in X_i. Let $f_i : X_i \to X_i$ be the extension of $\bar{f}_i$ by identity on the punctured disk. We have $K_{\bar{X}_i}(\bar{f}_i) = K_{X_i}(f_i)$. In other words, f_i is a $K_{\bar{X}_i}(\bar{f}_i)-$quasiconformal map. The reason why $K_{X_i}(f_i) = K_{\bar{X}_i}(\bar{f}_i)$ is because f_i restricts to identity around each of the punctured disk which is conformal locally.

The $\bar{\phi}_i(z)$ locally extends to the X_i and therefore defines a quadratic differential $\Phi_i = \phi_i(z)dz^2$ on X_i. We have

$$\int_{X_i} |\phi_i(z)|dx \wedge dy = \infty$$

since around each boundary curve in $\bar{X}_i$, the 'capping' gives rise to a 'cusp' of infinity length, which makes the area tend to infinity. Therefore, f_i are no longer Teichmüller maps since an infinite-area quadratic differential can not be an initial quadratic differential. Therefore, there exists an $f_i' : X_i \to X_i$ in the isotopy class of f_i with $K_{X_i}(f_i') < K_{X_i}(f_i)$.

By the definition of Teichmüller metric and A_i, we know

$$\begin{aligned}A_i = \tau(f_i) &= inf_{(X_i,\sigma)} d((X_i,\sigma), f_i \cdot (X_i,\sigma)) \\ &\leq \frac{1}{2} log(K_{X_i}(f_i')) \\ &< \frac{1}{2} log(K_{X_i}(f_i)) = \frac{1}{2} log(K_{\bar{X}_i}(\bar{f}_i)) = d((\bar{X}_i,\bar{\sigma}_i), \bar{f}_i(\bar{X}_i,\bar{\sigma}_i)).\end{aligned}$$

The first inequality comes by the definition of $\tau(f)$. The second inequlity comes from our discussion above. Since

$$d((\bar{X}_i,\bar{\sigma}_i), \bar{f}_i \cdot (\bar{X}_i,\bar{\sigma}_i)) = \frac{1}{2} log(K_{\bar{X}_i}(\bar{f}_i))$$

and $K_X(f)$ is taken to be the maximum of its local dilatation,

$$\frac{1}{2} log(K_{\bar{X}_i}(\bar{f}_i)) \leq \frac{1}{2} log(K_X(f)) = d((X,\sigma), f \cdot (X,\sigma)).$$

This implies that $A_i < d((X,\sigma), f \cdot (X,\sigma))$ for all i, and so $A < d((X,\sigma), f \cdot (X,\sigma))$.

(2) We call those punctures on S_i that correspond to c_i the inner punctures. Let $\epsilon > 0$ be arbitrary. Recall that on each of S_i, there exists (X_i,σ_i) such that $d((X_i,\sigma_i), f|_{S_i} \cdot (X_i,\sigma_i)) = \tau(f|_{S_i}) = A_i$. Therefore, by the Teichmüller theorems we can choose $g_i : X_i \to X_i$ such that g_i is isotopic to $f|_{S_i}$ and it is the Teichmüller mapping in that mapping class. Since we have assumed that f fixes each c_i and each component, we can assume that $f|_{S_i}$ and therefore g_i fixes each inner puncture. Let Φ_i be the initial differential of g_i.

We now consider an inner puncture P_0 of X_i. (Since there are only finitely many inner punctures on X_i, the following discussion on P_0 can be applied to each puncture without changing the argument.) Without loss of generality, assume that Φ_i does not vanish at P_0. Let z_0 denote the natural parameter of Φ_i around P_0. We choose a neighborhood Δ of P_0 so that z_0 maps Δ to a domain in $\mathbb{C}$. There exists a δ' such that $|z_0| < A_i\delta'$ is contained in Δ. Along with the segment $0 \leq Re(z_0) \leq \delta'$, $Im(z_0) = 0$, we have that g_i coincides with the map $z_0 \mapsto A_i z_0$ by the property of Teichmüller mapping. (Informally speaking, it stretches the horizontal direction of $\mathbb{C}$). Now we choose $h_i : X_i \to X_i$ such that

(1) h_i is the identity outside the disc $|z_0| < \delta'$.
(2) h_i coincides with $z_0 \mapsto A_i^{-1} z_0$ inside the disc $|z_0| < \delta$ for some δ chosen to be such that $\delta < \delta'$
(3) h_i is the affine map in the variables $\log|z_0|$ and $\arg(z_0)$ in the annulus $\delta \leq z_0 \leq \delta'$.

If the ratio δ/δ' is small enough, $K_X(h_i) < 1 + \epsilon/A_i$. Denoting $\bar{g}_i$ to be $h_i \circ g_i$, we have $K_X(\bar{g}_i) < K_X(h)K_X(g_i) < A_i + \epsilon$. Moreover, $\bar{g}_i$ restricts to the identity on the segment $0 \leq Re(z_0) \leq \delta$, $Im(z_0) = 0$ since g_i stretches this arc by A_i and h_i shrinks it by A_i. We cut X_i along the arc $0 \leq Re(z_0) \leq \delta$. These arcs would become boundaries of the newly obtained surface which is homeomorphic to $\bar{S}_i$ and $\bar{g}_i$ fixes these boundaries. The hyperbolic structure on X_i would give rise to a hyperbolic structure on the newly obtained $\bar{X}_i$. We denote them by $(\bar{X}_i,\bar{\sigma}_i)$.

Now we glue the $\bar{X}_i$ together by connecting the boundary curves that correspond to same inner punctures by an annulus A_R of height R. More precisely, let b_i denote the boundary curve on $\bar{X}_i$ corresponding to an inner puncture on X_i, b_j denote the boundary curve on $\bar{X}_j$ corresponding to an inner puncture on X_j and these two inner punctures correspond to the same c_k in S. We glue $\bar{X}_i$ and $\bar{X}_j$ by an annulus with one boundary identified with b_i and the other identified with b_j with height R. Applying this procedure to all $\bar{X}_i$, we obtain a new surface X_R that is homeomorphic to S. It has a hyperbolic metric (X_R, σ) that restricts to the a hyperbolic metric on $\bar{X}_i$ which is the same with $(\bar{X}_i, \bar{\sigma}_i)$. There exists a homeomorphism g with the following property:

(1) Its restriction to each $\bar{X}_i$ is $\bar{g}_i$.
(2) It is isotopic to the identity when it is restricted to each connecting annulus A_R.

We then have $K_X(g) < A + \epsilon$ since $K_{\bar{X}_i}(g|_{X_i}) = K_{\bar{X}_i}(\bar{g}_i) < A_i + \epsilon < A + \epsilon$ for all i and it is conformal on each of the annulus A_R.

However, it is possible for this g to be in a different mapping class from $[f]$. Fortunately, they only differ by a product of Dehn twists about A_R. This is because g's restriction on each $\bar{X}_i$ agrees with $[f]$'s restriction on it, and so the place where f and g are non-isotopic can only be the place where $\bar{X}_i$ are glued together, which is a product of Dehn twists around each curve $\{c_i\}$ in the reduction system.

We now let $f = g \circ T$ where T is a product of Dehn twists along the A_R so that $f \in [f]$. We have

$$\begin{aligned} K_X(f) &= K_X(g \circ T) \\ &\leq K_X(g) K_X(T) \\ &< (A + \epsilon) K_X(T) \end{aligned}$$

Recall that Dehn twist is given by $T : A_R \to A_R$

$$T(\theta, t) = (\theta + 2\pi t/R, t).$$

Moreover, its dilatation only depends on R. As $R \to \infty$, the dilatation of T approximates 1. (An intuitive explanation for this is that as the height of the annulus increases, the Dehn twist becomes more and more 'conformal' since the map $\theta \mapsto \theta + 2\pi t/R$ approximates to the identity as R increases) Therefore, if we increase the height of the annuli connecting each $\bar{X}_i$, $K_X(T) \to 1$. Thus,

$$K_X(f) < (A + \epsilon) K_X(T) = A + \epsilon$$

as the height of the connecting annuli increases. □

(1) and (2) imply that A is τ_f and it cannot be attained. Therefore, f induces parabolic action on S.

4.4. **Step 4: Hyperbolic action implies pseudo-Anosov.**

We will prove the following theorem.

Theorem 4.10. *If $f \in$ Mod(S) is a mapping class that induces hyperbolic action on Teich(S), then it is pseudo-Anosov.*

Let $(X, \tau) \in \text{Teich}(S)$ such that $d((X, \tau), f \cdot (X, \tau)) = \tau(f)$. Without loss of generality, suppose $\tau(f) > 0$ which means that $[f]$ is not elliptic. We begin by proving the following lemma.

Lemma 4.11. *If there exists (X, σ) such that $d((X, \sigma), f \cdot (X, \sigma)) = \tau(f)$, then the mapping class $[f]$ fixes a geodesic in Teich(S).*

Proof. Since $[f]$ is not elliptic, $(X,\sigma), f\cdot(X,\sigma), f^2\cdot(X,\sigma)$ are different. Let $(Y_1,\tau_1),(Y_2,\tau_2)$ be the midpoints of the geodesic in Teich(S) connecting $(X,\sigma), f\cdot(X,\sigma)$ and the geodesic connecting $f\cdot(X,\sigma), f^2\cdot(X,\sigma)$ respectively.

Notice that we have $d((Y_1,\tau_1), f\cdot(X,\sigma)) = \frac{1}{2}\tau(f)$ and $d(f\cdot(X,\sigma),(Y_2,\tau_2)) = \frac{1}{2}\tau(f)$. Therefore, $d((Y_1,\tau_1),(Y_2,\tau_2)) \leq \frac{1}{2}\tau(f)+\frac{1}{2}\tau(f) = \tau(f)$. However, we know $(Y_2,\tau_2)) = f\cdot(Y_1,\tau_1)$ and so $d((Y_1,\tau_1),(Y_2,\tau_2)) \geq \tau(f)$. Therefore, we have $d((Y_1,\tau_1),(Y_2,\tau_2)) = \tau(f)$, and the concatenation of the two segments connecting $(Y_1,\tau_1), f\cdot(X,\sigma)$ and $f\cdot(X,\sigma),(Y_2,\tau_2)$ is in fact a geodesic (a distance minimizing curve) connecting (Y_1,τ_1) and $(Y_2,\tau_2))$. Therefore, we have $(X,\sigma),(Y_1,\tau_1), f\cdot(X,\sigma),(Y_2,\tau_2), f^2\cdot(X,\sigma)$ lies on the same geodesic, which implies that $[f]$ fixes a geodesic passing (X,σ) □

Now we prove the following lemma to show that if $d((X,\sigma), f\cdot(X,\sigma)) = \tau(f)$, then the initial quadratic differential and the terminal quadratic differential of the Teichmüller map $f: X\to X$ are the same.

Lemma 4.12. *Let $f: X\to X$ be the Teichmüller map in the mapping class of $[f]$ and $d((X,\sigma), f\cdot(X,\sigma)) = \tau(f)$. Then*

(1) f^2 is also a Teichmüller map with $K_X(f^2) = K_X(f)^2$

(2) The initial and terminal quadraic differential of f are the same

Proof. (1) We know that $K_X(f^2) \leq K_X(f)^2$. Let K denote the dilatation of the Teichmüller map in the mapping class of f^2. We therefore have $K \leq K_X(f^2)$ and the inequality

$$K \leq K_X(f^2) \leq K_X(f)^2. \tag{4.13}$$

We know $[f]$ fixes the geodesic passing through $(X,\sigma), f\cdot(X,\sigma)$, and so $d((X,\sigma), f^2\cdot(X,\sigma)) = 2d((X,\sigma), f\cdot(X,\sigma)) = 2(\frac{1}{2}log(K_X(f))) = log(K_X(f))$. Thus, $log(K_X(f)) = \frac{1}{2}log(K)$, and so $K = K_X(f)^2$. Therefore, the inequalities are all equalities in (4.13). In particular, this implies f^2 is a Teichmüller map with $K_X(f^2) = K_X(f)^2$

(2) We use (1) to show (2). Let $q, q' \in Q(X)$ be the initial and terminal quadratic differentials for f. Choose $p\in X$ such that $q(p)\neq 0$. The unit circle in $T_p(X)$ is mapped to an ellipse E in $T_{f(p)}(X)$ by df. The direction of maximal stretch in $T_p(X)$ is the same as the direction of the horizontal foliation in X determined by q. The major axis of E in $T_{f(p)}(X)$ is the direction of the horizontal foliation of q' with length $\sqrt{K_X(f)}$. Let $E' = df(E)$ in $T_{f^2(p)}(X)$. It is the same as the image $d(f^2)$ applied to the unit circle, which is the same as an ellipse in $T_{f^2(p)}(X)$ which has maximal axis in the direction of the horizontal foliation of q' with length $\sqrt{K_X(f^2)} = K_X(f)$. This forces the direction of the maximal stretch of df in $T_{f(p)}(X)$ to be in the direction of the maximal axis of E for otherwise the length of the maximal axis in E' will not be attained. Thus, the direction of the maximal stretch of df, which is the direction of horizontal foliation of q, is the same as the direction of the maximal axis, which is the direction of horizontal foliation of q'. Therefore, the horizontal foliation for q and q' are the same. A similar argument would imply that the vertical foliation for q and q' are the same.

In the natural coordinate for q, we have $q = dz^2$. In the same coordinates, $q' = \phi(z)dz^2$. Notice that since the horizontal and vertical foliations for q, q' are the same, $\phi(z)$ has to be a constant C locally. Since the function $q'/q: X\to\mathbb{C}$ is a continuous function, it follows that C is constant on X. However, since initial and terminal quadratic differentials have norm 1, it follows that $C = 1$. Therefore, $q = q'$ □

The previous lemma implies that with the quadratic differential q and the horizontal and vertical foliation associated to it, $f : X \to X$ stretches the horizontal foliation by a factor of $\sqrt{K_X(f)}$ and shrinks the vertical foliation by a factor of $1/\sqrt{K_X(f)}$. Let $\mathcal{F}^s$ and $\mathcal{F}^u$ be the horizontal foliation and the vertical foliation induced by q on X and let μ^s, μ^u be the measures q induces. It follows that

$$f \cdot (\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \sqrt{K_X(f)}\mu^u)$$
$$f \cdot (\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, 1/\sqrt{K_X(f)}\mu^s).$$

This is exactly the definition of a pseudo-Anosov mapping class.

4.5. **Step 5: Conclusion.** In this section we synthesize the previous results and prove Nielson-Thurston classification.

Proof. Let $[f] \in Mod(S)$. It is either reducible or irreducible. If it is irreducible, then by Theorem 4.1, its action of Teich(S) is either elliptic or hyperbolic. If it induces an elliptic action, then it is periodic by Step 1. If it induces a hyperbolic action, then by Theorem 4.10, it is pseudo-Anosov. We have thus shown that $[f]$ has to be one of the three types.

Now we show the exclusivity. Suppose $[f]$ is a pseudo-Anosov element, then there is a paired transverse measured foliation $(\mathcal{F}^u, \mu^u)$, $(\mathcal{F}^s, \mu^s)$ and a representative f of $[f]$ such that $f \cdot (\mathcal{F}^u, \mu) = (\mathcal{F}^u, \sqrt{K}\mu)$, $f \cdot (\mathcal{F}^s, \mu) = (\mathcal{F}^s, 1/\sqrt{K}\mu)$. We can find a Riemann surface X homeomorphic to S with homeomorphism given by $\sigma : S \to X$ and a quadratic differential q on X that induces the two measured foliations.

With respect to the Riemann surface structure on X, $f : X \to X$ is a Teichmüller mapping on X with both initial and terminal quadratic differential q and with dilatation K.

Claim: Let $(X, \sigma) \in$ Teich(S). If $f : X \to X$ is a Teichmüller mapping and the initial and terminal quadratic differential of f coincide, we have

$$d((X, \sigma), f \cdot (X, \sigma)) = \tau(f).$$

Proof. Since the initial and terminal quadratic differential of f coincide, f^2 is also a Teichmüller mapping with dilatation $K_X(f)^2$. This implies that $d((X, \sigma), f^2 \cdot (X, \sigma)) = 2d((X, \sigma), f \cdot (X, \sigma))$. Thus, f fixes a geodesic in Teich(S) passing through (X, σ).

Now let $(Y, \tau) \in$Teich(S) be arbitrary.

We have that

$$\begin{aligned} nd((X, \sigma), f \cdot (X, \sigma)) &= d((X, \sigma), f^n(X, \sigma)) \\ &\leq d((X, \sigma), (Y, \tau)) + d((Y, \tau), f \cdot (Y, \tau)) + \cdots \quad + d(f^{n-1}(Y, \tau), f^n \cdot (Y, \tau)) \\ &\qquad + d(f^n \cdot (Y, \tau), f^n \cdot (X, \sigma)) \\ &= 2d((X, \sigma), (Y, \tau)) + nd((Y, \tau), f \cdot (Y, \tau)) \end{aligned}$$

If we let $n \to \infty$ and divide both sides by n, we have $d((X, \sigma), f \cdot (X, \sigma)) \leq d((Y, \tau), f \cdot (Y, \tau))$ Since (Y, τ) is chosen arbitrarily, we have that $d((X, \sigma), f \cdot (X, \sigma)) = \tau(f)$. □

Thus, f is not a parabolic element. By Theorem 4.7, $[f]$ cannot be reducible. This proves the exclusivity in Nielson-Thurston Classification. □

5. Acknowledgements

I would like to thank my mentor Howard Masur for his support and guidance throughout the program. He introduced me to this fascinating topic and provided meticulous feedback on the draft. I am also grateful to Peter May for organizing the REU and providing a great deal of helpful comments on the draft of this report.

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