ON RELATIVE FUNCTORS AND SIMPLICIAL LOCALIZATION

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Abstract. We provide an introduction to simplicial localization that requires as a prerequisite only basic knowledge of category theory. We begin with an introduction to the notion of $(\infty, 1)$-categories by describing quasicategories and simplicial categories, the model structures on those categories and the Quillen equivalence between them. Then we examine Dwyer and Kan's simplicial localization functor from relative categories to simplicial categories and we give some conditions for functors on relative categories to lift to equivalences of simplicial categories. We finish by giving some examples of simplicial localization.

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1. Introduction

Roughly an $(\infty, 1)$-category (or ∞-category for short) is a category with objects, morphisms between objects, 2-morphisms between morphisms, 3-morphisms between 2-morphisms and so on ad infinitum. Additionally for all $n > 1$, the n-morphisms are invertible. A motivating example is the ∞-category of topological spaces, with 1-morphisms being maps between spaces, 2-morphisms homotopies of maps, 3-morphisms homotopies of homotopies and so on. The 2-morphisms and above are invertible because homotopies can be reversed. It is clear that an ∞-category naturally encodes the data of homotopies. If we strengthen the definition so that the 1-morphisms are also invertible one obtains an ∞-groupoid. To any topological space one can associate its 'fundamental ∞-groupoid', a construction which extends the idea of the fundamental groupoid or 2-groupoid of a topological space in the canonical way. Grothendieck's homotopy hypothesis [AG] roughly conjectures that there is an equivalence between homotopy types of topological spaces and ∞-groupoids which is induced by the fundamental ∞-groupoid construction.

Date: November 17, 2025.

The hypothesis is true for some definitions of ∞-groupoids but remains unproven for others.

A natural question to ask is when a pair of ∞-categories is equivalent. One tool for this task is given by Quillen's [Q67] model categories. A model category is a category with distinguished classes of morphisms, most importantly a class of 'weak equivalences', which together satisfy certain properties that allow one to do homotopy theory. By defining a model category of ∞-categories, then a pair of ∞-categories is equivalent if they are weakly equivalent. However, model categories come with the limitation that their weak equivalences are not generally isomorphisms. One can resolve this by forming the homotopy category (or localization) which is a category in which the weak equivalences are isomorphisms. But the localization is created by quotienting maps under an equivalence relation, in particular doing this 'flattens' the higher homotopical data contained in the model category. Instead one can form the *simplicial localization*, an ∞-category which contains the data of the homotopy category while preserving the higher homotopical data of the model category. The purpose of this paper is to define the simplicial localization and discuss its properties and applications. We pay particular attention to the behavior of functors under simplicial localization.

There exist at least four distinct definitions or 'models' of ∞-categories. In section 2 we will define two of these models: quasicategories and simplicial categories.

In section 3 we will define relative categories, localizations, model categories and Quillen equivalences. Then in section 4 we will discuss model structures that can be defined on the category of quasicategories and the category of simplicial categories. We will introduce the homotopy coherent nerve which gives a Quillen equivalence of two of these model structures, thus establishing quasicategories and simplicial categories as equivalent models of ∞-categories.

In section 5 we will introduce Dwyer and Kan's [DK1] simplicial localization functor which produces a simplicial category from a relative category. We will show that the simplicial localization of a model category will contain the information of its homotopy category without sacrificing higher data.

In section 6 we will examine the simplicial localization on functors. We will introduce Dwyer and Kan's [DK2] hammock localization functor which is homotopy equivalent to the simplicial localization. Then we will use the hammock localization to prove that adjoint functors of relative categories whose unit and counit are weak equivalences lift to weak equivalences of simplicial categories– a fact which was stated but not proven in [DK2].

In section 7 we discuss some results which apply the simplicial localization; notably we precisely state the homotopy hypothesis in the case of Kan complexes and we discuss the model structure on relative categories.

2. Quasicategories and simplicial categories

Quasicategories are the easiest model of ∞-categories to define and benefit having a predominantly combinatorial definition. They were popularized by Joyal [J1], [J2] and his work along with the extensive writing of Lurie [KDN], [HA], [HTT] has greatly expanded the theory.

Definition 2.1 (Simplicial indexing category)**.** The simplicial indexing category, denoted Δ, is the full subcategory of Cat whose objects are the finite totally ordered sets $[n] := \{0, 1, ..., n\}$ for $n \in \mathbb{Z}_{\geq 0}$.

The morphisms in the above category are functors $F : [n] \to [m]$. The category $[n]$ has a morphism $f : n_1 \to n_2$ iff $n_1 \leq n_2$. Thus, since the image of f is a morphism $F(n_1) \to F(n_2)$ we must have $F(n_1) \leq F(n_2)$. A functor $F : [n] \to [m]$ therefore induces a unique order preserving map $[n] \to [m]$ and it is not hard to see that the converse is true.

Definition 2.2 (Simplicial set)**.** A simplicial set is a functor $\Delta^{\mathrm{op}} \to \mathsf{Set}$. Accordingly the category of simplicial sets, sSet, is the functor category $\mathsf{Set}^{\Delta^{\mathrm{op}}}$. For a simplicial set X, the set $X_n := X([n])$ is called the set of n-cells.

Definition 2.3 (Face and degeneracy maps)**.** We consider the order preserving maps $\delta_i^n : [n-1] \to [n]$ which 'skip' i and $\sigma_i^n : [n+1] \to [n]$ which 'double' i. These maps and their images in a simplicial set, denoted d_i^n and s_i^n, are called the face and degeneracy maps respectively.

Proposition 2.4. *Any map in the simplicial indexing category can be written as the composition of face and degeneracy maps. Thus a simplicial set X can be described uniquely in terms of its n-cells and its face and degeneracy maps.*

Proof. It suffices to show that any order preserving map $[n] \to [m]$ can be written as the composition of δs and σs. This can be shown explicitly via induction on n. □

Definition 2.5 (Degenerate cell)**.** We say that an n-cell in a simplicial set is degenerate if it is the image of a degeneracy map.

One useful characterizations is as follows: the degenerate 1-cells are the identity maps and the degenerate higher cell are those which contain an identity map as a 1-cell (i.e. as the image of some map $[0] \to [n]$).

Example 2.6 (Constant simplicial set)**.** Fix a set S, then define a simplicial set where the set of n-cells in each dimension is S and every map is the constant map id_S. This is called the consant simplicial set on S.

Example 2.7 (Nerve of a category)**.** Given a category C define a simplicial set as follows:

(1) The n-cells are chains of n morphisms in C, i.e. diagrams of the form
$$C_0 \longrightarrow C_1 \longrightarrow \cdots \longrightarrow C_{n-1} \longrightarrow C_n.$$
Note that this implies the zero-cells are the objects of C;
(2) The face maps in dimension $n > 0$ give the $n-1$ fold chain obtained from composing at the ith vertex (or simply omitting that vertex and the respective map when $i = 0, n$);
(3) The degeneracy maps give the $n+1$ fold chain obtained from repeating the ith vertex and adding an identity map between the appropriate vertices.

This simplicial set is called the nerve of the category C and it is denoted $N(C)$.

Definition 2.8 (Standard simplex)**.** The standard n-simplex, denoted by Δ^n, is given by the corepresentable functor $[m] \mapsto \mathrm{Hom}([m], [n])$.

The standard simplex in dimension n can be thought of as the prototypical example of an n-cell. It has exactly one non-degenerate n-cell and each m-cell is the image of that n-cell under exactly one of the maps $[m] \to [n]$. Indeed the n-cells of a simplicial set X are in bijective correspondance with maps $\Delta^n \to X$ by the Yoneda lemma.

When working with simplicial sets it is often helpful to draw diagrams in order to understand what an n-cell/standard simplex 'looks' like. One can imagine a zero-cell as a vertex. Then a 1-cell contains the information of an ordered pair of vertices and a single relation between them. One can draw this like a morphism:

$$0 \xrightarrow{\alpha} 1.$$

Of course there are more cells, but these are the only nondegenerate ones. In turn a 2-cell contains the information three vertices and three 1-cells being related follows:

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] \arrow[urr, phantom, "\sigma"] & & 2. \end{tikzcd}$$

The general procedure to produce an n-cell is to label vertices according to the maps $[0] \to [n]$ (evidently an n-cell will have $n+1$ vertices). The 1-cells can be produced by observing which 0-cells are contained under their face maps. Subsequently, the 2-cells can produced by observing which 1-cells they contain and so on.

Definition 2.9 (Subcomplex)**.** Given a simplicial set X we define a subcomplex S to be a simplicial set such that $S_n \subseteq X_n$ and the restrictions of the face and degeneracy maps in X to S coincide with those maps in S.

Definition 2.10 (Horn)**.** For $i < n$, the ith horn in dimension n, denoted Λ_i^n, is the largest subcomplex of Δ^n which does not contain $d_i^n(\mathrm{id}_{[n]})$. Explicitly,

$$(\Lambda_i^n)_m = \{\alpha \in \mathrm{Hom}_\Delta([m],[n]) \mid [n] \not\subseteq \alpha([m]) \cup \{i\}\}.$$

We say that a horn is inner if $0 < i < n$.

Definition 2.11 (Quasicategory)**.** A simplicial set S is called a quasicategory if it has inner horn extensions, i.e. for any inner horn, there is a map extending the diagram below

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d, hook] & S \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

In dimension 2, the horns are as follows:

$$\Lambda_0^2 : \begin{tikzcd} & 1 & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \qquad \Lambda_1^2 : \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd} \qquad \Lambda_2^2 : \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[rr] & & 2. \end{tikzcd}$$

Since only the center horn is inner, the above pictures should suggest that inner horn extension is something like composition. This is *almost* the case, rather inner horn extension gives non-unique composites.

Theorem 2.12 ([KDN], 1.3.4.1)**.** *A simplicial set has unique inner horn extensions iff it is isomorphic the nerve of a category.*

Corollary 2.13. *The nerve of a category is a quasicategory.*

Definition 2.14 (Kan complex)**.** A Kan complex is a simplicial set with all horn extensions.

Kan complexes are often called ∞-groupoids because their 1-morphisms are invertible up to homotopy. However, the terminology ∞-groupoid often refers to one of many different and nonequivalent definitions which are designed to represent an ∞-category where all the n-morphisms are invertible for $n > 0$.

Theorem 2.15 ([KDN], 1.3.5.2)**.** *A category C is a groupoid iff its nerve is a Kan complex.*

Simplicial categories (also called simplicially enriched categories) due to Quillen [Q67] are another important model of ∞-categories. They come into play as they are often the most natural model to lift to from ordinary categories (the simplicial localization lands in simplicial categories for example).

Definition 2.16 (Simplicial category)**.** A simplicial category C has the following data:

(1) A collection of objects $\mathrm{ob}(C)$;
(2) For each pair of objects $X, Y \in \mathrm{ob}(C)$ a simplicial set $\mathrm{Hom}_C(X, Y)$ or just $\mathrm{Hom}(X, Y)$ when C is obvious;
(3) For each triple of objects $X, Y, Z \in \mathrm{ob}(C)$ a composition map of simplicial sets $\mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$;
(4) For each simplicial set $\mathrm{Hom}(X, X)$ a 0-cell, id_X,

such that the following diagrams commute:

(i) Associativity:

$$\begin{tikzcd}
 & \mathrm{Hom}(Z,W) \times \mathrm{Hom}(Y,Z) \times \mathrm{Hom}(X,Y) \arrow[dl] \arrow[dr] & \\
\mathrm{Hom}(Z,W) \times \mathrm{Hom}(X,Z) \arrow[dr] & & \mathrm{Hom}(Y,W) \times \mathrm{Hom}(X,Y); \arrow[dl] \\
 & \mathrm{Hom}(X,W) &
\end{tikzcd}$$

(ii) Identity:

$$\begin{tikzcd}
\mathrm{Hom}(X,Y) \times \{\mathrm{id}_X\} \arrow[r, hook] \arrow[dr] & \mathrm{Hom}(X,Y) \times \mathrm{Hom}(X,X) \arrow[d] \\
 & \mathrm{Hom}(X,Y).
\end{tikzcd}$$

The diagram for post-composing by the identity should also commute.

Simplicial categories and simplicial sets are 'related,' but not quite equivalent concepts. In section 4, this relationship will be made explicit.

Example 2.17 (Constant simplicial category)**.** Given a category C, it can expressed as the simplicial category with the same objects and whose hom objects are the constant simplicial sets given by the hom sets of C.

Definition 2.18 (Locally Kan)**.** A simplicial category is said to be locally Kan if every hom object is a Kan complex.

Just as simplicial categories are related to simplicial sets, the constant simplicial category is related to the nerve and locally Kan simplicial categories are related to quasicategories.

Definition 2.19 (Simplicial functor)**.** A simplicial functor between simplicial categories contains the following information:

(1) A function $\mathrm{ob}(C) \to \mathrm{ob}(D)$;
(2) For every $X, Y \in C$ a map of simplicial sets $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$.

Such that the following conditions are satisfied:

(i) Identity: For $X \in C$ the map $\mathrm{Hom}(X, X) \to \mathrm{Hom}(F(X), F(X))$ takes $\mathrm{id}_X \mapsto \mathrm{id}_{F(X)}$;
(ii) Composition: For every triple $X, Y, Z \in C$ the following square commutes:

$$\begin{tikzcd}
\mathrm{Hom}(Y,Z) \times \mathrm{Hom}(X,Y) \arrow[r] \arrow[d] & \mathrm{Hom}(X,Z) \arrow[d] \\
\mathrm{Hom}(F(Y),F(Z)) \times \mathrm{Hom}(F(X),F(Y)) \arrow[r] & \mathrm{Hom}(F(X),F(Z)).
\end{tikzcd}$$

The category of simplicial categories, with simplicial categories as objects and simplicial functors as morphisms, is denoted $\mathsf{sSet\text{-}Cat}$.

3. Localizations and model categories

There are many scenarios in which one encounters what is called a 'weak equivalence': a map which preserves important structure of objects but which fails to be an isomorphism. A weak homotopy equivalence between topological spaces is a good example of such an equivalence. If one wishes for their weak equivalences to be isomorphisms, they can formally invert the weak equivalences in their category to form a new category called the localization. However, one might desire for all the morphisms in the localization to correspond to honest maps on objects. Additionally, such a procedure can introduce some issues with size.

Model categories were developed by Quillen [Q67] to provide a categorical framework in which to do homotopy theory. We will apply this framework to ∞-categories in section 5. Model categories also resolve some of the issues with taking localizations via a construction called the homotopy category (akin to the homotopy category of topological spaces) which is equivalent to the localization but more well behaved.

Definition 3.1 (Relative category; [BK1], 3.1)**.** A relative category is a pair (C, W) such that W is a subcategory containing all the objects of C. The class of morphisms in W is called the 'class of weak equivalences'. A functor of relative categories is a functor which takes weak equivalences to weak equivalences.

Definition 3.2 (Localization)**.** Given a relative category, we define a localization to be a category $C[W^{-1}]$ and a functor $\iota : C \to C[W^{-1}]$ satisfying the following:

(i) For every $w \in W$ the morphism $\iota(w)$ is an isomorphism;
(ii) For any category A, $- \circ \iota : A^{C[W^{-1}]} \to A^C$ is fully faithful, and essentially surjective on functors satisfying (i).

Theorem 3.3. *If a localization exists it is unique up to equivalence.*

Proof. Suppose $C[W^{-1}], C[W^{-1}]'$ are localizations of (C, W). Then we have functors $\iota : C \to C[W^{-1}]$ and $\iota' : C \to C[W^{-1}]'$ sending $w \in W$ to isomorphisms. By (ii) there is a functor $P : C[W^{-1}] \to C[W^{-1}]'$ such that $P \circ \iota \simeq \iota'$ and dually a functor $Q : C[W^{-1}] \to C[W^{-1}]'$ such that $Q \circ \iota' \simeq F$. So we have $P \circ \iota \simeq P \circ Q \circ \iota' \simeq \iota'$. Since $- \circ \iota'$ is fully faithful we have $P \circ Q \simeq 1$. Dually $Q \circ P \simeq 1$ so we are done. $\square$

Construction 3.4. Given a relative category there is a general procedure to produce its localization. One takes the set of morphism in C and W^{op} and considers the free category generated by these morphism. That is the category whose objects are the objects of C and whose morphisms are composable words of the above morphisms. Now on this category we can quotient each homset by the equivalence relation generated by, i.e. given by zig-zags and compositions of, the following relations:

(1) For $f : X \to Y$ we have $f \circ \mathrm{id}_X \sim f \sim \mathrm{id}_Y \circ f$;
(2) For $f : X \to Y$ and $g \in Y \to Z$ in C such that $g \circ f = h$ we have $g \circ f \sim h$;
(3) For $w : X \to Y$ in W we have $w \circ w^{\mathrm{op}} \sim \mathrm{id}_Y$ and $w^{\mathrm{op}} \circ w \sim \mathrm{id}_X$;
(4) Any of the above relations precomposed or postcomposed (or both) by some fixed morphism on both sides.

This construction gives a category and an obvious functor ι from C into this category. We will show that this pair gives the localization of C with respect to W and moving forward we will use the notation $C[W^{-1}]$ to refer to this category specifically.

Remark 3.5. Depending on your definition of a category, the above construction may fail to be a category as it is not always locally small ([HK], 4.15). We will disregard any size issues with regard to localization and later simplicial localization, however, they are given some treatment in [DK3].

Theorem 3.6. *The above construction is a localization.*

Proof. Note that the obvious functor $\iota : C \to C[W^{-1}]$ is faithful and bijective on objects. Further it is evident that this functor satisfies (i).

Now consider the functor $- \circ \iota$. Suppose we have functors $P, Q : C[W^{-1}] \to A$ and a natural transformation $\eta : P \to Q$ indexed by objects of $C[W^{-1}]$. Then we have a natural transformation $\eta \circ \iota : P \circ \iota \to Q \circ \iota$ indexed by objects of C and given by $(\eta \circ \iota)_X \coloneqq \eta_{\iota(X)}$. So $\eta \circ \iota = \eta' \circ \iota$ implies $\eta_{\iota(X)} = \eta'_{\iota(X)}$ and ι being surjective on objects implies $\eta = \eta'$. This gives that $- \circ \iota$ is faithful. Now suppose we have a natural transformation $\mu : P \circ \iota \to Q \circ \iota$ indexed by objects of C. I claim the set of maps $\eta_{\iota(X)} \coloneqq \mu_X$ is a natural transformation $P \to Q$. At the very least this is a collection of maps indexed by objects of $C[W^{-1}]$ since ι is bijective on objects.

Now since μ is a natural transformation we have

$$\begin{tikzcd} P\iota(X) \arrow[r, "P\iota(\alpha)"] \arrow[d, "\eta_{\iota(X)}"'] & P\iota(Y) \arrow[d, "\eta_{\iota(Y)}"] \\ Q\iota(X) \arrow[r, "Q\iota(\alpha)"'] & Q\iota(Y). \end{tikzcd}$$

So for the morphisms in the image of ι we have that μ satisfies naturality. Now we show this is the case for formal inverses of morphisms in the image of W under ι. We have

$$\begin{tikzcd} P\iota(X) \arrow[r, "P\iota(w)"] \arrow[d, "\eta_{\iota(X)}"'] & P\iota(Y) \arrow[r, "P\iota(w)^{\mathrm{op}}"] \arrow[d, "\eta_{\iota(Y)}"] & P\iota(X) \arrow[d, "\eta_{\iota(X)}"] \\ Q\iota(X) \arrow[r, "Q\iota(w)"'] & Q\iota(Y) \arrow[r, "Q\iota(w)^{\mathrm{op}}"'] & Q\iota(X). \end{tikzcd}$$

Since the whole square commutes the upper path commutes with the lower path. Then since the left square commutes, the upper path thus commutes with the middle path. So since $P\iota(w)$ is an isomorphism the right square thus commutes as desired. We have shown that for morphisms in $\iota(C)$ and formal inverses of $\iota(W)$ the appropriate squares for η commute. However, these sets of morphisms are exactly C and W^{op} which generate the morphisms of $C[W^{-1}]$ under composition. Since the composition of commuting squares also commutes we have that η satisfies naturality on all the morphisms of $C[W^{-1}]$. So we have shown $-\circ\iota$ is fully faithful.

Lastly suppose there is a functor $F : C \to A$ which sends W to isomorphisms. We will define a collection of maps of objects and morphisms $F' : C[W^{-1}] \to A$. First let $F'(\iota) = F$ on objects and morphisms in the image of ι (recall ι is bijective on objects and faithful). Further for $w \in \iota(W)$ let $F' : \iota(w)^{\mathrm{op}} \mapsto F(w)^{-1}$. It is evident that F' (which is not quite a functor yet) commutes with composition within $C \cup W^{\mathrm{op}}$ and since F and ι are functors it sends identities to identities. It is also evidently a well defined map on objects. Now for the remaining morphisms in $C[W^{-1}]$, which are composable words of morphisms in $C \cup W^{\mathrm{op}}$, define F' of a word to be F' applied to each map and then composed in A. Since it sends identities to identities, we already have that F' is well defined with respect to morphisms that are related by relation (1). Further the definition of F' gives that if it is well defined on morphisms related by zig-zags and compositions of $(1), (2), (3)$ then it is also well defined on morphisms related by zig-zags and compositions including (4) since F' is explicitly built to satisfy composition. But, since (2) and (3) live entirely in $C \cup W^{\mathrm{op}}$ where F' is already functorial, we are done. So F' is a well defined map and by construction a functor through which F factors. □

While a relative category is the most general setup for a category with some distinguished set of weak equivalences, we often desire something stronger. We introduce the definition of a category with weak equivalences which strengthens relative categories and then of a model category which stengthens categories with weak equivalences.

Definition 3.7 (Category with weak equivalences)**.** We say that a relative category (C, W) is a category with weak equivalences if:

(i) W contains all isomorphisms;

(ii) W satisfies the two-out-of-three property, i.e. given composable morphisms f, g if any two of $\{f, g, fg\}$ are in W then so is the third.

Definition 3.8 (Model category; [Q67], Defn I.1.1)**.** A model category (or a model structure on a category) is a category C together with three distinguished classes of morphisms, called weak equivalences, fibrations and cofibrations that satisfies a set of five axioms. Morphisms which are simultaneously weak equivalences and fibrations/cofibrations are called trivial fibrations/cofibrations. A model category satisfies the following five axioms:

(i) M is complete and cocomplete, i.e. it has all small limits and colimits;
(ii) The class of weak equivalences satisfies the two-out-of-three property;
(iii) Weak equivalences, fibrations and cofibrations are closed under retracts, i.e. for commuting diagrams in the form

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend left, "\mathrm{id}_X"] \arrow[d, "f"'] & A \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r] \arrow[rr, bend right, "\mathrm{id}_Y"'] & B \arrow[r] & Y \end{tikzcd}$$

if g is a weak equivalence, fibration or cofibration then so is f;
(iv) Every cofibration satisfies the left lifting property with respect to every trivial fibration, and every trivial cofibration satisfies the left lifting property with respect to every fibration, i.e. there is a morphism completing the following diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

whenever i is a cofibration and p a trivial fibration or i is a trivial cofibration and p a fibration;
(v) Every map has factorizations into a cofibration composed with a trivial fibration and a trivial cofibration composed with a fibration. Additionally these factorizations should give functors $C^{[1]} \to C^{[2]}$ which are sections of the composition functor $C^{[2]} \to C^{[1]}$ and with the morphisms living in the appropriate classes.

Remark 3.9. Quillen's original definition is slightly different in that he only requires finite limits and colimits. The above is what Quillen calls a 'closed model category'.

Example 3.10 (Quillen model structure on topological spaces; [Q67], II.3)**.** The category of topological spaces can be endowed with a model structure as follows:

(1) The weak equivalences are weak homotopy equivalences;
(2) The fibrations are Serre fibrations;
(3) The cofibrations are the maps which exhibit the left lifting property with respect to all the trivial fibrations.

Theorem 3.11. *A model category C with weak equivalences W is a category with weak equivalences.*

Proof. The idea is to show that any isomorphism is a retract of an identity, and then to use factorization to show that there is a weak equivalence p whose source is any object X so that $p = p \circ \mathrm{id}_X$. □

In general, when in the context of model categories, we will say that two things are 'weakly equivalent' if they are related in the equivalence relation generated by the weak equivalences.

Definition 3.12 (Fibrant and cofibrant objects)**.** Since model categories are complete and cocomplete, they have initial and terminal objects. We say an object is fibrant if the map to the terminal object is a fibration and cofibrant if the map from the initial object is a cofibration. An object satisfying both properties is called bifibrant.

Construction 3.13 (Fibrant replacement)**.** Consider an object X in a model category. There is a unique morphism $f : X \to T$ to the terminal object. This can be factored $f = p \circ i$ with i a trivial cofibration and p a fibration. We write:

$$X \xrightarrow{i} RX \xrightarrow{p} T.$$

So X is weakly equivalent to an object RX which is fibrant. This process is called fibrant replacement and the notation RX should be suggestive of the fact that fibrant replacement is a functor. This follows immediately from functorial factorization.

There is a dual notion called the cofibrant replacement of X, which is often denoted LX.

Corollary 3.14. *Every object in a model category is weakly equivalent to a bifibrant object.*

Remark 3.15. In a model category there is a suitable equivalence relation called 'homotopy' on maps of bifibrant objects such that weak equivalences are related to homotopy equivalences.

Definition 3.16 (Homotopy category)**.** Given a model category, one can form its homotopy category, the category whose objects are the bifibrant objects and whose morphisms are equivalence classes of morphisms under homotopy. We denote the homotopy category by $\mathrm{Ho}(C)$.

For more precise details regarding homotopy in model categories see [PH], 7.3-5.

Theorem 3.17 ([Q67], Thm I.1)**.** *The homotopy category is equivalent to the localization with respect to the weak equivalences:*

$$Ho(C) \simeq C[W^{-1}].$$

The above theorem is quite interesting. The fibrations and cofibrations are extra structure which allow a concrete construction of the localization of a model category with respect to its weak equivalences. However, this construction, up to equivalence, only relies on the underlying category of weak equivalences and not all the model structure.

Definition 3.18 (Quillen equivalence)**.** Let M, N be model categories and $F : M \leftrightarrows N : G$ be adjoint functors: $F \dashv G$. We say that the pair (F, G) is a pair of Quillen equivalences if:

(i) F preserves cofibrations and trivial cofibrations;
(ii) G preserves fibrations and trivial fibrations;
(iii) When $B \in M$ if cofibrant and $X \in N$ fibrant then $f : B \to GX$ is a weak equivalence iff $f^{\#} : FB \to X$ is a weak equivalence.

When the first two axioms are satisfied we call (F, G) a Quillen adjunction.

Quillen adjunctions are designed to induce adjunctions on the homotopy categories and Quillen equivalences are designed to induce equivalences on the homotopy categories. In this sense Quillen equivalences give that two model structures are equivalent up to homotopy.

In general we will say that model structures are Quillen equivalent if they can be connected via zig-zig and composite of Quillen equivalences.

Theorem 3.19 ([Q67], Thm I.3)**.** *A Quillen equivalence induces an equivalence on homotopy categories.*

4. Model structures of ∞-categories

As noted already, one should think of model categories as 'the right place to do homotopy theory in categories'; with some care one can translate many the notions used for homotopy theory of topological spaces into model categories. Accordingly, it is the framework that is used to study homotopy theory of ∞-categories. We will introduce the model structures on simplicial sets and simplicial categories and describe the Quillen equivalence between them which justifies the claim that quasi-categories and locally Kan-simplicial categories provide 'equivalent models of ∞-categories'.

Before describing the model structures we will need some additional definitions.

Definition 4.1 (Function complex)**.** Given simplicial sets X, Y, their function complex is the simplicial set:

$$\mathrm{Fun}(X, Y)_n := \mathrm{Hom}_{\mathsf{sSet}}(\Delta^n \times X, Y).$$

Here the product on simplicial sets is defined using the product on sets. For the maps in the function complex, note that any map $\alpha : [m] \to [n]$ gives an obvious map $\tilde{\alpha} : \Delta^m \to \Delta^n$ given by postcomposing with the elements of Δ^m. Then there is a map $\tilde{\alpha} \times \mathrm{id} : \Delta^m \times X \to \Delta^n \times X$. Precomposing then gives a map $\mathrm{Fun}(X, Y)_n \to \mathrm{Fun}(X, Y)_m$ as desired.

Definition 4.2 (Connected components)**.** Given a simplicial set X define an equivalence relation on X_0 given by $x \sim y$ if x can be connected to y via a zig-zag and composite of 1-cells. The set X_0 quotiented by the above relation is called the set of connected components of X and is denoted by $\pi_0 X$.

For simplicial sets X, Y we will write $[X, Y] := \pi_0 \mathrm{Fun}(X, Y)$.

Theorem 4.3 (Model structure on simplicial sets; [Q67], II.3)**.** *The Kan-Quillen model structure on simplicial sets is defined by:*

(1) Weak equivalences are maps $f : X \to Y$ which induce for every Kan complex K a bijection $f^* : [Y, K] \to [X, K]$;
(2) Cofibrations are monomorphisms in the category of simplicial sets, i.e. degreewise injections;
(3) Fibrations are Kan fibrations, i.e. maps f which have the lifting property with respect to horn inclusions:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

Theorem 4.4 ([Q67], Prop II.3.3)**.** *All objects in the above structure are cofibrant and the fibrant objects are Kan complexes.*

Remark 4.5. When we say weak equivalences of simplicial sets it will be in the sense defined above unless stated otherwise.

Definition 4.6 (Homotopy category of a simplicial category)**.** Given a simplicial category C one can define a category, denoted $\pi_0 C$ whose objects are the objects of C and whose hom sets are given $\mathrm{Hom}_{\pi_0 C}(X, Y) \coloneqq \pi_0 \mathrm{Hom}_C(X, Y)$.

Theorem 4.7 (Model structure on simplicial categories; [B1])**.** *The Dwyer-Kan-Bergner model structure on simplicial categories is defined by:*

(1) Weak equivalences are functors which induce weak equivalences on the hom simplicial sets and equivalences on the homotopy categories, called Dwyer-Kan equivalences;
(2) Fibrations are maps $F : C \to D$ which induce fibrations on the hom simplicial sets and such that if there is a map $e : Fa_1 \to b$ in D which is an isomorphism in $\pi_0(D)$ then there is a map $d : a_1 \to a_2$ in C that is an isomorphism in $\pi_0(C)$ and that $Fd = e$;
(3) Cofibrations are maps satisfying the left lifting property with respect to the trivial fibrations.

Theorem 4.8. *The fibrant objects in this model structure are locally Kan simplicial categories.*

Proof. One can easily check that the terminal simplicial set is the constant simplicial set on one object and the terminal simplicial category is the category with one object with the hom object being the terminal simplicial set. So note that a fibration $F : C \to T$ in simplicial categories induces fibrations from the hom simplicial sets to the terminal simplicial set. It follows that any fibrant object is locally Kan. Conversely, if C is locally Kan and $F : C \to T$ a map to the terminal object then the induced maps on the hom simplicial sets are maps from Kan complexes to the terminal simplicial set. The maps on the hom simplicial sets are thus fibrations of simplicial sets. In turn for any $a_1 \in C$ there is a unique map $e : Fa_1 \to b$ which is an equivalence in the homotopy category since the terminal simplicial category has only one morphism. Evidently the map id_{a_1} is an equivalence in the homotopy category of C and maps to e. So C is fibrant. $\square$

Construction 4.9. We will now describe a functor Path : $\Delta \to \mathsf{sSet\text{-}Cat}$. It is not hard to check that such a functor will induce a functor $\mathsf{sSet\text{-}Cat} \to \mathsf{sSet}$ by

$C \mapsto \mathrm{Hom}(\mathrm{Path}(-), C)$. Consider the toset $[n]$, for any $x, y \in [n]$ there is a poset whose elements are totally ordered subsets beginning at x and terminating at y under reverse inclusion. So for $[n]$ there is a simplicial category $\mathrm{Path}([n])$ whose objects are the objects of $[n]$, whose hom objects are the nerves of the posets defined above and with composition given by taking the union of tosets. This construction is functorial as follows: for a map $\alpha : [n] \to [m]$ each object is mapped to its image under α and each n-cell, which is a totally ordered subset of $[n]$, is mapped to its image under α.

Definition 4.10 (Homotopy coherent nerve, [JC])**.** The homotopy coherent nerve, denoted as $N^{hc} : \text{sSet-Cat} \to \text{Cat}$ is exactly the functor

$$C \mapsto \mathrm{Hom}(\mathrm{Path}(-), C).$$

Theorem 4.11 ([KDN], 2.4.3.12)**.** *Given a category C let $\tilde{C}$ denote the associated constant simplicial category. Then $N(C)$ is isomorphic to $N^{hc}(\tilde{C})$.*

Theorem 4.12 ([KDN], 2.4.4.4)**.** *The homotopy coherent nerve admits a left adjoint.*

The adjoint to the homotopy coherent nerve is often called rigidification. Following Kerodon we call this the path functor. One can think of this an extension to the path functor defined on the $[n]$s [KDN], 2.4.4.15.

Theorem 4.13 (Model structure on quasicategories; [J2], 9.3)**.** *The Joyal model structure on simplicial sets, often called the model structure on quasicategories is defined as follows,*

(1) The weak equivalences are maps which go to Dwyer-Kan equivalences under the path functor;
(2) The cofibrations are monomorphisms;
(3) The fibrations are maps satisfying right lifting with respect to the trivial cofibrations.

Additionally all objects are cofibrant and the fibrant objects in this model structure are quasicategories.

Altogether we can state the correspondance between quasicategories and simplicial categories

Theorem 4.14 ([J3], 2.10)**.** *The homotopy coherent nerve and the path functor are Quillen equivalences between the model structure on quasicategories and the model structure on simplical categories.*

Corollary 4.15. *The homotopy coherent nerve of a locally Kan simplicial category is an quasicategory.*

Lemma 4.16 ([DS], 5.9)**.** *The counit $\epsilon_D : Path(N^{hc}(D)) \to D$ is a weak equivalence when D is locally Kan.*

Theorem 4.17. *If C, D are locally Kan simplicial categories and $f : C \to D$ is a weak equivalence, then $N^{hc}(f)$ is a weak equivalence in the model structure on quasicategories.*

Proof. We begin with the composite,

$$\mathrm{Path}(N^{hc}(C)) \xrightarrow{\epsilon_C} C \xrightarrow{f} D.$$

This is a weak equivalence. Also, $N^{hc}(C)$ is cofibrant and D is fibrant. Since the homotopy coherent nerve and path functor are Quillen equivalences we have that $(f \circ \epsilon_C)^{\#}$ is a weak equivalence. Explicitly this is the composite:

$$N^{hc}(C) \xrightarrow{\eta_{N^{hc}(C)}} N^{hc}(\mathrm{Path}(N^{hc}(C))) \xrightarrow{N^{hc}(\epsilon_C)} N^{hc}(C) \xrightarrow{N^{hc}(f)} N^{hc}(D)$$

By the triangle identity this is just $N^{hc}(f)$ so we have the desired statement. □

Remark 4.18. Note that this theorem does *not* say that if two locally Kan simplicial categories are weakly equivalent then so are their corresponding quasicategories. Being weakly equivalent merely means that they can connected by a zig-zag of weak equivalences, and in particular this zig-zag need not pass through locally Kan simplicial categories. Only when this condition is actually met, does the theorem guarentee equivalent quasicategories.

There exist more models of ∞-categories, all of which can be connected back to quasicategories and simplicial categories with Quillen equivalences. Bergner has a good paper on this matter, though it is not fully up to date [B2]. More recently Riehl and Verity have done work which explores this question from an axiomatic perspective [RV].

5. Simplicial localization

We follow Dwyer and Kan's original construction of the simplicial localization which promotes a relative category to a simplicial category. Subsequently, we provide relevant results that show the simplicial localization encodes interesting information–both the homotopy category and higher data–about a model category

Definition 5.1 (O-Cat; [DK1], 1.4.ii)**.** Fix a set O. An O-category is a category whose set of objects is O. The category O-Cat is the category whose objects are O-categories and whose morphisms are functors which are the identity on objects.

Construction 5.2. Let sO-Cat denote the category of simplicial objects in O-Cat, i.e. functors $\Delta^{\mathrm{op}} \to O$-$\mathsf{Cat}$ with morphisms being natural transformations between them. To every $S \in sO$-Cat a simplicial category can be associated whose set of objects is O and hom objects are $\mathrm{Hom}(X,Y)_n := \mathrm{Hom}_{S([n])}(X,Y)$. The face and degeneracy maps induced by the face and degeneracy functors in S, while the composition maps are given by composition in S.

Definition 5.3 (Standard resolution; [DK1], 2.5)**.** Given a category $C \in O$-Cat, define $FC \in O$-Cat to be the category which for every non-identity morphism $c \in C$ has a morphism Fc and whose non-identity morphisms are freely generated by these Fc. This construction comes with functors $\varphi : FC \to C$ given by $Fc \mapsto c$ and $\psi : FC \to F^2C$ given by $Fc \to F(Fc)$. Now the standard resolution of C, denoted $F_*C \in sO$-Cat, is defined as follows:

(1) $F_nC := F^{n+1}C$;
(2) $d_i^n := F^i\varphi F^{n-i} : F^{n+1}C \to F^nC$;
(3) $s_i^n := F^i\psi F^{n-i} : F^{n+1}C \to F^{n+2}C$.

Definition 5.4 (Simplicial localization; [DK1], 4.1)**.** Given a relative category (C, W) with objects O, the simplicial localization, denoted $L(C, W) \in sO$-cat, is given by the levelwise localization $F_*C[F_*W^{-1}]$. At every level there is a map

$F_nC \to F_nC[F_nW^{-1}]$ and the face and degeneracy maps are given by precomposing by the associated map in F_*C and factoring via the definition of localizations.

In general, when we refer to the simplicial localization we will mean the associated simplicial category. In some proofs we will switch between these definitions as necessary when one is more convenient.

Now we consider the simplicial localization in the case when it is applied to model categories.

Theorem 5.5 ([DK1], 4.2)**.** *The homotopy category of the simplicial localization is equivalent to the localization obtained by inverting weak equivalences:*

$$\pi_0 L(C,W) \simeq C[W^{-1}].$$

Theorem 5.6 ([DK3], 4.8)**.** *Given a simplicial model category S_* ([Q67], II.2), let $S \coloneqq S_0$. The simplicial categories $L(S,W)$ and S_*^{cf} are weakly equivalent, where the latter is the restriction on bifibrant objects.*

The above theorem admits a generalization:

Theorem 5.7 ([DK3], 4.4)**.** *Given a model category M and $X, Y \in M$ one can take the (co)simplicial resolutions ([DK3], 4.3), X^* and Y_*. Then diag$M(X^*,Y_*)$ is weakly equivalent to $L^H M(X,Y)$. Here $M(X^*,Y_*)$ is a bifunctor $\Delta^{op} \to \mathsf{Set}$ given by taking the hom sets levelwise and the diagonal is given by precomposition with the product map $\Delta^{op} \to \Delta^{op} \times \Delta^{op}$.*

The above three theorems justify the claim that has been made throughout this paper, that the simplicial localization of a model category is some sort of generalization of the homotopy category which contains higher homotopy data from the model category.

We also have some other nice results:

Theorem 5.8 ([AM]; [VH])**.** *The homotopy coherent nerve of (the fibrant replacement of) the simplicial localization of a relative category is weakly equivalent in the Joyal model structure to the localization of the nerve in the sense of ([HA], 1.3.4.1):*

$$\mathbf{R}N^{hc}L(C,W) \sim N(C)[W^{-1}].$$

The expression on left side is often referred to as the underlying quasicategory of a relative/model category.

Theorem 5.9 ([DK4], 2.5)**.** *Every simplicial category is weakly equivalent to the simplicial localization of a category with weak equivalences.*

6. Equivalences under the hammock localization

We introduce the hammock localization functor which was invented by Dwyer and Kan to make the hom objects of the simplicial localization easier to calculate. Subsequently, we discuss two types of functors that lift to weak equivalences: adjoint relative functors with unit and counit in weak equivalences and Quillen equivalences. The former of these facts is stated, but not proven, in [DK2]; we provide a proof.

Definition 6.1 (Hammock localization; [DK2], 2.1)**.** Let (C, W) be a relative category. We define a simplicial category $L^H(C, W)$ (or $L^H C$ when W is clear) whose objects are the objects of C and whose hom objects are defined as follows:

The n-cells of $L^H C(X, Y)$ are 'reduced hammocks of height n and any length k', i.e. a commuting diagram of the form

$$\begin{tikzcd}
 & C_{1,0} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & C_{k-1,0} \arrow[d] \arrow[dr, dash] & \\
X \arrow[ur, dash] \arrow[r, dash] \arrow[ddr, dash] & C_{1,1} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & C_{k-1,1} \arrow[r, dash] \arrow[d] & Y \arrow[ddl, dash] \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & C_{1,n} \arrow[r, dash] & \cdots \arrow[r, dash] & C_{k-1,n} &
\end{tikzcd}$$

such that

(i) Arrows in adjacent columns point in opposing directions;
(ii) Leftward and downward pointing arrows are in W,
(iii) No column consists only of identity maps.

The length k indexes the number of horizontal arrows in a single row and is a non-negative integer. In the simplicial set $L^H C(X, X)$ there is a height zero hammock with two copies of X and the identity map between them (the direction does not matter). This is not reduced, but it can be reduced to the diagram with a single vertex X. This is the identity 0-cell id_X.

The ith degeneracy map is given by repeating the ith row of vertices and the ith face map is given by collapsing the ith row of vertices (or just removing it if $i = 0, n$). If the result is not reduced it can be made so by repeatedly composing adjacent columns pointing in the same direction and removing any columns that consist only of identity maps.

Finally the composition map $L^H C(Y, Z) \times L^H C(X, Y) \to L^H C(X, Z)$ is given by concatenating hammocks and expanding the vertex Y into n vertices with vertical identity maps between them, then reducing as necessary.

Remark 6.2. One must be careful because writing the diagram with a single vertex X is ambigous. There is, in each dimension, a hammock with two copies of X and $n+1$ copies of the identity (again the directions give the same thing) which reduces to this diagram. So one really needs to specify both the diagram and the dimension in this case.

Proposition 6.3 (Hammock localization is a functor)**.** *Given a relative functor $F : (C, W) \to (D, V)$, there is a simplicial functor $L^H F : L^H C \to L^H D$ such that on objects $L^H F(A) = F(A)$ and on morphisms the map $L^H F(X, Y) : L^H C(X, Y) \to L^H D(F(X), F(Y))$ is given by applying F to a hammock and reducing as necessary.*

Now we have defined the hammock localization functor, which we have promised will allow us to more easily calculate the simplicial localization, we ought to ensure that these functors are indeed equivalent. This is one of the main results of [DK2]:

Theorem 6.4 ([DK2], 2.2)**.** *The obvious functors*[1]

$$L^H(C,W) \leftarrow diag L^H F_* C \rightarrow F_* C[F_* W^{-1}] = L(C,W)$$

are weak equivalences. Here $L^H F_* C$ is a bifunctor $\Delta^{op} \rightarrow O\text{-}\mathsf{Cat}$ given by taking the hammock localization with respect to $F_* W$ levelwise.

This theorem will prove to be useful, as some facts are easier to show using the free resolutions, while others are easier using hammocks.

Lemma 6.5 ([DK2], 3.1)**.** *The homotopy category of the hammock localization is equivalent to the localization obtained by inverting weak equivalences:*

$$\pi_0 L^H(C,W) \simeq C[W^{-1}].$$

Proof. This fact follows by homotopy equivalence above, however, we need the explicit description of this functor for a later proof. The map on objects is evident and for morphisms we first have a rule for sending 0-cells $X \rightarrow Y$ to morphisms in $C[W^{-1}]$. Since all the left facing arrows are in W their inverses can be taken and composing gives a morphism $X \rightarrow Y$. To show this is a rule for sending morphisms of $\pi_0 L^H C(X,Y)$ into $C[W^{-1}]$ it suffices to show that the image of the faces is the same for any 1-cell, since 0-cells that can be connected by composite and zig-zag will thus go to the same place. Reversing the left facing maps of a 1-cell gives a diagram of the form

$$\begin{tikzcd}
& \bullet \arrow[r] \arrow[dd] & \cdots \arrow[r] & \bullet \arrow[dd] \arrow[dr] & \\
X \arrow[ur] \arrow[dr] & & & & Y. \\
& \bullet \arrow[r] & \cdots \arrow[r] & \bullet \arrow[ur] &
\end{tikzcd}$$

Each square and the initial and terminal triangles in this 'hammock' will commute thus the entire thing will commute, giving the desired result. Since composition is given by composing hammocks, this rule evidently satisfies composition. Further it sends the identity 0-cell to the identity. So our rule defines a (essentially surjective) functor. I claim it is also fully faithful. The full part is obvious since every morphism in $C[W^{-1}]$ is the composite of morphisms in C and W^{op}. For faithfullness we suppose that two 0-cells go to the same morphism. That is the 0-cells give composable words which are related by the equivalence relation on $C[W^{-1}]$. We need to show that the 0-cells live in the same connected component when they give composable words related by relations (1), (2),(3) and (4) of 3.4. We will implicitly use the fact that 0-cells map injectively onto composable words. Relation (1) is irrelevant because reduced hammocks of height zero contain no identity maps. Now note that for a 0-cell to give the composable word $f \circ g$ one of f or g is the formal inverse of some weak equivalence. So now we have the diagrams:

[1] See proof of lemma 6.7.

$$\begin{tikzcd}
& Z \arrow[dd, "g^{-1}"] & & & Y \arrow[dl, "f^{-1}"'] \arrow[dd, "f^{-1}"] \arrow[dr, "g"] & \\
X \arrow[ur, "h"] \arrow[dr, "f"'] & & Z, \arrow[ul, "\mathrm{id}_Z"'] \arrow[dl, "g^{-1}"] & X & & Z. \\
& Y & & & X \arrow[ul, "\mathrm{id}_X"] \arrow[ur, "h"'] &
\end{tikzcd}$$

Then for (3) we have the following diagrams:

$$\begin{tikzcd}
& X \arrow[dd, "w"] & & & X \arrow[dl, "w"'] \arrow[dd, "w"] \arrow[dr, "w"] & \\
X \arrow[ur, "\mathrm{id}_X"] \arrow[dr, "w"'] & & X, \arrow[ul, "\mathrm{id}_X"'] \arrow[dl, "w"] & Y \arrow[dr, "\mathrm{id}_Y"'] & & Y. \arrow[dl, "\mathrm{id}_Y"] \\
& Y & & & Y &
\end{tikzcd}$$

Finally (4) can be obtained by concatenating the pictures above with the appropriate morphisms. $\square$

Lemma 6.6 ([DK1], 4.4)**.** *If $u : X \to Y \in W$ then u induces weak equivalences*

$$LC(Y,Z) \xrightarrow{u^*} LC(X,Z), \qquad LC(Z,X) \xrightarrow{u_*} LC(Z,Y).$$

Proof. Explicitly, the above maps are given by pre and postcomposing by $F^{n+1}u$ in the nth cell. It suffices to show that u induces a map of simplicial sets since it will have the obvious inverse map u^{-1}. So we want that for a simplicial map α the following diagram commutes:

$$\begin{tikzcd}
F^kC[F^kW^{-1}](Y,Z) \arrow[r, "F^ku^*"] \arrow[d, "\alpha"'] & F^kC[F^kW^{-1}](X,Z) \arrow[d, "\alpha"] \\
F^mC[F^mW^{-1}](Y,Z) \arrow[r, "F^mu^*"'] & F^mC[F^mW^{-1}](X,Z).
\end{tikzcd}$$

The maps α on the homsets are given by a functor $F^kC[F^kW^{-1}] \to F^mC[F^mW^{-1}]$ in O-cat. Thus, the diagram will commute if given a morphism $f \in F^kC[F^kW^{-1}](Y,Z)$ we have $\alpha(f) \circ F^mu = \alpha(f \circ F^ku)$. By functoriality, $\alpha(f \circ F^ku) = \alpha(f) \circ \alpha(F^ku)$, so it suffices to show that $\alpha(F^ku) = F^mu$. Since α is a simplicial map this only needs to be checked on the face and degeneracy maps. Evidently $F^i\varphi F^{k-1-i} : F^ku \mapsto F^{k-1}u$ and $F^i\psi F^{k-1-i} : F^ku \mapsto F^{k+1}u$. So u^* and dually u_* are isomorphisms (and thus weak equivalences) of simplicial sets. $\square$

Lemma 6.7 ([DK2], 3.3)**.** *If $u : X \to Y \in W$ then u induces weak equivalences*

$$L^HC(Y,Z) \xrightarrow{u^*} L^HC(X,Z), \qquad L^HC(Z,X) \xrightarrow{u_*} L^HC(Z,Y).$$

Proof. The goal is to produce a map making the following diagram commute:

$$\begin{tikzcd}
L^H C(Y,Z) \arrow[d, "u^*"'] & \mathrm{diag} L^H F_* C(Y,Z) \arrow[l] \arrow[r] \arrow[d, dashed] & LC(Y,Z) \arrow[d, "u^*"] \\
L^H C(X,Z) & \mathrm{diag} L^H F_* C(X,Z) \arrow[l] \arrow[r] & LC(X,Z).
\end{tikzcd}$$

The left map is given by precomposing hammocks with the hammock that has $n+1$ copies of u^*. It is obvious that this is a map of simplicial sets since the face and degeneracy maps merely repeat or compose/remove rows in a hammock. Now, it helps to think about what $\mathrm{diag} L^H F_* C$ actually looks like. As a simplicial category the n-cells are given by hammocks of height n with maps in $F^{n+1}C$ and weak equivalences in $F^{n+1}W$. The natural thing to do is to precompose a hammock in the nth level with the map $F^{n+1}u$. Now the face and degeneracy maps on $\mathrm{diag} L^H F_* C(Y,Z)$ are given by first applying the associated map on $L^H F_* C(Y,Z)$ to the hammock and then applying then applying the associated map on $LC(Y,Z)$ on all the maps in the hammock. In the previous proof we showed that for a map α on the simplicial localization, we have $\alpha(F^k u) = F^m u$, so it is not hard to see that the mapping we have given is a map of simplicial sets. This map is the desired map completing the diagram and by the two-out-of-three property we are done.

In theorem 6.4 we asserted that the functors of simplicial categories were 'obvious'. This is the case in that there are canonical maps that one can write down, though it is not necessarily clear what those maps are. We provide some explicit details. The maps are evidently constant on objects so we need only consider what they do to the simplicial sets. The map $\mathrm{diag} L^H F_* C \to L^H C$ is given by taking the map φ^{n+1} on every map in each hammock of height n. In turn the map $\mathrm{diag} L^H F_* C \to LC$ is given at each level by inverting the left facing arrows in $F^{n+1}W$ and composing all the arrows (all the rows of the hammock will produce that same resulting map). In particular these functors will take the hammock with $n+1$ copies of $F^{n+1}u$ to the hammock with $n+1$ copies of u and the map $F^{n+1}u$ respectively. In turn since these are indeed functors, this gives exactly the statement that the desired diagram commutes. ☐

Lemma 6.8 ([DK2], 3.5)**.** *Let $F, F' : (C,W) \to (D,V)$ be relative functors and $\eta : F \to F'$ a natural transformation such that $\eta_Z \in V$ for all $Z \in D$. Then for all $X, Y \in C$ the following diagram commutes up to homotopy*

$$\begin{tikzcd}
& L^H D(F(X),F(Y)) \arrow[dr, "\eta_{Y*}"] & \\
L^H C(X,Y) \arrow[ur] \arrow[dr] & & L^H D(F(X),F'(Y)). \\
& L^H D(F'(X),F'(Y)) \arrow[ur, "\eta_X^*"'] &
\end{tikzcd}$$

Proof. To show the above is to show a map $h : \Delta^1 \times L^H C(X,Y) \to L^H D(F(X),F'(Y))$ such that $h(0,-)$ gives the top path and $h(1,-)$ gives the lower path. It helps to write down an explicit description of the simplicial set Δ^1. There is a single nondegenerate 1-cell which is an arrow $0 \to 1$. Now any n-cell has $n+1$ ordered vertices and it will contain the data of a 1-cell from each vertex to the next one. The n-cells of Δ^1 can be thought of as paths between $n+1$ vertices that begin at 0 and end at 1 (and the ones that are the identity on 0 and on 1). In particular they can be characterized by the position i of the first non-zero vertex (we write $i = n+1$ when there is none). This completely characterizes the n cells. So we consider n-cell in $\Delta^1 \times L^H C(X,Y)$ and suppose that $i \neq 0, n+1$ is the first nonzero vertex on the component in Δ^1. Let the hammock of the component in $L^H C(X,Y)$ be the following:

$$\begin{tikzcd}
& C_{1,0} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & C_{k-1,0} \arrow[d] \arrow[dr, dash] & \\
F(X) \arrow[ur, dash] \arrow[dr, dash] & \vdots \arrow[d] & & \vdots \arrow[d] & Y. \\
& C_{1,n} \arrow[r, dash] & \cdots \arrow[r, dash] & C_{k-1,n} \arrow[ur, dash] &
\end{tikzcd}$$

Then we define a map h whose image is given by:

$$\begin{tikzcd}
& F(X) \arrow[r, dash] \arrow[d] & F(C_{1,0}) \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & F(C_{k-1,0}) \arrow[r] \arrow[d] & F(Y) \arrow[d] \arrow[dddr, "\eta_Y"] & \\
& \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \\
& F(X) \arrow[r, dash] \arrow[d, "\eta_X"] & F(C_{1,i}) \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & F(C_{k-1,i}) \arrow[r] \arrow[d] & F(Y) \arrow[d, "\eta_Y"'] \arrow[dr, "\eta_Y"] & \\
F(X) \arrow[uuur] \arrow[ur] \arrow[r, "\eta_X"'] \arrow[ddr, "\eta_X"] & F'(X) \arrow[r, dash] \arrow[d] & F'(C_{1,i+1}) \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & F'(C_{k-1,i+1}) \arrow[r] \arrow[d] & F'(Y) \arrow[r] \arrow[d] & F'(Y). \\
& \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \\
& F'(X) \arrow[r, dash] & F'(C_{1,n}) \arrow[r, dash] & \cdots \arrow[r, dash] & F'(C_{k-1,n}) \arrow[r] & F'(Y) \arrow[uur] &
\end{tikzcd}$$

When the component in Δ^1 is the constant path on 0 or on 1 we simply send the hammock to the upper and lower paths in the commutative diagram respectively.

Now, given a degeneracy map, applying it to an n-cell in $\Delta^1 \times L^H C(X,Y)$ will double the jth vertex in Δ^1, inserting an identity map, and double the jth row of the hammock, again inserting identities. Applying h evidently leads to the jth row being doubled, and since F and F' are functors, there are identity maps between. Of course this is what one would get if they applied h first and then the degeneracy map.

The same thing is true for the face maps. For $j \neq 0, n$, a face map applied to an n-cell in $\Delta^1 \times L^H C(X,Y)$ will compose at the jth vertex in Δ^1 and at the jth row of the hammock. When applying h the result will be that the jth row has been collapsed. Here functoriality of F, F' will make sure that the composite morphisms creating by collapsing the jth row will be the same whether h or the face map was applied first. There is a bit more to check when one of the maps being composed is the map $0 \to 1$ but it follows immediately from the definition of a natural transformation. The cases when $j = 0, n$ are easy to check as well. Altogether h is the desired homotopy. $\square$

Theorem 6.9 ([DK2], 3.6)**.** *An adjoint pair of relative functors $F \dashv G$ between $(C, W) \rightleftharpoons (D, V)$ such that $\eta_X : X \to GF(X) \in W$ for $X \in C$ and $\epsilon_Y : FG(Y) \to Y \in V$ for $Y \in D$ induces a weak equivalence $L^H(C, W) \simeq L^H(D, V)$.*

Proof. There are two things to show here.

(1) We begin with showing that F and G induce equivalences on $\pi_0 L^H C$ and $\pi_0 L^H D$. First, the functors F, G induce obvious functors $\tilde{F} : C[W^{-1}] \leftrightarrows D[V^{-1}] : \tilde{G}$ which agree with F, G on C, D and send formal inverses to formal inverses. η and ϵ will give natural transformation $1 \to \tilde{G}\tilde{F}$ and $\tilde{F}\tilde{G} \to 1$ and since they live in $C[W^{-1}]$ and $D[V^{-1}]$ they are in particular natural isomorphisms. So $\tilde{F}, \tilde{G}$ are equivalences.

Now consider the functors $\pi_0 L^H F : \pi_0 L^H C \leftrightarrows \pi_0 L^H D : \pi_0 L^H G$. I claim the following diagram commutes, giving us what we want:

$$\begin{tikzcd} \pi_0 L^H C \arrow[r, "\pi_0 L^H F"] \arrow[d] & \pi_0 L^H D \arrow[d] \\ C[W^{-1}] \arrow[r, "\tilde{F}"'] & D[V^{-1}]. \end{tikzcd}$$

In the lower path, the left facing arrows are inverted and the morphism are composed, then $\tilde{F}$ is applied to the composite. Meanwhile in the upper path, F is applied to each morphism then the left facing arrows are inverted and composed. Since $\tilde{F}(w^{\text{op}}) = F(w)^{\text{op}}$ the result is the same.

(2) Now we show that $F : L^H C(X, Y) \to L^H D(F(X), F(Y))$ and dually for G are weak equivalences. First we have the diagram

$$\begin{tikzcd} & L^H C(GF(X), GF(Y)) \arrow[dr, "\eta_X^*"] & \\ L^H C(X, Y) \arrow[ur] \arrow[dr] & & L^H C(X, GF(Y)). \\ & L^H C(X, Y) \arrow[ur, "\eta_{Y*}"'] & \end{tikzcd}$$

Since the bottom left map is the identity map this is really the diagram

$$\begin{tikzcd} L^H C(GF(X), GF(Y)) \arrow[dr, "\eta_X^*"] & \\ & L^H C(X, GF(Y)). \\ L^H C(X, Y) \arrow[uu, "FG"] \arrow[ur, "\eta_{Y*}"'] & \end{tikzcd}$$

By the two-out-of-three property FG is homotopic to a weak equivalence and thus itself a weak equivalence. So for any Kan complex K we have

$$[L^H C(GF(X), GF(Y)), K] \xrightarrow{G^*} [L^H D(F(X), F(Y)), K] \xrightarrow{F^*} [L^H C(X, Y), K]$$

is a bijection for all X, Y. So the first map is an injection and the second a surjection. By duality we conclude that F^* is a bijection for all K and $G(A), G(B)$. So the map $F|_G$ induces a weak equivalence on simplicial sets. Finally consider the diagram

$$\begin{tikzcd}
 & L^H D(F(X), F(Y)) \arrow[r, "FG"] & L^H D(FGF(X), FGF(Y)) \arrow[d, equal] \\
L^H C(X, Y) \arrow[ur, "F"] \arrow[dr, "GF"'] & & \\
 & L^H C(GF(X), GF(Y)) \arrow[r, "F|_G"'] & L^H D(FGF(X), FGF(Y)).
\end{tikzcd}$$

We know every map in this diagram other than F is a weak equivalence, so by the two-out-of-three property we conclude that F itself is also a weak equivalence. By duality G is as well.

□

Theorem 6.10 ([DK3], 5.6)**.** *If M, N are model categories that are Quillen equivalent, then $L^H M$ and $L^H N$ are weakly equivalent.*

Note that the Quillen adjunctions are generally not relative functors, so we should not expect them to induce functors on simplicial categories under hammock localization. Mazel-Gee showed that despite this obstruction a Quillen adjunction gives an adjoint pair of functors on underlying quasicategories [MG].

7. Examples

We give some examples where the simplicial localization can be used to produce interesting simplicial categories.

- Note that the function complex makes the category of simplicial sets enriched in itself. This becomes a simplicial model category under the Kan-Quillen and Joyal model structures. So we have:
  (1) By theorems 4.4 and 5.6 the localization of simplicial sets under the Kan-Quillen model structure is weakly equivalent to the simplicial category of Kan complexes. By [KDN], 3.1.3.4 this is enriched in Kan complexes, for this reason we call this the ∞*-category of* ∞*-groupoids* and is often denoted by $\mathcal{S}$ or $\infty\mathsf{Grpds}$.
  (2) By theorems 4.13 and 5.6 the localization of simplicial sets under the Joyal model structure is weakly equivalent to the simplicial category of quasicategories. By [CR1], 22.4 this is enriched in quasicategories and thus forms an '$(\infty, 2)$-category'.

- The *Dold-Kan correspondance* states that there is an equivalence of categories between simplicial abelian groups and nonnegative chain complexes in Ab.
$$N : \mathsf{Ab}^{\Delta^{op}} \leftrightarrows \mathrm{Ch}(\mathsf{Ab})_+ : \Gamma.$$
These are in particular model categories so they are categories with weak equivalences. The weak equivalences are, weak equivalences on the underlying simplicial sets in the case of simplicial abelian groups, and quasi-isomorphisms in the case of non-negative chain complexes. Additionally, the functors N and Γ preserve weak equivalences. This is enough to say that $N \dashv \Gamma$ satisfy the conditions of theorem 6.9, thus the associated simplicial categories of simplicial abelian groups and of chain complexes are weakly equivalent, induced by the functors N, Γ under hammock localization. Exposition on the Dold-Kan correspondance can be found in [GJ], II.3.
- A classical result by Quillen [Q67] (Ex. 2, pp. 64) is that the Quillen model structure on topological spaces and the Kan-Quillen model structure on simplicial sets have a Quillen equivalence between them. In particular this equivalence is given by the *singular simplicial set functor* (which gives the fundamental ∞-groupoid in Kan complexes) and its left adjoint *geometric realization*, the contructions of which can be found in [KDN], 1.2.2-3. This is usually written:
$$|-| : \mathsf{sSet}_{\text{Kan-Quillen}} \leftrightarrows \mathsf{Top}_{\text{Quillen}} : \mathrm{Sing}(-).$$
This quite a remarkable statement because it tells us that simplicial sets and topological spaces have equivalent homotopy categories. However, with the machinery of simplicial localization we can expand this result.

  Theorem 6.10 gives a weak equivalence $L^H(\mathsf{Top}_{\text{Quillen}}) \sim L^H(\mathsf{sSet}_{\text{Kan-Quillen}})$. Further, we have a weak equivalence $RL^H(\mathsf{Top}_{\text{Quillen}}) \sim L^H(\mathsf{sSet}_{\text{Kan-Quillen}})$ where R is the fibrant replacement. This statement is the *homotopy hypothesis* when our ∞-groupoids are Kan complexes; the singular simplicial set functor induces an equivalence of the ∞-category of topological spaces and the ∞-category of ∞-groupoids.

Interestingly, in more recent work Barwick and Kan showed that relative categories admit a model structure which is equivalent to the model structures on the models of ∞-categories.

Theorem 7.1 (Model structure on relative categories; [BK1], 6.1)**.** *There is a model structure on relative categories that is Quillen equivalent to the model structure on complete Segal spaces as defined in [CR2], 7.2.*

Complete Segal spaces are a model for ∞-categories with the model structure Quillen equivalent to the models of ∞-categories which we have discussed. This gives the desired statement.

The simplicial localization factors into this as follows.

Theorem 7.2 ([BK2], 1.8)**.** *A functor of relative categories maps to a weak equivalence under simplicial localization if and only if it is a weak equivalence in the model structure on relative categories.*

However, the simplicial localization itself is not known to induce an Quillen equivalence on models of ∞-categories.

Acknowledgments

I would like to thank Peter May and Ajay Srinivasan for their mentorship on this project.

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