INTRODUCTION TO GEOMETRIC MEASURE THEORY

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Abstract. Measure theory is a powerful tool in analysis used to assign a notion of size to sets in a suitable manner. For example, the Lebesgue measure assigns n-dimensional volume to subsets of $\mathbb{R}^n$, such as area in $\mathbb{R}^2$ or volume in $\mathbb{R}^3$. The Hausdorff measure generalizes this by assigning s-dimensional volume to subsets of any metric space, leading to the concept of the Hausdorff dimension of a set. This paper will begin with an overview of abstract measure theory, followed by an introduction to Hausdorff measure and dimension, its applications to fractal geometry, and the Minkowski dimension.

Contents

1. Measure Theory

For certain simple subsets of $\mathbb{R}^n$, n-dimensional volume follows what we would intuitively expect; a circle in $\mathbb{R}^2$ will have area πr^2, an interval in $\mathbb{R}$ has length equal to the starting point minus the ending point, and the volume of two disjoint sets should be the sum of their volumes. However, for more complicated sets, there may not be an intuitive answer for the volume, such as $\mathbb{Q}$ in $\mathbb{R}$, an unbounded and infinite set which actually has length (Lebesgue measure in this case) 0 in $\mathbb{R}$. In general, for sets in a metric space, measure theory is needed to assign a notion of size to a set. For this section, we follow the approach of Folland [1].

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1.1. **σ-algebras.** We begin by defining σ-algebras, the type of collection of sets which measures are defined on.

Definition 1.1. Let X be a nonempty set. An **algebra** $\mathcal{A}$ on X is a nonempty collection of subsets of X containing $\emptyset$ which has the following properties:

(1) $\mathcal{A}$ is closed under finite unions: for any finite sequence of sets $E_1 \dots\ E_n$ in $\mathcal{A}$, the union $\bigcup_{i=1}^{n} E_i$ is also in $\mathcal{A}$.
(2) $\mathcal{A}$ is closed under complements: for any $E \in \mathcal{A}$, we have that $E^c \in \mathcal{A}$.

Definition 1.2. Let X be a nonempty set. A **σ-algebra** $\mathcal{M}$ on X is an algebra which is closed under countable unions: for any sequence of sets $\{E_i\}_{i=1}^{\infty}$ in $\mathcal{A}$, the union $\bigcup_{i=1}^{\infty} E_i$ is also in $\mathcal{A}$.

Note that since $\bigcap_i E_i = (\bigcup_i E_i^c)^c$, algebras are closed under finite intersections and σ-algebras are closed under countable intersections.

If $\mathcal{E}$ is any collection of subsets of X, the intersection of all σ-algebras containing $\mathcal{E}$ denoted by $\mathcal{M}(\mathcal{E})$ is still a σ-algebra: for any $E \in \mathcal{M}(\mathcal{E})$, E is in every σ-algebra containing $\mathcal{E}$, hence E^c is too, thus $E^c \in \mathcal{M}(\mathcal{E})$. Also, for any sequence of sets $\{E_i\}_{i=1}^{\infty}$ in $\mathcal{M}(\mathcal{E})$, since each E_i is in every σ-algebra containing $\mathcal{E}$, the union $\bigcup_{i=1}^{\infty} E_i$ is as well. $\mathcal{M}(\mathcal{E})$ is thus the smallest σ-algebra containing $\mathcal{E}$, which we call the **σ-algebra generated by** $\mathcal{E}$.

One important example of a σ-algebra in a given metric space X is the Borel σ-algebra, which is the σ-algebra generated by open sets in X denoted by $\mathcal{B}_X$. This contains all open sets in X, as well as all closed sets (since any closed set is the complement of an open set) and countable intersections of open and closed sets, etc.

1.2. **Measures.** We now define the properties of measures.

Definition 1.3. Let X be a set with a σ-algebra $\mathcal{M}$. A **measure** on $(X, \mathcal{M})$ is a set function $\mu : \mathcal{M} \to [0, \infty]$ such that:

(1) $\emptyset \in \mathcal{M}$ and $\mu(\emptyset) = 0$.
(2) μ is **countably additive**: for any sequence of disjoint sets $\{E_i\}_{i=1}^{\infty}$ in $\mathcal{M}$, $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$.

Note that countable additivity implies finite additvity since we can let all later sets be empty after some point in the sequence. We call the sets in $\mathcal{M}$ **measurable sets** and we call $(X, \mathcal{M}, \mu)$ a **measure space**.

Theorem 1.4. *Let $(X, \mathcal{M}, \mu)$ be a measure space.*

*(1) μ is **monotonic**: If $E, F \in \mathcal{M}$ and $E \subset F$, then $\mu(E) \le \mu(F)$.*
*(2) μ is **countably subadditive**: If $\{E_i\}_{i=1}^{\infty}$ is a sequence of sets in $\mathcal{M}$, then $\mu(\bigcup_{i=1}^{\infty} E_i) \le \sum_{i=1}^{\infty} \mu(E_i)$.*
*(3) μ is **continuous from above**: If $\{E_i\}_{i=1}^{\infty}$ is a sequence of sets in $\mathcal{M}$ and $E_1 \subset E_2 \subset \dots$, then $\mu(\bigcup_{i=1}^{\infty} E_i) = \lim_{i\to\infty} \mu(E_i)$.*
*(4) μ is **continuous from below**: If $\{E_i\}_{i=1}^{\infty}$ is a sequence of sets in $\mathcal{M}$ and $E_1 \supset E_2 \supset \dots$ with $\mu(E_1) < \infty$ then $\mu(\bigcap_{i=1}^{\infty} E_i) = \lim_{i\to\infty} \mu(E_i)$.*

Proof. (1) $\mu(F) = \mu(E) + \mu(F \setminus E) \ge \mu(E)$ for $E \subset F$

(2) Let $F_1 = E_1$, $F_k = E_k \setminus (\bigcup_{i=1}^{k-1} E_i)$ for $k > 1$. The F_k's are disjoint and $\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i$ for all $n \in \mathbb{N}$. Thus

$$\mu(\bigcup_{i=1}^{\infty} E_i) = \mu(\bigcup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} \mu(F_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

where the last inequality follows from (1).

(3) Setting $E_0 = \emptyset$,

$$\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i \setminus E_{i-1}) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(E_i \setminus E_{i-1}) = \lim_{n\to\infty} \mu(E_n).$$

The first equality follows from countable additivity and for the last equality note that $\sum_{i=1}^{n} \mu(E_i \setminus E_{i-1}) = \mu(E_n)$ from finite additivity.

(4) Let $F_i = E_1 \setminus E_i$, so we have $F_1 \subset F_2 \subset \dots$, $\mu(E_1) = \mu(F_i) + \mu(E_i)$, and $\bigcup_{i=1}^{\infty} F_i = E_1 \setminus (\bigcap_{i=1}^{\infty} E_i)$. Then

$$\mu(E_1) = \mu(\bigcap_{i=1}^{\infty} E_i) + \lim_{i\to\infty} \mu(F_i) = \mu(\bigcap_{i=1}^{\infty} E_i) + \lim_{i\to\infty} (\mu(E_1) - \mu(E_i)).$$

The first equality is because $\mu(\bigcup_{i=1}^{\infty} F_i) = \lim_{i\to\infty} \mu(F_i)$ from (c). Subtracting $\mu(E_1) < \infty$ from both sides yields the desired result.

□

A set $E \in \mathcal{M}$ is called a null set if $\mu(E) = 0$. As these sets have no size in the sense of the measure being used, if a statement is true for all $x \in X$ except for some x in a null set, we say that this statement is true **almost everywhere**.

For E such that $\mu(E) = 0$, it follows by monotonicity that for $F \subset E$, $\mu(F) = 0$ if F is measurable. However, subsets of null sets are not necessarily measurable. If all subsets of null sets are measurable, we call the measure **complete**.

1.3. **Outer Measures.** Outer measures are a weaker notion of size defined on all subsets of a set, not just on a σ-algebra of measurable subsets. While useful on their own, outer measures can also be used to construct a measure.

Definition 1.5. An outer measure μ^* on a nonempty set X is a set function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ such that:

(1) $\mu^*(\emptyset) = 0$.
(2) μ^* is **monotonic**: $\mu^*(A) \leq \mu^*(B)$ if $A \subset B \subset X$.
(3) μ^* is **countably subadditive**: $\mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$ for any sequence $\{A_i\}_{i=1}^{\infty} \subset X$.

Note that unlike measures, these properties hold on all subsets of X, not just measurable ones. Outer measures are typically constructed by starting with a family of sets and a set function on that family of sets, then taking the infimum of the set function over all covers of sets in the family.

Proposition 1.6. *Let $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0, \infty]$ such that $\emptyset \in \mathcal{E}$, $X \in \mathcal{E}$, and $\rho(\emptyset) = 0$. For any $A \subset X$, define*

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : E_i \in \mathcal{E} \text{ and } A \subset \bigcup_{i=1}^{\infty} E_i \right\}.$$

Then μ^* is an outer measure.

Proof. $\mu^*(\emptyset) = 0$ since $\rho(\emptyset) = 0$. Monotonicity follows since any cover of A is also a cover of B if $A \subset B$. To see countable subadditivity, consider any $\{A_i\}_{i=1}^{\infty} \subset X$ and fix any $\epsilon > 0$. For each i, there exists a sequence $\{E_{i,k}\}_{k=1}^{\infty} \subset \mathcal{E}$ such that $A_i \subset \bigcup_{k=1}^{\infty} E_{i,k}$ and $\sum_{k=1}^{\infty} \rho(E_{i,k}) \leq \mu^*(A_i) + \epsilon 2^{-i}$ from how we define $\mu^*(A_i)$. Since $\bigcup_{i=1}^{\infty} A_i \subset \bigcup_{i=1}^{\infty} \bigcup_{k=1}^{\infty} E_{i,k}$, by summing over each A_i, we have $\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \rho(E_{i,k}) \leq \sum_{i=1}^{\infty} \mu^*(A_i) + \epsilon$. As the sequence of $E_{i,k}$'s covers $\bigcup_{i=1}^{\infty} A_i$, we have that $\mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i) + \epsilon$. Since ϵ was arbitrary, μ^* is countably subadditive. □

Definition 1.7. Let μ^* be an outer measure on X and $A \subset X$. A is **μ^*-measurable** if $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ for all $E \subset X$.

Intuitively, this means that for a well behaved set A, if $E \supset A$, the outer measure $\mu^*(A) = \mu^*(E \cap A)$ is equal to the inner size $\mu^*(E) - \mu^*(E \cap A^c)$. For any $A \subset X$, it follows immediately that $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ for any $E \subset X$ by subadditivity, thus to prove A is measurable, it suffices to show the reverse inequality. Using this definition of measurable sets, we can then construct a measure from an outer measure with the following theorem.

Theorem 1.8. ***Carathéodory's Theorem.*** *If μ^* is an outer measure on X, the collection of μ^*-measurable sets $\mathcal{M}$ forms a σ-algebra. Moreover, the measure formed by restricting μ^* to $\mathcal{M}$ is a complete measure.*

Proof. We first prove that $\mathcal{M}$ is an algebra. $\mathcal{M}$ is closed under complements from the definition of μ^*-measurable sets as the definition is the same for A and A^c. To show $\mathcal{M}$ is an algebra, consider any $A, B \in \mathcal{M}$ and $E \subset X$.

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \end{aligned}$$

Since $A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$, by subadditivity,

$$\mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) \geq \mu^*(E \cap (A \cup B)),$$

thus

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c),$$

hence $A \cup B \in \mathcal{M}$ and $\mathcal{M}$ is an algebra.

To show that $\mathcal{M}$ is a σ-algebra, it suffices to show that it is closed under countable disjoint unions since we can express any countable union of sets in $\mathcal{M}$ as a countable disjoint union of sets in $\mathcal{M}$. Consider any sequence of disjoint sets $\{A_i\}_{i=1}^{\infty}$ in $\mathcal{M}$ and let $B_n = \bigcup_{i=1}^{n} A_i$ and $B = \bigcup_{i=1}^{\infty} A_i$. For any $E \subset X$,

$$\begin{aligned} \mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}), \end{aligned}$$

and induction shows $\mu^*(E \cap B_n) = \sum_{i=1}^{n} \mu^*(E \cap A_i)$. Thus

$$\mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \geq \sum_{i=1}^{n} \mu^*(E \cap A_i) + \mu^*(E \cap B^c)$$

and letting $n \to \infty$, we have

$$\begin{aligned} \mu^*(E) &\geq \sum_{i=1}^{\infty} \mu^*(E \cap A_i) + \mu^*(E \cap B^c) \geq \mu^*(\bigcup_{i=1}^{\infty}(E \cap A_i)) + \mu^*(E \cap B^c) \\ &= \mu^*(E \cap B) + \mu^*(E \cap B^c) \geq \mu^*(E), \end{aligned}$$

hence all of the inequalities in the equation are equalities. We then obtain that $B = \bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$, and letting $E = B$, $\mu^*(B) = \sum_{i=1}^{\infty} \mu^*(A_i)$, so $\mathcal{M}$ is a σ-algebra and μ^* is countably additive on $\mathcal{M}$.

To see that the restriction of μ^* to $\mathcal{M}$ is complete, for any A such that $\mu^*(A) = 0$ and for any $E \subset X$,

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c) = \mu^*(E \cap A^c) \leq \mu^*(E),$$

so $A \in \mathcal{M}$. $\square$

Although the construction of an outer measure in Proposition 1.6 does not impose many requirements on ρ or $\mathcal{E}$, we often want $\mu^*(A)$ to equal $\rho(A)$ for $A \in \mathcal{E}$. This is not always guaranteed; consider the case in $\mathbb{R}$ where $\mathcal{E}$ is the family of intervals in $\mathbb{R}$ and $\rho((a,b)) = (b-a)^2$ for any interval (a,b) without loss of generality. $\rho((0,1)) = 1$ but $\mu^*((0,1)) < 1$ since $(0, 1/2]$ and $[1/2, 1)$ cover $(0,1)$ but $\rho((0,1/2]) + \rho([1/2,1)) = 1/2$. Hence to ensure that ρ and μ^* agree on sets $A \in \mathcal{E}$, we want ρ to be a premeasure.

Definition 1.9. A set function $\mu_0 : \mathcal{A} \to [0, \infty]$ on an algebra $\mathcal{A}$ is called a **premeasure** if:

(1) $\mu_0(\emptyset) = 0$
(2) For any sequence of disjoint sets in A such that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$, $\mu_0(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu_0(A_i)$

The key difference between a measure and a premeasure is that measures are defined on a σ-algebra while premeasures are defined on an algebra. However, with the following proposition, we see that premeasures can be extended to an outer measure and thus a measure.

Proposition 1.10. *Let μ_0 be a premeasure on $\mathcal{A} \subset \mathcal{P}(X)$. μ_0 induces an outer measure μ^* on X defined as*

$$\mu^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu_0(A_i) : A_i \in \mathcal{A} \text{ and } E \subset \bigcup_{i=1}^{\infty} A_i \right\}$$

Then for every $A \in \mathcal{A}$, A is measurable and $\mu^*(A) = \mu_0(A)$.

Proof. It follows from Proposition 1.6 that μ^* is an outer measure. To see that every $A \in \mathcal{A}$ is measurable, consider any $E \subset X$ and $\epsilon > 0$. Then from the definition of $\mu^*(E)$, there exists a sequence $\{A_i\}_{i=1}^{\infty} \subset \mathcal{A}$ such that $E \subset \bigcup_{i=1}^{\infty} A_i$ and $\sum_{i=1}^{\infty} \mu_0(A_i) \leq \mu^*(E) + \epsilon$. Since μ_0 is additive on $\mathcal{A}$,

$$\mu^*(E) + \epsilon \geq \sum_{i=1}^{\infty} \mu_0(A_i \cap A) + \sum_{i=1}^{\infty} \mu_0(A_i \cap A^c) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

The first inequality follows because $\mathcal{A}$ is an algebra and μ_0 is additive on A and the second inequality follows from the definition of μ^*. Since ϵ was arbitrary, A is measurable.

To show that $\mu^*(A) = \mu_0(A)$, consider any $A \in \mathcal{A}$. $A \subset \bigcup_{i=1}^{\infty} A_i$ for some sequence of sets $\{A_i\}_{i=1}^{\infty} \subset \mathcal{A}$ and let $B_n = A \cap (A_n \setminus \bigcup_{i=1}^{n-1} A_i)$. The B_n's are disjoint members of $\mathcal{A}$ such that $\bigcup_{i=1}^{\infty} B_n = A$. Then $\mu_0(A) = \sum_{i=1}^{\infty} \mu_0(B_i) \leq \sum_{i=1}^{\infty} \mu_0(A_i)$, hence $\mu_0(A) \leq \mu^*(A)$ and the reverse inequality follows because $A \in \mathcal{A}$ is a cover of itself, thus $\mu_0(A) = \mu^*(A)$. $\square$

With this, we can construct a measure on X by starting with a premeasure μ_0 defined on an algebra $\mathcal{A}$ and using this to construct an outer measure μ^* as shown in Proposition 1.10. Then applying Carathéodory's Theorem, the collection of μ^*-measurable sets $\mathcal{M}$ is a σ-algebra, giving a measure μ on $(X, \mathcal{M})$ where $\mu = \mu^*$ on the sets in $\mathcal{M}$ and $\mu = \rho$ on the sets in $\mathcal{A}$, which are all measurable.

One of the most important measures in $\mathbb{R}$ is the Lebesgue measure, which we define as follows. Let $\mathcal{A}$ be the algebra of half open intervals in $\mathbb{R}$ of the form $(a, b]$ or (a, ∞) with $-\infty \leq a \leq b < \infty$ and let $\nu(A) = \sum_{i=1}^{n}(b_i - a_i)$ where $A = \bigcup_{i=1}^{n}(a_i, b_i]$. Then ν is a premeasure on $\mathcal{A}$ [1], hence it induces an outer measure μ^* on X defined as

$$\mu^*(E) = \inf\left\{\sum_{i=1}^{\infty}\nu(A_i) : A_i \in \mathcal{A} \text{ and } E \subset \bigcup_{i=1}^{\infty} A_i\right\},$$

which we call the Lebesgue outer measure. From here, we apply Carathéodory's theorem which gives the **Lebesgue measure** $\mathcal{L}$.

An interesting result of the Lebesgue measure is that $\mathcal{L}(\mathbb{Q}) = 0$, or that the rationals have 0 length in $\mathbb{R}$. The Lebesgue measure of any single point is 0 since we can cover a point with arbitrarily small half open intervals, and since measures are countably additive, $\mathcal{L}(\mathbb{Q}) = 0$ (as well as for any countable set).

We now introduce some important properties of measures.

Definition 1.11. Let μ be an outer measure on a metric space X.

(1) μ is locally finite if for every $x \in X$ there exists $r > 0$ such that $\mu(B_r(x)) < \infty$
(2) μ is a Borel measure if all Borel sets are μ measurable
(3) μ is Borel regular if it is a Borel measure and if for every $A \subset X$ there exists a Borel set $B \subset X$ such that $A \subset B$ and $\mu(A) = \mu(B)$
(4) μ is a Radon measure if it is a Borel measure and:
  (a) $\mu(K) < \infty$ for compact sets $K \subset X$
  (b) $\mu(V) = \sup\{\mu(K) : K \subset V \text{ is compact }\}$ for open sets $V \subset X$
  (c) $\mu(A) = \inf\{\mu(V) : V \supset A \text{ is open }\}$ for sets $A \subset X$

The Lebesgue measure defined earlier is one example of a Borel measure. To see this, note that every half open interval is measurable, hence the σ-algebra of measurable sets includes all Borel sets since $\mathcal{B}_{\mathbb{R}}$ is generated by the family of half open intervals described in the construction of Lebesgue measure.

Theorem 1.12. *Let μ be an outer measure on a metric space X. Then μ is a Borel measure if and only if $\mu(A \cup B) = \mu(A) + \mu(B)$ for all sets $A, B \subset X$ with $d(A, B) > 0$.*

Proof. See Matilla [2]. □

2. Hausdorff Measure and Dimension

We now begin our discussion of the Hausdorff measure which leads to the Hausdorff dimension, a way to define the dimension of subsets of a metric space. We follow the approach of Matilla [2] for this section.

2.1. **Construction.** For this section, we let X be a metric space, $\mathcal{F}$ be a family of subsets of X and ζ be a non-negative function on $\mathcal{F}$ with the following properties:

(1) For every $\delta > 0$, there exists a sequence of sets $\{E_i\}_{i=1}^{\infty} \subset \mathcal{F}$ with $d(E_i) < \delta$ for each E_i such that $X = \bigcup_{i=1}^{\infty} E_i$
(2) For every $\delta > 0$, there exists $E \in \mathcal{F}$ such that $\zeta(E) \leq \delta$ and $d(E) \leq \delta$

For every $0 < \delta \leq \infty$ and $A \subset X$, we define

$$\psi_\delta = \inf \left\{ \sum_{i=1}^{\infty} \zeta(E_i) : A \subset \bigcup_{i=1}^{\infty} E_i, d(E_i) \leq \delta, E_i \in \mathcal{F} \right\}$$

From Proposition 1.6, ψ_δ is an outer measure. However, ψ_δ is not always a Borel measure [2]. Note that ψ_δ increases monotonically as δ decreases, hence we can define ψ using ζ on $\mathcal{F}$ by

$$\psi(A) = \lim_{\delta \to 0^+} \psi_\delta(A) = \sup_{\delta > 0} \psi_\delta(A) \text{ for } A \subset X \tag{2.1}$$

We define ψ this way as it has better measure-theoretic properties than ψ_δ.

Theorem 2.2. *ψ is a Borel measure and if the members of $\mathcal{F}$ are Borel sets, then ψ is Borel regular.*

Proof. ψ is an outer measure as ψ_δ is for every $\delta > 0$, thus taking limits all the properties still hold. To see that ψ is a Borel measure, we use Theorem 1.12. Consider any $A, B \subset X$ with $d(A, B) > 0$ and pick δ such that $0 < \delta < d(A, B)/2$. For any sequence of sets $\{E_i\}_{i=1}^{\infty} \subset \mathcal{F}$ that covers $A \cup B$ and satisfies $d(E_i) \leq \delta$ for all i, then no E_i can intersect both A and B. Thus

$$\sum_{i=1}^{\infty} \zeta(E_i) \geq \sum_{A \cap E_i \neq \emptyset}^{\infty} \zeta(E_i) + \sum_{B \cap E_i \neq \emptyset}^{\infty} \zeta(E_i) \geq \psi_\delta(A) + \psi_\delta(B),$$

and by taking the infimum over all covers $\{E_i\}_{i=1}^{\infty} \in \mathcal{F}$, it follows that $\psi_\delta(A \cup B) \geq \psi_\delta(A) + \psi_\delta(B)$. Taking limits as $\delta \to 0^+$, $\psi(A \cup B) = \psi(A) + \psi(B)$.

We see that ψ is Borel regular as for any $A \subset X$, for every $i \in \mathbb{N}$, we can pick a sequence of sets $\{E_{i,j}\}_{j=1}^{\infty} \subset \mathcal{F}$ such that $A \subset \bigcup_{j=1}^{\infty} E_{i,j}$, $d(E_{i,j}) \leq 1/i$, and $\sum_{j=1}^{\infty}(E_{i,j}) \leq \psi_{1/i}(A) + 1/i$. Then $B = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} E_{i,j}$ is a Borel set with $A \subset B$ and $\psi(A) = \psi(B)$ by taking limits. ☐

2.2. **Hausdorff Measure.** Let X be a separable metric space, $0 \leq s < \infty$, and let $\mathcal{F} = \mathcal{P}(X)$ and $\zeta(E) = d(E)^s$. Using the construction in (2.1), we get the measure $\mathcal{H}^s_\delta$ defined as

$$\mathcal{H}^s_\delta(A) = \inf \left\{ \sum_{i=1}^{\infty} d(E_i)^s : A \subset \bigcup_{i=1}^{\infty} E_i, d(E_i) \leq \delta, E_i \subset X \right\} \tag{2.3}$$

This gives the measure $\mathcal{H}^s$ defined by $\mathcal{H}^s(A) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(A)$, which we call the **$s$-dimensional Hausdorff measure**. For integer values of s, the Hausdorff measure gives a notion of s-dimensional volume. For example, for $s = 0$, $\mathcal{H}^0(A)$ gives the cardinality of A and $\mathcal{H}^1$ is a length measure. In particular, in $\mathbb{R}^n$, we have that $\mathcal{H}^n = \mathcal{L}^n$ up to a constant [2].

Theorem 2.4. *Let $0 \leq s < n$ and define $\zeta(E) = d(E)^s$ for $E \subset X$. If $\mathcal{F} = \{U \subset X : U \text{ is open}\}$, then $\psi(\mathcal{F}, \zeta) = \mathcal{H}^s$.*

Proof. For any $E \subset X$ and $\epsilon > 0$, the set $U = \{x : d(x, E) < \epsilon\}$ is open and $d(U) \leq d(E) + 2\epsilon$. Since any cover of E can be turned into an open cover using only slightly larger open sets, it follows that $\psi = \mathcal{H}^s$. ☐

Theorem 2.5. *For $0 \leq s < t < \infty$ and $A \subset X$,*

(1) $\mathcal{H}^s(A) < \infty$ *implies* $\mathcal{H}^t(A) = 0$
(2) $\mathcal{H}^t(A) > 0$ *implies* $\mathcal{H}^s(A) = \infty$

Proof. To prove (1), fix any $\delta > 0$ and consider $\{E_i\}_{i=1}^{\infty}$ with $d(E_i) \leq \delta$ such that $A \subset \bigcup_{i=1}^{\infty} E_i$ and $\sum_i^{\infty} d(E_i)^s \leq \mathcal{H}_\delta^s(A) + 1$. Note that we can pick such $\{E_i\}_{i=1}^{\infty}$ from the definition of $\mathcal{H}_\delta^s(A)$. Then

$$\mathcal{H}_\delta^t(A) \leq \sum_{i=1}^{\infty} d(E_i)^t = \sum_{i=1}^{\infty} d(E_i)^s * d(E_i)^{t-s} \leq \delta^{t-s} \sum_{i=1}^{\infty} d(E_i)^s \leq \delta^{t-s}(\mathcal{H}_\delta^s(A) + 1).$$

Since this holds for any $\delta > 0$, taking limits, we see that as $\delta \to 0^+$, $\mathcal{H}^t(A) = 0$ if $\mathcal{H}^s(A)$ is finite. (2) follows as it is the contrapositive of (1), though it has been restated as it leads to the concept of the Hausdorff dimension. ☐

2.3. Hausdorff Dimension.

Definition 2.6. The Hausdorff dimension of a set $A \subset X$ is defined as

$$\begin{aligned} \dim A &= \sup\{s : \mathcal{H}^s(A) > 0\} = \sup\{s : \mathcal{H}^s(A) = \infty\} \\ &= \inf\left\{t : \mathcal{H}^t(A) < \infty\right\} = \inf\left\{t : \mathcal{H}^t(A) = 0\right\} \end{aligned} \tag{2.7}$$

In other words, $\dim A$ is the unique number such that $s < \dim A$ implies $\mathcal{H}^s(A) = \infty$ and $t > \dim A$ implies $\mathcal{H}^t(A) = 0$. Such a number exists and is equivalent to the definition above as a consequence of Theorem 2.5.

Proposition 2.8. *Let X be a metric space.*

(1) *Hausdorff dimension is monotonic:* $\dim A \leq \dim B$ *for* $A \subset B \subset X$
(2) *Hausdorff dimension is* ***countably stable****:* $\dim \bigcup_1^{\infty} A_i = \sup \dim A_i$ *for any sequence of sets* $A_i \subset X$

Proof. To see (1), note that $\mathcal{H}^s(A) \leq \mathcal{H}^s(B)$ for any s since $\mathcal{H}^s$ is an outer measure, thus $\dim A \leq \dim B$.

It follows that $\sup \dim A_i \leq \dim \bigcup_1^{\infty} A_i$. For the reverse inequality, $\mathcal{H}^s(\bigcup_1^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mathcal{H}^s(A_i)$, hence if $s > \sup \dim A_i$ then $\mathcal{H}^s(\bigcup_1^{\infty} A_i) = 0$, thus $\sup \dim A_i \geq \dim \bigcup_1^{\infty} A_i$. ☐

When $s = \dim A$, this does not necessarily imply that $\mathcal{H}^s(A)$ is nonzero or finite. For example, $\dim \mathbb{R}^n = n$ but $\mathcal{H}^n(\mathbb{R}^n) = \infty$. However, if there exists s such that $\mathcal{H}^s(A)$ is nonzero and finite, then s must equal $\dim A$.

2.4. Cantor Set.

Let $0 < \lambda < 1/2$ and $I_{0,1} = [0, 1]$. We form the intervals $I_{1,1} = [0, \lambda]$ and $I_{1,2} = [1 - \lambda, 1]$ by removing the middle from the previous interval. Continue this way by removing the open middle λ sized portion from each previous intervals, so that for any k, we have $I_{k,1} \ldots I_{k,2^k}$ intervals of size λ^k.

We define the Cantor set by

$$C(\lambda) = \bigcap_{k=0}^{\infty} \bigcup_{j=1}^{2^k} I_{k,j}.$$

The Cantor set is an uncountable compact set without any interior points and has zero Lebesgue measure, typically constructed with $\lambda = 1/3$.

FIGURE 1. The first 7 interations of the Cantor set [4]

We now calculate the Hausdorff dimension of the Cantor set. For any $k \in \mathbb{N}$, $C(\lambda) \subset \bigcup_{j=1}^{2^k} I_{k,j}$, hence

$$\mathcal{H}^s_{\lambda^k}(C(\lambda)) \leq \sum_{j=1}^{2^k} d(I_{k,j})^s = 2^k \lambda^{ks} = (2\lambda^s)^k.$$

To make this a useful upper bound, we want it to be bounded as $k \to \infty$, and the smallest s where this holds is when $2\lambda^s = 1$, i.e when $s = \log 2/\log(1/\lambda)$. This gives $\mathcal{H}^s(C(\lambda)) = \lim_{k\to\infty} \mathcal{H}^s_{\lambda^k}(C(\lambda)) \leq 1$, thus $\dim C(\lambda) \leq s$.

Now we show

$$\mathcal{H}^s(C(\lambda)) \geq 1/4, \tag{2.9}$$

which means s is the unique value such that $\mathcal{H}^s(C(\lambda))$ is finite, hence that $\dim C(\lambda) = s$. To prove (2.9), from Theorem 2.4, it suffices to show that for any sequence of open intervals $I_1, I_2 \ldots$ that covers $C(\lambda)$, we have

$$\sum_j d(I_j)^s \geq 1/4. \tag{2.10}$$

To show (2.10), it suffices to show that for any open interval I and any fixed l,

$$\sum_{I_{l,i} \subset I} d(I_{l,i})^s \leq 4d(I)^s. \tag{2.11}$$

It suffices to show this because for any sequence of open intervals I_j that covers $C(\lambda)$,

$$4\sum_j d(I_j)^s \geq \sum_j \sum_{I_{k,i}\subset I_j} d(I_{k,i})^s \geq \sum_{i=1}^{2^k} d(I_{k,i})^s = 1.$$

The first inequality is by assumption since for each open interval I_j we assumed that $4d(I_j)^s \geq \sum_{I_{k,i}\subset I_j} d(I_{k,i})^s$.

The second inequality comes from the fact that each $I_{k,i}$ is contained in some I_j. To see this, note that since $C(\lambda)$ is compact, there exists a finite subcover of I_j, which we can make an arbitrarily small amount larger so that the endpoints of each interval are outside $C(\lambda)$ since the Cantor set has no interior points. Then there exists $\delta > 0$ such that the distance from the endpoints of each interval to $C(\lambda)$ is at least δ, hence for sufficiently large k such that $\delta > \lambda^k$, every interval $I_{k,i}$ is contained in some I_j. Finally, the last equality follows by construction.

To show (2.11), suppose there is at least one interval $I_{l,i}$ in I (otherwise the statement holds since $I_{l,i} \subset I$ is empty) and let n be the smallest integer such that I contains some $I_{n,i}$, hence $n \leq l$. Let $I_{n,1} \dots I_{n,p}$ be all the n-th generation intervals which intersect I. We have that $p \leq 4$ because otherwise there would be some previous generation interval $I_{n-1,i}$ contained in I. Hence

$$4d(I)^s \geq \sum_{m=1}^{p} d(I_{n,m})^s = \sum_{m=1}^{p} \sum_{I_{l,i} \subset I_{n,m}} d(I_{l,i})^s \geq \sum_{I_{l,i} \subset I} d(I_{l,i})^s.$$

2.5. **Self-similar Sets.** We can also calculate the Hausdorff dimension of the Cantor set with a method that works for any self-similar set. In $\mathbb{R}^n$, a set is self-similar if it can be split into parts that are geometrically similar to the entire set. For example, the Cantor set previously can be split into two halves, each geometrically similar to the whole set.

Definition 2.12. A mapping $S : \mathbb{R}^n \to \mathbb{R}^n$ is called a **similitude** if for some $0 < r < 1$, $|S(x) - S(y)| = r|x - y|$ for $x, y \in R^n$.

In other words, a similitude is a map that can be expressed as $S(x) = rg(x) + z$ for $x \in \mathbb{R}^n$ for some orthogonal transformation g, $z \in \mathbb{R}^n$ and $0 < r < 1$. The Cantor set can be expressed in terms of similitudes as $C(\lambda) = S_1(C(\lambda)) \cup S_2(C(\lambda))$, where $S_1, S_2 : \mathbb{R} \to \mathbb{R}$, $S_1(x) = \lambda x$, $S_2(x) = \lambda x + 1 - \lambda$.

Definition 2.13. Let $S = \{S_1, \dots S_N\}$ for $N \geq 2$ be a finite sequence of similitudes with contraction ratios $r_1 \dots r_N$. We say a non-empty compact set K is **invariant** under S if $K = \bigcup_{i=1}^{N} S_i K$

The Cantor set is an example of an invariant set under the sequence of similitudes described previously. Another example of an invariant set under a sequence of similitudes is the Koch snowflake, where at each iteration one replaces the shape with four of the same shapes scaled down by 1/3. The Koch snowflake can be described as $K = S_1K \cup S_2K \cup S_3K \cup S_4K$, where S_1, S_2, S_3, S_4 are similitudes that rotate and translate the set with contraction ratios 1/3.

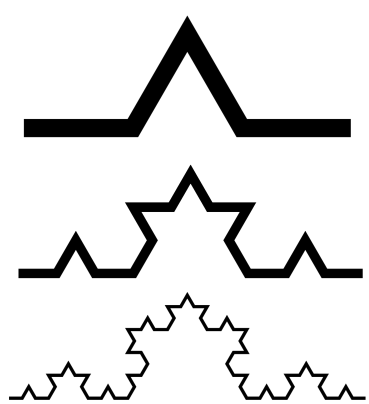

FIGURE 2. The first 3 iterations of the Koch snowflake [5]

For any finite sequence of similitudes, there exists a unique invariant compact set. One way to see this is to note that the set of all non-empty compact subsets of $\mathbb{R}^n$ forms a complete metric space with the Hausdorff metric

$$\rho(E,F) = \max\{d(x,F), d(y,E) : x \in E, y \in F\}.$$

A finite sequence of similitudes S is a contraction mapping with the Hausdorff metric [2], hence it has a unique fixed point by the Banach fixed point theorem, which is the unique invariant set by definition. Furthermore, applying the Banach fixed point theorem again, for any compact set $F \subset \mathbb{R}^n$ we start with, repeatedly applying S to F will make the resulting set $S^n(F)$ converge to K in the Hausdorff metric.

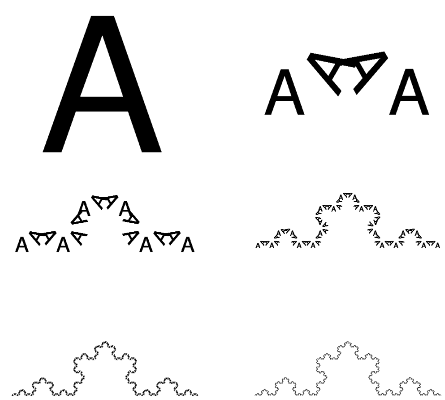

FIGURE 3. The letter A converging to the Koch snowflake [5]

An invariant set K under S is called **self-similar** if for $s = \dim K$, we have $\mathcal{H}^s(S_i(K) \cap S_j(K)) = 0$ for $i \neq j$, which roughly speaking means that K doesn't overlap under the different similitudes in S. However, a stronger separation condition is typically used to define self-similarity called the open set condition.

Definition 2.14. A finite sequence of similitudes S is said to satisfy the **open set condition** if there exists a non-empty open set O such that $\bigcup_{i=1}^N S_i(O) \subset O$ and $S_i(O) \cap S_j(O) = \emptyset$ for $i \neq j$.

The sequence of similitudes for the Koch snowflake is an example that satisfies the open set condition; the open triangle formed by the leftmost, top, and rightmost points satisfies this condition. The Cantor set can also be seen to satisfy the open set condition by considering the interval $(0,1)$. This condition is useful as for a finite sequence of similitudes S that satisfies the open set condition, the dimension of K is uniquely determined by the contraction ratios of S.

Theorem 2.15. *Let S be a finite sequence of similitudes satisfying the open set condition. Then the invariant set K is self-similar and $0 < \mathcal{H}^s(K) < \infty$ where $s = \dim K$. Furthermore, s is the unique number such that $\sum_{i=1}^N r_i^s = 1$.*

Proof. See Matilla [2]. □

This gives a much simpler method of calculating the Hausdorff dimension of a self-similar set. For S such that $r_1 \dots r_N = r$, we have that $\dim K = \log N / \log(1/r)$, which matches with what was previously proved about the Cantor set. For the Koch snowflake, this gives $\dim K = \log 4 / \log 3$.

3. Minkowski Dimension

Another common way to define the dimension of a set is the Minkowski dimension, which is defined using coverings with open balls instead of coverings of arbitrarily small sets. While the Minkowski dimension can be applied to subsets of any metric space, for this section we restrict our discussion to subsets of $\mathbb{R}^n$.

Definition 3.1. Let A be a non-empty bounded subset of $\mathbb{R}^n$ and for $0 < \epsilon < \infty$, define $N(A, \epsilon)$ to be the smallest number of ϵ-balls needed to cover A, or

$$N(A, \epsilon) = \min \left\{ k : A \subset \bigcup_{i=1}^{k} B_\epsilon(x_i) \right\}.$$

The **upper** and **lower Minkowski dimensions** of A are defined as

$$\overline{\dim}_M A = \inf \left\{ s : \limsup_{\epsilon \to 0^+} N(A, \epsilon)\epsilon^s = 0 \right\}$$

and

$$\underline{\dim}_M A = \inf \left\{ s : \liminf_{\epsilon \to 0^+} N(A, \epsilon)\epsilon^s = 0 \right\}$$

Equivalently, we have that

$$\begin{aligned} \overline{\dim}_M A &= \inf \left\{ s : \limsup_{\epsilon \to 0^+} N(A, \epsilon)\epsilon^s < \infty \right\} \\ &= \sup \left\{ s : \limsup_{\epsilon \to 0^+} N(A, \epsilon)\epsilon^s = \infty \right\} \\ &= \sup \left\{ s : \limsup_{\epsilon \to 0^+} N(A, \epsilon)\epsilon^s > 0 \right\}, \end{aligned}$$

and similarly for $\underline{\dim}_M A$.

Minkowski dimension can also equivalently be formulated as

$$\overline{\dim}_M A = \limsup_{\epsilon \to 0^+} \frac{\log N(A, \epsilon)}{\log(1/\epsilon)},$$

$$\overline{\dim}_M A = \liminf_{\epsilon \to 0^+} \frac{\log N(A, \epsilon)}{\log(1/\epsilon)}.$$

Intuitively, Minkowski dimension is defined this way because one would expect the number of ϵ-balls required to cover A to grow proportionally to the dimension of A; for some constant c, a line would require c/ϵ ϵ-balls, a sphere would require c/ϵ^2, etc.

It follows that

$$\dim A \leq \underline{\dim}_M A \leq \overline{\dim}_M A.$$

To see the first inequality, note that for any s such that $\liminf_{\epsilon \to 0} N(A, \epsilon)\epsilon^s = 0$, we also have that $\mathcal{H}^s(A) = 0$ since any covering that works for the Minkowski dimension also works for the Hausdorff dimension.

Interestingly, these inequalities can be strict even for rather uncomplicated sets. Consider $A = \{0\} \cup \{1/k : k \in \mathbb{N}\}$, a countable compact set. $\dim A = 0$ since A is countable but $\underline{\dim}_M A = 1/2$. To see this, fix any $0 < \epsilon < 1/2$ and consider $n \in \mathbb{N}$

such that $1/(n+1)^2 < 2\epsilon < 1/n^2$. The points in $\{0\} \cup \{1/k : k > n\}$ can be covered in $n+1$ ϵ-balls and the remaining n points can be covered in n ϵ-balls. Then

$$N(A,\epsilon) \leq 2n+1 \leq \frac{2n+1}{n}(1/\epsilon)^{1/2},$$

hence $N(A,\epsilon)\epsilon^s \leq \frac{2n+1}{n}\epsilon^{s-1/2}$, so $\overline{\dim}_M A \leq 1/2$.

To see the reverse inequality, note that the distance between neighboring points is

$$\frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)} \geq \frac{1}{(k+1)^2},$$

so at least $n-1$ ϵ-balls are required to cover A because the first n points must be covered individually. Since

$$N(A,\epsilon) \geq n-1 \geq \frac{n-1}{n+1}(1/\epsilon^{1/2}),$$

we have $\underline{\dim}_M A \geq 1/2$. One can then observe that Minkowski dimension is not countably stable like Hausdorff dimension as described in Proposition 2.8.

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