Chicago REU - cobordism II

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Summary $\Omega_{n} = \{n\text{-mflds}\}^{\text{closed}} / \sim \text{bordism}$

$$\Omega_* \cong \varinjlim \pi^s_{*+k}(Th(V^k)) = \pi_* MO$$

$$MO = \{ Th(V^k),\ \Sigma Th(V^k) \approx Th(V^k \oplus \mathbb{R}) \to Th(V^{k+1}) \}$$

ASS $\quad Ext^s_{A_p}(\widetilde{H\mathbb{F}_p}^* X, \mathbb{F}_p)_t \Rightarrow \pi_{t-s}(X)^\wedge_p$

$\left(\text{e.g. } A_2 = \mathbb{F}_2\langle Sq^i \rangle / \sim \quad Sq^a Sq^b = \sum_i \binom{b-i-1}{a-2i} Sq^{a+b-i} Sq^i \right)$

$Sq^0 = 1$

$\binom{n}{k} := \begin{cases} 1, & n=-1,\ k \geq 0 \\ 0, & n \geq 0,\ k > n \end{cases}$

$n \geq -1$, $k \geq 0$

b \ a										
-1	1	1	1	1	1	1	1	1		
0	1	0	0	0	0	0	0	0		
1	1	1	0	0	0	0	0	0		
2	1	0	1	0	0	0	0	0		
3	1	1	1	1	0	0	0	0		
4	1	0	0	0	1	0	0	0		
5	1	1	0	0	1	1	0	0		
6	1	0	1	0	1	0	1	0		
7	1	1	1	1	1	1	1	1		
8	1	0	0	0	0	0	0	0	1	
	1	1	0	0	0	0	0	0	1	1
				0	0	0	0	0		
					0	0	0	0		
						0	0	0		
							0	0		
								0		

$b \downarrow$ $\quad \longrightarrow a$ $\quad \binom{b}{a}$

A_p^{*} is a Hopf algebra $\quad A_p \otimes A_p \longrightarrow A_p$

$= N_*^{st}(H\mathbb{F}_p^*, H\mathbb{F}_p^{**})$

$= H\mathbb{F}_p^{n} H\mathbb{F}_p$

Quillen's description of Ω^u

$$\mathrm{Spec}(P) \cong \mathrm{Aut}(\hat{\mathbb{G}}_a / \mathbb{F}_p)$$

$P = \mathbb{F}_p[\xi_1, \ldots] \subseteq (H\mathbb{F}_p)_* H\mathbb{F}_p \qquad \xi_i$

$\uparrow \qquad \uparrow$

$\mathbb{F}_p\{\beta_0, \beta_1, \ldots = (\widetilde{H\mathbb{F}_p})_{*+2} \mathbb{C}P^\infty \qquad \beta_{p^i}$

$H\mathbb{Z}_* MU \qquad b_i$

$\uparrow \qquad \uparrow$

$H\mathbb{Z}_{*+2} \mathbb{C}P^\infty \qquad \beta_{i+1}$

$= (H\mathbb{Z})_{*+2} BU(1)$

$H\mathbb{F}_p^* \mathbb{C}P^\infty = \mathbb{F}_p[x] \qquad |x| = 2$

$x: \mathbb{C}P^\infty \longrightarrow \Sigma^2 H\mathbb{F}_p$

$x = c_1(V^1_{\mathbb{C}})$

$$\begin{array}{c} \mathrm{Spec}(E) \\ \downarrow \\ \mathrm{Spec}(A_p^\wedge) \circlearrowright H\mathbb{F}_p^*(\mathbb{C}P^\infty) \\ \downarrow \\ \mathrm{Spec}(P) \end{array}$$

Claim $\quad H\mathbb{F}_p^*(\mathbb{C}P^\infty)$ "is $\hat{\mathbb{G}}_a / \mathbb{F}_p$"

$H^*(\mathbb{C}P^\infty) = \mathbb{Z}[x]$

$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{V^1 \otimes V^1} \mathbb{C}P^\infty$

$\mathbb{Z}[x,y] \longleftarrow \mathbb{Z}[x]$

$x + y \longleftarrow x$

$c_1(v \otimes w) = c_1(v) + c_1(w)$

E^* cx orientable $\Leftrightarrow \tilde{E}^{*+2n}(Th(V)) \cong E^*(B)$

$Th(V^1_{\mathbb{C}}) \approx \mathbb{C}P^\infty$

$\tilde{E}^2(\mathbb{C}P^\infty) \cong E^0(\mathbb{C}P^\infty)$

$\ni \qquad \psi$

$$\begin{array}{ccc} \tilde{E}^2(\mathbb{C}P^\infty) & \cong & E^0(\mathbb{C}P^\infty) \\ \cup & & \cup \\ x_E & \longleftarrow & 1 \end{array}$$

$c_1(v \otimes w) = c_1(v) + c_1(w)$

$E^*(\mathbb{C}P^\infty) \cong E^*(pt)[[x]]$

Prop E cx orient. iff $\exists\, x \in \tilde{E}^2(\mathbb{C}P^\infty) \longrightarrow \tilde{E}^2(S^2)$

$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty$

$E^*(pt)[[x,y]] \longleftarrow E^*(pt)[[x]]$

$x +_E y \longleftarrow x$

$\vdots$

FGL.

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e.g. additive, multiplicative, elliptic.

$Th(V_1') \xrightarrow{x} \underline{MU}_2$

$= \mathbb{C}P^\infty$

$x \in \tilde{MU}^2(\mathbb{C}P^\infty)$ cx orientation.

Q: what is $x +_{MU} y$?

What are the options?

Lazard classified FGL's

$x +_F y\ /R$ $\qquad \exists\ \log_F \in R \otimes \mathbb{Q}[[x]]$, $\log_F = x + m_1 x^2 + m_2 x^3 + \cdots$

R has no torsion $\qquad \log_F(x +_F y) = \log_F(x) + \log_F(y)$

$\log_F : F/_{R\otimes\mathbb{Q}} \longrightarrow \hat{\mathbb{G}}_a/_{R\otimes\mathbb{Q}}$

Def: Iso of FGL's over R

Note: $F/_R$ might NOT be isomorphic to $\hat{\mathbb{G}}_a/R$

$\exp_F(x) = \log_F^{-1}(x)$

$\exp_F(x) = \log_F^{-1}(x)$
$\qquad\qquad = x + b_1 x^2 + b_2 x^3 + \cdots$

$x +_F y = \exp_F(\log_F(x) + \log_F(y))$

pullback of FGL

$x +_{\varphi^* F} y \longleftarrow\!\shortmid x +_F y$

$\mathrm{Spec}(T) \longrightarrow \mathrm{Spec}(R)$

$R \xrightarrow{\varphi} T$

could F/R ... to $\mathbb{G}_a^1/R$

So $\quad FGL(R \otimes \mathbb{Q}) \cong \{\exp_F(x) = x + b_1 x^2 + \cdots\}$

$\qquad = \mathrm{Spec}(\mathbb{Z}[b_1, b_2, \ldots])(R \otimes \mathbb{Q})$

$x +_F y/_{R\otimes\mathbb{Q}} \longleftarrow\!\shortmid \exp(\log(x) + \log(y))/_{\mathbb{Z}[b_1, \ldots]} = x +_{F_{add}} y$

$\mathrm{Spec}(R \otimes \mathbb{Q}) \longrightarrow \mathrm{Spec}(\mathbb{Z}[b_1, \ldots])$

Lazard: $\exists\, L \subseteq \mathbb{Z}[b_1, b_2, \ldots]$

s.t. $\quad FGL(R) \cong \mathrm{Spec}(L)(R)$

$x +_F y/_R \longleftarrow\!\shortmid x +_{F_{univ}} y/_L$

$\mathrm{Spec}(R) \longrightarrow \mathrm{Spec}(L)$

$x +_{F_{add}} y \longleftarrow\!\shortmid x +_{F_{univ}} y$

$\mathrm{Spec}(\mathbb{Z}[b_1, \ldots]) \longrightarrow \mathrm{Spec}(L)$

L torsion free $\qquad L \hookrightarrow \mathbb{Z}[b_1, b_2, \ldots]$

$\qquad\qquad \mathbb{Q}$-equiv.

(since any FGL over a $\mathbb{Q}$-alg is of the form $\exp_F(\log_F(x) + \log_F(y))$)

Thm: (Lazard) $\quad L \cong \mathbb{Z}[x_1, x_2, \ldots]$

$I = (b_1, b_2, \ldots)$

$x_i \equiv \begin{cases} p b_i & \text{mod } I^2, \quad i = p^k - 1 \\ b_i & \text{mod } I^2, \quad i \neq p^k - 1 \end{cases}$

Quillen

$\Omega^U_* = MU_* \longrightarrow H\mathbb{Z}_* MU \overset{\text{thom iso}}{=} \mathbb{Z}[b_1, b_2, \ldots]$

$$\begin{tikzcd} \Omega^U_* = MU_* \arrow[rr] \arrow[dr, hook] & & HZ_* MU = \mathbb{Z}[b_1, b_2, \dots] \arrow[dl, hook'] \\ & HQ_* MU & \end{tikzcd}$$

(thm iso)

$$HQ_* MU \cong (MU_*) \otimes \mathbb{Q}$$

<u>Thm</u> (Quillen)

(1) $MU_* \subseteq \mathbb{Z}[b_1, b_2, \dots]$, $MU_* = L$

(2) $x +_{MU} y = x +_{F_{univ}} y$

(3) If E is any cx oriented thy

$\exists \; MU \to E$

$$x +_E y \longleftarrow x +_{F_{univ}} y$$

$$\mathrm{Spec}(E_*) \to \mathrm{Spec}(MU_*) = \mathrm{Spec}(L)$$

<u>Vistas</u>: • coh theories associated to FGL's

$H, K, MU, Ell, K(n), E(n),$

• Ht classification of FGL's.

• ANSS $\quad H^*(\mathrm{Spec}(\mathbb{Z}[b_1, b_2, \dots]), MU_*) \Longrightarrow \pi^s_{t-s}$

$MU_* MU$

$$F \rightsquigarrow f(f^{-1}(x) +_F f^{-1}(y)) := x +_{f \cdot F} y$$

<u>ht</u> $\quad F \in FGL(R)$ $\quad ht(F) \le n \iff$ over R/p $\quad C_p$

p-local

$$[p](F) = \underbrace{x +_F \cdots +_F x}_{p} = px + w_1 x^2 + w_2 x^3 + \cdots$$

Let $z = f^{-1}(x)$, $w = f^{-1}y$ $\quad f(z +_F w) = f(x) +_{f \cdot F} f(y)$

$$\mathrm{Spec}(MU_*)_{\mathbb{Z}_{(p)}} = FGL_{/\mathbb{Z}_{(p)}} = \bigcup FGL^{ht \leq n}$$

$\mathrm{Aut}(F_n)$ $\shortparallel$

$$FGL^{ht_p \leq n} = \mathrm{Spec}\left(v_{n,p}^{-1}(MU)_{\mathbb{Z}_{(p)}}\right) \qquad FGL^{ht \leq n} \setminus FGL^{ht \leq n-1} \cong \{F_n\} \circlearrowleft \{f\}$$

$f : F_n \longrightarrow F_n$

$$H^*\left(\mathrm{Spec}(\mathbb{Z}(b_1, \ldots)) ; FGL^{ht \leq n} \setminus FGL^{ht \leq n-1}\right)$$

$$\cong$$

$$H^*(\mathrm{Aut}(F_n))$$

$FGL^{ht \leq n} \setminus FGL^{ht}$

- ANSS / v_n-periodicity.