Chicago REU - cobordism

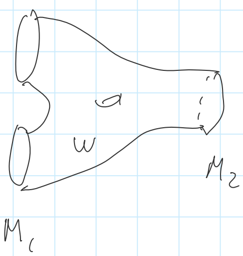
Sunday, July 6, 2025 10:17 PM

<u>Problem</u> Classify all closed smooth n-mflds

Hard

<u>Easier ver.</u> Classify them up to cobordism.

M_1^n cobordant to M_2^n if $\exists$ $\underbrace{W^{n+1}}_{\text{cpt mfld w/ } \partial}$ s.t. $\partial W = M_1 \sqcup M_2$

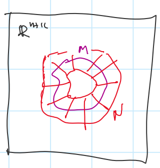


$$\Omega_n = \{\text{Diffeo classes } M^n\} / \underbrace{\sim}_{\text{bordism.}}$$

<u>Thm:</u> Ω_n can be computed as the homotopy groups of something.

$$M^n \hookrightarrow \mathbb{R}^{n+k} \rightsquigarrow$$

$$\begin{array}{ccc} \nu_M^k & \xrightarrow{\alpha_M} & V^k = \{(V,x) \mid V \subseteq \mathbb{R}^\infty,\ x \in V,\ \dim V = k\} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\exists} & BO(k) = Gr_k(\mathbb{R}^\infty) = \{V \subseteq \mathbb{R}^\infty \mid \dim V = k\} \end{array}$$



$$N \cong D(\nu_M^k) \xrightarrow{\alpha_M} D(V^k)$$

$$S^{n+k} = (\mathbb{R}^{n+k})^+ \xrightarrow{\tilde{\alpha}_M} D(V^k)/S(V^k) =: Th(V^k)$$

$$y \longmapsto \begin{cases} \alpha_M(y), & y \in N \\ *, & y \notin N \end{cases}$$

$$M \rightsquigarrow \tilde{\alpha}_M \in \pi_{n+k} Th(V^k)$$

$$M \leftsquigarrow \tilde{\alpha}_M$$

$$M = \tilde{\alpha}_M^{-1}(0_{V^k})$$

$$0_{V^k} \hookrightarrow D(V^k)$$
zero section.

cobordism of mflds $\leftrightsquigarrow$ homotopy of $\tilde{\alpha}_M$ (transverse to 0 sect.)

$M \hookrightarrow \mathbb{R}^{n+k} \to \mathbb{R}^{n+k+1}$

$$\pi_{n+k} Th(V^k) \longrightarrow \pi_{n+k+1}(Th(V^{k+1}))$$

$$\begin{array}{ccc} V^k \oplus \mathbb{R} & \longrightarrow & V^{k+1} \\ \downarrow & & \downarrow \\ BO(k) & \longrightarrow & BO(k+1) \end{array}$$

Th (Thm)

$$\begin{array}{ccc} \Omega_n & \xrightarrow{\cong} & \boxed{\varinjlim_k \pi_{n+k} Th(V^k)} \\ [M] & \longmapsto & [\alpha_M] \end{array}$$

Thom also computed this

Revisionist viewpoint.

<u>Computational tool:</u> Adams spectral sequence.

$X \in Top_*$

π_n super duper hard don't satisfy excision	$\pi_n^s(X) := \varinjlim_k \pi_{n+k}(\Sigma^k X)$ still hard, but satisfy excision.	$\tilde{H}_n(X)$ satisfies excision $\tilde{H}_n(S^0)$ easy. $\Downarrow$ easy

reduces to $\pi^s_{n+k}(S^k)$

$\cong$

$\pi_n^s(S^0)$ $\xLeftarrow{\text{ASS}}$

Hard.

<u>stability:</u> $$\varinjlim_k \pi_{n+k} Th(V^k) \cong \varinjlim_k \pi^s_{n+k} Th(V^k)$$

<u>Adams spectral sequence</u>

$$\mathcal{A}_p^k = Nat^*\left(\tilde{H}^*(-;\mathbb{F}_p), \tilde{H}^{*+k}(-;\mathbb{F}_p)\right)$$

$$\begin{array}{ccc} \tilde{H}^*(-;\mathbb{F}_p) & \xrightarrow{\varphi} & \tilde{H}^{*+k}(-;\mathbb{F}_p) \\ \cong \text{ susp} & & \cong \text{ susp} \\ \tilde{H}^{*+1}(\Sigma(-);\mathbb{F}_p) & \xrightarrow[\varphi.]{} & \tilde{H}^{*+1+k}(\Sigma - ;\mathbb{F}_p) \end{array}$$

<u>Idea:</u>

$$\begin{array}{c} L \\ \Big| \; G = Aut_K(L) \\ K \end{array} \qquad \begin{array}{c} \tilde{H}^*(-;\mathbb{F}_p) \\ \Big\downarrow \; \mathcal{A}_p \\ \pi_*^s(-) \end{array}$$

$E_2 \Rightarrow A$

$gr\, A = \dfrac{\bigoplus \ker d_r}{\sum_r \operatorname{im} d_r}$

<u>ASS</u>

Galois descent

Alg constructions$/K$ $\cong$ G-equivariant Alg constructions ~~over~~ $/L$

obstructions: Galois coh.

[illegible]

$$Ext^{s,t}_{\mathcal{A}_p}\left(\tilde{H}^*(X;\mathbb{F}_p), \mathbb{F}_p\right) \Rightarrow \pi^s_{t-s}(X)^\wedge_p$$

$d_r: \; s \longmapsto s+r$
$\quad\;\; t-s \longmapsto t-s-1$

Complicated

e.g. $p=2$

$$\mathcal{A}_2 = \mathbb{F}_2\langle Sq^i \rangle_{i \geq 1} / \sim \qquad |Sq^i| = i$$

obstructions: Galois coh.

$$H^*(G;M) \cong \mathrm{Ext}_{\mathbb{Z}[G]}(\mathbb{Z}, M)$$

e.g. $p=2$ $\quad A_2 = \mathbb{F}_2\langle Sq^i\rangle_{i\geq 1}/\sim \qquad |Sq^i| = i$

$$Sq^a Sq^b = \sum_i \binom{b-i-1}{a-2i} Sq^{a+b-i} Sq^i$$

A_p is a cocommutative Hopf algebra

e.g. $A_p \otimes A_p \xrightarrow{\mu} A_p$

$A_p \xrightarrow{\psi} A_p \otimes A_p$ map of algebras

[pic of ASS for sphere]

Milnor

$$A_p^\vee = \mathrm{Hom}(A, \mathbb{F}_p)$$

is a commutative Hopf alg.

really nice structure

Wrong! p odd this means graded commutative

$$A_2^\vee = \mathbb{F}_2[\xi_1, \xi_2, \dots] \qquad |\xi_i| = 2^i - 1$$

comm ring

$$\psi(\xi_k) = \sum_{i+j=k} \xi_i^{2^j} \otimes \xi_j$$

$$(f, f') \mapsto \xi_i'\left(\sum_j \xi_j t^{2^j}\right)^{2^i} = f' \circ f$$

$$\sum \xi_i'(\xi_j)^{2^i} t^{2^{i+j}}$$

$$Sq^i = (\xi_1^i)^\vee$$

$$\mathrm{Spec}(A) \times \mathrm{Spec}(A) \cong \mathrm{Spec}(A \otimes A) \longrightarrow \mathrm{Spec}(A)$$

makes $\mathrm{Spec}(A)$ into an affine gp scheme.

$$\mathrm{Spec}(A_2^\vee)(R) = \begin{cases} \phi, & R \text{ is not an } \mathbb{F}_2\text{-alg} \\ f(t) \in R[[t]]^\times, \; f(t_1+t_2) = f(t_1) + f(t_2), & R \text{ is an } \mathbb{F}_2 \text{ alg} \end{cases}$$

$\mathrm{Alg}(A_2, R)$

$$f(t) = \sum_i \xi_i t^{2^i} \qquad \xi_i \in R, \; \xi_0 = 1$$

$$\mathrm{Spec}(A_2^\vee) \cong A.$$

$$A_p \otimes H^*(X; \mathbb{F}_p) \longrightarrow H^*(X;\mathbb{F}_p) \quad A_p\text{-module}$$

$$H_*(X;\mathbb{F}_p) \longrightarrow A_p \otimes H_*(X;\mathbb{F}_p) \quad A_p^\vee\text{-comodule.}$$

ASS

$$\mathrm{Ext}^{s,t}_{A_p^\vee\text{-comod}}(\mathbb{F}_p, \tilde{H}_*(X;\mathbb{F}_p)) \Rightarrow \pi^s_{t-s}(X)^\wedge_p$$

$$\| \quad H^*(\mathrm{Spec}(A_p^\vee); \tilde{H}_*(X;\mathbb{F}_p))$$

Fact Ω_* is an $\mathbb{F}_2$-alg $\qquad M \amalg M$ $\qquad M\times I$ $\qquad \phi$

$$\Rightarrow \Omega_* = (\Omega_*)^\wedge_2 \qquad \Rightarrow 2[M] = 0$$

ASS

$$\mathrm{Ext}_{A_2^\vee}(\mathbb{F}_2, \varinjlim \tilde{H}_{*+k}(Th(\gamma^k); \mathbb{F}_2)) \Rightarrow (\varinjlim \pi^s_{*+k} Th(V^k))^\wedge_2 \cong \Omega_*$$

$$\tilde{H}_{*+k}(Th(V^k); \mathbb{F}_2) \cong H_*(BO(k)) \qquad \varprojlim_k (H^*(BO(k); \mathbb{F}_2) \cong \mathbb{F}_2[w_1, w_2, \dots, w_k])$$

Thom iso

[Another one is

$$\tilde{H}_{*+k}(Th(V^k);\mathbb{F}_2) \cong H_*(BO(k)) \qquad \varinjlim_k \left(H^*(BO(k);\mathbb{F}_2) \cong \mathbb{F}_2[w_1, w_2, \dots, w_k] \right)$$

Thm iso

$\boxed{\text{Aut } \cdots \text{ is of } A_*\text{-comodules}}$

$$\mathbb{F}_2[w_1, w_2, \dots]$$

$$X \xrightarrow{V} BO(k)$$

$$H^*(X) \longleftarrow H^*(BO(k))$$

$$w_i(V) \longleftarrow w_i$$

Hopf algebra $\Psi(w_k) = \sum_{i+j=k} w_i \otimes w_j$

$$w_k(V \oplus W) = \sum w_i(V) \cup w_j(W)$$

Crazy fact:

$$\begin{array}{ccccc} \varinjlim H_* Th(V^k) & \cong & \varinjlim H_*(BO(k);\mathbb{F}_2) & \cong & \mathbb{F}_2[a_1, a_2, \dots] \\ \uparrow & & \uparrow & & \uparrow \; \alpha_i \mapsto a_i \\ \tilde{H}_{*+1} \underbrace{Th(V^1)}_{\mathbb{RP}^\infty} & \cong & H_*(\underbrace{BO(1)}_{\mathbb{RP}^\infty}) & \cong & \mathbb{F}_2\{\alpha_0, \alpha_1, \dots\} \end{array}$$

$$Sq^i(x^j) = \binom{j}{i} x^{i+j}$$

coaction $\mathbb{F}_2\{\alpha_0, \alpha_1, \alpha_2, \dots\} \longrightarrow \mathbb{F}_2[\xi_1, \xi_2, \dots] \otimes \mathbb{F}_2\{\alpha_0, \alpha_1, \alpha_2, \dots\}$

$$(g \circ f) \longleftarrow \Big(\underbrace{\sum \xi_i t^{2^i}}_{f}, \underbrace{\sum \alpha_n t^n}_{g} \Big)$$

$$\alpha_k \longmapsto \operatorname{coef}_{t^k}(g \circ f) \qquad \alpha_0 + \alpha_1(t + \xi_1 t^2 + \xi_2 t^4) + \alpha_2(t + \xi_1 t^2 + \xi_2 t^4)^2 + \cdots$$

$$\boxed{\text{e.g.} \quad (t + \xi_1 t^2)^j = \sum_i \binom{j}{i} \xi_1^i t^{2i+j-i}}$$

$$= \alpha_0 + \alpha_1 t + (\alpha_1 \xi_1 + \alpha_2) t^2 + \cdots$$

$$\mathbb{F}_2[a_1, a_2, \dots] \longrightarrow \mathbb{F}_2[\xi_1, \xi_2, \dots] \otimes \mathbb{F}_2[a_1, a_2, \dots]$$

$\Psi(\alpha_0) = 1 \otimes \alpha_0$

$\Psi(\alpha_1) = 1 \otimes \alpha_1$

$\Psi(\alpha_2) = 1 \otimes \alpha_2 + \xi_1 \otimes \alpha_1$

$\Big\downarrow$ Spec$(-)(R)$

$$R[[t]]^\times \longleftarrow \underbrace{\{t + \xi_1 t^2 + \xi_2 t^4 + \cdots\}}_{\text{Aut}(\hat{\mathbb{G}}_a/\mathbb{F}_2)(R)} \times \underbrace{\{t + a_1 t^2 + \cdots\}}_{R[[t]]^\times}$$

$$f \circ g \longleftarrow (f, g)$$

takeaway Spec$(A_2^\vee)$ is a subgp of Spec$(\mathbb{F}_2[a_1, a_2, \dots])$

$$\mathbb{F}_2[a_1, a_2, \dots] \cong \underbrace{\mathbb{F}_2[\xi_1, \xi_2, \dots]}_{\text{comodule}} \otimes \mathbb{F}_2[a_n \mid n \neq 2^i - 1]$$

$$\operatorname{Ext}^s_{A_2^\vee}(\mathbb{F}_2, A_2^\vee \otimes \mathbb{F}_2[a_n \mid n \neq 2^i - 1]) \cong \begin{cases} \mathbb{F}_2[a_n \mid n \neq 2^i], & s = 0 \\ 0, & o/w \end{cases}$$

e.g. $H^*(G; \mathbb{Z}[G] \otimes M) \cong \begin{cases} M, & * = 0 \\ 0, & o/w \end{cases}$ "

$\Rightarrow$ ASS collapses $\Omega_* = \mathbb{F}_2[a_n \mid n \neq 2^i]$

Milnor's ... Ω w/ Ω^U ν has $\mathbb{C}$ structure

Milnor: replace Ω_* w/ Ω_*^u ν_M has cx structure

$$\Omega_*^u \cong \varinjlim \pi_{*+2k}^s Th(V_{\mathbb{C}}^k)$$

$$V_{\mathbb{C}}^k \downarrow BU(k) = Gr_k(\mathbb{C}^\infty) = \{V \le \mathbb{C}^\infty \mid \dim_{\mathbb{C}} V = k\}$$

need to compute

$$\begin{array}{ccc} \Omega_*^u & \longrightarrow & \prod_p (\Omega_*^u)_p^\wedge \\ \downarrow & & \downarrow \\ (\Omega_*^u)_{\mathbb{Q}} & \longrightarrow & \left(\prod_p (\Omega_*^u)_p^\wedge\right)_{\mathbb{Q}} \end{array}$$

$\Rightarrow [M] + [\bar{M}] = 0$

$\Rightarrow [\bar{M}] = -[M]$

Fact:

$$\pi_*^s(X) \otimes \mathbb{Q} \cong \tilde{H}_*(X;\mathbb{Q})$$

follows from $\pi_*^s(S^0)_{\mathbb{Q}} \cong \mathbb{Q}$ rational stable homotopy satisfies dimension axiom.

$|b_i| = |c_i| = 2i$

$$\varprojlim H^*(BU(k)) \cong \mathbb{Z}[c_1, c_2, \ldots] \qquad \Psi(c_k) = \sum_{i+j} c_i \otimes c_j$$

$$\varinjlim H_*(BU(k)) \cong \mathbb{Z}[b_1, b_2, \ldots] \longleftarrow H_* BU(1)$$

$b_i \longleftrightarrow \beta_i$

$$\Rightarrow (\Omega_*^u)_{\mathbb{Q}} \cong \mathbb{Q}[b_1, b_2, \ldots]$$

$|\xi_i| = 2(p^i - 1) \qquad |\tau_i| = 2p^i - 1$

$$\mathcal{A}_p^\vee \cong \underbrace{\mathbb{F}_p[\xi_1, \xi_2, \ldots]}_{P} \otimes \underbrace{\Lambda[\tau_0, \tau_1, \tau_2, \ldots]}_{E}$$

generated by odd degree elts

$p = 2$ $\quad P = \mathbb{F}_p[\xi_1^2, \xi_2^2, \ldots]$

$\mathrm{Spec}(P) = \mathrm{Aut}(\hat{\mathbb{G}}_a / \mathbb{F}_2)$

comodules

$$H_*(Th(V_{\mathbb{C}}^k); \mathbb{F}_p) \longrightarrow \mathcal{A}_*^\vee \otimes H_*(Th(V_{\mathbb{C}}^k); \mathbb{F}_p)$$

$$\searrow \quad P_* \otimes H_*(Th(V_{\mathbb{C}}^k)) \quad \nearrow$$

$$\varinjlim H_{*+2k}(Th(V_{\mathbb{C}}^k); \mathbb{F}_p) \cong \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes \mathbb{F}_p[b_n \mid n \ne p^i - 1]$$

$$\mathrm{Ext}_{P \otimes E}(\mathbb{F}_p, P \otimes \mathbb{F}_p[b_n]_{n \ne p^i - 1}) \cong \mathrm{Ext}_E(\mathbb{F}_p, \mathbb{F}_p) \otimes \mathbb{F}_p[b_n]_{n \ne p^i - 1}$$

$\mathrm{Ext}_E(\mathbb{F}_p, \mathbb{F}_p) \cong \mathbb{F}_p[v_0, v_1, v_2, \ldots]$

$$H^*(G; \mathbb{Z}[G/K] \otimes M) \cong H^*(K) \otimes M$$

fixed action

$|v_i| = (1, 2p^i - 1)$

$|b_n| = (0, 2n)$

even $\Rightarrow$ no diff'ls.

$t-s$

in other words $\quad gr\, \mathbb{Z}_p^\wedge \cong \mathbb{F}_p[v_0]$

$$(\Omega_*^u)_p^\wedge \cong \mathbb{Z}_p[x_1, x_2, \ldots]$$

$$(\Omega_*^u)_{\mathbb{Q}} \cong \mathbb{Q}[x_1, x_2, \ldots]$$

$\vdots$

$$\Omega_*^u \cong \mathbb{Z}[x_1, x_2, \ldots]$$

$\hat{\mathbb{G}}_a$ "additive formal gp"

$\vdots$