Phase Transitions in Random-Cluster Models

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Abstract

This expository paper studies phase transitions in Bernoulli bond percolation and the Fortuin–Kasteleyn random-cluster model on transitive graphs. Criteria for the existence of a supercritical phase are established: we prove that $p_c < 1$ if and only if the number of minimal cutsets from the origin grows at most exponentially with their size, and in particular that $p_c < 1$ for every uniformly transient infinite graph. Next, we prove sharpness of the phase transition – namely, exponential decay of connection probabilities for $p < p_c$ and linear growth of the infinite-cluster density for $p > p_c$ – by an OSSS-informed decision tree approach. Finally, using a recent differential inequality for cluster volumes, we derive new inequalities relating critical exponents and show that in the entire subcritical regime the cluster-size distribution has an exponential tail.

Contents

1 Introduction

This article provides an exposition of several recent advances in the study of phase transitions in Bernoulli bond percolation and random-cluster models. We have two main goals: (i) to present rigorous criteria for the existence of a nontrivial supercritical phase; and (ii) to provide proofs of the sharpness of the phase transition, together with new inequalities relating the critical exponents. To this end, we combine geometric arguments (Peierls'-type estimates), probabilistic inequalities (the OSSS decision-tree method), and analytic techniques (differential inequalities for cluster volumes).

Classical results on percolation include Peierls' 1936 argument establishing $p_c < 1$ for $\mathbb{Z}^2$ [Pei36], the works of Menshikov [Men86] and Aizenman-Barsky [AB87] proving sharpness of the transition on $\mathbb{Z}^d$. More recently, Duminil-Copin, Raoufi, and Tassion [DRT19] introduced the decision-tree method (based on the OSSS inequality [OSSS05]), which can be extended to random-cluster models. On the geometric side, Babson and Benjamini (1999) conjectured that $p_c < 1$ if and only if the number of minimal cutsets grows at most exponentially; this conjecture was finally confirmed by Easo, Severo, and Tassion [EST24]. On the analytic side, Hutchcroft [Hut20] developed new volume-based differential inequalities, from which follow both exponential subcritical decay of cluster volumes and new universal inequalities between critical exponents.

Organization of the paper. In Section 2 we recall the definitions and basic properties of Bernoulli bond percolation and the random-cluster model. Section 3 is devoted to the existence of phase transitions: we review Peierls" classical argument, state and explain the converse Peierls' theorem of Easo-Severo-Tassion, and deduce that all uniformly transient graphs satisfy $p_c < 1$. Section 4 turns to the sharpness of the transition with respect to radii of large open clusters: we present the decision-tree/OSSS approach, and indicate the extension to random-cluster models with $q \ge 1$. Section 5 discusses Hutchcroft's volume-based differential inequality and derives the critical exponent inequalities $\gamma \le \delta - 1$ and $\Delta \le \gamma + 1$, together with the exponential decay of cluster volumes in the subcritical phase.

2 Background

In this section, we introduce Bernoulli bond percolation, random-cluster models, and related terminologies and state a few well-known properties that will be important to us. An interested reader may look into

Grimmett's books [Gri99][Gri06] for more details.

2.1 Bernoulli Percolation

Definition 2.1. Consider an infinite, connected, transitive graph G with maximal degree$\leq d$. Let E be its edge set and V be its vertex set. For a given $p \in [0,1]$, each edge $e \in E$ has a probability of p to be open and a chance of $1-p$ to be closed. Formally, there is a family of i.i.d. random variables $\{X_e^p\}_{e\in E}$ with $X_e^p \sim Ber_p$ for all e, so that $X_e^p = 1$ implies edge e is open and $X_e^p = 0$ implies edge e is closed. Define O as the collection of all open edges. Such (G, O) is called a (bond) Bernoulli percolation model.

Notice that in this process of constructing O, there will be (random) open paths (for example, there may be vertices x and y connected by a path consisting solely of open edges) and thus open clusters (sets of vertices each pair of which is connected by an open path).

Denote x, y connected by an open path as $x \leftrightarrow y$ and open cluster containing x as $C(x)$.

The rest of the section often assumes without loss of generality that $G = \mathbb{Z}^d$ (the d-dimensional cubic lattice).

Definition 2.2. (Percolation probability space). Call a state of percolation processes a configuration (which is to assign a 0/1 (closed/open) value to each edge in E). Let $\Omega = \prod_{e\in E}\{0,1\}$ be the space of all configurations.

Let $\mathcal{F}$ be the $\sigma-$algebra generated by finite cylinder sets $\left\{\prod_{e\in F}\{0,1\} : F \subseteq E, |F| < \infty\right\}$. The probability measure on $(\Omega, \mathcal{F})$ is the product measure P_p characterized by $P_p(\omega : \omega(e) = 1) = p$ and $P_p(\omega : \omega(e) = 0) = 1-p$ for each edge e (independently). In other words, $P_p = \bigotimes_{e\in E} \text{Bernoulli}(p)$ on Ω. We will work with the probability space $(\Omega, \mathcal{F}, P_p)$ for a fixed value of p.

Definition 2.3. Let $\Lambda(x, n)$ denote the box (cube) of side length $2n$ centered at x in G (in particular, $\Lambda(0, n)$ is the box of radius n around the origin). We abbreviate $\Lambda_n := \Lambda(0, n) = [-n, n]^d$.

Define $\theta_n(p) = P_p(0 \leftrightarrow \partial\Lambda_n), \theta(p) = P_p(0 \leftrightarrow \infty)$. That is, $\theta(p)$ is the probability that 0 is in an infinite open cluster.

Define $\psi(p) = P[\bigcup_{x\in\mathbb{Z}^d}\{|C^p(x)| = \infty\}]$. That is, $\psi(p)$ is the probability that there exists an infinite open cluster.

It is easy to see that $\theta(p)$ is a nondecreasing function of p, since increasing p (making each edge more likely to be open) can only increase the chance of an infinite cluster.

With the help of Kolmogorov's 0-1 Law, we can prove the following lemma which says that the positivity of $\theta(p)$ is enough to determine the existence of an infinite open cluster.

Lemma 2.4. *For any* $p \in [0,1]$, $\psi(p) = \begin{cases} 0 & (\text{if } \theta(p) = 0) \\ 1 & (\text{if } \theta(p) > 0) \end{cases}$

Since $p \mapsto \theta(p)$ is monotone increasing, $\psi(p)$ stays 0 until p increases to some point such that $\theta(p)$ is nonzero, and then $\psi(p)$ stays 1. Therefore, it is natural to use infimum to define that threshold.

Definition 2.5. (Critical value). The critical value $p_c(d)$ for the percolation model in $\mathbb{Z}^d$ is defined as $p_c(d) := \inf\{p \in [0,1] : \theta(p) > 0\} = \inf\{p \in [0,1] : \psi(p) = 1\}$.

Theorem 2.6. *(Existence of phase transitions in Bernoulli percolation models). Given $G = \mathbb{Z}^d$ with $d \geq 2$. There exists a critical probability $p_c \in (0,1)$ such that for all $p > p_c$, $\theta(p) = 1$ (supercritical phase); for all $p < p_c$, $\theta(p) = 0$ (subcritical phase). Thus, we say that the Bernoulli bond percolation has a unique phase transition.*

We will prove this theorem in a more general form in Section 3.1.

2.2 Random Cluster Model

We will first work on a finite graph $G = (V, E)$ and later pass to the limit $G \nearrow \mathbb{Z}^d$.

For a configuration $\omega \in \{0,1\}^E$ write $o(\omega) = |\{e \in E : \omega_e = 1\}|$ and $c(\omega) = |E| \setminus o(\omega)$ for the numbers of open and closed edges. For a boundary condition $\# \in \{\mathrm{f}, \mathrm{w}\}$ (free or wired), let $k_\#(\omega)$ be the number of connected components ("clusters") of the open subgraph. In the wired case $\# = \mathrm{w}$, all vertices on the boundary of the finite graph G are identified as a single "wire" vertex, which counts as one component if it is occupied; In the free case $\# = \mathrm{f}$, the boundary condition is unconstrained.

Definition 2.7 (Random-cluster measure)**.** Fix $q > 0$ and $p \in [0,1]$. The (finite-volume) random-cluster measure with boundary condition $\#$ is the probability measure

$$\phi^{\#}_{G,p,q}(\omega) = \frac{1}{Z^{\#}_{G,p,q}} p^{o(\omega)}(1-p)^{c(\omega)} q^{k_\#(\omega)}, \qquad \omega \in \{0,1\}^E,$$

where $Z^{\#}_{G,p,q}$ is the normalizing constant. When $q = 1$, the factor $q^{k_\#(\omega)} \equiv 1$ and $\phi^{\#}_{G,p,1}$ is the Bernoulli bond percolation measure with edge-parameter p (independent edges).

On $\mathbb{Z}^d$ with $d \geq 2$ and $q \geq 1$, the measures $\phi^{\mathrm{f}}_{\Lambda,p,q}$ and $\phi^{\mathrm{w}}_{\Lambda,p,q}$ on boxes $\Lambda \subset \mathbb{Z}^d$ have the FKG property and are monotone in Λ. Hence the thermodynamic limits $\phi^{\mathrm{f}}_{p,q}$ and $\phi^{\mathrm{w}}_{p,q}$ exist. Define the (percolation) order parameter

$$\theta^{\#}(p,q) := \phi^{\#}_{p,q}(0 \leftrightarrow \infty), \qquad \# \in \{\mathrm{f}, \mathrm{w}\}.$$

Since the event $\{0 \leftrightarrow \infty\}$ is increasing, $p \mapsto \theta^{\#}(p,q)$ is nondecreasing for fixed $q \geq 1$.

Definition 2.8 (Critical value)**.** For $q \geq 1$, set

$$p_c(q) := \inf\{p \in [0,1] : \theta^{\mathrm{w}}(p,q) > 0\}.$$

Theorem 2.9 (Existence of a phase transition for the Random-Cluster Model on $\mathbb{Z}^d$)**.** *Fix $d \geq 2$ and $q \geq 1$. Then $p_c(q) \in (0,1)$ and*

$$\theta^{\mathrm{w}}(p,q) = 0 \text{ for } p < p_c(q), \qquad \theta^{\mathrm{w}}(p,q) > 0 \text{ for } p > p_c(q).$$

In particular, the RCM exhibits a (unique) phase transition at $p_c(q)$.

Proof. For $q = 1$ (Bernoulli bond case) we know $p_c(1) \in (0, 1)$ on $\mathbb{Z}^d$. A comparison inequality (Theorem 5.5 in [Gri06]) states that for $1 \leq q' \leq q$,

$$\frac{1}{p_c(q)} \leq \frac{1}{p_c(q')} \leq \frac{q/q'}{p_c(q)} - \frac{q}{q'} + 1.$$

Taking $q' = 1$ gives $1/p_c(q) \leq 1/p_c(1)$, hence $p_c(q) \geq p_c(1) > 0$, and also $\frac{1}{p_c(1)} \leq \frac{q}{p_c(q)} - q + 1$, which implies $\frac{1}{p_c(q)} > 1$ (since $p_c(1) < 1$), so $p_c(q) < 1$. Monotonicity of $\theta^{\#}(p, q)$ in p and the definition of $p_c(q)$ yield the subcritical and supercritical statements. □

Remark 2.10. The terminology "percolation probability" refers only to the connectivity event under the RCM measure; the model itself is not Bernoulli unless $q = 1$. The $p(1-p)$ parametrization is equivalent to the usual $\prod_{xy}(e^{\beta J_{xy}} - 1)^{\omega_{xy}}$ form via the change of variables $p_{xy} = 1 - e^{-\beta J_{xy}}$ on unweighted graphs.

The following alternate definition is often used in analytical contexts.

Definition 2.11. (Alternate definition for RCM). Given a finite subgraph $G = (V, E)$ of a weighted lattice $(\mathbb{G}, \{J_{xy}\}_{xy \in \mathbb{E}})$. For a configuration $\omega \in \{0, 1\}^E$, let $k_f(\omega)$ be the number of connected components in the graph induced by ω, and $k_w(\omega)$ be the number of connected components in the graph induced by ω by considering all vertices in ∂G as one single vertex.

Fix $q, \beta > 0$. Define the random-cluster measure on G with free boundary conditions as the probability measure satisfying, for all $\omega \in \{0, 1\}^E$,

$$\phi^{\mathrm{f}}_{G,\beta,q}(\omega) = \frac{q^{k_f(\omega)}}{Z} \prod_{xy \in E} (e^{\beta J_{xy}} - 1)^{\omega_{xy}}$$

where Z is a normalizing constant. Similarly, we define the random-cluster measure on G with wired boundary conditions $\phi^{\mathrm{w}}_{G,\beta,q}$ by replacing $k_f(\omega)$ with $k_w(\omega)$.

3 Existence of Phase Transitions

We first present the classical Peierls' argument in Section 3.1, which is $0 < p_c < 1$ in certain graphs. We then show the converse of the Peierls' argument and a very general sufficient graph condition for $p_c < 1$ in Section 3.2.

3.1 Peierls' Argument

Throughout this subsection let $G = (V, E)$ be an infinite connected graph with $\deg v \leq D < \infty$ for all $v \in V$, and let $C(x)$ be the open cluster of x in Bernoulli bond percolation with parameter $p \in [0, 1]$.

This subsection aims to provide a bound for the critical value in percolation.

Theorem 3.1 (Lower bound)**.** *Let G be an infinite connected graph with maximal degree D. Then*

$$p_c(G) \geq \frac{1}{D-1}.$$

Proof. Fix a vertex x. For $n \geq 1$, let S_n be the set of self-avoiding paths of length n starting at x, and put $\mu_n := |S_n|$. We have the crude bound $\mu_1 \leq D$ and $\mu_n \leq D(D-1)^{n-1}$ for $n \geq 1$, hence $\mu := \limsup_{n\to\infty} \mu_n^{1/n} \leq D - 1$.

Let $B(x, n)$ be the ball of graph distance $\leq n$ about x (not box $\Lambda(x, n)$. If $x \leftrightarrow \partial B(x, n)$ occurs, then there exists an open self-avoiding path of length exactly n from x, so by a union bound

$$P_p\big(x \leftrightarrow \partial B(x, n)\big) \leq \mu_n p^n.$$

Since $\{x \leftrightarrow \infty\} \subseteq \{x \leftrightarrow \partial B(x, n)\}$ for every n and the latter events decrease to $\{x \leftrightarrow \infty\}$, continuity from above gives

$$P_p(x \leftrightarrow \infty) = \lim_{n\to\infty} P_p\big(x \leftrightarrow \partial B(x, n)\big) \leq \limsup_{n\to\infty} \mu_n p^n.$$

If $p < \mu^{-1}$ (in particular, if $p < 1/(D - 1)$), the right-hand side is 0. Hence $P_p(x \leftrightarrow \infty) = 0$ and so $p_c(G) \geq 1/(D - 1)$. □

Definition 3.2 (Cut-set and minimal cut-set)**.** Fix $x \in V$. A <u>cut-set</u> (for x) is a set of edges $\Pi \subset E$ such that every infinite self-avoiding path starting at x uses at least one edge of Π. A cut-set Π is <u>minimal</u> if no proper subset of Π is a cut-set.

Remark 3.3. Every finite cut-set contains a minimal cut-set (remove edges one by one while preserving the cut-set property).

Lemma 3.4. *For Bernoulli bond percolation on G and $x \in V$,*

$$x \leftrightarrow \infty \quad \Longleftrightarrow \quad \textit{every finite minimal cut-set } \Pi \textit{ contains at least one open edge.}$$

Equivalently, $C(x)$ is finite if and only if there exists a finite cut-set all of whose edges are closed.

Proof. If $C(x)$ is finite, let Π be the set of edges with exactly one endpoint in $C(x)$:

$$\Pi = \{yz \in E : y \in C(x), z \notin C(x)\}.$$

Then Π is finite, all its edges are closed, and it is a cut-set for x. Conversely, if there exists a finite cut-set Π all of whose edges are closed, then no open path from x can cross Π, so $C(x)$ is contained in the finite component of $G \setminus \Pi$ containing x, hence finite. Taking contrapositives yields the stated equivalence. □

Theorem 3.5 (Peierls' upper bound via cut-set counting)**.** *Suppose there exist $n_0 \in \mathbb{N}$ and $M \geq 1$ such that, for all $n \geq n_0$, the number $\mathcal{C}_n$ of minimal cut-sets for x of size n satisfies $|\mathcal{C}_n| \leq M^n$. Then*

$$p_c(G) \leq 1 - \frac{1}{M}.$$

Proof. Fix $p > 1 - \frac{1}{M}$, so $M(1-p) < 1$. For $N \geq n_0$, let

$$A_N := \Big\{\exists n > N, \exists \Pi \in \mathcal{C}_n \text{ with all edges of } \Pi \text{ closed}\Big\}.$$

By a union bound,

$$P_p(A_N) \leq \sum_{n>N} |\mathcal{C}_n|(1-p)^n \leq \sum_{n>N} \big(M(1-p)\big)^n.$$

Since $M(1-p) < 1$, choose N so large that $P_p(A_N) \leq \frac{1}{2}$.

Let $S = \{\Pi \in \mathcal{C}_n : n \leq N\}$ and $E_S = \bigcup_{\Pi \in S} \Pi$, a finite set of edges. Define the increasing event $B := \{\text{all edges in } E_S \text{ are open}\}$. Then $P_p(B) > 0$, and by FKG (both B and A_N^{c} are increasing),

$$P_p(B \cap A_N^{\mathrm{c}})_p(B)P_p(A_N^{\mathrm{c}}) = P_p(B)(1 - P_p(A_N)) \geq \tfrac{1}{2}P_p(B) > 0.$$

On $B \cap A_N^{\mathrm{c}}$ there is no finite minimal cut-set with all edges closed (sizes $\leq N$ are ruled out by B, and sizes $> N$ by A_N^{c}). By Lemma 3.4, $x \leftrightarrow \infty$ on $B \cap A_N^{\mathrm{c}}$. Hence $P_p(x \leftrightarrow \infty) > 0$, so $p > p_c(G)$. This holds for every $p > 1 - \frac{1}{M}$, giving $p_c(G) \leq 1 - \frac{1}{M}$. □

Example 3.6 (Regular tree)**.** Let $\mathbb{T}_d$ be the d-regular tree with $d \geq 2$. Then

$$p_c(\mathbb{T}_d) = \frac{1}{d-1}.$$

Proof. The lower bound $p_c \geq 1/(d-1)$ follows from Theorem 3.1. For the upper bound, explore $C(o)$ away from the root o. Off the edge to its parent, each vertex has $d-1$ children, and the subtree edges are independent; thus $|C(o)|$ is dominated by a Galton–Watson process with offspring $\binom{d-1}{p}$. The survival probability is positive if and only if $(d-1)p > 1$, i.e. $p > 1/(d-1)$. Hence $p_c \leq 1/(d-1)$, and the two bounds match. □

3.2 Counting Minimal Cutsets and $p_c < 1$

This section is based on the paper [EST24] by P. Easo, F. Severo, and V. Tassion, which develops a connection between the geometry of cutsets and the existence of a nontrivial phase transition. Two main contributions are made:

1. The classical Peierls' argument shows that if the number of minimal cutsets grows at most exponentially in their size, then $p_c(G) < 1$. The authors prove the exact **converse**: whenever $p_c(G) < 1$, the number of minimal cutsets from the root to infinity grows at most exponentially.

2. The authors give a very general sufficient condition for the critical probability $p_c < 1$. This condition is phrased in terms of **uniform transience** of the underlying graph, and it applies to a broad family of infinite connected locally finite graphs.

In what follows, we introduce the key definitions (minimal cutsets, exposed boundaries, and the growth constant $\kappa(G)$) and then present the main theorems together with the essential ideas of their proofs.

Let $G = (V, E)$ be an infinite, connected, locally finite graph.

Definition 3.7. A set of edges $E' \subseteq E$ is called a cutset from a vertex o to infinity if removing E' disconnects o from infinity – equivalently, o lies in a finite connected component of $G \setminus E'$. Such a cutset E' is minimal if no proper subset of E' is still a cutset from o to infinity.

We denote by $Q_n(v)$ the set of all minimal cutsets from v to infinity of size n, and $\mathcal{C}_n(v) = |Q_n(v)|$. Let $q_n = \sup_{v \in V} \mathcal{C}_n(v)$. The growth rate of the number of minimal cutsets is measured by the quantity

$$\kappa(G) := \sup_{n \geq 1} q_n^{1/n}$$

which may be finite or $+\infty$. In particular, $\kappa(G) < \infty$ means that the number of minimal cutsets grows at most exponentially in n.

Definition 3.8. Define the critical percolation threshold (for bond percolation) by

$$p_c(G) = \inf\{p \in [0,1] : P_p(o \leftrightarrow \infty) > 0\}$$

where $P_p(o \leftrightarrow \infty)$ is the probability (with edges open independently with probability p) that o lies in an infinite open cluster. We say $p_c(G) < 1$ if there exists some $p < 1$ for which an infinite open cluster occurs with positive probability (a supercritical percolation phase on G).

Definition 3.9. For a finite set of vertices $A \subset V$, the exposed boundary $\partial_\infty A$ is the set of all edges with one endpoint in A and one endpoint in $V \setminus A$.

3.2.1 Converse of Peierls' Argument

Our first main result is that the exponential growth of cutsets provides a sharp criterion for the existence of a supercritical percolation phase

Theorem 3.10 (Criterion for $p_c < 1$ via Cutset Counting)**.** *For every infinite, locally finite graph G, the critical probability $p_c(G) < 1$ if and only if $\kappa(G) < \infty$. In other words, G has a supercritical percolation phase if and only if the number of minimal cutsets from o to infinity grows at most exponentially in n.*

Remark 3.11 (Intuition behind Theorem 3.10). The equivalence $p_c(G) < 1 \iff \kappa(G) < \infty$ can be understood as follows.

(1) The easy direction (Peierls' argument). If the number of minimal cutsets grows at most exponentially ($\kappa(G) < \infty$), then a union bound shows that for p close to 1, the probability that the cluster of the origin o is surrounded by a closed cutset is very small. Thus the origin percolates with positive probability, implying $p_c(G) < 1$.

(2) The difficult direction (converse). Suppose $p_c(G) < 1$. Then for some $p < 1$ one has $P_p(o \leftrightarrow \infty) > 0$. If there were "too many" (super-exponentially many) minimal cutsets, then with high probability o would be trapped inside one of them, contradicting survival. The heart of the proof is to quantify this heuristic.

- Lemma 3.12. The exposed boundary of any finite connected set $A \ni o$ is a minimal cutset. Thus minimal cutsets can always be realized as exposed boundaries of clusters.

- Lemma 3.13. For a minimal cutset Π, let A be the finite component of o in $G \setminus \Pi$ and let B be the set of inner vertices of Π. Then for any S with $B \subset S \subset A$ one has $\partial_\infty S = \Pi$. Hence minimal cutsets can be detected through the set B of their inner vertices.

- Lemma 3.14. In a finite graph with positive association, if each vertex connects to B with probability at least θ, and each edge is open with probability at least p, then the origin simultaneously connects to all vertices of B with probability at least $c^{|B|}$ for some $c(p, \theta) > 0$. The proof constructs a maximal chained sequence of vertices, serving as a probabilistic bottleneck. This guarantees two facts: (P2) every vertex connects to the sequence with probability $\geq \theta/2$, and (P3) the sequence is not too long ($\leq 2|B|/\theta$). Combining these yields a uniform exponential lower bound for the probability that o connects to all of B.

(3) Conclusion. For a minimal cutset Π of size n with inner vertices B, the probability that $\partial_\infty C(o) = \Pi$ is at least $c^n(1-p)^n$. Since these boundary events are disjoint, summing over all $\Pi \in \mathcal{C}_n$ gives

$$1 \geq \sum_{\Pi \in \mathcal{C}_n} P_p(\partial_\infty C(o) = \Pi) \geq |\mathcal{C}_n| (c(1-p))^n.$$

Thus $|\mathcal{C}_n| \leq (c(1-p))^{-n}$, showing that the number of minimal cutsets grows at most exponentially, i.e. $\kappa(G) < \infty$.

Lemma 3.12 (Exposed Boundary as a Minimal Cutset)**.** *If $A \subset V$ is a finite connected set of vertices containing o, then $\partial_\infty A$ is a minimal cutset from o to infinity.*

Proof. Any path from o to infinity must exit A, so it uses some edge of $\partial_\infty A$. Thus $\partial_\infty A$ separates o from infinity, i.e. it is a cutset. It is minimal because if any edge $e \in \partial_\infty A$ were removed (opened), then since A is connected there would be a path from o through A and then through e to the outside of A, allowing o to reconnect to infinity. So no proper subset of $\partial_\infty A$ can disconnect o from infinity. □

Lemma 3.13 (Inner/outer description of a minimal cutset)**.** *Let $u \in V$ and Π a minimal cutset from u to ∞. Let A be the connected component of u in $(V, E \setminus \Pi)$ and $B = \{e \cap A, e \in \Pi\}$ be the set of inner vertices of Π. For all $S \subset V$, when $B \subset S \subset A$, we have $\partial_\infty S = \Pi$.*

Proof. Since $A \subseteq (V, E \setminus \Pi)$, we have $\partial_\infty A \subseteq \partial A \subseteq \Pi$. By Lemma 3.12, because Π separates u from ∞ and is minimal, we have $\partial_\infty A = \partial A = \Pi$.

Fix S with $B \subset S \subset A$.

(i) $\Pi \subset \partial_\infty S$. Let $e = xy \in \Pi$ with $x \in A$ the inner endpoint and $y \notin A$. Since $x \in B \subset S$, we have $x \in S$ while $y \notin A \supset S$, so $y \in V \setminus S$. In $(V, E \setminus S)$ the vertex y still lies in the unbounded component (because every path from y to ∞ avoids A, hence avoids $S \subset A$). Thus $y \leftrightarrow \infty$ in $V \setminus S$ and therefore $e \in \partial_\infty S$ by definition of the exposed boundary. Hence $\Pi \subset \partial_\infty S$.

(ii) $\partial_\infty S \subset \Pi$. Let $e = xy \in \partial_\infty S$ with $x \in S$ and $y \notin S$ and such that $y \leftrightarrow \infty$ in $V \setminus S$. If $y \in A$, then y would lie in the finite set $A \setminus S$ and hence could not be connected to ∞ in $V \setminus S$—a contradiction. Thus $y \notin A$, which forces $x \in A$ (since $S \subset A$). Therefore $e \in \partial_\infty A = \Pi$. This shows $\partial_\infty S \subset \Pi$.

Combining (i) and (ii) yields $\partial_\infty S = \Pi$. □

Lemma 3.14. *Let G be a finite, connected graph. Let P be a positively associated percolation measure on G. Let $B \subset V$ and $\theta, p \in (0, 1]$, and suppose that*

$$P(v \leftrightarrow B) \geq \theta \quad \textit{for all } v \in V, \qquad P(e \textit{ is open}) \geq p \quad \textit{for all } e \in E. \tag{1}$$

Then for all $o \in V$,

$$P(\bigcap_{b \in B} \{o \leftrightarrow b\}) \geq c^{|B|}$$

where $c = (\frac{p\theta}{2})^{3/\theta}$

Proof. Say that a finite sequence of vertices $x_1, \ldots, x_k$ is chained if $x_1 = o$ and, for all $i \geq 2$,

$$\frac{p\theta}{2} \leq P\big(x_i \leftrightarrow \{x_1, \ldots, x_{i-1}\}\big) \leq \frac{\theta}{2}. \tag{2}$$

There exists at least one chained sequence (take $k = 1$), and since V is finite we can take a maximal chained sequence $x_1, \ldots, x_k$ in the sense that for every $x_{k+1} \in V$ the sequence $x_1, \ldots, x_{k+1}$ fails (2). Set $X := \{x_1, \ldots, x_k\}$ and let $n := |B|$. We claim the following two properties:

(P1) For every $v \in V$, $P(v \leftrightarrow X) \geq \frac{\theta}{2}$

(P2) One has $k \leq \dfrac{2n}{\theta}$.

Proof of (P1). Let $W := \{v \in V : P(v \leftrightarrow X) \geq \theta/2\}$. Clearly W is nonempty (it contains X). If $W \neq V$, since G is connected there exists an edge uv with $u \in W$ and $v \notin W$. By positive association,

$$P(v \leftrightarrow X) \geq P(uv \text{ open}) P(u \leftrightarrow X) \geq p \cdot \frac{\theta}{2} = \frac{p\theta}{2}.$$

Because $v \notin W$, we also have $P(v \leftrightarrow X) < \theta/2$. Thus $x_1, \ldots, x_k, v$ would satisfy (2), contradicting the maximality of the chained sequence. Hence $W = V$ and (P1) holds.

Proof of (P2). For $i \in \{1, \ldots, k\}$ let N_i be the number of (open) clusters that intersect both $\{x_1, \ldots, x_i\}$ and B. Then $N_1 = \mathbf{1}_{\{x_1 \leftrightarrow B\}}$ and, for $i \geq 2$,

$$N_i - N_{i-1} \geq \mathbf{1}_{\{x_i \leftrightarrow B\}} - \mathbf{1}_{\{x_i \leftrightarrow \{x_1, \ldots, x_{i-1}\}\}}.$$

Taking expectations and using the hypotheses together with (2),

$$\mathbb{E}[N_i] - \mathbb{E}[N_{i-1}] \geq \theta - \frac{\theta}{2} = \frac{\theta}{2} \qquad (i \geq 2),$$

and $\mathbb{E}[N_1] \geq \theta/2$. Summing gives $\mathbb{E}[N_k] \geq \theta k/2$. Deterministically $N_k \leq |B| = n$, hence $k \leq 2n/\theta$, proving (P2).

By FKG and (2),

$$P\Big(\bigcap_{u\in X}\{o \leftrightarrow u\}\Big) \geq \prod_{i=2}^{k} P\big(x_i \leftrightarrow \{x_1,\dots,x_{i-1}\}\big) \geq \Big(\frac{p\theta}{2}\Big)^{k-1} \geq \Big(\frac{p\theta}{2}\Big)^{2n/\theta}.$$

By (P1) and FKG again,

$$P\Big(\bigcap_{b\in B}\{b \leftrightarrow X\}\Big) \geq \Big(\frac{\theta}{2}\Big)^{n}.$$

Since the intersection of the two events implies $\bigcap_{b\in B}\{o \leftrightarrow b\}$, we obtain

$$P\Big(\bigcap_{b\in B}\{o \leftrightarrow b\}\Big) \geq \Big(\frac{p\theta}{2}\Big)^{2n/\theta}\Big(\frac{\theta}{2}\Big)^{n} \geq \Big(\frac{p\theta^2}{2}\Big)^{3n/\theta} = c^{|B|}.$$

(The last inequality uses $(\theta/2) \geq (\theta^2/2)^{1/\theta}$ for $\theta \in (0,1]$, which is elementary and recorded in the paper.) $\square$

Remark 3.15. The "maximal chained sequence" is like probabilistically growing a fence until further growth would violate the connection bounds. This ensures that X is a reliable bottleneck: every vertex must attach to it with decent probability, but X itself is not too large, so probabilities do not collapse.

Proof of Theorem 3.10. The Peierls' argument states that if $\kappa(G) < \infty$, $p_c(G) < 1$, so we will mainly focus on the forward direction.

We will prove the form using $K = M$ instead of κ.

Fix $\theta, p \in (0,1)$ s.t. $P_p(u \leftrightarrow \infty) \geq \theta$ for all $u \in V$. Fix $o \in V$ and $n \geq 1$. Writing $C = C(o)$. We claim that for all minimal cutset Π from o to ∞ of size n satisfies

$$P_p(\partial_\infty C = \Pi) \geq K^{-n}$$

where $K = K(p,\theta) \in (0,\infty)$ is a finite constant. Then we will have

$$1 \geq \sum_{\Pi\in Q_n(o)} P_p(\partial_\infty C = \Pi) \geq |Q_n(o)|/K^n.$$

Let A be the connected component of o in $(V, E\setminus\Pi)$ and B the set of inner vertices of Π. Any infinite open path from a vertex $u \in A$ must intersect B before exiting A, so

$$\forall u \in A : P_p(u \overset{A}{\leftrightarrow} B) \geq P_p(u \leftrightarrow \infty) \geq \infty$$

Let $E = \{\forall v \in B : v \overset{A}{\leftrightarrow} o\}$, so by Lemma 3.14 on the subgraph induced by A, we have $P_p(E) \geq c^n$ where $c = (\frac{p\theta}{2})^{3/\theta} > 0$. Let $F = \{\text{all edges of } \Pi \text{ closed}\}$, so

$$P_p(E\cap F) = P_p(E)P_p(F) \geq c^n(1-p)^n$$

If $E\cap F$ occurs, $B \subset C(o) \subset A$, so by Lemma 3.13, $\partial_\infty C = \Pi$. Thus,

$$P_p(\partial_\infty C = \Pi) = P_p(E\cap F) \geq c^n(1-p)^n$$

$\square$

3.2.2 $p_c < 1$ for All Uniformly Transient Graphs

The main result of this subsection is that on any infinite, connected, locally finite, and uniformly transient graph, one has $p_c(G) < 1$. In particular, we will use a Markov chain covering lemma to show that the number of minimal cutsets grows at most exponentially, which is equivalent to the existence of a supercritical percolation phase.

Definition 3.16 (Uniformly transient)**.** There is an $\epsilon > 0$ such that a simple random walk started at every vertex v has probability at least $\frac{\epsilon}{deg(v)}$ of never returning to its start. Write $deg(v)$ as d_v. Alternatively, for all v,

$$d_v P_v(\forall t \geq 1 : X_t \neq v) \geq \epsilon$$

Theorem 3.17. *If G is infinite, connected, locally finite, and uniformly transient graph, then $\kappa(G) < \infty$.*

Lemma 3.18 (Markov Covering Lemma)**.** *Let $n \geq 1$ and let $P = (p(i,j))_{i,j \in [n]}$ be a substochastic Markov transition matrix, i.e. $\sum_j p(i,j) \leq 1$ for all i. Let Γ be the set of finite sequences $\gamma = (\gamma_0, \dots, \gamma_k)$ with $\gamma_0 = \gamma_k = 1$ that visit every state $\{2, \dots, n\}$. Set $p(\gamma) := \prod_{t=1}^{k} p(\gamma_{t-1}, \gamma_t)$. Assume that for some $\epsilon > 0$,*

$$\sum_{i \in I} \sum_{j \in [n] \setminus I} p(i,j) \geq \epsilon \qquad \text{for every nonempty proper } I \subset [n].$$

Then

$$\sum_{\gamma \in \Gamma} p(\gamma) \geq \delta^n, \qquad \text{where } \delta := \frac{\epsilon^2}{16e^2}.$$

Proof. Sample i.i.d. edges $e_1, \dots, e_{2n-2} \in ([n] \times [n]) \cup \{\varnothing\}$ with distribution

$$P(e_t = (u,v)) = \frac{p(u,v)}{n}, \qquad P(e_t = \varnothing) = 1 - \frac{1}{n} \sum_{u,v} p(u,v) \geq 0$$

Let H be the undirected multigraph on $[n]$ obtained by keeping the non-$\varnothing$ samples (forgetting orientation). Let H_1 be the spanning subgraph of H using $e_1, \dots, e_{n-1}$ and H_2 the one using $e_n, \dots, e_{2n-2}$.

Step 1: connectivity of H_1. Expose $e_1, \dots, e_{n-1}$ sequentially. If $C_1, \dots, C_r$ are the components after $k-1$ steps, the conditional probability that e_k joins two distinct components is

$$\frac{1}{n} \sum_{i=1}^{r} \sum_{x \in C_i} \sum_{y \notin C_i} p(x,y) \geq \frac{r\epsilon}{n}.$$

Multiplying over the successive merges from $r = n$ down to 1 gives

$$P(H_1 \text{ connected}) \geq \prod_{r=2}^{n} \frac{r\epsilon}{n} = \frac{\epsilon^{n-1}}{n^{n-1}} n! \geq \left(\frac{\epsilon}{2e}\right)^n.$$

By independence, the same bound holds for H_2, so

$$P(H_1 \text{ and } H_2 \text{ connected}) \geq \left(\frac{\epsilon}{2e}\right)^{2n}.$$

Step 2: presence of a covering walk. Say that a sequence $\gamma \in \Gamma$ is present if each directed edge (γ_{t-1}, γ_t) (or its reverse) appears among $e_1, \ldots, e_{2n-2}$. If H_1 and H_2 are connected, the union contains two edge-disjoint spanning trees, hence an Eulerian spanning subgraph; thus some $\gamma \in \Gamma$ is present.

For fixed γ of length m, the probability it is present is at most

$$(2n-2)^m \left(\frac{2}{n}\right)^m \prod_{t=1}^{m} p(\gamma_{t-1}, \gamma_t) \leq 4^m p(\gamma) \leq 4^{2n} p(\gamma).$$

Hence

$$\left(\frac{\epsilon}{2e}\right)^{2n} \leq \sum_{\gamma\in\Gamma} P(\gamma \text{ present}) \leq 4^{2n} \sum_{\gamma\in\Gamma} p(\gamma),$$

which gives the claim. □

Proof of Theorem 3.17. Replace each undirected edge $e = uv \in E$ by a path $u - m(e) - v$ by inserting a new vertex $m(e)$ (the "midpoint"). Denote the expanded graph by $G' = (V \cup m(E), E')$.

Let $(X_t)_{t\geq 0}$ be the simple random walk on G'. For any state z, let

$$\tau = \tau(z) := \sup\{t \geq 0 : X_t = z\}$$

be the last return time to z. We write P'_z for the law of (X_t) started at z.

Set $\epsilon_1 := \frac{2\epsilon}{4+\epsilon} \in (0,1)$. One checks two cases:

(i) For $v \in V$, $P'_v(\tau = 0) \geq \frac{1}{2} P_v(T_v^+ = \infty) \geq \frac{\epsilon}{2d_v} \geq \frac{\epsilon_1}{2}$

(ii) For $z = m(uv) \in m(E)$,

$$\begin{aligned}
\frac{1}{P'_z(\tau = 0)} =& \sum_{n\geq 0} P'_z(\tau > 0)^n = \mathbb{E}'_z[N_z] \\
=& \sum_{t\geq 0} P'(X_t = z) = 1 + \sum_{t\geq 1} [P'_z(X_{t-1} = u)\frac{1}{d_u} + P'_z(X_{t-1} = v)\frac{1}{d_v}] \leq 1 + \mathbb{E}'_z[N_u]\frac{1}{d_u} + \mathbb{E}'_z[N_v]\frac{1}{d_v} \\
\leq& 1 + \mathbb{E}'_u[N_u]\frac{1}{d_u} + \mathbb{E}'_v[N_v]\frac{1}{d_v} \\
=& 1 + \frac{1}{d_u P'_u(\tau = 0)} + \frac{1}{d_v P'_v(\tau = 0)}
\end{aligned}$$

and $P'_z(\tau = 0) \geq \epsilon_1 = \dfrac{2\epsilon}{4+\epsilon}$. (Here N_z is the number of visits to z and we use uniform transience on G together with the coupling of the walk on G' with a lazy walk on G.) In particular,

$$P'_{o'}(\tau = 0) \geq \frac{\epsilon_1}{2} \tag{3}$$

for any midpoint $o' = m(\{o, w\})$ adjacent to a fixed root $o \in V$ (the extra factor $1/2$ is a slack we keep for later SMP factorization).

Now fix a minimal cutset Π from o to ∞ of size n. Let A be the finite connected component of o in $(V, E \setminus \Pi)$. Define the interior and the boundary midpoints by

$$I := A \cup m\big(E[A]\big), U := m(\Pi) \cup \{o'\},$$

where $E[A]$ denotes the set of edges with both endpoints in A, and o' is a fixed midpoint incident to o.

We consider the walk on G' started at o' and killed when first hitting a vertex in $(V \cup m(E)) \setminus (I \cup U)$ (i.e., when exiting $I \cup U$). Let P be the induced sub-stochastic transition matrix on the finite state space U obtained by recording successive visits to U before the killing time.

As proved in Lemma 3.18 (Markov Covering Lemma), there exists a constant $\epsilon_2 = \epsilon_2(\epsilon) > 0$ such that for every nonempty proper $J \subset U$,

$$\sum_{i \in J} \sum_{j \in U \setminus J} P(i,j) \geq \epsilon_2. \tag{4}$$

(Informally: by uniform transience, once the walk leaves a point in J, with probability bounded below it reaches $U \setminus J$ before either returning or being killed; summing gives (4).)

Let Γ be the set of closed walks on U that start and end at o' and visit every state of U. By Lemma 3.18 applied to P and the expansion constant ϵ_2, we have

$$P'_{o'}(\mathcal{E}) \geq \delta^{|U|} \text{ with } \delta := \frac{\epsilon_2^2}{16e^2}$$

where $\mathcal{E}$ is the event that the walk visits every vertex in U and returns to o' before exiting $I \cup U$. Since $|U| = n+1$, we have

$$P'_{o'}(\mathcal{E}) \geq \delta^{n+1} =: \epsilon_3^{n+1} \text{ with } \epsilon_3 := \delta \in (0,1) \tag{5}$$

On $\mathcal{E}$, the walk never touches the exterior $(V \cup m(E)) \setminus (I \cup U)$ and visits every midpoint in $m(\Pi)$ from the interior side (because it visits all of U). Let

$$C := \{X_t : 0 \leq t \leq \tau(o')\}$$

where $\tau(o')$ is the last return time to o'. Then the exposed boundary of C in G' is exactly $m(\Pi)$:

$$\mathcal{E} \cap \{\tau(o') = 0\} \subseteq \{\partial_\infty C = m(\Pi)\} \tag{6}$$

By the strong Markov property at the (random) time of the last visit to o',

$$P'_{o'}\big(\partial_\infty C = m(\Pi)\big) \geq P'_{o'}(\mathcal{E}) \inf_{z \in U} P'_z(\tau = 0) \geq P'_{o'}(\mathcal{E}) \cdot \frac{\epsilon_1}{2}, \tag{7}$$

using (3) in the last step.

Combining (5) and (7), for every minimal cutset Π of size n we have

$$P'_{o'}\big(\partial_\infty C = m(\Pi)\big) \geq \frac{\epsilon_1}{2}\epsilon_3^{n+1}.$$

The events $\{\partial_\infty C = m(\Pi)\}$ are disjoint as Π ranges over $Q_n(o)$ (the family of minimal cutsets of size n), hence

$$1 \geq \sum_{\Pi \in Q_n(o)} P'_{o'}\big(\partial_\infty C = m(\Pi)\big) \geq |Q_n(o)| \frac{\epsilon_1}{2} \epsilon_3^{n+1}.$$

Therefore

$$|Q_n(o)| \leq \frac{2}{\epsilon_1\epsilon_3}\Big(\frac{1}{\epsilon_3}\Big)^n \leq K(\epsilon)^n \qquad (n \geq 1),$$

with $K(\epsilon) := \max\big\{\epsilon_3^{-1}, 2/(\epsilon_1\epsilon_3)\big\} < \infty$ depending only on ϵ. This shows

$$\kappa(G) = \limsup_{n\to\infty} \frac{1}{n} \log |Q_n(o)| \leq \log K(\epsilon) < \infty.$$

$\square$

4 Sharp Phase Transitions for Percolation and the Random-Cluster Models

This section aims to present a new proof of the sharp phase transitions with respect to radii of large open clusters. The proof uses a generalized OSSS inequality due to H. Duminil-Copin, A. Raoufi, and V. Tassion (2019) [DRT19]. The result imposes very few conditions on the graph, and the strategy is valid for random-cluster models with cluster weight $q \geq 1$ (whose sharp phase transition problem had not been solved for long) on locally finite vertex-transitive graphs. The main tool called OSSS inequality will be applied in Section 5 too.

4.1 Decision Tree and OSSS Inequality

In computer science, a decision tree is a flowchart-like tree structure where each internal node represents a feature, each edge represents the outcome of a query, and each leaf node represents a class label.

Now, let's develop a mathematical description of the algorithm encoded by a decision tree. Consider a set E with $|E| = n$. Write $e_{[t]} = (e_1, \ldots, e_t)$.

Definition 4.1. A decision tree is defined by $(e_1, \{\phi_t\}_{2\leq t\leq n})$, where $e_1 \in E$ is the fixed "root" for the tree, and ϕ_t are "decision rules" that deterministically map $(e_{[t-1]}, \omega_{e_{[t-1]}})$ to an element in $E \setminus \{e_1, \ldots, e_{t-1}\}$. The input for the algorithm is $\omega \in \{0,1\}^E$. The algorithm first queries the value of ω_{e_1}. For all $t \geq 2$, set $e_t = \phi_t(e_{[t-1]}, \omega_{e_{[t-1]}})$, and we query the value of ω_{e_t}. In this way, we will obtain an ordering on E, $(e_1, \ldots, e_n)$, from the input ω.

We associate decision tree $T = (e_1, \{\phi_t\})$ with a (boolean) function $f : \{0,1\}^E \to \mathbb{R}$, which corresponds to, in the computer science sense, what the decision tree is computing. Define stopping time

$$\tau(\omega) = \tau_{f,T}(\omega) := \min\{t \geq 1 : \forall \omega' \in \{0,1\}^E, \omega'_{e[t]} = \omega_{e[t]} \Rightarrow f(\omega) = f(\omega')\}$$

That is, once we reach the depth of $\tau(\omega)$, the outcome is determined.

The class of measures we will mainly consider throughout this section is monotonic measures.

Definition 4.2. A measure μ on $\{0,1\}^E$ is monotonic if for all $e \in E$ and $F \subset E$ and any $\xi, \zeta \in \{0,1\}^E$, $\xi \leq \zeta$, $\mu[\omega_e = \xi_e, \forall e \in F] > 0$ and $\mu[\omega_e = \zeta_e, \forall e \in F] > 0$ imply

$$\mu[\omega_e = 1 | \omega_e = \xi_e, \forall e \in F] \leq \mu[\omega_e = 1 | \omega_e = \zeta_e, \forall e \in F]$$

A strictly positive probability measure is monotonic if and only if it satisfies the FKG lattice property (Theorem 2.27 in [Gri06]).

Definition 4.3. The revealment of f is defined by $\delta_e(f, T) := \mu\{e \text{ is revealed}\} = \mu\{\exists t \leq \tau(\omega) : e_t = e\}$.

The OSSS inequality was originally introduced for product measure by R. O'Donnell, M. Saks, O. Schramm, and R. Servedio [OSSS05]. It bounds the variance of boolean functions using revealments and "influence" (corresponding to $I_e[f] := \mu(f|\omega_e = 1) - \mu(f|\omega_e = 0)$). Duminil-Copin, Raoufi, and Tassion generalize it to monotonic measures by an appropriate coupling.

Denote $\vec{E}$ the set of lists $(e_1, \ldots, e_n)$ where each element of E occurs exactly once. For $\{U_t\}$ a sequence of i.i.d. uniform random variables on $[0,1]$ and e a $\vec{E}-$valued random variable, define $\boldsymbol{X} = F_e(U)$ by

$$X_{e_t} := \begin{cases} 1 & (U_t \geq \mu(\omega_{e_t} = 0 | \omega_{e_{[t-1]}} = X_{e_{[t-1]}})) \\ 0 & (else) \end{cases}$$

It can be proved that $\boldsymbol{X}$ has law μ (See Lemma 2.1 in [DRT19] for details).

Theorem 4.4. *(Generalized OSSS). Fix an increasing function $f : \{0,1\}^E \to [0,1]$ on a finite set E. For any monotonic measure μ and any decision tree T,*

$$\mathrm{Var}_\mu(f) \leq \sum_{e \in E} \delta_e(f, T) \mathrm{Cov}_\mu(f, \omega_e) \tag{8}$$

Proof. Let $\{\boldsymbol{U}_t\}, \{\boldsymbol{V}_t\}$ be two independent sequences of i.i.d. uniform $[0,1]$ random variables. Write $\mathbb{E}^{U,V}$ for the expectation of the coupling between U, V. Construct $(\boldsymbol{e}, \boldsymbol{X}, \tau)$ as follows: for $t \geq 1$,

$$\boldsymbol{e}_t := \begin{cases} \boldsymbol{e}_1 & (t = 1) \\ \phi_t(\boldsymbol{e}_{[t-1]}, \boldsymbol{X}_{\boldsymbol{e}_{[t-1]}}) & (t > 1) \end{cases}, \qquad \boldsymbol{X}_{e_t} := \begin{cases} 1 & (\boldsymbol{U}_t \geq \mu(\omega_{\boldsymbol{e}_t} = 0 | \omega_{\boldsymbol{e}_{[t-1]}} = \boldsymbol{X}_{\boldsymbol{e}_{[t-1]}})) \\ 0 & (else) \end{cases}$$

and $\tau := \min\{t \geq 1 : \forall x \in \{0,1\}^E, x_{e_{[t]}} = \boldsymbol{X}_{e_{[t]}} \Rightarrow f(x) = f(\boldsymbol{X})\}$. Then, $\boldsymbol{X}$ has law μ.

Define $\boldsymbol{W}^t := (\boldsymbol{V}_1, \ldots, \boldsymbol{V}_t, \boldsymbol{U}_{t+1}, \ldots, \boldsymbol{U}_\tau, \boldsymbol{V}_{\tau+1}, \ldots, \boldsymbol{V}_n$ and set $\boldsymbol{Y}^t := F_{\boldsymbol{e}}(\boldsymbol{W}^t)$. Notice that $\boldsymbol{W}^t = \boldsymbol{V}$ if $t \geq \tau$, so $\boldsymbol{Y}^n$ is independent of $\boldsymbol{U}$ and also has law μ. Since f is valued in $[0,1]$, $\mu[|f - \mu(f)| > \frac{1}{2}] \leq \frac{1}{2}$, so

$$\begin{aligned}
\mathrm{Var}_\mu(f) \leq & \frac{1}{2}\mu[|f - \mu[f]|] = \frac{1}{2}\mathbb{E}^{U,V}[|\mathbb{E}^{U,V}[f(\boldsymbol{X})|\boldsymbol{U}] - \mathbb{E}^{U,V}[f(\boldsymbol{Y}^n)|\boldsymbol{U}]|] \\
\leq & \frac{1}{2}\mathbb{E}^{U,V}[|f(\boldsymbol{X}) - f(\boldsymbol{Y}^n)|] = \frac{1}{2}\mathbb{E}^{U,V}[|f(\boldsymbol{Y}^0) - f(\boldsymbol{Y}^n)|] \\
\leq & \frac{1}{2}\sum_{t=1}^n \mathbb{E}^{U,V}[|f(\boldsymbol{Y}^t) - f(\boldsymbol{Y}^{t-1})|] = \frac{1}{2}\sum_{t=1}^n \mathbb{E}^{U,V}[|f(\boldsymbol{Y}^t) - f(\boldsymbol{Y}^{t-1})|1_{t\leq\tau}] \\
= & \frac{1}{2}\sum_{e \in E}\sum_{t=1}^n \mathbb{E}^{U,V}\left[\mathbb{E}^{U,V}\left[\left|f(\boldsymbol{Y}^t) - f(\boldsymbol{Y}^{t-1})\right|\middle| \boldsymbol{U}_{[t-1]}\right] 1_{t\leq\tau, \boldsymbol{e}_t = e}\right]
\end{aligned}$$

where the equality on the third line is by $f(\boldsymbol{Y}^0) = f(\boldsymbol{X})$ (the entries for $t > \tau$ in $\boldsymbol{Y}^0$ will not affect f); the equality on the fourth line is by $\boldsymbol{Y}^t = \boldsymbol{Y}^{t-1}$ for all $t > \tau$.

It is left to show that $\mathbb{E}^{U,V}\left[\left|f(\boldsymbol{Y}^t) - f(\boldsymbol{Y}^{t-1})\right| \boldsymbol{U}_{[t-1]}\right] \leq 2\mathrm{Cov}_\mu(f, \omega_e)$ on the event $\{t \leq \tau, \boldsymbol{e}_t = e\}$, because $\sum_{t=1}^n \mathbb{E}^{U,V}[1_{t\leq\tau, \boldsymbol{e}_t = e}] = \delta_e(f, T)$.

Now restrict ourselves to $\{t \leq \tau, \boldsymbol{e}_t = e\}$. Since $\boldsymbol{Y}^t = \boldsymbol{Y}^{t-1}$ whenever $\boldsymbol{Y}^t_e = \boldsymbol{Y}^{t-1}_e$,

$$|f(\boldsymbol{Y}^t) - f(\boldsymbol{Y}^{t-1})| = (f(\boldsymbol{Y}^t) - f(\boldsymbol{Y}^{t-1}))(\boldsymbol{Y}^t_e - \boldsymbol{Y}^{t-1}_e) = f(\boldsymbol{Y}^{t-1})\boldsymbol{Y}^{t-1}_e + f(\boldsymbol{Y}^t)\boldsymbol{Y}^t_e - f(\boldsymbol{Y}^{t-1})\boldsymbol{Y}^t_e - f(\boldsymbol{Y}^t)\boldsymbol{Y}^{t-1}_e \tag{9}$$

It can be proved that $\mathbb{E}^{U,V}[g(\boldsymbol{Y}^t)|\boldsymbol{U}_{[t]}] = \mu[g(\omega)]$ for all measurable g and $t \leq n$, so

$$\mathbb{E}^{U,V}[f(\boldsymbol{Y}^{t-1})\boldsymbol{Y}^{t-1}_e|\boldsymbol{U}_{[t-1]}] = \mu[f(\omega)\omega_e] = \mathbb{E}^{U,V}[f(\boldsymbol{Y}^t)\boldsymbol{Y}^t_e|\boldsymbol{U}_{[t-1]}] \tag{10}$$

where for the second equality, we first condition on $\boldsymbol{U}_{[t]}$ and apply the tower law.

For fixed $\boldsymbol{U}_{[n]}$ and s, since μ is monotonic, $\boldsymbol{Y}^s = F_{\boldsymbol{e}}(\boldsymbol{W}^s)$ is an increasing function of $\boldsymbol{V}$. Since $f, \boldsymbol{W}_{\boldsymbol{e}}$ are also increasing functions of $\boldsymbol{V}$, so are $f(\boldsymbol{Y}^{t-1}), \boldsymbol{Y}^t_e$. Apply Harris-FKG to i.i.d. random variables $\boldsymbol{V}$:

$$\mathbb{E}^{U,V}[f(\boldsymbol{Y}^{t-1})\boldsymbol{Y}^t_e|\boldsymbol{U}_{[n]}] \geq \mathbb{E}^{U,V}[f(\boldsymbol{Y}^{t-1})|\boldsymbol{U}_{[n]}]\mathbb{E}^{U,V}[\boldsymbol{Y}^t_e|\boldsymbol{U}_{[n]}]$$

Average over $\boldsymbol{U}_{[t-1]}$ and apply $\mathbb{E}^{U,V}[g(\boldsymbol{Y}^{t-1})|\boldsymbol{U}_{[t-1]}] = \mu[g(\omega)]$:

$$\mathbb{E}^{U,V}[f(\boldsymbol{Y}^{t-1})\boldsymbol{Y}^t_e|\boldsymbol{U}_{[t-1]}] \geq \mathbb{E}^{U,V}[f(\boldsymbol{Y}^{t-1})|\boldsymbol{U}_{[t-1]}]\mathbb{E}^{U,V}[\boldsymbol{Y}^t_e|\boldsymbol{U}_{[t-1]}] = \mu[f(\omega)]\mu[\omega_e] \tag{11}$$

Similarly,

$$\mathbb{E}^{U,V}[f(\boldsymbol{Y}^t)\boldsymbol{Y}^{t-1}_e|\boldsymbol{U}_{[t-1]}] \geq \mu[f(\omega)]\mu[\omega_e] \tag{12}$$

Substituting (10), (11), (12) into (9) concludes the proof. □

For a subgraph $G = (V, E)$, define its boundary ∂G as the collection of vertices x such that there exists $y \notin G$ with xy an edge. Set Λ_n as the graph induced by $\{x \in V : d(x, 0) \leq n\}$, where $d(\cdot, \cdot)$ is the graphical distance.

Similar to the proof in [DCT15], we are also going to deduce a "differential inequality" (which will be $f'_n \geq \frac{n}{\sum_{k=1}^n f_n} f_n$) to obtain information of $\{0 \leftrightarrow \partial\Lambda_n\}$. To do that, we will view the indicator function of connectedness, $1_{0\leftrightarrow\partial\Lambda_n}$, as a boolean function on the edge space, where 0 means an edge is closed, and 1 means an edge is open. A boolean function on the edge space is a function map from $\{0,1\}^E$ to $\mathbb{R}$.

We are going to bound the probability of $\{0 \leftrightarrow \partial\Lambda_n\}$ by constructing a random decision tree to "compute" $1_{0\leftrightarrow\partial\Lambda_n}$, and a crucial tool is the "OSSS inequality".

This section is organized as follows: subsection 4.1 defines decision trees and proves the generalized OSSS inequality for monotonic measures (8); subsection 4.2 proves the sharp phase transition in nearest-neighbor Bernoulli bond percolation (Theorem 4.8); subsection 4.3 discusses a few modifications of percolation's proof to make it also adapt to random-cluster models (Theorem 4.9).

4.2 Sharp Phase Transitions of Bernoulli Percolation

We first prove this (non-probabilistic) lemma. Our goal is to match β_1 with the critical value of our model.

Lemma 4.5. *Consider a sequence of increasing differentiable functions $f_n : [0, \beta_0] \to [0, M]$ such that $f = \lim_{n\to\infty} f_n$ is a real function, and $f_n' \geq \frac{n}{\Sigma_n} f_n$ for all $n \geq 1$, where $\Sigma_n = \sum_{k=0}^{n-1} f_k$. Then, there exists $\beta_1 \in [0, \beta_0]$ such that*

(A) For any $\beta < \beta_1$, there exists $c_\beta > 0$ such that for any n large enough, $f_n(\beta) \leq \exp(-c_\beta n)$

(B) For any $\beta > \beta_1$, $f(\beta) \geq \beta - \beta_1$

Proof. Claim the following choice of β_1 suffices:

$$\beta_1 := \inf\left\{\beta : \limsup_{n\to\infty} \frac{\log \Sigma_n(\beta)}{\log(n)} \geq 1\right\}.$$

For part (A), take $\beta < \beta_1$. Set $\delta = \frac{\beta_1 - \beta}{3}$, and set $\beta' = \beta + \delta, \beta'' = \beta + 2\delta$.

Integrate $f_n' \geq \frac{n}{\Sigma_n} f_n$ on $[\beta', \beta'']$,

$$-\log(f_n(\beta')) \geq \log(f_n(\beta'')) - \log(f_n(\beta')) \geq C + (\beta'' - \beta')\frac{n}{\Sigma_n(\beta'')} = C + \delta\frac{n}{\Sigma_n(\beta'')}$$

so $f_n(\beta') \leq M \exp\left(-\delta \frac{n}{\Sigma_n(\beta'')}\right)$.

By definition of β_1, there exists $N \in \mathbb{N}$ and $\alpha > 0$ such that $\Sigma_n(\beta) \leq n^{1-\alpha}$ for all $n \geq N$, so $f_n(\beta') \leq e^{-C}\exp(-\delta n^\alpha)$, so $\sum_{k=0}^\infty f_n(\beta') < \infty$. That is, there exists $\Sigma(\beta') \in \mathbb{R}$ such that $\Sigma_n(\beta') \leq \Sigma(\beta')$ for all n. Integrate $f_n' \geq \frac{n}{\Sigma(\beta')} f_n$ on $[\beta, \beta']$, we get $f_n(\beta) \leq M \exp\left(-\delta \frac{n}{\Sigma(\beta')}\right)$.

For part (B), take $\beta > \beta_1$. Define $T_n = \frac{1}{\log(n)} \sum_{k=1}^n \frac{f_k}{k}$. Since for all $k \geq 1$,

$$\frac{f_k}{\Sigma_k} \geq \int_{\Sigma_k}^{\Sigma_{k+1}} \frac{dt}{t} = \log(\Sigma_{k+1}) - \log(\Sigma_k)$$

we obtain

$$T_n' = \frac{1}{\log(n)} \sum_{k=1}^n \frac{f_k'}{k} \geq \frac{1}{\log(n)} \sum_{k=1}^n \frac{kf_k}{k\Sigma_k} \geq \frac{1}{\log(n)} \sum_{k=1}^n \log(\Sigma_{k+1}) - \log(\Sigma_k) = \frac{\log(\Sigma_{n+1}) - \log(\Sigma_1)}{\log(n)}. \tag{13}$$

Take arbitrary $\beta' \in (\beta_1, \beta)$. Integrate (13) on $[\beta', \beta]$,

$$T_n(\beta) - T_n(\beta') \geq (\beta - \beta')\frac{\log(\Sigma_n(\beta')) - \log(M)}{\log(n)}. \tag{14}$$

If we can show that $T_n(\beta) \xrightarrow{n\to\infty} f(\beta)$, taking limsup on both sides of (14) yields

$$f(\beta) - f(\beta') \geq (\beta - \beta')\left[\limsup_{n\to\infty} \frac{\log(\Sigma_n(\beta'))}{\log(n)}\right] \geq \beta - \beta'. \tag{15}$$

Then taking $\beta' \searrow \beta_1$ suffices.

To see the convergence of $T_n(\beta)$, notice that for any sequence $\{a_n\} \in \mathbb{R}^\mathbb{N}$ with $a := \lim_{n\to\infty} a_n \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{a_n/n}{\log(n) - \log(n-1)} = \lim_{n\to\infty} \frac{a_n}{n\log(\frac{n}{n-1})} = \lim_{n\to\infty} \frac{a_n}{\log(1 + \frac{1}{n-1})^n} = \frac{a}{\ln(e)} = a.$$

By Stoltz-Cesaro theorem, $\frac{1}{\log(n)} \sum_{k=1}^n \frac{a_k}{k} \to a$ as $n \to \infty$. $\square$

Lemma 4.6. *Consider a finite graph $G=(V,E)$ containing 0. For any monotonic measure μ on $\{0,1\}^E$, and any $n \geq 1$,*

$$\sum_{xy\in E} \mathrm{Cov}_\mu(1_{0\leftrightarrow\partial\Lambda_n}, \omega_e) \geq \frac{n}{4\max_{x\in\Lambda_n}\sum_{k=0}^{n-1}\mu(x\leftrightarrow\partial\Lambda_k(x))}\mu(0\leftrightarrow\partial\Lambda_n)(1-\mu(0\leftrightarrow\partial\Lambda_n))$$

Remark 4.7. This lemma is where we apply the OSSS inequality. To do that, we must choose an appropriate decision tree to compute $f := 1_{0\leftrightarrow\partial\Lambda_n}$.

If we let the algorithm naively check every edge in Λ_n, the revealment $\delta_e(f)$ will be 1 for each edge e. We then obtain $\mathrm{Var}_\mu(f) \leq \sum_{e\in E}\mathrm{Cov}_\mu(f,\omega_e)$, which is known as the Poincaré inequality. If we substitute this into (17), we will get

$$\theta_n'(p) \geq \frac{1}{p(1-p)}\theta_n(p)(1-\theta_n(p))$$

This is not strong enough to invoke Lemma 4.5 to prove the sharp phase transition.

To fix that, we will define a list of decision trees $T_1,\ldots,T_n$ such that each T_k computes $1_{0\leftrightarrow\partial\Lambda_n}$ and only explores the connected components of $\partial\Lambda_k$. As a result, the average of the revealment of each edge will be small.

Proof. For each k, we define the algorithm associated with the decision tree as follows. Let V be the collection of edges that have been found by the decision tree to be connected to $\partial\Lambda_n$, but the edges themselves have not been revealed. Let F be the collection of revealed edges. Initialize V to $\{xy\in E : x\in\partial\Lambda_k \vee y\in\partial\Lambda_k\}$ and F to emptyset. Fix an ordering of the edges.

```
while V is nonempty:
    e = the smallest unrevealed edge in the ordering
    Reveal the state of e
    V <- V - {e}
    F <- F + {e}
    V <- V + {all unrevealed edges that are connected to the k-box boundary
        using edges in F}
    if 0 can be connected to the k-box boundary using edges in F
        return 1
return 0
```

where the "k-box" is Λ_k.

Such a decision tree T_k successfully computes $1_{0\leftrightarrow\partial\Lambda_n}$ by only discovering the connected components of $\partial\Lambda_k$. Hence, if $e=uv$ is revealed by T_k, either $u\leftrightarrow\partial\Lambda_k$ or $v\leftrightarrow\partial\Lambda_k$, so

$$\delta_e(T) \leq \mu(u\leftrightarrow\partial\Lambda_k)+\mu(v\leftrightarrow\partial\Lambda_k)$$

Also, the event $\{u\leftrightarrow\partial\Lambda_k\}$ is contained in $\{u\leftrightarrow\partial\Lambda_{|k-d(u,0)|}(u)\}$, so

$$\mu(u\leftrightarrow\partial\Lambda_k) \leq \mu(u\leftrightarrow\partial\Lambda_{|k-d(u,0)|}(u)) \leq 2\mu(u\leftrightarrow\partial\Lambda_k(u))$$

Set $M := \max_{x\in\Lambda_n}\sum_{k=0}^{n-1}\mu(x\leftrightarrow\partial\Lambda_k(x))$, so $\sum_{k=1}^{n}\delta_e(T) \leq \sum_{k=1}^{n}\mu(u\leftrightarrow\partial\Lambda_k)+\sum_{k=1}^{n}\mu(v\leftrightarrow\partial\Lambda_k) \leq 4M$.

Apply the generalized OSSS inequality (8) to T_k and $f = 1_{0\leftrightarrow\partial\Lambda_n}$ and sum on k:

$$\sum_{k=1}^{n} \mu(0 \leftrightarrow \partial\Lambda_n)(1-\mu(0 \leftrightarrow \partial\Lambda_n)) \leq \sum_{k=1}^{n}\sum_{e} \delta_e(T_k)\mathrm{Cov}(1_{0\leftrightarrow\partial\Lambda_n}, \omega_e) \leq \sum_{e} 4M\mathrm{Cov}(1_{0\leftrightarrow\partial\Lambda_n}, \omega_e)$$

LHS is $n\mu(0 \leftrightarrow \partial\Lambda_n)(1-\mu(0 \leftrightarrow \partial\Lambda_n))$, so we're done. □

Theorem 4.8. *For Bernoulli bond percolation with critical value p_c,*

(i) There exists $c > 0$ such that $\theta(p) \geq c(p - p_c)$ for all $p \geq p_c$ close enough to p_c

(ii) For $p < p_c$, there exists $c_p > 0$ such that

$$\forall n \geq 0 : P_p[0 \leftrightarrow \partial\Lambda_n] \leq \exp(-c_p n)$$

Proof. Fix $p_0 \in [0, 1]$. For $p \leq p_0$ and $n \geq 1$, define

$$\mu_n \equiv P_p, \qquad \theta_n(p) = \mu_n[0 \leftrightarrow \partial\Lambda_n], \qquad S_n := \sum_{k=0}^{n-1} \theta_k \tag{16}$$

By Russo's formula,

$$\frac{d}{dp}\theta_n(p) = \sum_{e\in E} \frac{1}{p(1-p)}\mu_n\left(f_n(\omega) \neq f_n(\omega^{(e)})\right) = \sum_{e\in E} \frac{1}{p(1-p)}\mathrm{Cov}(1_{0\leftrightarrow\partial\Lambda_n}, \omega_e) \tag{17}$$

Since for all $x \in \Lambda_n$

$$\sum_{k=1}^{n} \mu_n[x \leftrightarrow \partial\Lambda_k(x)] = S_n \tag{18}$$

by Lemma 4.6 (note that μ_n is monotonic),

$$\frac{d}{dp}\theta_n(p) = \sum_{e\in E} \frac{1}{p(1-p)}\mathrm{Cov}(1_{0\leftrightarrow\partial\Lambda_n}, \omega_e) \geq \frac{1}{p(1-p)}\frac{n}{4S_n}\theta_n(1-\theta_n) \geq \frac{1-\theta_1(p_0)}{p(1-p)}\frac{n}{4S_n}\theta_n \tag{19}$$

because $\theta_n \leq \theta_1$ and $p \leq p_0$. Set $c := \frac{1-\theta_1(p_0)}{4p(1-p)} > 0$. Since $\theta_n \to \theta$, we can apply Lemma 4.5 to $f_n = \theta_n/c$, which yields a p_1 such that (A) and (B) occur.

Then, for all $n \geq 1$ and $p < p_1$, $P_p(0 \leftrightarrow \partial\Lambda_n) = \theta_n(p) \leq \exp(-c_p n)$, which also implies $p_1 \leq p_c$. By (15), for all $p > p_1$, $\theta(p) \geq \theta(p) - \theta(p_1) \geq p - p_1 > 0$, which implies $p_1 \geq p_c$. Thus, $p_1 = p_c$, which concludes both (i) and (ii) of the theorem. □

4.3 Sharp Phase Transitions of Random-Cluster Models

Theorem 4.9. *For $q \geq 1$ and a random cluster model on a weighted lattice $(\mathbb{G}, J)$ (suppose J_{xy} are finite-range),*

(i) There exists $c > 0$ such that $\theta(\beta) \geq c(\beta - \beta_c)$ for all $\beta \geq \beta_c$ close enough to β_c

(ii) For all $\beta < \beta_c$, there exists $c_\beta > 0$ such that

$$\forall n \geq 0 : \phi^{\mathrm{w}}_{\Lambda_n,\beta,q}[0 \leftrightarrow \partial\Lambda_n] \leq \exp(-c_\beta n)$$

Proof. The proofs of Lemma 4.5 and Lemma 4.6 do not depend on specific models, so we can keep them.

Fix $q \geq 1$ and $\beta_0 > 0$. Take $\beta \leq \beta_0$.

In (16), we replace $\mu_n = \phi^{\mathrm{w}}_{\Lambda_{2n},\beta,q}$, which is still a monotonic measure by Theorem 3.8 and Theorem 2.27 in [Gri06].

In (17), we do not have Russo's formula anymore, but Theorem 3.12 in [Gri06] offers a similar differential equality:

$$\theta_n'(\beta) = \sum_{xy\in E} \frac{J_{xy}}{e^{\beta J_{xy}}-1} \mathrm{Cov}(1_{0\leftrightarrow\partial\Lambda_n}, \omega_e) \geq \min\left\{\frac{J_{xy}}{e^{\beta_0 J_{xy}}-1}\right\} \sum_{xy\in E} \mathrm{Cov}(1_{0\leftrightarrow\partial\Lambda_n}, \omega_e)$$

In (18), we do not necessarily have equality, but by the comparison between boundary conditions (Lemma 4.14(b) in [Gri06]) and transitivity of $\mathbb{G}$, for all $x \in \Lambda_n$,

$$\sum_{k=1}^{n-1} \mu_n[x \leftrightarrow \partial\Lambda_k(x)] \leq 2 \sum_{k\leq n/2} \mu_n[x \leftrightarrow \partial\Lambda_k(x)] \leq 2 \sum_{k\leq n/2} \mu_k[0 \leftrightarrow \partial\Lambda_k] \leq 2S_n$$

Still, apply Lemma 4.6, we obtain

$$\frac{d}{d\beta}\theta_n(\beta) \geq \min\left\{\frac{J_{xy}}{e^{\beta_0 J_{xy}}-1}\right\} \frac{n}{8S_n}\theta_n(1-\theta_n) = c\frac{n}{S_n}\theta_n \tag{20}$$

where $c := \min\left\{\frac{J_{xy}}{e^{\beta_0 J_{xy}}-1}\right\} \frac{1-\theta_1(\beta_0)}{8} > 0$. Applying Lemma 4.5 to $f_n = \theta_n/c$ and repeating the rest of the proof in Theorem 4.8 will give what we want. ☐

5 Hutchcroft's New Critical Exponent Inequalities for Percolation and the Random-Cluster Models

Similar to the previous section, this section presents a differential inequality based on the OSSS inequality from Hutchcroft's paper [Hut20], but instead of radius, we focus on volumes. The differential inequality leads to two main results:

1. Scaling relation inequalities: $\gamma \leq \delta - 1$ and $\Delta \leq \gamma + 1$
2. A sharpness theorem: the distribution of cluster size has an exponential tail. In addition to [DRT19], the theorem also works for long-range/infinite-range interactions.

5.1 Standard Critical Exponents and Volume-Scaling Relations

In percolation theory (and more generally in the FK random-cluster model), critical exponents are used to characterize the behavior of cluster observables near the phase transition.

Let $G = (V, E)$ be an infinite, connected, transitive graph, and consider either Bernoulli bond percolation or the random-cluster model on G with edge parameter $p \in [0, 1]$ (and cluster weight $q \geq 1$ for the random-cluster model). We write P_p and E_p for probabilities and expectations with respect to the percolation/RCM measure at edge parameter p. Let K denote the cluster of an arbitrary fixed vertex (say the origin $o \in V$). The critical probability p_c is the threshold at which an infinite cluster appears ($p_c := \inf p : P_p(|K| = \infty) > 0$). Near p_c, the following power-law behaviors are conjectured (and in some cases proven) to hold. They serve to define the critical exponents β, γ, δ, and Δ (if they exist):

1. For $p > p_c$, $P_p(|K| = \infty) \approx (p - p_c)^\beta$ as $p \downarrow p_c$
2. For $p < p_c$, $\mathbb{E}_p[|K|] \approx (p_c - p)^{-\gamma}$ as $p \uparrow p_c$
3. $P_{p_c}(|K| \geq n) \approx n^{-1/\delta}$ as $n \to \infty$
4. For each $k \geq 1$, $\mathbb{E}_p[|K|^k] \approx (p_c - p)^{-[(k-1)\Delta + \gamma]}$ as $p \uparrow p_c$

Here the notation $f(p) \approx g(p)$ means $\frac{\log f}{\log g} \to 1$ in the given limit. Rigorously establishing existence and exact values of these exponents in general dimensions remains an open problem in mathematical physics.

Two scaling relations of these exponents are conjectured:

$$\gamma = \beta(\delta - 1), \beta\delta = \Delta \tag{21}$$

These relations are believed to hold universally for continuous (second-order) phase transitions, including percolation on $\mathbb{Z}^d$ for each $d \geq 2$. (21) is consistent with the heuristic that the various ways of measuring cluster "size" should not be independent of one another. For example, if $P_{p_c}(|K| \geq n) \sim n^{-1/\delta}$, one can integrate this tail to recover the divergence of $\mathbb{E}_p[|K|]$ as $p \uparrow p_c$, yielding $\gamma = \delta - 1$ in a heuristic sense (indeed, we will rigorously prove an inequality in this direction shortly). Likewise, one expects $\Delta = \beta\delta$ because $\mathbb{E}_p[|K|^2]$ can be related to the product $P_p(|K| = \infty) \cdot P_{p_c}(|K| \geq n)$ in a scaling argument, etc.

These scaling relations have been rigorously proved only in special cases. For percolation in two dimensions, Kesten proved (21) holds [Kes87] as a consequence of conformal invariance techniques. In high dimensions ($d > 6$) and on certain tree-like or mean-field graphs, it is known that percolation exhibits mean-field critical behavior, meaning $\beta = 1, \gamma = 1, \delta = 2, \Delta = 2$ (and other exponents ν, η, α take their mean-field values as well). In those cases one can check that (21) is indeed satisfied (e.g. $1 \cdot 2 = 2$ for $\beta\delta = \Delta$, etc.). Aside from these special situations, the full set of scaling relations remains unproven in $3 \leq d \leq 6$ and for most non-Euclidean transitive graphs.

Instead, progress in general settings has come in the form of inequalities between critical exponents, often derived via clever differential inequalities or other rigorous techniques. In subsections 5.3.1 and 5.3.2, we will present three new exponent inequalities, providing *upper bounds* on γ and Δ in terms of the others. Specifically, we will show that under very general conditions:

$$\gamma \leq \delta - 1, \Delta \leq \gamma + 1 \tag{22}$$

These inequalities were not known previously even for classical Bernoulli percolation on $\mathbb{Z}^d$. They are consistent with the conjectural scaling laws (21) – in fact (22) saturates to equality in the mean-field regime. Our derivation of (22) will follow from a powerful new Russo-type differential inequality that we introduce below.

5.2 Differential inequality

Definition 5.1 (Lower right Dini derivative)**.** For $f : [0, \infty) \to \mathbb{R}$, the lower-right Dini derivative at β is

$$(D_+ f)(\beta) \;:=\; \liminf_{h \downarrow 0} \frac{f(\beta + h) - f(\beta)}{h}.$$

We will write $\frac{d}{d\beta}_+$ when taking Dini derivatives.

In the following lemma, we use the alternate RCM definition (Definition 2.11): on a transitive weighted graph (G, J) with edge weights $J = \{J_e\}_{e \in E}$, cluster weight $q \geq 1$, and boundary condition $\# \in \{f, w\}$, the random-cluster measure at inverse temperature $\beta \geq 0$ is denoted by $\phi^{\#}_{\beta,q}$. For a fixed vertex v, let K_v be its open cluster.

Lemma 5.2. *(A Russo-type formula). For an increasing function $F : \{0, 1\}^E \to \mathbb{R}$ and boundary condition $\# \in \{\mathrm{f}, \mathrm{w}\}$, for all $\beta \geq 0$,*

$$\left(\frac{d}{d\beta}\right)_+ \phi^{\#}_{\beta,q}[F(\omega)] \geq \sum_{e \in E} \frac{J_e}{e^{\beta J_e} - 1} \mathrm{Cov}_{\phi^{\#}_{\beta,q}}[F(\omega), \omega(e)]] \tag{23}$$

Proof. We give the details for $\# = f$ (free boundary); the wired case is similar.

For any finite weighted subgraph $G_n = (V_n, E_n)$ of G and any two parameters $\alpha, \beta \geq 0$ we define the measure $\phi_{G_n,\beta,\alpha,q,A}$ in which edges in a finite set $A \subseteq E_n$ are assigned weight β while edges in $E_n \setminus A$ are assigned α. Because the state space is finite, the map $\beta \mapsto \phi_{G_n,\beta,\alpha,q,A}[F]$ is differentiable and the usual finite–volume Russo formula (Thm 3.12 in [Gri06]) gives

$$\frac{d}{d\beta} \phi_{G_n,\beta,\alpha,q,A}[F] = \sum_{e \in A} \frac{J_e}{e^{\beta J_e} - 1} \mathrm{Cov}_{\phi_{G_n,\beta,\alpha,q,A}}\big[F(\omega), \omega(e)\big].$$

Let G_n be an increasing sequence of finite subgraphs of G that exhausts the infinite graph, and let E_n be the edge set of G_n. For a fixed finite edge set A and $\beta \geq \alpha$, the measures $\phi_{G_n,\beta,\alpha,q,A}$ converge monotonically to a limiting measure $\phi^f_{\beta,\alpha,q,A}$ obtained by standard FKG arguments, as $n \to \infty$. The covariance $\mathrm{Cov}_{\phi_{G_n,\beta,\alpha,q,A}}\big[F(\omega), \omega(e)\big]$ also converges, and therefore when $\beta > \alpha$

$$\frac{d}{d\beta} \phi^f_{\beta,\alpha,q,A}[F] = \sum_{e \in A} \frac{J_e}{e^{\beta J_e} - 1} \mathrm{Cov}_{\phi^f_{\beta,\alpha,q,A}}\big[F(\omega), \omega(e)\big] \tag{24}$$

Because F is increasing and $\phi^f_{\beta,q}$ stochastically dominates $\phi^f_{\beta,\alpha,q,A}$ for every $\beta \geq \alpha$ (only edges in A have the larger weight β), we have

$$\frac{\phi^f_{\beta,q}[F] - \phi^f_{\alpha,q}[F]}{\beta - \alpha} \geq \frac{\phi^f_{\beta,\alpha,q,A}[F] - \phi^f_{\alpha,q,A}[F]}{\beta - \alpha}.$$

Letting $\alpha \uparrow \beta$ turns the left-hand side into the Dini derivative $\left(d/d\beta\right)_{+}\phi^{f}_{\beta,q}[F]$, whereas the right-hand side tends (by (24)) to $\sum_{e\in A}\frac{J_e}{e^{\beta J_e}-1}\mathrm{Cov}_{\phi^{f}_{\beta,\alpha,q,A}}[F,\omega(e)]$. Finally, take the supremum over all finite $A \subseteq E$. Positive association ensures that enlarging A only increases each covariance term, and then the supremum equals the full infinite sum over E. $\square$

To convert (23) into a logarithmic derivative for the tail event $F(\omega) = 1(|K_v| \geq n)$, we need a uniform positive lower bound on the total covariance $\sum_e \mathrm{Cov}(F, \omega(e))$. This is provided by an OSSS/ghost-field argument.

Lemma 5.3. *(OSSS for decision forests). Let μ be a monotonic measure on $\{0,1\}^E$. For measurable, $\mu-$integrable $f, g : \{0,1\}^E \to \mathbb{R}$ with f increasing and every decision forest $F = \{T_1, \dots\}$ computing g, we have*

$$\frac{1}{2}|\mathrm{Cov}_\mu[f,g]| \leq \sum_{e\in E} \delta_e(F,\mu)\mathrm{Cov}_\mu[f,\omega(e)]$$

Proof. By (8), we only need to construct a decision tree T that computes the same functions as F and has $\delta_e(T,\mu) = \delta_e(F,\mu)$ for all $e \in E$. Indeed, A possible construction is to

1. At time p_i^j (p_i is the ith prime), execute the jth step of T^i
2. At non-prime-power time, re-query the first input queried by T^1

$\square$

The following lemma is where we apply the OSSS inequality, and we need to choose an appropriate decision forest.

Lemma 5.4. *Let $G = (V, E)$ be a countable graph and μ be a monotonic measure on $\{0,1\}^E$ (in particular, $\mu = \phi^{\#}_{\beta,q}$ with $q \geq 1$). Then for all $v \in V, n \geq 1, \lambda > 0$*

$$\sum_{e\in E} \mathrm{Cov}_\mu[1(|K_v| \geq n), \omega(e)] \geq \left[\frac{(1 - e^{-\lambda} - \mu\left[1 - e^{-\lambda|K_v|/n}\right]}{2\sup_{u\in V}\mu\left[1 - e^{-\lambda|K_u|/n}\right]}\right]\mu(|K_v| \geq n)$$

Proof. Let the ghost field $\eta \in \{0,1\}^V$ be a random subset of V where each vertex is included independently with probability $h = 1 - e^{-\lambda/n} \leq \lambda/n$.

Let P and $\mathbb{E}$ denote probability and expectation with respect to the joint law of (ω, η). Fix $v \in V$, and define the increasing indicator functions

$$f(\omega,\eta) := 1\{|K_v| \geq n\}, \qquad g(\omega,\eta) := 1\{K_v \text{ contains at least one green vertex of } \eta\}$$

For each $u \in V$, define a **decision tree** T^u as follows:

1. Query $\eta(u)$.

2. If $\eta(u) = 0$ then stop and output u forever.

3. If $\eta(u) = 1$, then explore the cluster of u in ω in some predetermined ordering.

Formally, fix an enumeration of E and let $u \in V$. Set $T_1^u(\omega, \eta) = u$. If $\eta(u) = 0$, set $T_n^u = u$ for all $n \geq 2$. Otherwise, at step n:

1. Maintain sets U_n^u (revealed vertices), O_n^u (revealed open edges), and C_n^u (revealed closed edges).

2. Initialize $U_1^u = \{u\}$, $O_1^u = C_1^u = \emptyset$.

3. If every edge adjacent to U_n^u has been revealed, stop. Otherwise, reveal all such edges and update T_{n+1}^u with the smallest among them in the enumeration.

4. Update C_{n+1}^n, O_{n+1}^n based on $\omega(T_{n+1}^n)$

Such decision tree T^u satisfies

$$\{x \in V \cup E : T_n^u(\omega, \eta) = x \text{ for some } n \geq 1\} = \begin{cases} \{u\} & (\eta(u) = 0) \\ \{u\} \cup E(K_u(\omega)) & (\eta(u) = 1) \end{cases} \tag{25}$$

In particular, $F = \{T^u : u \in v\}$ computes g.

By the two-function OSSS inequality,

$$\mathrm{Cov}_{\mu \otimes \nu}[f, g] \leq \sum_{e \in E} \delta_e(F, \mu) \mathrm{Cov}_\mu[f, \omega(e)],$$

where ν denotes the ghost law. Moreover,

$$\mathrm{Cov}[f, g] = 2\mathrm{Cov}[f, g]_\mu = 2\mu(f = 1, g = 1) - 2\mu(f = 1)\mu(g = 1).$$

Since $\mathbb{E}[g||K_v| = m] = 1 - e^{-\lambda m/n}$, we obtain

$$\mathrm{Cov}_\mu(f, g) = \mu(|K_v| \geq n)(1 - e^{-\lambda}) - \mu[1 - e^{-\lambda|K_v|/n}].$$

An edge e is revealed by F if and only if the cluster containing e contains a green vertex. Thus

$$\delta_e(F, \mu) \leq 2 \sup_{u \in V} \mu(\eta(u) = 1 \text{ and } u \in K_e) \leq 2 \sup_{u \in V} \mu\big[1 - e^{-\lambda|K_u|/n}\big].$$

Combining with OSSS gives the claim. □

We are now ready to prove the differential inequality.

Theorem 5.5 (Differential inequality for percolation)**.** *For each $n \geq 1$, $\lambda > 0$, and $0 < p < 1$, we have*

$$\frac{d}{dp} \log P_p(|K| \geq n) \geq \frac{1}{2p(1-p)} \left[\frac{(1 - e^{-\lambda})n}{\lambda \sum_{m=1}^{\lceil n/\lambda \rceil} P_p(|K| \geq m)} - 1 \right] \tag{26}$$

Proof of the differential inequality. Lemma 5.2 gives

$$\frac{d}{dp}P_p(|K|\geq n)\geq\frac{1}{2p(1-p)}\sum_{e\in E}\mathrm{Cov}_p(1_{\{|K|\geq n\}},\omega(e)).$$

By Lemma 5.4,

$$\sum_{e\in E}\mathrm{Cov}_p(1_{\{|K|\geq n\}},\omega(e))\geq\frac{(1-e^{-\lambda})-P_p\big[1-e^{-\lambda|K|/n}\big]}{2\sup_u P_p\big[1-e^{-\lambda|K_u|/n}\big]}P_p(|K|\geq n) \tag{27}$$

$$\geq\frac{1}{2}\left[\frac{(1-e^{-\lambda})}{\frac{\lambda}{n}\sum_{m=1}^{\lceil n/\lambda\rceil}P_p(|K|\geq m)}-1\right]P_p(|K|\geq n) \tag{28}$$

Apply Lemma 5.2 to $F=1(|K|\geq n)$, we have

$$\left(\frac{d}{dp}\right)_+P_p(|K|\geq n)\geq\sum_{e\in E}\frac{1}{p(1-p)}\mathrm{Cov}[F,\omega(e)]\geq\frac{1}{2p(1-p)}\left[\frac{(1-e^{-\lambda})}{\frac{\lambda}{n}\sum_{m=1}^{\lceil n/\lambda\rceil}P_p(|K|\geq m)}-1\right]P_p(|K|\geq n). \tag{29}$$

It is known that $(\log f)'=\frac{1}{f}f'$, and this rule also applies to Dini's derivatives. Thus,

$$\left(\frac{d}{dp}\right)_+\log P_p(|K|\geq n)\geq\frac{1}{2p(1-p)}\left[\frac{(1-e^{-\lambda})}{\frac{\lambda}{n}\sum_{m=1}^{\lceil n/\lambda\rceil}P_p(|K|\geq m)}-1\right]$$

□

Corollary 5.6 (Differential inequality for RCM)**.** Given that (G,J) is an infinite transitive weighted graph, $q\geq 1$ and $\#\in\{f,w\}$. Then for $\beta\geq 0,\lambda>0,n\geq 1$,

$$\max_{e\in E}[\frac{e^{\beta J_e}-1}{J_e}]\left(\frac{d}{d\beta}\right)_+\log\phi^{\#}_{\beta,q}(|K|\geq n)\geq\frac{1}{2}\left[\frac{(1-e^{-\lambda})n}{\lambda\sum_{m=1}^{\lceil n/\lambda\rceil}\phi^{\#}_{\beta,q}(|K|\geq m)}-1\right]$$

5.3 Results

5.3.1 Critical Exponents Inequalities $\gamma\leq\delta-1,\Delta\leq\delta$

The inequalities $\gamma\leq\delta-1,\Delta\leq\delta$ are direct consequences of Theorem 5.8.

Lemma 5.7 (Integrated differential inequality)**.** *Let (G,J) be an infinite transitive weighted graph, $q\geq 1$, $\#\in\{f,w\}$, and write $\psi_\beta(n):=\phi^{\#}_{\beta,q}(|K|\geq n)$ and $C(\beta):=\max_{e\in E}\frac{e^{\beta J_e}-1}{J_e}$. For every $n\geq 1$, $\lambda>0$, and $0\leq\beta\leq\beta_0$, we have*

$$\psi_\beta(n)\ \leq\ \psi_{\beta_0}(n)\exp\left\{-\frac{\beta_0-\beta}{2C(\beta_0)}\left[\frac{(1-e^{-\lambda})n}{\lambda\sum_{m=1}^{\lceil n/\lambda\rceil}\psi_{\beta_0}(m)}-1\right]\right\} \tag{30}$$

and, since $\sum_{m=1}^{\lceil n/\lambda\rceil}\psi_{\beta_0}(m)\leq\phi^{\#}_{\beta_0,q}[|K|]$,

$$\psi_\beta(n)\ \leq\ \psi_{\beta_0}(n)\exp\left\{-\frac{\beta_0-\beta}{2C(\beta_0)}\left[\frac{(1-e^{-\lambda})n}{\lambda\phi^{\#}_{\beta_0,q}[|K|]}-1\right]\right\}. \tag{31}$$

Proof. By Corollary 5.6, for all $\beta \geq 0$,

$$C(\beta)\left(\frac{d}{d\beta}\right)_+ \log \psi_\beta(n) \;\geq\; \frac{1}{2}\left[\frac{(1-e^{-\lambda})n}{\lambda \sum_{m=1}^{\lceil n/\lambda \rceil} \psi_\beta(m)} - 1\right].$$

Both $\psi_\beta(m)$ and $C(\beta)$ are nondecreasing in β. Hence for $\beta \leq t \leq \beta_0$,

$$\sum_{m=1}^{\lceil n/\lambda \rceil} \psi_t(m) \;\leq\; \sum_{m=1}^{\lceil n/\lambda \rceil} \psi_{\beta_0}(m), C(t) \;\leq\; C(\beta_0).$$

so for $\beta \leq t \leq \beta_0$,

$$C(\beta_0)\left(\frac{d}{dt}\right)_+ \log \psi_t(n) \;\geq\; \frac{1}{2}\left[\frac{(1-e^{-\lambda})n}{\lambda \sum_{m=1}^{\lceil n/\lambda \rceil} \psi_{\beta_0}(m)} - 1\right].$$

Integrate from $t = \beta$ to $t = \beta_0$ and use the identity $\log \psi_{\beta_0}(n) - \log \psi_\beta(n) = \int_\beta^{\beta_0} \frac{d}{dt} \log \psi_t(n) dt$ to deduce

$$C(\beta_0)\big(\log \psi_{\beta_0}(n) - \log \psi_\beta(n)\big) \;\geq\; \frac{\beta_0 - \beta}{2}\left[\frac{(1-e^{-\lambda})n}{\lambda \sum_{m=1}^{\lceil n/\lambda \rceil} \psi_{\beta_0}(m)} - 1\right].$$

Exponentiating gives (30). Since $\sum_{m=1}^{\lceil n/\lambda \rceil} \psi_{\beta_0}(m) \leq \sum_{m \geq 1} \psi_{\beta_0}(m) = \phi^{\#}_{\beta_0,q}[|K|]$, (31) follows immediately. □

Theorem 5.8 (Exponential bound for Bernoulli percolation)**.** *Let G be an infinite, connected, locally finite transitive graph, and suppose that there exist constants $C > 0$ and $\delta > 1$ such that for all $n \geq 1$*

$$P_{p_c}(|K| \geq n) \leq C n^{-1/\delta}.$$

Then the following hold:

1. *There exist constants $c, C' > 0$ such that for all $0 \leq p < p_c, n \geq 1$*

$$P_p(|K| \geq n) \leq C' n^{-1/\delta} \exp\big[-c(p_c - p)^\delta n\big]$$

2. *There exists a constant $C'' > 0$ such that for all $k \geq 1, 0 \leq p < p_c$,*

$$\mathbb{E}_p[|K|^k] \leq k! \left[\frac{C''}{p_c - p}\right]^{(\delta-1)+(k-1)\delta}$$

Theorem 5.9 (Exponential bound for RCM)**.** *Let (G, J) be an infinite transitive weighted graph. Let $\beta_0 > 0$, $q \geq 1$, and $\# \in \{f, w\}$. Suppose there exist constants $C > 0$ and $\delta > 1$ such that for all $n \geq 1$*

$$\phi^{\#}_{\beta_0,q}(|K| \geq n) \leq C n^{-1/\delta}$$

Then the following hold:

1. *There exist constants $c_1, C_1 > 0$ such that for all $0 \leq \beta < \beta_0, n \geq 1$*

$$\phi^{\#}_{\beta,q}(|K| \geq n) \leq C_1 n^{-1/\delta} \exp\big[-c_1(\beta_0 - \beta)^\delta n\big] \tag{32}$$

2. *There exists a constant $c_2 > 0$ such that for all $k \geq 1, 0 \leq \beta < \beta_0$,*

$$\phi^{\#}_{\beta,q}[|K|^k] \leq k!\big[c_2(\beta_0 - \beta)\big]^{-\delta k+1} \tag{33}$$

Remark 5.10. Clearly, Theorem 5.9 follows as a corollary of Theorem 5.8 by specializing to the case $q = 1$ and uniform weights $J_e \equiv 1$. Therefore, in what follows we only prove Theorem 5.9.

Proof of Theorem 5.9. Fix $\beta_0 > 0$, and suppose there exist constants $C > 0$ and $\delta > 1$ such that

$$\phi^{\#}_{\beta_0,q}(|K| \geq n) \leq Cn^{-1/\delta}, \qquad \text{for every } n \geq 1.$$

Denote inequalities that hold up to positive multiplicative constants depending only on $(G, J), \delta, C, \beta_0$ by $\lesssim$ or $\gtrsim$.

Summing over $m \leq n$ gives

$$\sum_{m=1}^{n} \phi^{\#}_{\beta_0,q}(|K| \geq m) \lesssim n^{1-1/\delta}.$$

By Lemma 5.7, for every $0 \leq \beta_1 < \beta_0$ and $n \geq 1$,

$$\phi^{\#}_{\beta_1,q}(|K| \geq n) \lesssim n^{-1/\delta} \exp\Big[-c_1(\beta_0 - \beta_1)n^{1/\delta}\Big].$$

Now summing over $n \geq 1$ yields

$$\phi^{\#}_{\beta_1,q}[|K|] \lesssim (\beta_0 - \beta_1)^{-\delta+1}.$$

In particular, for every $0 \leq \beta < \beta_0$ and $n \geq 1$, we obtain

$$\phi^{\#}_{\beta,q}(|K| \geq n) \lesssim n^{-1/\delta} \exp\left[-c_2 \frac{(\beta_0 - \beta)n}{(\beta_0 - \beta)^{\delta-1}}\right] \lesssim n^{-1/\delta} \exp\big[-c_2(\beta_0 - \beta)^{\delta} n\big].$$

which proves (32).

For (33), note that for any $x > 0$,

$$\phi^{\#}_{\beta,q}[|K| \geq x] \lesssim x^{-1/\delta} \exp\big[-c_2(\beta_0 - \beta)^{\delta} x\big].$$

Let $\varepsilon = c_2(\beta_0 - \beta)^{\delta}$ and $\alpha = k - 1 - 1/\delta$, we get

$$\phi^{\#}_{\beta,q}[|K|^k] = k \int_0^{\infty} x^{k-1} \phi^{\#}_{\beta,q}(|K| \geq x)dx \lesssim k \int_0^{\infty} x^{\alpha} e^{-\varepsilon x} dx.$$

By change of variable $y = \varepsilon x$, we obtain

$$\phi^{\#}_{\beta,q}[|K|^k] \lesssim k\varepsilon^{-\alpha-1}\Gamma(\alpha + 1) \lesssim k![c_2(\beta_0 - \beta)]^{-\delta k+1}.$$

which proves (33). □

5.3.2 Critical Exponents Inequalities $\Delta \le \gamma + 1$

The inequality $\Delta \le \gamma + 1$ is a direct consequence of Theorem 5.11.

Theorem 5.11 (Bernoulli Percolation Version)**.** *Let G be an infinite, connected, locally finite transitive graph, and suppose there exist constants $C > 0$ and $\gamma \ge 1$ such that for all $0 \le p < p_c$*

$$\chi(p) := \mathbb{E}_p[|K|] \le C(p_c - p)^{-\gamma}$$

Then there exists $C' > 0$ such that, for every $k \ge 1$ and $0 \le p < p_c$,

$$\mathbb{E}_p[|K|^k] \le k! \left[\frac{C'}{p_c - p}\right]^{\gamma + (k-1)(\gamma+1)}.$$

Theorem 5.12 (RCM Version)**.** *Let (G, J) be an infinite transitive weighted graph. Fix $\beta_0 > 0$, $q \ge 1$, and $\# \in \{f, w\}$. Suppose there exist constants $C > 0$ and $\gamma \ge 1$ such that for all $0 \le \beta < \beta_0$*

$$\chi^{\#}_{\beta,q} := \phi^{\#}_{\beta,q}[|K|] \le C(\beta_0 - \beta)^{-\gamma}. \tag{34}$$

Then there exists $c > 0$ such that, for every $k \ge 1$ and $0 \le \beta < \beta_0$,

$$\phi^{\#}_{\beta,q}[|K|^k] \le k! \big[c(\beta_0 - \beta)\big]^{-(k-1)(\gamma+1)-\gamma}.$$

Remark 5.13. Still, Theorem 5.12 follows as a corollary of Theorem 5.11 by specializing to the case $q = 1$ and uniform weights $J_e \equiv 1$.

Proof of Theorem 5.12. Fix $0 \le \beta < \beta_0$ and let $\beta < \beta_1 < \beta_0$ be arbitrary (to be optimized later). By the integrated differential inequality with $\lambda = 1$ (Lemma 5.7), for some constant $a > 0$ depending only on $(G, J), q, \#$,

$$\phi^{\#}_{\beta,q}(|K| \ge n) \le \phi^{\#}_{\beta_1,q}(|K| \ge n) \exp\left\{-a\frac{\beta_1 - \beta}{\chi^{\#}_{\beta_1,q}} n\right\} \exp\{O(\beta_1 - \beta)\}. \tag{35}$$

Use Markov's inequality at level β_1 and absorb the harmless factor $\exp\{O(\beta_1 - \beta)\}$ into the constants:

$$\phi^{\#}_{\beta_1,q}(|K| \ge n) \ \le\ \frac{\chi^{\#}_{\beta_1,q}}{n}.$$

Substituting into (35) yields

$$\phi^{\#}_{\beta,q}(|K| \ge n) \le \frac{\chi^{\#}_{\beta_1,q}}{n} \exp\left\{-a\frac{\beta_1 - \beta}{\chi^{\#}_{\beta_1,q}} n\right\}. \tag{36}$$

By (34), $\chi^{\#}_{\beta_1,q} \le C(\beta_0 - \beta_1)^{-\gamma}$, so

$$\phi^{\#}_{\beta,q}(|K| \ge n) \ \le\ \frac{C'}{n}(\beta_0 - \beta_1)^{-\gamma} \exp\{-c_1(\beta_1 - \beta)(\beta_0 - \beta_1)^{\gamma} n\}.$$

Choose the balancing midpoint $\beta_1 = (\beta+\beta_0)/2$. Then $(\beta_1-\beta) \asymp (\beta_0-\beta_1) \asymp (\beta_0-\beta)$ and hence

$$\phi^{\#}_{\beta,q}(|K| \ge n) \le \frac{C''}{n}(\beta_0-\beta)^{-\gamma}\exp\bigl\{-c_2(\beta_0-\beta)^{\gamma+1}n\bigr\}.$$

Finally, for $k \ge 1$,

$$\phi^{\#}_{\beta,q}[|K|^k] = k\int_0^\infty x^{k-1}\phi^{\#}_{\beta,q}(|K|\ge x)dx \;\le\; kC''(\beta_0-\beta)^{-\gamma}\int_0^\infty x^{k-2}e^{-c_2(\beta_0-\beta)^{\gamma+1}x}dx.$$

so

$$\phi^{\#}_{\beta,q}[|K|^k] \;\le\; k!\bigl[c(\beta_0-\beta)\bigr]^{-(k-1)(\gamma+1)-\gamma},$$

□

5.3.3 Sharp Phase Transitions via Volumes

Theorem 5.14 (Sharpness of Phase Transitions). *Let (G,J) be an infinite transitive weighted graph, let $q \ge 1$ and $\# \in \{f,w\}$, and let $\beta_c^{\#} = \inf\{\beta \ge 0 : \phi^{\#}_{\beta,q}(|K|=\infty) > 0\}$. Then:*

1. *For every $0 \le \beta < \beta_c^{\#}$ there exist constants $C_\beta, c_\beta > 0$ such that for all $n \ge 1$*

$$\phi^{\#}_{\beta,q}\bigl(|K| \ge n\bigr) \;\le\; C_\beta e^{-c_\beta n} \tag{37}$$

2. *For every $\beta > \beta_c^{\#}$,*

$$\phi^{\#}_{\beta,q}(|K|=\infty) \;\ge\; \frac{\beta-\beta_c^{\#}}{2\max_{e\in E}\frac{e^{\beta J_e}-1}{J_e}+\beta-\beta_c^{\#}} \tag{38}$$

Proof. Define the volume–sharpness threshold

$$\widetilde{\beta}_c^{\#} := \sup\Bigl\{\beta\ge 0:\ \exists\, c,C>0 \text{ with } \phi^{\#}_{\beta,q}(|K|\ge n)\le Cn^{-c} \text{ for all } n\ge 1\Bigr\} = \inf\Bigl\{\beta\ge 0: \limsup_{n\to\infty}\frac{\log\phi^{\#}_{\beta,q}(|K|\ge n)}{\log n}\ge 0\Bigr\}$$

Obviously $\widetilde{\beta}_c^{\#} \le \beta_c^{\#}$. We need $\widetilde{\beta}_c^{\#} \ge \beta_c^{\#}$ and deduce (37) and (38).

First, fix $0 \le \beta < \widetilde{\beta}_c^{\#}$. By definition of $\widetilde{\beta}_c^{\#}$ there exists $\beta_0 \in (\beta, \widetilde{\beta}_c^{\#})$ and $\delta > 1$ such that $\phi^{\#}_{\beta_0,q}(|K|\ge n) \lesssim n^{-1/\delta}$. Applying Theorem 5.9 with this β_0 gives that for all $n \ge 1$

$$\phi^{\#}_{\beta,q}(|K|\ge n) \;\le\; C_\beta e^{-c_\beta n} \tag{39}$$

Hence (37) holds for every $\beta < \widetilde{\beta}_c^{\#}$.

Write

$$P_n(\beta) := \phi^{\#}_{\beta,q}(|K|\ge n), \qquad \Sigma_n(\beta) := \sum_{m=0}^{n-1}P_m(\beta), \qquad T_k(\beta) := \frac{1}{\log k}\sum_{n=1}^{k}\frac{P_n(\beta)}{n}$$

Then $T_k(\beta) \to \phi^{\#}_{\beta,q}(|K|=\infty)$ as $k\to\infty$ (standard renewal–type argument). Applying Corollary 5.6 with $\lambda = 1$ and summing over $n \le k$ gives

$$\Bigl(\frac{d}{d\beta}\Bigr)_+ T_k(\beta) \;\ge\; \frac{1}{2C(\beta)\log k}\sum_{n=1}^{k}\Biggl[\frac{(1-e^{-1})P_n(\beta)}{\Sigma_n(\beta)} - \frac{P_n(\beta)}{n}\Biggr], \qquad C(\beta) := \max_{e\in E}\frac{e^{\beta J_e}-1}{J_e}.$$

Using $u \geq \log(1+u)$ for $u \geq 0$ we obtain

$$\sum_{n=1}^{k} \frac{P_n(\beta)}{\Sigma_n(\beta)} \;\geq\; \sum_{n=1}^{k} \big(\log \Sigma_{n+1}(\beta) - \log \Sigma_n(\beta)\big) = \log \Sigma_{k+1}(\beta),$$

so that

$$\Big(\frac{d}{d\beta}\Big)_+ T_k(\beta) \;\geq\; \frac{(1-e^{-1}) \log \Sigma_{k+1}(\beta)}{2C(\beta)\log k} \;-\; \frac{T_k(\beta)}{2C(\beta)} \qquad (k \geq 2). \tag{40}$$

Fix $\widetilde{\beta}_c^{\#} < \beta_1 < \beta_2$. By the definition of $\widetilde{\beta}_c^{\#}$, $\Sigma_{k+1}(\beta_1) \geq k^{1-o(1)}$, whence

$$\lim_{k\to\infty} \frac{\log \Sigma_{k+1}(\beta_1)}{\log k} = 1.$$

Taking $\inf_{\beta\in[\beta_1,\beta_2]}$ of the left-hand side of (40), $\sup_{\beta\in[\beta_1,\beta_2]}$ inside $C(\beta)$, and then $\limsup_{k\to\infty}$ yields

$$\limsup_{k\to\infty} \inf_{\beta_1\leq\beta\leq\beta_2} \Big(\frac{d}{d\beta}\Big)_+ T_k(\beta) \;\geq\; \frac{1-e^{-1}}{2C(\beta_2)} \;-\; \frac{\phi^{\#}_{\beta_2,q}(|K|=\infty)}{2C(\beta_2)}.$$

Integrating this differential inequality over $\beta \in [\beta_1,\beta_2]$ and letting $k\to\infty$ gives

$$\phi^{\#}_{\beta_2,q}(|K|=\infty) \;\geq\; \frac{(1-e^{-1})(\beta_2-\beta_1)}{2C(\beta_2)+\beta_2-\beta_1} \;>\; 0.$$

As $\beta_1 \downarrow \widetilde{\beta}_c^{\#}$ and $\beta_2 > \beta_1$ are arbitrary, this shows $\phi^{\#}_{\beta,q}(|K|=\infty) > 0$ for every $\beta > \widetilde{\beta}_c^{\#}$, hence $\widetilde{\beta}_c^{\#} \geq \beta_c^{\#}$. Since $\widetilde{\beta}_c^{\#} \leq \beta_c^{\#}$, we conclude $\widetilde{\beta}_c^{\#} = \beta_c^{\#}$ and (39) holds for all $\beta < \beta_c^{\#}$, **proving** (37).

Apply Corollary 5.6 with general $\lambda > 0$ to obtain, for $k \geq 2$,

$$\Big(\frac{d}{d\beta}\Big)_+ T_k(\beta) \;\geq\; \frac{1}{2C(\beta)\log k} \sum_{n=1}^{k} \left[\frac{(1-e^{-\lambda})P_n(\beta)}{\lambda \sum_{m=1}^{\lfloor n/\lambda \rfloor} P_m(\beta)} - \frac{P_n(\beta)}{n} \right].$$

Similarly, we get that for every $\beta_1 < \beta_2$,

$$\limsup_{k\to\infty} \inf_{\beta_1\leq\beta\leq\beta_2} \Big(\frac{d}{d\beta}\Big)_+ T_k(\beta) \;\geq\; \frac{1-e^{-\lambda}}{2C(\beta_2)} \;-\; \frac{\phi^{\#}_{\beta_2,q}(|K|=\infty)}{2C(\beta_2)}.$$

Let $\lambda\to\infty$ so that $1-e^{-\lambda} \uparrow 1$, and integrate over $\beta \in [\beta_c^{\#}, \beta]$. Since $T_k(\beta_c^{\#}) \to 0$ (by sharpness just proved), we obtain

$$\phi^{\#}_{\beta,q}(|K|=\infty) \;\geq\; \frac{\beta-\beta_c^{\#}}{2C(\beta)+\beta-\beta_c^{\#}},$$

which **proves** (38). $\square$

6 Future Work and Open Problems

While recent works have broadened the range of models where inputs like the existence of critical exponents are available, several questions remain open. We collect a few directions below.

Existence (and values) of critical exponents on $\mathbb{Z}^d$, $3 \leq d \leq 6$. For Bernoulli percolation in high dimensions ($d > 6$), the lace expansion proves mean-field critical behavior and the triangle condition, yielding $\beta = \gamma = 1$ and $\delta = 2$, among other exponents [HS93, HH17]. In contrast, for $3 \leq d \leq 6$, the existence and values of most exponents remain open (even for $q = 1$). Rigorous identification of exponents in this range, and verification of scaling/hyperscaling, is a central challenge.

Planar FK model: nature of the phase transition. On $\mathbb{Z}^2$, the critical point is at the self-dual value $p_{\mathrm{sd}}(q) = \sqrt{q}\,(1+\sqrt{q})^{-1}$ for all $q \geq 1$ [BDC12]. The transition is continuous for $1 \leq q \leq 4$ (uniqueness of the infinite-volume Gibbs state at criticality and polynomial decay) [DST17], while it is discontinuous for $q > 4$ [DC17]. A mature critical exponent theory is available for $q = 1$ (critical site percolation on the triangular lattice via SLE) and much is known for $q = 2$ (Ising), but for $1 < q < 2$ a complete rigorous determination of exponents (and full conformal invariance statements) remains open. Progress here includes parafermionic-observable and RSW-type inputs [DST17]; turning these into full exponent identities for $1 < q < 2$ is a promising avenue.

Higher dimensions for FK with $q \geq 3$. In sufficiently high dimensions (or for sufficiently strong mean-field signatures), first-order transitions in Potts/FK with $q \geq 3$ are rigorously established via reflection positivity and mean-field bounds [BC03]. Sharpening these results—lowering the required dimension, quantifying the discontinuity, and mapping the d–q phase diagram—remains an active direction. In particular, the case $q = 3$ on intermediate dimensions (e.g., $d = 3, 4$) is a natural target for new techniques.

Random-cluster model for $1 < q < 2$ in $d \geq 3$. For nonplanar lattices, very little is known about the precise nature of the transition when $1 < q < 2$. Determining whether the transition is always continuous, identifying the near-critical scaling window, and proving the existence/values of critical exponents are major open problems. Extending lace-expansion or decision-tree/OSSS methods to this regime (possibly with new multi-scale inputs) is a concrete research path.

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References

[Men86] M. V. Menshikov. *Coincidence of critical points in percolation problems.* Dokl. Akad. Nauk SSSR, **288 (6)** (1986), 1308–1311.

[AB87] M. Aizenman, and D. J. Barsky, *Sharpness of the Phase Transition in Percolation Models*, Commun. Math. Phys. **108** (1987), 489-526.

[Pei36] R. Peierls, *On Ising's Model of Ferromagnetism.* (1936).

[Ham57] J. M. Hammersley. *Percolation processes: Lower bounds for the critical probability*. Ann. Math. Statist. **28** (1957), 790–795.

[Gri99] G. Grimmett, *Percolation*, Vol. 321 of *Grundlehren der mathematischen Wissenschaften*. Springer, Berlin, 1999.

[DCT15] H. Duminil-Copin, and V. Tassion, *A new proof of the sharpness of the phase transition for Bernoulli percolation and the Ising model*, Commun. Math. Phys. **343** (2016), 725-745.

[DRT19] H. Duminil-Copin, A. Raoufi, and V. Tassion, *Sharp phase transition for the random-cluster and Potts models via decision trees*, Ann. Math. **189** (2019), 75–99.

[Gri06] G. Grimmett, *The Random-Cluster Model*, Vol. 333 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin, 2006.

[EST24] P. Easo, F. Severo, and V. Tassion, *Counting Minimal Cutsets and* $p_c < 1$. (2024).

[Hut20] T. Hutchcroft, *New critical exponent inequalities for percolation and the random cluster model*, Prob. Math. Phys. **1(1)** (2020), 147-165.

[Kes87] H. Kesten, *Scaling relations for 2D-percolation*, Comm. Math. Phys. **109(1)** (1987), 109–156.

[OSSS05] R. O'Donnell, M. Saks, O. Schramm, and R. Servedio, *Every decision tree has an influential variable* Pittsburgh, IEEE FOCS (2005).

[BDC12] V. Beffara and H. Duminil-Copin, *The self-dual point of the two-dimensional random-cluster model is critical for* $q \geq 1$, Probab. Theory Related Fields **153** (2012), no. 3–4, 511–542.

[DST17] H. Duminil-Copin, V. Sidoravicius, and V. Tassion, *Continuity of the phase transition for planar random-cluster and Potts models with* $1 \leq q \leq 4$, Comm. Math. Phys. **349** (2017), 47–107.

[DC17] H. Duminil-Copin, *Lectures on the Ising and Potts models on the hypercubic lattice*, 2016/2017. Available at `https://arxiv.org/abs/1707.00520`. See also follow-up works proving first-order transitions for $q > 4$ in two dimensions.

[BC03] M. Biskup and L. Chayes, *Rigorous analysis of discontinuous phase transitions via mean-field bounds*, Comm. Math. Phys. **238** (2003), 53–93.

[HS93] T. Hara and G. Slade, *Mean-field behaviour and the lace expansion*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **354** (1993).

[HH17] M. Heydenreich and R. van der Hofstad, *Progress in high-dimensional percolation and random graphs*, Springer, Berlin, 2017.