

# Chicago REU - cobordism II

Wednesday, July 9, 2025 7:22 AM

Summary  $\Omega_n = \frac{\{\text{closed } n\text{-manifolds}\}}{\sim \text{cobordism}}$

$$\Omega_n \cong \varinjlim \pi_{n+k}^S(T\mathbb{H}(V^k)) = \pi_n MO$$

$$MO = \{ T\mathbb{H}(V^k), \sum T\mathbb{H}(V^k) \approx T\mathbb{H}(V^k \oplus \mathbb{R}) \rightarrow T\mathbb{H}(V^{k+1}) \}$$

Ass  $Ext_{A_p}^S(\tilde{H}\tilde{F}_p^* X, F_p)_c \Rightarrow \pi_{k-s}(X)_p^{\wedge}$

e.g.  $A_2 = \mathbb{F}_2 \langle S_2^1 \rangle / \sim$   
 $S_2^a S_2^b = \sum_i \binom{b-i-1}{a-2i} S_2^{a+b-i} S_2^i$

$S_2^0 = 1$

$\binom{n}{k} := \begin{cases} 1, & n=k=0 \\ 0, & n \geq 0, k > n \end{cases}$

$A_p^u$  is a Hopf algebra  $A_p \otimes A_p \rightarrow A_p$

$N_{A_p^u}(\tilde{H}\tilde{F}_p^*, \tilde{H}\tilde{F}_p^{*+u})$   
 $= R\tilde{F}_p^n \tilde{H}\tilde{F}_p$

Quilbic description of  $\Omega^u$

$$Spec(P) \cong A_{\mathcal{H}}(G_n^{\wedge}/F_p)$$

$P = F_p[s_1, \dots] \subseteq (H\tilde{F}_p)_* H\tilde{F}_p$

$F_p[s_1, s_2, \dots] = (H\tilde{F}_p)_{*+2} \mathbb{C}P^{\infty}$

$H\tilde{F}_p \mathbb{C}P^{\infty} = F_p[x] \quad |x|=2$

$x: \mathbb{C}P^{\infty} \rightarrow \Sigma^2 H\tilde{F}_p$

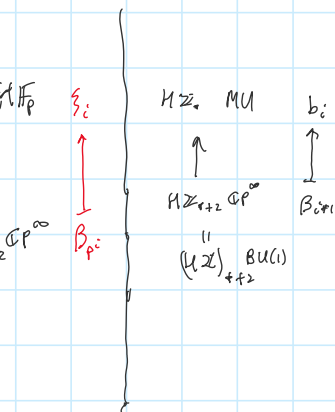
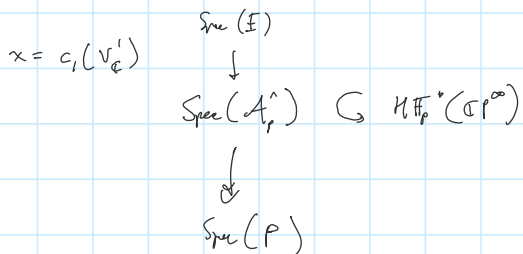


Table with 9 rows and 9 columns. Rows are indexed 1-9 on the left. Columns are indexed 1-9 on the top. The table contains 0s and 1s. A green diagonal line runs from the top-left to the bottom-right. A yellow box highlights the first 8 rows and columns. A green arrow points from the bottom-right towards the right.



Claim  $H\tilde{F}_p^*(\mathbb{C}P^{\infty}) \cong G_n^{\wedge}/F_p$

$H^*(\mathbb{C}P^{\infty}) \cong \mathbb{Z}[x]$

$\mathbb{C}P^{\infty} \times \mathbb{C}P^{\infty} \xrightarrow{V' \otimes V'} \mathbb{C}P^{\infty}$

$\mathbb{Z}[x, y] \leftarrow \mathbb{Z}[x]$

$x+y \leftarrow x$

$c_1(v \otimes w) = c_1(v) + c_1(w)$

$E^* \text{ cox orientable } \Leftrightarrow \tilde{E}^{*+2n}(T\mathbb{H}(V)) \cong E^*(B)$

$T\mathbb{H}(V_c) \approx \mathbb{C}P^{\infty}$

$\tilde{E}^2(\mathbb{C}P^{\infty}) \cong E^0(\mathbb{C}P^{\infty})$

$$\tilde{E}^2(\mathbb{CP}^\infty) \cong E^0(\mathbb{CP}^\infty)$$

$$\begin{array}{ccc} \psi & & \psi \\ \chi_E & \longleftarrow & 1 \end{array}$$

$$c_1(v \oplus w) = c_1(v) + c_1(w)$$

$$E^*(\mathbb{CP}^\infty) \cong E^*(pt)[[x]] \quad \text{Prop } E \propto \text{rank} \iff \frac{1}{x} \in \tilde{E}^2(\mathbb{CP}^\infty) \longrightarrow \tilde{E}^2(S^2)$$

$$\mathbb{CP}^\infty \times \mathbb{CP}^\infty \longrightarrow \mathbb{CP}^\infty$$

$$E^*(pt)[[x, y]] \longleftarrow E^*(pt)[[x]]$$

$$\begin{array}{ccc} x + y & & x \\ \downarrow & \longleftarrow & \downarrow \\ x_E & & x \end{array}$$

FGL

$\downarrow$

eq.

additive  
multiplicative  
elliptic

$$Th(V_0') \xrightarrow{x} MU_2$$

"  
 $\mathbb{CP}^\infty$

$$x \in \tilde{MU}^2(\mathbb{CP}^\infty) \quad c \propto \text{orientation}$$

Q: what is  $x +_{MU} y$ ?

What are the options?

Lazard classified FGL's

$$x + m_1 x^2 + m_2 x^3 + \dots$$

$$x +_F y \in R$$

$$\exists \log_F \in R \otimes \mathbb{Q}[[x]]$$

R has no divisors

$$\log_F(x +_F y) = \log_F(x) + \log_F(y)$$

$$\log_F: F/R \otimes \mathbb{Q} \longrightarrow G_a^1/R \otimes \mathbb{Q}$$

$$\exp_F(x) = \log_F^{-1}(x)$$

Def: Fro of FGL's over R

Note:  $F/R$  might NOT be isomorphic to  $G_a^1/R$

$$\exp_F(x) = \log_F^{-1}(x)$$

$$x + b_1 x^2 + b_2 x^3 + \dots$$

map  $T/R$  map to  $G_a^1/R$

$$x +_F y = \exp_F(\log_F(x) + \log_F(y))$$

parallel of FGL

$$x +_F y \longleftarrow x +_F y$$

$$\mathrm{Spec}(T) \longrightarrow \mathrm{Spec}(R)$$

$$R \xrightarrow{q} T$$

$$\text{So } \mathrm{FGL}(R \otimes \mathbb{Q}) = \{ \exp_F(x) = x + b_1 x^2 + \dots \}$$

$$= \mathrm{Spec}(\mathbb{Z}[b_1, b_2, \dots])(R \otimes \mathbb{Q})$$

$$\begin{array}{ccc} x +_F y / R \otimes \mathbb{Q} & \xleftarrow{\exp(\log(x) + \log(y)) / \mathbb{Z}[b_1, \dots]} & x +_{F_{\mathrm{add}}} y \\ \mathrm{Spec}(R \otimes \mathbb{Q}) & \longrightarrow & \mathrm{Spec}(\mathbb{Z}[b_1, \dots]) \end{array}$$

$$\text{Lazard: } \exists L \subseteq \mathbb{Z}[b_1, b_2, \dots]$$

$$\text{s.t. } \mathrm{FGL}(R) \cong \mathrm{Spec}(L)(R)$$

$$x +_F y / R \longleftarrow x +_{F_{\mathrm{add}}} y / L$$

$$\mathrm{Spec}(R) \longrightarrow \mathrm{Spec}(L)$$

$$x +_{F_{\mathrm{add}}} y \longleftarrow x +_{F_{\mathrm{add}}} y$$

$$\mathrm{Spec}(\mathbb{Z}[b_1, \dots]) \longrightarrow \mathrm{Spec}(L)$$

$L$  form for

$$L \hookrightarrow \mathbb{Z}[b_1, b_2, \dots]$$

$\mathbb{Q}$ -equiv.

(since any FGL over a  $\mathbb{Q}$ -alg is of the form  $\exp_F(\log_F(x) + \log_F(y))$ )

$$\text{Th: (Lazard)} \quad L \cong \mathbb{Z}[x_1, x_2, \dots]$$

$$I = (b_1, b_2, \dots)$$

$$x_i \equiv \begin{cases} p b_i \text{ mod } I^2, & i = 1 - 1 \\ b_i \text{ mod } I^2, & i \neq 1 - 1 \end{cases}$$

Orth

$$\Sigma_{\star}^n = MU_{\star} \xrightarrow{\text{this is}} H\mathbb{Z}_{\star} MU = \mathbb{Z}[b_1, b_2, \dots]$$

can be

$$\begin{array}{ccc} \Omega^q = MU_* & \xrightarrow{\quad} & H\mathbb{Z}_*, MU_* = \mathbb{Z}[b_1, b_2, \dots] \\ & \searrow \quad \swarrow & \\ & HQ_* MU & \\ & (12) & \\ & (MU_*) \otimes \mathbb{Q} & \end{array}$$

Thm (a.1.1)  $MU_* \subseteq \mathbb{Z}(b_1, b_2, \dots)$   
 (1)  $\downarrow$   
 $L$

(2)  $x +_{MU} y = x +_{Fun} y$

(3) If  $I$  is an ideal in  $\mathbb{Z}$

$$\exists MU \rightarrow E$$

$$x +_E y \longleftarrow x +_{Fun} y$$

$$Spec(E_*) \rightarrow Spec(MU_*)$$

$$Spec(L)$$

Visits: • coh theory associated to FGL's  
 $H, K, MU, ER, \mathbb{Z}(1), E(1),$

• H+ classification of FGL's.

• ANSS  $H^*(Spec(\mathbb{Z}(b_1, b_2, \dots)), MU_*) \Rightarrow \pi_{t-s}^s$

$MU_* MU$

$$F \rightsquigarrow f(f^{-1}(x) +_F f^{-1}(y)) := x +_{f \cdot F} y$$

ht

$$F \in FGL(R)$$

$\uparrow$   
 $p \cdot 1 = 0$

$$ht(F) \leq n \Leftrightarrow$$

or  $R/p$

$G_p$

$$h_F(F) = \underbrace{x_F \cdots x_F}_p x = p x + w_2 x^2 + w_3 x^3 + \dots$$

$$\text{Let } \begin{matrix} z = f(x) \\ w = f(y) \end{matrix} \quad f(z +_F w) = f(x) +_{f(F)} f(y)$$

$$S_{\text{per}}(MU_p)_{\mathbb{Z}(p)} = FGL / \mathbb{Z}(p) = \bigcup FGL^{ht \leq n}$$

$$Aut(F_n)$$

$$FGL^{ht \leq n} = S_{\text{per}}(v_{n,p}^{-1}(MU_p))$$

$$FGL^{ht \leq n} \setminus FGL^{ht \leq n-1} \cong \{F_n\} \hookrightarrow \{f\}$$

$$H^*(S_{\text{per}}(\mathbb{Z}(h, \dots))) ; FGL^{ht \leq n} \sim FGL^{ht \leq n-1}$$

$$f: F_n \rightarrow F_n$$

$$\begin{matrix} \parallel \\ H^*(Aut(F_n)) \end{matrix}$$

$$FGL^{ht \leq n} \setminus FGL^{ht}$$

• AWSS /  $v_n$ -periodicity.