CONVEXITY AND DUALITY IN OPTIMAL CONTROL: KUHN-TUCKER COEFFICIENTS, THE MINIMAX THEOREM AND FENCHEL DUALITY

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Abstract. This paper explores the two ways in which the addition of convexity conditions enhances solution methods in optimal control theory. First, in an unconstrained optimization problem, convexity is necessary for the construction of the dual problem, whose solutions describe the existence conditions for those of the primal problem. Second, the application of such convexity-induced duality to constrained optimization problems allows for the treatment of more dynamic constraints as those in the Karush-Kuhn-Tucker, Minimax and Fenchel Duality Theorems.

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Introduction

We begin with the proof of the Lagrange Multiplier Theorem, which characterizes the constrained extrema for functions not necessarily convex. After introducing convexity conditions, we show that they induce a dual problem, whose solutions possess existence conditions connected to those of the primal problem. We then present variations of duality in constrained optimization problems, including the Karush-Kuhn-Tucker and the Minimax Theorems. Finally, in the exposition of Fenchel duality, we show that the invertibility of conjugation is a special property of convex functions that allows for the dualization of optimization problems, and present convex duality in its most general form.

Only knowledge of analysis in $\mathbb{R}^n$ is assumed, except for one or two theorems imported from elementary topology which will be named. Readers are introduced to calculus on Banach spaces and convex analysis at the beginning of relevant sections.

1. Characterization of Constrained Extrema

In this section, we deduce the necessary conditions for the existence of solutions to the general constrained optimization problem without convexity conditions. In the process, we introduce some fundamental theorems from analysis on Banach spaces and discuss their applications in solution algorithms.

1.1. **Calculus on Banach spaces.** This is a brief exposition on the extension of calculus in $\mathbb{R}^n$ to Banach spaces that provides the notation and vocabulary for the first section, including the notions of dual spaces and types of differentiability.

Definition 1.1 (The operator space and its norm). $B(X,Y)$ denotes the space of all continuous linear operators from X to Y. Its norm for any $g : X \longrightarrow Y$ is defined by

$$\|g\|_{B(X,Y)} \coloneqq \sup_{\|x\|_X=1} \|g(x)\|_Y = \sup_{x\in X} \frac{\|g(x)\|_Y}{\|x\|_X}.$$

In cases where $Y = \mathbb{R}$, the space $B(X,\mathbb{R})$ is also denoted as χ^* and called the **dual space** of X. The elements in $\chi^* = B(X,\mathbb{R})$ are called **functionals**.

Similarly, by replacing absolute values with norms on respective Banach spaces, we generalize the notions of differentiability and continuity.

Definition 1.2 (Fréchet-derivative). A function $f : U \longrightarrow Y$, where $U \subset X$ is open and X and Y are normed linear spaces, is **Fréchet differentiable** (or **strongly differentiable**) at x if there is a linear operator $A \in B(X,Y)$ such that

$$\frac{1}{\|h\|_X}\|f(x+h) - f(x) - Ah\|_Y \longrightarrow 0 \text{ as } h \longrightarrow 0.$$

When A exists, it is called a **Fréchet-derivative** of f at x and denoted variously as

$$A = A_x = f'(x) = Df(x).$$

Definition 1.3 (Lipschitz). The **Lipschitz** property of a function describes the boundedness of the change in function values with respect to a movement in the domain. A mapping $f : (X, d_X) \longrightarrow (Y, d_Y)$ is Lipschitz if there is $K \geq 0$ such that $d_Y(f(x), f(y)) \leq Kd_X(x,y)$, where K is called a **Lipschitz constant**.

Note that continuously differentiable functions are a subset of Lipschitz functions, which are then a subset of uniformly continuous functions. Lipschitz functions are automatically continuous: For any $x \in X$ and $\epsilon > 0$, there is an open ball of radius $r = \epsilon/2K$ around x such that any $y \in B_r(x)$ will have $d_Y(f(x), f(y)) \leq Kd_X(x,y) \leq \epsilon/2 < \epsilon$. We will use this property in the following proof of the Banach Contraction-Mapping Principle.

Another approach to describe the boundedness of change within a function is through the boundedness of the Fréchet-derivative, relying on the Banach-space variant of the Mean-Value Theorem.

Theorem 1.4 (The Mean-Value Theorem). Let X and Y be normed linear spaces and U an open subset of X. Suppose that $f : U \to Y$ is Fréchet differentiable everywhere in U, in particular in the line segment $l = \{tx_2 + (1-t)x_1 | 0 \le t \le 1\}$ contained in U for some fixed $x_1, x_2 \in U$. Then the supremum of the Fréchet-derivatives on l are Lipschitz constants for f on l:

$$\|f(x_2) - f(x_1)\|_Y \le \sup_{x \in l} \|Df(x)\|_{B(X,Y)} \|x_2 - x_1\|_X.$$

Proof. See [1, Theorem 9.6]. □

1.2. **Properties of the contraction mapping.**

Definition 1.5 (Contraction). Let (X, d) be a complete metric space and $G : X \to X$. The mapping G is a **contraction** if there is $0 \le \theta < 1$ such that

$$\forall x, y \in X, d(G(x), G(y)) \le \theta d(x, y).$$

That is, a Lipschitz function with Lipschitz constant less than 1 is a contraction. A **fixed point** of the mapping is an $x \in X$ such that $x = G(x)$.

Theorem 1.6 (Banach Contraction-Mapping Principle). Let (X, d) be a complete metric space and G a contraction mapping on X. Then there is a unique fixed point of G in X.

Proof. For any given $x_0 \in X$, we iteratively define a sequence $\{x_k\}_{k=0}^{\infty}$ by $x_{k+1} = G(x_k)$, which is Cauchy:

$$\begin{aligned} 0 \le d(x_m, x_n) &\le d(x_m, x) + d(x, x_n) \\ &\le \theta^{m-1} d(x_0, G(x_0)) + \theta^{n-1} d(x_0, G(x_0)) \longrightarrow 0 \text{ as } m, n \to \infty. \end{aligned}$$

Since X is complete, this is equivalent to $x_k \to \bar{x}$ for some $\bar{x} \in X$. G is Lipschitz, hence continuous, so $G(x_k) \to G(\bar{x})$. Note that the sequence $\{x_k\}_{n=0}^{\infty}$ is the same as $\{G(x_k)\}$ except for the addition of x_0 at the beginning, so they converge to the same limit $\bar{x} = G(\bar{x})$. □

Corollary 1.7 (Fixed Point Iteration). Suppose that (X, d) is a complete metric space and G is a contraction mapping of X with contraction constant θ. Choose $x_0 \in X$ and define the sequence $\{x_n\}_{n=0}^{\infty}$ by $x_{n+1} = G(x_n)$. Then $x_n \to x$, where x is the unique fixed point of G in X. Moreover,

$$d(x_n, x) \le \frac{\theta^n}{1-\theta} d(x_0, x_1).$$

Proof. For the first part of the corollary, assume that the fixed point obtained is not unique. That is, there is $x' \in X$ such that $x' \ne x$ and $G(x') = x'$. A contradiction follows:

$$d(G(x), G(x')) \le \theta d(x, x') < d(x, x') = d(G(x), G(x')).$$

The second part follows from triangular inequality of metric spaces used in the proof that $\{x_k\}_{k=0}^{\infty}$ is Cauchy in the preceding theorem:

$$d(x_n, x_0) \le \sum_{k=1}^{n} d(x_k, x_{k-1}) \le \sum_{k=1}^{n} \theta^{k-1} d(x_1, x_0) = \frac{\theta^n}{1-\theta} d(x_0, x_1).$$

□

Since the result usefully proves existence and uniqueness at the same time, we want to have a similar statement for operators that send their domains to a different space.

Theorem 1.8 (Simplified Newton Method). Let X, Y be Banach spaces and $f : X \to Y$ a Fréchet-differentiable mapping. Suppose that there is $x_0 \in X$ such that $A = Df(x_0)$ has a bounded inverse and that there exists $r > 0$ such that

$$\forall x \in B_r(x_0), \quad \left\|I - A^{-1}Df(x)\right\|_{B(X,X)} \leq \kappa < 1.$$

Then the equation $f(x) = y$ has a unique solution $x \in B_r(x_0)$ if $y \in B_\delta(f(x_0))$, where

$$\delta = \frac{(1-\kappa)r}{\left\|A^{-1}\right\|_{B(X,Y)}}.$$

Proof. For given $y \in B_\delta(f(x_0))$, we define the mapping $g_y(x) : X \to X$ by $g_y(x) = x - A^{-1}(f(x) - y)$ such that $g_y(x) = x$ if and only if $f(x) = y$. This is because if A^{-1} were the zero mapping, it would contradict our hypothesis that it is invertible. Now that the problem has a similar structure to that of the fixed point, we check that our conditions are equivalent to those of a contractive map.

By the chain rule, the derivative $Dg_y(x) = I - A^{-1}Df(x)$, so our assumption states that $\left\|Dg_y(x)\right\|_{B(X,X)} \leq \kappa < 1$. This implies that g_y is contractive by the Mean Value Theorem:

$$\forall x_1, x_2 \in B_r(x_0), \frac{\left\|g_y(x_1) - g_y(x_2)\right\|}{\|x_1 - x_2\|} \leq \sup_{x \in B_r(x_0)} \left\|Dg_y\right\|_{B(X,Y)} \leq \kappa < 1.$$

Also note that by our choice of y and δ,

$$\begin{aligned}\left\|g_y(x_0) - x_0\right\|_X = \left\|A^{-1}(f(x_0 - y))\right\| &\leq \|f(x_0) - y\|_Y \left\|A^{-1}\right\|_{B(X,Y)} \\ &\leq \delta \left\|A^{-1}\right\|_{B(X,Y)} \leq (1-\kappa)r.\end{aligned}$$

As a result, g_y is a contraction which maps $B_r(x_0)$ to itself:

$$\begin{aligned}\forall x \in B_r(x_0), \left\|g_y(x) - x_0\right\|_X &\leq \left\|g_y(x) - g_y(x_0)\right\|_X + \left\|g_y(x_0) - x_0\right\|_X \\ &\leq \kappa \|x - x_0\|_X + (1-\kappa)r < r.\end{aligned}$$

The last equality follows from the definition of operator norm.

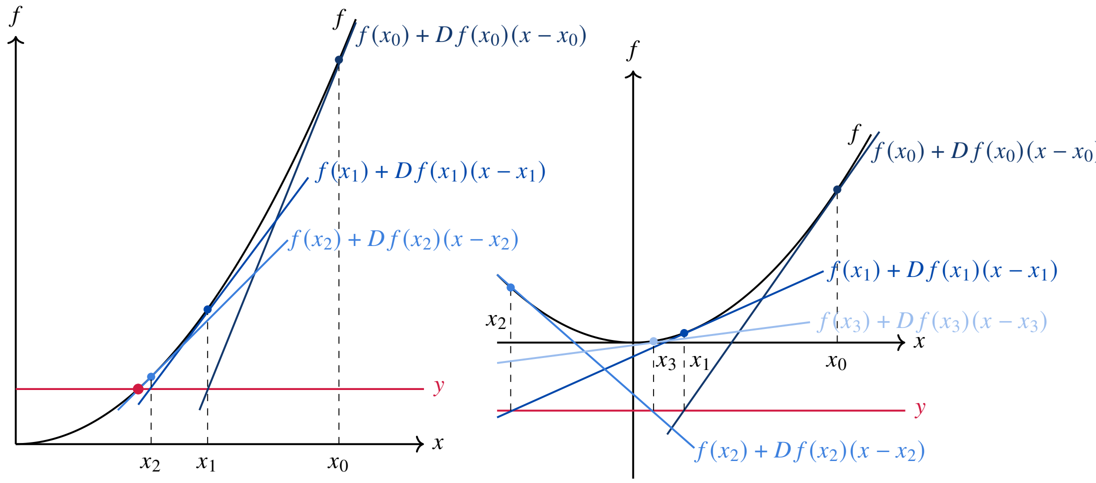
With the hypotheses in the Banach Contraction-Mapping Principle verified for g_y, we conclude that there exists unique $x \in B_r(x_0)$ such that $g_y(x) = x$, which is the unique solution for $f(x) = y$. □

1.3. **The Inverse Function Theorem.**

Lemma 1.9 (Inverse Function Theorem). Suppose that f satisfy the hypotheses of the Simplified Newton Method. Then the inverse mapping $f^{-1} : B_\delta(f(x_0)) \to B_r(x_0)$ is Lipschitz.

Note that the conditions above are identical to those in the Simplified Newton Method, so our proof of the Inverse Function Theorem is near identical to the preceding proof. We only need the invertibility condition to convert the boundedness of change in the inverse function to that of movement within the domain.

Proof. Let $y_1, y_2 \in B_\delta(f(x_0))$. The Simplified Newton Method states that there exist unique $x_1, x_2 \in B_r(x_0)$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. For given $y \in B_\delta(f(x_0))$,



(a) $x_{t+1} = \frac{y - f(x_t) + Df(x_t)x_t}{Df(x_t)}$ is contractive. **(b)** $x_{t+1} = \frac{y - f(x_t) + Df(x_t)x_t}{Df(x_t)}$ is not contractive.

Figure 1. An application of Simplified Newton's Method is in the numerical solution to systems of equations. To find the point $\bar{x}$ such that $f(\bar{x}) = y$, we iteratively update the estimate for $\bar{x}$ by the Taylor approximation $f(x_{t+1}) \approx f(x_t) + Df(x_t)(x_{t+1} - x_t)$. Whether $f(x) = y$ has a solution depends on whether this mapping is contractive. If so, we would arrive at a precise numerical solution with few iterations.

recall our definition of $g_y(x) = x - A^{-1}(f(x) - y)$, which maps $B_r(x_0)$ onto itself with Lipschitz constant $\kappa < 1$:

$$\begin{aligned}\left\|f^{-1}(y_1) - f^{-1}(y_2)\right\| = \|x_1 - x_2\|_X &= \left\|g_y(x_1) - g_y(x_2) + A^{-1}(f(x_2) - f(x_1))\right\|_X \\ &\leq \left\|g_y(x_1) - g_y(x_2)\right\| + \left\|A^{-1}\right\|_{B(Y,X)} \|f(x_2) - f(x_1)\|_Y \\ &\leq \kappa\|x_1 - x_2\|_X + \left\|A^{-1}\right\|_{B(Y,X)} \|y_1 - y_2\|.\end{aligned}$$

Upon rewriting the relationship as $\left\|f^{-1}(y_1) - f^{-1}(y_2)\right\|X = \|x_1 - x_2\|_X \leq \dfrac{\left\|A^{-1}\right\|_{B(Y,X)}}{1 - \kappa}$, we see that f^{-1} is Lipschitz with constant of at most $\dfrac{\left\|A^{-1}\right\|_{B(Y,X)}}{1 - \kappa}$. □

A stronger statement concerns the invertibility of the derivative, formalized as:

Definition 1.10 (Diffeomorphism). Let X, Y be Banach spaces and $U \subset X, V \subset Y$ open. Suppose that the bijective function $f : U \to V$ and its inverse f^{-1} are Fréchet-differentiable with continuous derivatives on U and V. That is, the maps $x \mapsto Df(x)$ and $y \mapsto Df^{-1}$ are continuous from U to $B(X, Y)$ and from V to $B(X, Y)$. Then f is called a **diffeomorphism** on U and U is **diffeomorphic** to V.

Lemma 1.11. Let X, Y be Banach spaces and $GL(X, Y)$ denote the set of all isomorphisms of X onto Y. Then $GL(X, Y)$ is an open subset of $B(X, Y)$. Moreover, for $GL(X, Y) \neq \emptyset$, the mapping $J_{X,Y} : GL(X, Y) \to GL(Y, X)$ given by $\forall A \in B(X, Y), J_{X,Y}(A) = A^{-1}$, the map of an invertible function onto its inverse, is bijective and continuous.

Proof. Let $A \in GL(X,Y)$ and $H \in B(X,Y)$ be such that $\|H\|_{B(X,Y)} < \theta/\left\|A^{-1}\right\|_{B(Y,X)}$ for some $\theta < 1$. We will show that the operator $A + H$ is bijective, that is, an element of $GL(X,Y)$.

Define $S_N = A^{-1}\sum_{n=0}^{N}(-HA^{-1})^n$ in $B(Y,X)$ such that the sequence $\{S_N\}_{N=1}^{\infty}$ is Cauchy in $B(Y,X)$. By the Cauchy-Schwartz Inequality,

$$\forall M > N, \quad \|S_M - S_M\|_{B(Y,X)} \leq \left\|A^{-1}\right\|_{B(Y,X)} \sum_{n=N+1}^{M} \left\|(HA^{-1})^n\right\|$$
$$\leq \left\|A^{-1}\right\|_{B(Y,X)} \sum_{n=N+1}^{M} (\|H\|_{B(X,Y)}\left\|A^{-1}\right\|_{B(Y,X)})$$
$$\leq \left\|A^{-1}\right\|_{B(Y,X)} \sum_{n=N+1}^{M} \theta^n \to 0 \text{ as } N \to \infty.$$

Hence S_N converges, say to S, in $B(Y,X)$ such that

$$(A+H)S = \lim_{N\to\infty}(A+H)S_N = \lim_{N\to\infty}\sum_{n=0}^{N}[(-HA^{-1})^n - (-HA^{-1})^{n+1}]$$
$$= \lim_{N\to\infty}[I - (-HA^{-1})^{N+1}] = I$$

By a similar calculation, $S(A+H) = I$, so $A + H$ has an inverse $S : Y \to X$, which makes it bijective. As a result, the operator $A = (A + H) - H$ is bijective.

For continuity of the map $A \mapsto A^{-1}$, we will show that $(A+H)^{-1} \to A^{-1}$, or equivalently, $S - A^{-1} \to 0$, as $\|H\|_{B(X,Y)} \to 0$. Recall that for $|x| < 1$, $(1+x)^{-1} = \sum_{n=0}^{\infty}(-x)^n$. Analogously, our construction of $\{S_N\}$ gives

$$\|S\|_{B(Y,X)} = \lim_{N\to\infty}\|S_N\|_{B(Y,X)} \leq \left\|A^{-1}\right\|_{B(Y,X)}/(1-\theta),$$

such that

$$\left\|S - A^{-1}\right\|_{B(Y,X)} \leq \|S\|_{B(Y,X)}\|H\|_{B(Y,X)}\left\|A^{-1}\right\|_{B(Y,X)} \to 0 \text{ as } \|H\|_{B(X,Y)} \to 0,$$

with $\left\|A^{-1}\right\|_{B(Y,X)}$ fixed and $\|S_N\|_{B(Y,X)}$ bounded. □

Theorem 1.12 (Inverse Function Theorem). Let X, Y be Banach spaces. Let $x_0 \in X$ be such that f is Fréchet differentiable in a neighborhood of $x_0 \in X$ and $Df(x_0)$ is an isomorphism. Then there is an open set $U \subset X$ containing x_0 and open set $V \subset Y$ containing $f(x_0)$ such that $f : U \to V$ is a diffeomorphism. Moreover, for all $y \in V$ and $x \in U$ such that $y = f(x)$,

$$D(f^{-1})(y) = (Df(x))^{-1}.$$

Proof. Let $A = Df(x_0)$. Since f is continuously differentiable, $Df(x) \to A$ in $B(X,Y)$, which implies $A^{-1}Df(x) \to I$ on $B(X,X)$, as $x \to x_0$ in X, so there is $r' > 0$ such that $\left\|I - A^{-1}Df(x_0)\right\| \leq \kappa < 1$ for any $x \in B_{r'}(x_0)$. At the same time, lemma 2.7 implies that there is $0 < r'' \leq r'$ such that $Df(x)$ has a bounded inverse with $\left\|Df(x)^{-1}\right\|_{B(Y,X)} \leq \left\|A^{-1}\right\|/(1-\kappa)$, for all $x \in B_{r''}(x_0)$. Now that $Df(x)$ satisfies the hypotheses of the Simplified Newton Method, there is $r, \delta > 0$ such that $f : U \to V$ is isomorphic, where

$$V = B_\delta(f(x_0)) \text{ with } \delta = \frac{r}{2\left\|A^{-1}\right\|_{B(Y,X)}}$$

$$U = B_r(x_0) \cap f^{-1}(V).$$

U and V are then potential diffeomorphic subsets of the domain and the codomain. To verify the that $y \mapsto Df^{-1}$ is indeed continuous, we only need to prove the latter equality $D(f^{-1}(y)) = (Df)^{-1}(x)$ whenever $x \in U, y \in V$ and $f(x) = y$, which allows us to write $y \mapsto Df^{-1}(y)$ as the composition of three continuous functions:

$$y \mapsto f^{-1}(y) \mapsto Df(f^{-1}(y)) \mapsto Df(f^{-1}(y)) = Df^{-1}(y)$$

$$Y \xrightarrow{f^{-1}} X \xrightarrow{Df} B(X,Y) \xrightarrow{J} B(Y,X),$$

where J is defined and shown to be continuous in Lemma 1.11, while the continuity of f^{-1} and Df is given by Lemma 1.9 and the assumption that f is continuously differentiable.

It remains to check that $Df^{-1} = (Df)^{-1}$. For given $y \in V$, let $k \in Y$ be small enough such that $y + k \in V$ and $h = f^{-1}(y+k) - x$, then

$$\begin{aligned}
&\left\|f^{-1}(y+k) - f^{-1}(y) - Df(x)^{-1}k\right\|_X = \left\|h - Df(x)^{-1}[f(x+h) - f(x)]\right\|_X \\
=&\left\|Df(x)^{-1}[f(x+h) - f(x) - Df(x)h]\right\|_X \\
\le& 2\left\|A^{-1}\right\|_{B(X,Y)} \|f(x+h) - f(x) - Df(x)h\|_Y \to 0 \text{ as } h \to 0,
\end{aligned}$$

since f is differentiable at x. As $k \to 0$, it follows from the Lipschitz property of f^{-1} that

$$\|h\|_X = \left\|f^{-1}(y+k) - f^{-1}(y)\right\|_X \le M\|k\|_Y \to 0.$$

Then, as $k \to 0$, $\left\|f^{-1}(y+k) - f^{-1}(y) - Df(x)^{-1}k\right\|_X \to 0$, so f^{-1} is differentiable at $y = f(x)$ and $Df^{-1}(y) = Df(x)^{-1}$. □

1.4. **The Lagrange Multiplier Theorem.** Rather than directly solve the constrained minimization problem, we will convert it into an unconstrained problem whose critical points, characterized by the zero gradient, automatically satisfy the equality constraints. Consider $L : X \times \mathbb{R}^m \to \mathbb{R}$, defined as

$$L(x, \lambda) = f(x) + \lambda_1 g_1(x) + \cdots + \lambda_m g_m(x) \tag{1.1}$$

such that its gradient over λ describes every constraint in the minimization problem:

$$DL(\lambda; x) = (g_1(x), g_2(x), \ldots, g_m(x)).$$

Indeed, there is a one-to-one correspondence between constrained and unconstrained minimization problems, meaning that when asked to minimize L, we can construct an equivalent problem with m constraints by plugging in a vector of the standard basis in $\mathbb{R}^n$. The Lagrange Multiplier Theorem comments on the feasibility of this approach.

Theorem 1.13 (Lagrange Multiplier Theorem). Let X be a Banach space, $U \subset X$ open, and $f, g_i : U \to \mathbb{R}, i = 1, \ldots, m$ continuously differentiable. Suppose $x \in U$ maximizes the value of f under the constraints $g_i(x) = 0$ for all $i = 1, \ldots, m$. That is, $x \in M$, where

$$M = \{x \in U : \forall i = 1, \ldots, m, g_i(x) = 0\}$$

is a relative extrema for $f|_M$. Then there exists $\lambda = (\lambda_0, \ldots, \lambda_m) \in \mathbb{R}^{m+1}$ such that $\lambda \neq \mathbf{0}$ and

$$\langle \lambda, Df(x) \rangle = \lambda_0 Df(x) + \lambda_1 Dg_1(x) + \cdots + \lambda_m Dg_m(x) = 0.$$

Proof. Let x be a local minimum of $f|_M$ such that there exists an open neighborhood V containing x with $f(x) \leq f(y)$ for every $y \in M \cap V$. If we define $F : V \longrightarrow \mathbb{R}^{m+1}$ by

$$F(y) = (f(y), g_1(y), \dots, g_m(y)),$$

then it is not possible for its first term to be less than $f(x)$ and satisfy the conditions defining M at the same time. In other words,

$$\forall \epsilon > 0, \forall y \in V, \quad (f(x) - \epsilon, 0, \dots, 0) \neq F(y).$$

F cannot map an open subset of V containing x onto an open set around $F(x) = (f(x), 0, \dots, 0)$.

From this result, we use the contrapositive of the generalized Inverse Function Theorem to show that $DF(x)$ cannot map X onto $\mathbb{R}^{m+1}$. Assume, for the sake of contradiction, that it does. Then construct the space $\tilde{X} = \text{span}\{v_1, \cdots, v_{m+1} | DF(v_i) = e_i\}$ where e_i is the standard unit vector in the i-th direction in $\mathbb{R}^{m+1}$. That is, $\tilde{X}$ consists of the preimage of the standard basis for $\mathbb{R}^{m+1}$. We can do this because F is a surjective mapping, so there is at least one choice of v_i for each e_i, albeit not necessarily unique. For our purposes, any choice of v_i yields an invertible and Lipschitz map $\overline{DF(x)}$ defined by the restriction of $DF(x)$ to a subset of X isomorphic to $\mathbb{R}^{m+1}$.

For the local extremum x, define $\hat{X} = \{v \in \tilde{X} | x + v \in V\}$ and $h(v) : \hat{X} \longrightarrow \mathbb{R}^{m+1}, h(v) = F(x + v)$ such that $Dh(0) = DF(x)$ by the chain rule. Defined by the further restriction of $\overline{DF(x)}$ to the intersection between its domain and $V - x$, the function h preserves its invertibility and Lipschitz property. By the Inverse Function Theorem, F maps an open subset of $\hat{X}$, which is also open in V, onto an open subset of $\mathbb{R}^m$, contradicting our previous observation.

Rather, DF maps onto a proper subspace of $\mathbb{R}^{m+1}$ and there is some nonzero vector λ orthogonal to $DF(x)(X)$. That is,

$$\forall y \in X, \lambda_0 Df(x)(y) + \lambda_1 Dg_1(x)(y) + \cdots + \lambda_m Dg_M(x)(y) = 0.$$

□

The Lagrange Multiplier Theorem confirms that solutions to the initial constrained optimization problem are indeed the unconstrained extrema of some function in the form

$$L(x, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i g_i(x). \tag{1.2}$$

Later, we shall see that L is an example of the Lagrangian of an optimization problem.

2. Duality in Convex Optimization

2.1. **Properties of convex sets and functions.**

This subsection introduces the fundamental concepts in convex analysis and several operations between the domain and the dual space that are particularly useful when applied to convex functions.

Definition 2.1 (Convex). A subset $C \in X$ is **convex** if every line segment formed by connecting two points in C is contained by C. That is,

$$\forall x, y \in A, \forall t \in [0, 1], \quad tx + (1 - t)y \in A.$$

The **epigraph** of the function f is denoted and defined as

$$\text{epi}\, f = \{(x, y) \in X \times \mathbb{R} | f(x) \leq y\}.$$

A function f is convex if epi f is a convex set. A function g is concave if $-g$ is convex.

The epigraph also induces the notion of **closed** functions, specifically:

Definition 2.2 (Lower semicontinuous). If $f : X \to \mathbb{R}$ satisfy any of the three equivalent conditions, then it is **lower semicontinuous**, abbreviated as **l.s.c.**, at point x_0 :

(i) For every $y < f(x_0)$, there exists a neighborhood around x_0 such that $f(x) > y$, or

$$\liminf_{x \to x_0} f(x) \geq f(x_0);$$

(ii) For every $\gamma \in \mathbb{R}$, the **level set** $V_\gamma = \{x | f(x) \leq \gamma\}$ is closed;

(iii) The epigraph of f is closed.

See [2, Proposition 1.1.2.] for proof of equivalence. A function g is **upper semicontinuous**, or **u.s.c.**, if $-g$ is l.s.c.

Convexity and lower semicontinuity are preserved in several operations:

Proposition 2.3.

(a) For the convex set $C \subset X$, the interior of C, denoted as C°, and the closure of C, denoted as $\overline{C}$, are convex. If $C^\circ \neq \emptyset$, then $\overline{C} = \overline{C^\circ}$;

(b) The pointwise supremum of a family of convex functions is convex;

(c) The pointwise supremum of a family of l.s.c. functions is l.s.c.

Proof. (a) Here we define $\overline{C}$ as the set containing the limit points of all sequences $\{c_n\}_{n=0}^\infty$ formed by elements in C. If $c, d \in \overline{C}$, then there is $(c_n) \to c$ and $(d_n) \to d$, so $(\lambda c_n + (1-\lambda)d_n) \to \lambda c + (1-\lambda)d$. For all $\lambda \in [0,1], \lambda c_n + (1-\lambda)d_n \in C$ by convexity, so $\lambda c + (1-\lambda)d$ is a limit point of a sequence in C and $\lambda c + (1-\lambda)d \in \overline{C}$, which proves the convexity of $\overline{C}$. We define C° as the set of points in C that has an open ball around it contained within C. Take the open balls $B_{\epsilon_1}(x), B_{\epsilon_2}(y) \subset C^\circ$. By convexity, the open set $\lambda B_{\epsilon_1}(x) + (1-\lambda)B_{\epsilon_2}(y)$ is a subset of C, so there exists an open ball around $\lambda x + (1-\lambda)y$ contained in C and the convexity of C° is proven.

Since $C^\circ \subset C$, the inclusion $\overline{C^\circ} \subset \overline{C}$ follows from the definition of closure. To show that $\overline{C} \subset \overline{C^\circ}$, we write $c \in C$ as the limit point of sequence (c_n) and try to find a sequence $\{x_n\}_{n=0}^\infty \subset C^\circ$ such that $(x_n) \to c$. Note that the convex combination of any $c \in C$ and $x \in C^\circ$ is in C° :

$$\begin{aligned} \exists \epsilon > 0, &\quad B_\epsilon(x) \subset C, \\ \forall \lambda \in [0,1], &\quad \lambda c + (1-\lambda)B_\epsilon(x) = B_{(1-\lambda)\epsilon}(\lambda c + (1-\lambda)x) \subset C. \end{aligned}$$

We can now pick any $c_0 \in C$ to construct a sequence $x_n = \lambda_n c_0 + (1-\lambda_n)c_n$, where $(\lambda_n) \to 0$, so that $(x_n) \to \lim_{n\to\infty} c_n = c$ and $\{x_n\}_{n=0}^\infty \subset C^\circ$. Thus c is also a limit point in $\overline{C^\circ}$.

(b) If $f : X \to \mathbb{R}$ is the pointwise supremum of the collection of functions $\{f_i\}$, then its epigraph is the intersection of epi f_i. Since convexity is preserved under infinite intersection, epi f is convex and f is a convex function.

(c) The result is analogous to (b): The pointwise supremum of a family of l.s.c. functions has an epigraph formed by the intersection of their epigraphs, which are closed by Definition 2.2. Since closedness is preserved under infinite intersection, the epigraph of superior envelop is also closed and its corresponding function is l.s.c. □

We now define the functional operation that is critical to our later construction of dual problems. Specifically, it transform a functional in X one in χ^* with the desirable properties of convexity and lower semicontinuity.

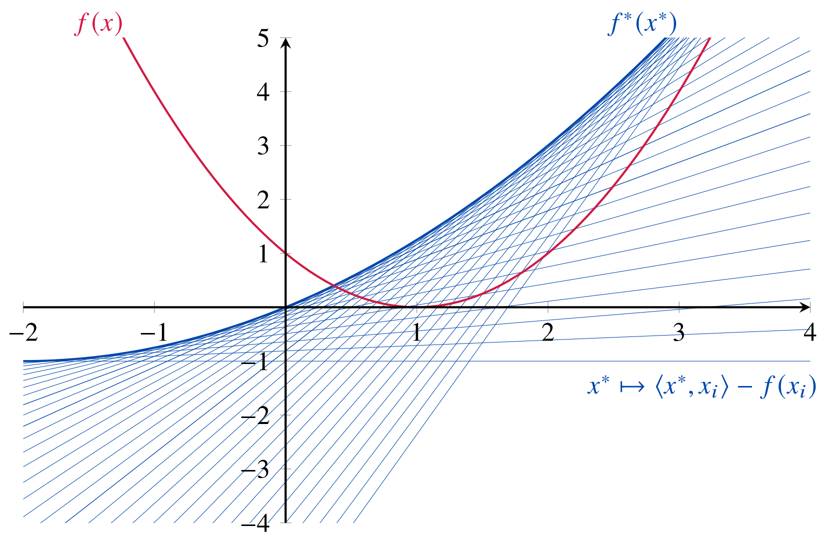


Figure 2. The conjugate function is the superior envelop of the family of affine maps $\{x^* \mapsto \langle x^*, x_i \rangle - f(x_i) | x_i \in X\}$.

Definition 2.4 (Conjugate function). For any $F : V \to \overline{\mathbb{R}}$ that is **proper**, meaning $F \neq +\infty$ and has value $-\infty$ nowhere in its domain, its **conjugate function** $F^* : V^* \to \overline{\mathbb{R}}$ is defined as

$$F^*(x^*) = \sup_{x \in X} \{\langle x^*, x \rangle - F(x)\}.$$

As the pointwise supremum of the family of convex l.s.c., or more precisely, affine maps $x^* \mapsto \langle x, x^* \rangle - F(x)$, F^* is convex and lower semicontinuous on χ^*. In fact, any convex l.s.c. function $f : X \to \overline{\mathbb{R}}$ can be expressed as the pointwise supremum of all affine functions h such that $h \leq f$ in X. The existence of affine minorants of any convex function is later shown in Corollary 3.4, while the construction of convex functions as the superior envelop of affine maps is formalized in the regularized space.

Definition 2.5 (Γ−**regularization).** If the function $F : V \to \overline{\mathbb{R}}$ is the pointwise supremum of a family of continuous affine functions, then $F \in \Gamma(V)$. In particular, $\Gamma_0(V)$ denotes $\Gamma(V)$ excluding the constant functions $+\infty$ and $-\infty$.

From our observation, $\Gamma(V)$ is equivalent to the class of convex l.s.c. functions in V, while $\Gamma_0(V)$ is the further restriction of that set to proper functions. For a geometric proof of this result, see [6, Theorem 20.3.].

The localized set of such affine functions gives another interpretation of differentiability adapted for testing convexity.

Definition 2.6 (subgradient). A function $f : X \to \overline{\mathbb{R}}$ is **subdifferentiable** at point x if there is at least one affine function $l : X \to \overline{\mathbb{R}}$ such that $l \leq f$ in X and $l(x) = f(x)$. The slope $x^* \in X$ of such an affine function is the **subgradient** of F at x, and the collection of subgradients at x form the **subdifferential** of F at x, denoted as $\partial F(x)$.

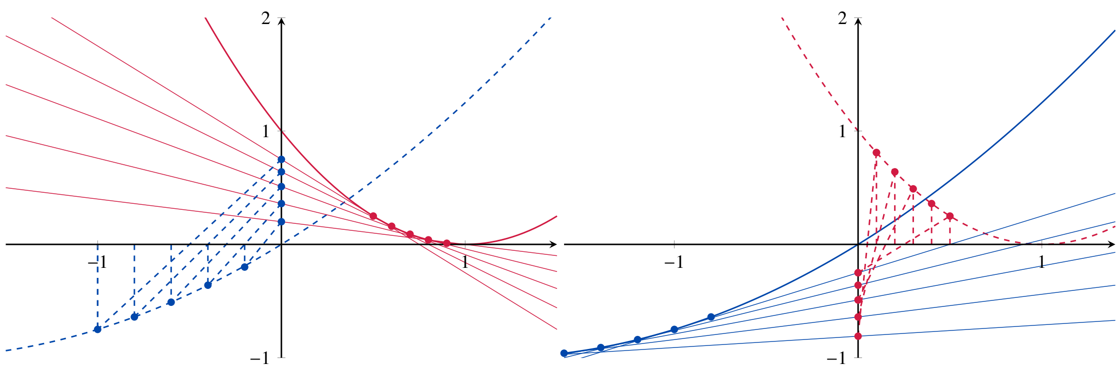
Formally, $x^* \in \partial F(x)$ if and only if $F(x)$ is finite and $\langle x' - x, x^* \rangle + F(x) \leq F(x')$ for every $x' \in X$ and $F(x) = \langle x^*, x \rangle$, or equivalently, $F(x) + F^*(x^*) = \langle x, x^* \rangle$.

Like conjugation, subdifferentiability introduces a dual relation:

Proposition 2.7. For the function $F : V \to \mathbb{R}$. if $u^* \in \partial F(u)$, then $u \in \partial F^*(u^*)$. If, furthermore, $F \in \Gamma(V)$, then the two statements are equivalent.

Proof. See [4, Corollary I.5.2.]. □

In fact, the one-to-one correspondence between the greatest affine minorants of a convex function and those of its conjugate suggests that conjugation on convex functions is invertible. From the subgradients of the conjugate function, it is possible to recover the affine minorants that frame the original function, should they exist. Convexity is a necessary condition for the existence of affine minorants and the invertibility of conjugation, as later proven analytically in Theorem 3.5 and Corollary 3.4.



(a) Construction of the conjugate function.

(b) Restoration of a convex function from its conjugate using subgradient duality.

Figure 3. The dual matching between the subgradients of a convex function and its conjugate induces a bijection between the points on their graphs.

The definition of the subgradient captures convexity conditions. While subdifferentiability constrains the set of differential points to those with a derivative from below, it relaxes the requirement for derivatives if we consider any l.s.c. convex function with a discontinuity, or a continuous convex function such as $|x|$ that is not Fréchet differentiable. Specifically,

Proposition 2.8. If the function $F : V \to \overline{\mathbb{R}}$ is convex, finite and continuous at the point $u \in V$, then $\partial F(v) \neq \varnothing$ for all v in the interior of its **effective domain**, the subset of $\mathcal{D}f$ where $F \neq +\infty$.

Proof. See [4, Proposition I.5.2.]. □

2.2. **Lagrangian duality.** Using the duality of the conjugate and subgradients of a convex function, we will show the equivalence between a convex minimization problem and its dual maximization problem. Other than the possibility of simplifying solution algorithms, the advantage of this approach is that the existence conditions of a maximization problem is now linked to those of its dual minimization problem, characterizing the sufficient existence conditions that the Lagrange Multiplier Theorem alone fails to address.

Definition 2.9 (perturbed problem). Consider the problem $\mathcal{P}$ for $F : X \to \overline{\mathbb{R}}$: $\inf_{x\in X} F(x)$, where X is a Hausdorff topological vector space with dual χ^*. For another Hausdorff topological vector space Y and its dual Y^*, we construct the **perturbation function** $\Phi : X \times Y \to \overline{\mathbb{R}}$ such that $\Phi(x, 0) = F(x)$ and the **perturbed problem** $\mathcal{P}_y$ for given $y \in Y$ as $\inf_{x\in X} \Phi(x, y)$.

In an unconstrained maximization problem, the maximizing arguments in the primal and perturbed problems are identical, regardless of choice of Φ. The following discussion on the Karush-Kuhn-Tucker Theorem shows that the perturbation function is useful (and aptly named) in constrained optimization where the addition of the second argument punishes the first argument for violating any constraint.

Definition 2.10 (dual problem). For **primal problem** $\mathcal{P}$ and its perturbed problems $\{\mathcal{P}_y | y \in Y\}$, the **dual problem**, denoted as $\mathcal{P}^*$, is

$$\sup_{y\in Y^*} \{-\Phi^*(0, y^*)\}, \text{ where } \Phi^*(0, y^*) = \sup_{\substack{x\in X\\ y\in Y^*}} [\langle y^*, y\rangle - \Phi(x, y)].$$

We denote the minimal value in $\mathcal{P}$ as $\inf \mathcal{P}$ and the maximal value in $\mathcal{P}^*$ as $\sup \mathcal{P}^*$.

The inner product pairing for product topology, computed as $\langle (x^*, y^*), (x, y)\rangle = \langle x^*, x\rangle + \langle y^*, y\rangle$, verifies that for $\Phi^*(0, y^*)$ is obtained by the fixing $x^* = 0$ in conjugation of Φ. Fixing $y = 0$ in the sup expression, we have

$$\forall x \in X, -\Phi^*(0, y^*) = -\sup_{\substack{x\in X\\ y\in Y^*}} \{\langle y^*, y\rangle - \Phi(x, y)\} \le -[\langle y^*, 0\rangle - \Phi(x, 0)] = \Phi(x, 0).$$

This implies the weak duality result $\sup \mathcal{P}^* \le \inf \mathcal{P}$, which gives an extremal relation between the solutions of the primal and dual problems, should they exist.

Proposition 2.11 (extremal relation). If solutions exist for both $\mathcal{P}$ and $\mathcal{P}^*$ such that $\inf \mathcal{P}$ and $\sup \mathcal{P}^*$ are equal with a finite value, then all solutions $\bar{x}$ of $\mathcal{P}$ and $\bar{y}^*$ of $\mathcal{P}^*$ satisfy the **extremal relation** $\Phi(\bar{x}, 0) + \Phi^*(0, \bar{y}^*) = 0$.

Conversely, if there are $\bar{x} \in X$ and $\bar{y}^* \in Y^*$ that satisfy the above condition, then they are solutions to $\mathcal{P}$ and $\mathcal{P}^*$, and $\inf_{x\in X} \mathcal{P} = \sup_{y^*\in Y} \mathcal{P}^*$ is finite.

Proof. Suppose that $\inf \mathcal{P} = \Phi(\bar{x}, 0) = \sup \mathcal{P}^* = -\Phi^*(0, \bar{y}^*)$. Upon rearrangement,

$$\Phi(\bar{x}, 0) + \Phi^*(0, \bar{y}^*) = 0.$$

Conversely, the weak duality

$$-\Phi^*(0, y^*) \le \sup_{y^*\in Y^*} \{-\Phi^*(0, y^*)\} \le \inf_{x\in X} \Phi(x, 0) \le \Phi(\bar{x}, 0)$$

argues that attainment of extrema is necessary for equality between $-\Phi^*(0, y^*)$ and $\Phi(\bar{x}, 0)$, which is equivalent to the extremal relation. □

We now investigate the conditions under which the strong duality $\inf \mathcal{P}^* = \sup \mathcal{P}$ with its extremal relation holds, beginning with the existence conditions for the general convex minimization problems.

Theorem 2.12 (existence conditions). Consider the optimization problem $\inf_{u\in C} F(u)$, where C is convex and $F : C \to \overline{\mathbb{R}}$ is convex, l.s.c. and proper. The set of its solutions is a convex set in the reflexive Banach space X. In addition, suppose that one of the following conditions is assumed:

(i) C is bounded;
(ii) F is **coercive** over C, meaning $F(u) \to +\infty$ for $u \in C$ as $\|u\| \to \infty$.

Then the problem has at least one solution, and the set of all solutions is convex. If it is further assumed that F is strictly convex, then the problem has a unique solution.

Proof. For the convexity result, construct a problem with an identical solution set, but extended to X:

$$\inf_{x\in X} \hat{F}(x) \text{ where } \hat{F}(x) = \begin{cases} F(x) & x \in C \\ +\infty & x \notin C \end{cases}.$$

Note that $\hat{F}$ is also convex and l.s.c. Suppose $\inf_{u\in C} F(u) = \alpha \neq +\infty$, which follows from the assumption that F is proper. Now the set of minimizing arguments for F can be rewritten as the epigraph $\{u \in X | \hat{F}(u) \leq \alpha\}$, which is convex by since $\hat{F}$ is convex.

For the existence result, construct a sequence in C denoted $\{u_n | u_n \in C\}_{n=0}^{\infty}$ such that $F(u_n) \to \alpha$. If either condition is met, $\{u_n\}$ is bounded in X. In the case of condition (ii), $\|u\| \to \infty$ will lead to $\|F(u) - \alpha\| \to \infty$ and contradict the construction of $\{u_n\}$. There is then a subsequence $\{u_{nk}\}$ of $\{u_n\}$ that converges weakly to some $\bar{u} \in C$. Since lower semicontnuity is preserved on the weak topology, $F(\bar{u}) \leq \liminf_{u_{nk}\to\infty} F(u_{nk}) = \alpha$, so $\bar{u}$ is the solution to $\inf_{u\in C} F(u)$, which takes on a value greater than $-\infty$.

If there are two different solutions, u_1 and u_2, the convexity of the solution set argues that $\bar{u} = (u_1 + u_2)/2$ is also a solution, which leads to the contradiction that $F(\bar{u}) \leq \frac{1}{2}(F(u_1) + F(u_2)) = \alpha$ if F is strictly convex. □

Theorem 2.13 (dual existence conditions). Let X be a reflexive Banach space and $\Phi \in \Gamma_0(X \times Y)$ while satisfying the following conditions:

(i) Φ is **stable**: $\exists x_0 \in X, \quad y \mapsto \Phi(x_0, y)$ is continuous and finite at $0 \in Y$;
(ii) Φ is coercive: $\Phi(x, 0) \to \infty$ as $\|x\|_X \to \infty$.

Then there is $\mathcal{P}$ and $\mathcal{P}^*$ each have at least one solution.

Proof. From condition (ii), $\Phi(u, 0)$ satisfies the hypotheses in Theorem 2.12, so $\mathcal{P}$ has at least one solution. For the solution to $\mathcal{P}^*$, consider the **value function** $h : Y \to \overline{\mathbb{R}}$, $h(y) = \inf_{x\in X} \Phi(x, y)$. If $\Phi \in \Gamma_0(X \times Y)$, then h is convex:

$$\forall p, q \in Y, \quad h(\lambda p + (1-\lambda)q) = \inf_{x\in X} \Phi(x, \lambda p + (1-\lambda)q),$$

$$\forall u, v \in X, \inf_{x\in X} \Phi(x, \lambda p + (1-\lambda)q) \leq \Phi(\lambda u + (1-\lambda)v, \lambda p + (1-\lambda)q).$$

By the convexity of Φ,

$$h(\lambda p + (1-\lambda)q) \leq \lambda\Phi(u, p) + (1-\lambda)\Phi(v, q) \leq \lambda h(p) + (1-\lambda)h(q).$$

Because Φ is continuous at 0, there is a neighborhood W around 0 in Y such that $\Phi(x_0, y)$ is continuous and bounded above, say by $M < +\infty$. By definition, the minimized value

$h(y) = \inf_{x \in X} \Phi(x, y) \leq \Phi(x_0, y) \leq M$, which implies h is continuous at 0. Take the ball $V = -W \cap W$centered around 0 and there is the ball δW for every $\delta \in (0, 1)$ such that for any $x \in \epsilon W$, one of the following cases follows:

$$\frac{x}{\delta} \in V \implies h(x) \leq (1 - \delta)h(0) + \delta h(x/\delta) \leq h(0) + \delta M, \text{ or}$$
$$-\frac{x}{\delta} \in V \implies h(x) \geq (1 + \delta)h(0) - \delta h(x/\delta) \geq h(0) - \delta M.$$

Since M is constant, there is always small enough δ such that $|h(x) - h(0)| < \epsilon$ and h is continuous as 0. Since h is convex, Proposition 2.7 then implies $\partial h(0) \neq \emptyset$. Furthermore, the subgradient $\partial h^{**}(0) = \partial h(0) \neq \emptyset$ by Theorem 3.5. We will complete the proof by showing the solution set to $\mathcal{P}^*$ is exactly $\partial h^{**}(0)$.

If $\bar{y}^*$ is a solution of $\mathcal{P}^*$, then for every $y^* \in Y^*$, $-\Phi^*(0, \bar{y}^*) \geq -\Phi^*(0, y^*)$, or equivalently, $-h(\bar{y}^*) \geq -h(y^*)$, which is then rewritten as:

$$-h^*(\bar{y}^*) = \sup_{y^* \in Y^*} [\langle 0, y^* \rangle - h^*(y^*)] = h^{**}(0).$$

This is equivalent to $\bar{y}^* \in \partial h^{**}(0)$, since $\bar{y}^*$ is an affine minorant:

$$\begin{aligned} h^{**}(y) - h^{**}(0) &= \sup_{y^* \in Y^*} [\langle y^*, y \rangle - h^*(y^*)] + h^*(\bar{y}^*) \\ &= \sup_{y^* \in Y^*} [\langle y^*, y \rangle - h^*(y^*)] - \sup_{y^* \in Y^*} [\langle 0, y^* \rangle - h^*(y^*)] \\ &\geq \sup_{y^* \in Y^*} [\langle y^*, y \rangle] \geq \langle \bar{y}^*, y \rangle . \end{aligned}$$

□

Now we characterize the solution set using duality. We introduce the Lagrangian, which incorporates the arguments of the primal and dual problems into one equation.

Definition 2.14 (Lagrangian). The **Lagrangian** of problem $\mathcal{P}$ relative to perturbations $\{\Phi : X \times Y \to \overline{\mathbb{R}}\}$ is $L : X \times Y^* \to \overline{\mathbb{R}}$ with

$$\forall x \in X, y^* \in Y, \quad -L(u, y^*) = \sup_{y \in Y} \{\langle y^*, y \rangle - \Phi(x, y)\},$$

or equivalently, $L(x, y) = -\Phi_x^*(y^*)$, where Φ_x is the map $y \mapsto \Phi(x, y)$ for fixed $x \in X$. The **saddle points** of $L : X \times Y^* \to \overline{\mathbb{R}}$ are the set of points $(\bar{x}, \bar{y}^*) \in X \times Y^*$ such that

$$\forall x \in X, \forall y \in Y^*, \quad L(\bar{x}, y^*) \leq L(\bar{x}, \bar{y}^*) \leq L(x, \bar{y}^*).$$

Note that for any $x_0 \in X$, the Lagrangian $L(x_0, y^*)$ is concave and u.s.c. in Y^*, since $-L(x_0, y^*) = \Phi_{x_0}^*(y^*)$ is convex and l.s.c. as a conjugate function. If Φ is convex, then dually, for $y_0^* \in Y^*$, the Lagrangian $L(x, y_0^*)$ is convex in X. Furthermore,

Theorem 2.15. For $\Phi \in \Gamma_0(V \times Y)$, $(\bar{x}, \bar{y}^*)$ is a saddle point of L if and only if $\bar{x}$ solves $\mathcal{P}$, $\bar{y}^*$ solves $\mathcal{P}^*$ and $\inf \mathcal{P} = \sup \mathcal{P}^*$.

Proof. We will express the perturbation functions and then the affiliated problems using the Lagrangian:

$$\begin{aligned}\Phi^*(x^*, y^*) &= \sup_{x\in X}[\langle x^*, x\rangle + \langle y^*, y\rangle - \Phi(x, y)]\\ &= \sup_{x\in X}\{\langle u^*, u\rangle + \sup_{y\in Y}[\langle y^*, y\rangle - \Phi(x, y)]\}\\ &= \sup_{x\in X}[\langle x^*, x\rangle - L(x, y^*)].\end{aligned}$$

In particular, $-\Phi^*(0, y^*) = -\sup_{x\in X}[-L(x, y^*)] = \inf_{x\in X} L(x, y^*)$, so the dual problem $\mathcal{P}^*$ becomes $\sup_{y^*\in Y^*}\inf_{x\in X} L(x, y^*)$ for any perturbation function Φ. If it is further assumed that $\Phi \in \Gamma_0(X \times Y)$, the invertibility of conjugation implies an analogous relationship

$$\Phi(x, y) = \Phi_x^*(y) = \sup_{y^*\in Y^*}[\langle y, y^*\rangle - \Phi_x^*(y^*)] = \sup_{y^*\in Y}[\langle y, y^*\rangle + L(x, y^*)],$$

such that $\Phi(x, 0) = \sup_{y^*\in Y^*} L(x, y^*)$ and the problem $\mathcal{P}$ becomes $\inf_{x\in X}\sup_{y^*\in Y^*} L(x, y^*)$. If $(\bar{x}, \bar{y}^*)$ is a saddle point of L in $X \times Y^*$,

$$L(\bar{x}, \bar{y}^*) = \inf_{x\in X} L(x, y^*) = -\Phi^*(0, \bar{p}^*) = \sup_{y^*\in Y^*} L(\bar{x}, y^*) = \Phi(\bar{x}, 0)$$

satisfies the extremal relation, which is equivalent to $\bar{x}$ and $\bar{y}^*$ being solutions of $\mathcal{P}$ and $\mathcal{P}^*$, respectively, and $\inf \mathcal{P} = \sup \mathcal{P}^*$ by Proposition 2.11. □

Using the Lagrangian, we can show that the existence conditions of solutions to the dual problem and those of the primal problems are equivalent.

Theorem 2.16. Suppose that $\Phi \in \Gamma_0(X \times Y)$ and $\mathcal{P}$ is stable, meaning that it has a value function $h(y) = \inf_{x\in X} \Phi(x, y)$ that is finite and subdifferentiable at 0. Then $\bar{x}$ is a solution of $\mathcal{P}$ and only if there exists $\bar{y}^* \in Y^*$ such that $(\bar{x}, \bar{y}^*)$ is a saddle point of L.

Proof. By Theorem 2.15, if $(\bar{x}, \bar{y}^*)$ is a saddle point, then $\bar{x}$ solves $\mathcal{P}$ and $\bar{y}^*$ solves $\mathcal{P}^*$. To prove the converse direction, we have shown in the proof of Theorem 2.13 that if L is stable, then $\mathcal{P}^*$ has at least one solution $\bar{y}^*$ and $\inf \mathcal{P} = \sup \mathcal{P}^*$. Again, by Theorem 2.15, $(\bar{x}, \bar{y}^*)$ is a saddle point of L.

□

2.3. **The Minimax Theorem.** The perturbation function can be written as the pointwise infimum of the Lagrangian, which in turn contains the supremum expression. The existence of oppositional optimizing arguments relates the computation of the Lagrangian to the solution of zero-sum games, which direct us to the study of minimax theory.

The Minimax Theorem investigates conditions on function $L : X \times Y \to \overline{\mathbb{R}}$ necessary for equality to be attained in the **minimax inequality**

$$\sup_{y\in Y}\inf_{x\in X} L(x, y) \le \inf_{x\in X}\sup_{y\in Y} L(x, y),$$

which follows from the definitions of the supremum and infimum: Since every element in the set $\{\inf_{x\in X} L(x, y) | y \in Y\}$ is less than or equal to any element in the set$\{\sup_{y\in Y} K(x, y) | x \in X\}$, the supremum of the former is less than or equal to the infimum of the latter.

To begin the investigation, we show that the attainment of equality is equivalent to the existence of a saddle point $(\bar{x}, \bar{y})$, which takes the same value as the above expression: By the definition of the saddle point,

$$\inf_{x\in X}\sup_{y\in Y} L(x, y) \le \max_{y\in Y} L(\bar{x}, y) = L(\bar{x}, \bar{y}) = \min_{x\in X} L(x, \bar{y}) \le \sup_{y\in Y}\inf_{x\in X} L(x, y),$$

which when combined with the minimax inequality, is equivalent to

$$\sup_{y\in Y}\inf_{x\in X} L(x,y) = \inf_{x\in X}\sup_{y\in Y} L(x,y).$$

Conversely, if the extrema are equal and attained in each argument, say at $\bar{x}$ and $\bar{y}$, $(\bar{x},\bar{y})$ obeys the definition of a saddle point:

$$\sup_{y\in Y}\inf_{x\in X} L(x,y) = \max_{y\in Y}\inf_{x\in X} L(x,y) = L(\bar{x},\bar{y}) = \min_{x\in X}\sup_{y\in Y} L(x,y) = \inf_{x\in X}\sup_{y\in Y} L(x,y).$$

Then the hypotheses in the Minimax Theorem are equivalent to the existence conditions for saddle points. A set of assumptions that form the basis for the existence conditions for function $K : C\times D \to \overline{\mathbb{R}}$ is that C and D are convex, closed and non-empty subsets of reflexive Banach spaces. At the same time, $v\mapsto L(u,v)$ is concave and u.s.c. for any $u\in C$ and $u\mapsto L(u,v)$ is convex and l.s.c. for any $v\in D$, or L is **concave-convex** and closed for brevity.

Proposition 2.17. If $L : C\times D \to \overline{\mathbb{R}}$ is a concave-convex function, where C and D are convex subsets of reflexive Banach spaces, then the set of saddle points of L, denoted as $C_0\times D_0$, is convex. Furthermore, if strict concavity holds for every function in $\{v\mapsto L(u,v)|u\in C\}$, then D_0 contains at most one element. Similarly, if strict convexity holds for every function in $\{u\mapsto L(u,v)|v\in D\}$, then C_0 contains at most one element.

Proof. Assume that $C_0\times D_0$ is non-empty and $\alpha\in C_0\times D_0$. Then by the definition of the saddle point, which takes value α, $L(u_1,v)\le\alpha$ and $L(u_2,v)\le\alpha$ for any $u_1,u_2\in C_0, \lambda\in(0,1)$ and given $v\in D$. By the convexity of $u\mapsto L(u,v)$, $L(\lambda u_1+(1-\lambda)u_2,v)\le \lambda L(u_1,v)+(1-\lambda)L(u_2,v)\le\alpha$. At the same time, $L(u,v_0)\ge\alpha$ for $v_0\in D_0$, so $L(\lambda u_1+(1-\lambda)u_2,v_0)=\alpha$ is another saddle point. If concavity is strict and there are more than one saddle points u_1, the $L(\lambda u_1+(1-\lambda)u_2,v)<\alpha$ and $\lambda u_1+(1-\lambda)u_2\notin C_0\times D_0$, contradicting this observation. □

Theorem 2.18 (von Neumann). Suppose that $L : C\times D\to\overline{\mathbb{R}}$ is closed and concave-convex and C and D are convex subsets of the reflexive Banach spaces X and Y. If C and D are bounded, then K has at least one saddle point, say $(\bar{u},\bar{v})$, and the extrema are attained:

$$L(\bar{u},\bar{v}) = \min_{u\in C}\max_{v\in D} L(u,v) = \max_{v\in D}\min_{u\in C} L(u,v).$$

Proof. By the Banach-Alaogu Theorem, C and D are compact for the weak topologies of X and Y. The lower semicontinuity of $u\mapsto K(u,v)$ and upper semicontinuity $v\mapsto L(u,v)$ are also preserved in the weak topologies, so the infimum of $u\mapsto L(u,v)$ is attained for any $v\in D$. First consider the case where $u\mapsto L(u,v)$ is strictly convex such that the minimizing argument is unique. Let $e: D\to C$ be the well-defined map from a given v to the minimizing argument of $L(u,v)$ and $f(v)=L(e(v),v)=\min_{u\in C}L(u,v)$. As the lower envelop of concave u.s.c. functions, $f(v)$ is concave and u.s.c. and its maximum is attained, say at $\bar{v}$, so $f(\bar{v})=\max_{v\in D}\min_{u\in C}L(u,v)$.

We now show that $\bar{u}=e(\bar{v})$. By the concavity of $v\mapsto L(u,v)$, $L(u,e_\lambda)\ge\lambda L(u,v)+(1-\lambda)L(u,\bar{v})$. For $0<\lambda<1$, define $e_\lambda=e((1-\lambda)\bar{v}+\lambda v)$. Then by the concavity of f

and $v \mapsto L(u, v)$,

$$\begin{aligned} f(\bar{v}) &\geq f((1-\lambda)\bar{v} + \lambda v) = L(e_\lambda, (1-\lambda)\bar{v} + \lambda v) \\ &\geq (1-\lambda)L(e_\lambda, \bar{v}) + \lambda L(e_\lambda, v) \geq (1-\lambda) f(\bar{v}) + L(e_\lambda, v). \end{aligned}$$

Upon rearrangement, $f(\bar{v}) \geq \frac{1}{\lambda} L(e_\lambda, v) \geq L(e_\lambda, v)$. In particular, Since C is weakly sequentially compact, the sequence e_{λ_n} converges to some point $\bar{u} \in C$ as $\lambda_n \to 0$. By the lower semicontinuity of $u \mapsto L(u, v)$ (first inequality), the definition of e, and the concavity and the upper semicontinuity of $v \mapsto L(u, v)$ (second inequality),

$$\begin{aligned} \forall u \in C, \quad L(\bar{u}, \bar{v}) &\leq \liminf_{\lambda_n \to 0} L(e_{\lambda_n}, \bar{v}) \leq \lim_{\lambda_n \to 0} f((1-\lambda_n)\bar{v} + v) \\ &\leq \limsup_{\lambda_n \to 0} L(u, (1-\lambda_n)\bar{v} + \lambda_n v), \end{aligned}$$

so $L(\bar{u}, \bar{v}) = \lim_{\lambda_n \to 0} L(e(\lambda(1-\lambda_n)\bar{v} + \lambda_n v)$ and $(1-\lambda_n)\bar{v} + \lambda_n v) = L(e\bar{v}, \bar{v})$. Since the minimizing argument in C for $u \mapsto L(u, v)$ is unique, $\bar{u} = e(\bar{v})$. Combine the statement contained in the above inequality

$$\forall u \in C, \quad L(u, \bar{v}) \geq f(\bar{v}) \geq L(\bar{u}, \bar{v}),$$

with the definition of $\bar{v}$:

$$\forall v \in C, \quad f(\bar{v}) \geq f(v) = \min_{u \in C} L(u, v) \geq L(\bar{u}, v),$$

and $(\bar{u}, \bar{v})$ is a saddle point of L in $C \times D$, which is equivalent to:

$$\max_{v \in D} \min_{u \in C} L(u, v) = \sup_{v \in D} \inf_{u \in C} L(u, v) = \inf_{u \in C} \sup v \in DL(u, v) = \min_{u \in C} \max_{v \in D} L(u, v).$$

The attainment of extrema follows from the compactness of C and D, invoked in the construction of $(\bar{u}, \bar{v})$.

In cases where $u \mapsto L(u, v)$ is convex but not strictly so, apply the observation to the perturbed Lagrangian $L_\epsilon : C \times D \to \overline{\mathbb{R}}$, defined as

$$L_\epsilon(u, v) = L(u, v) + \epsilon \|u\|_C,$$

which is strictly convex in C and has a unique saddle point $(\bar{u}_\epsilon, \bar{v}_\epsilon)$. By the weak compactness of C and D, there is some sequence $\epsilon_n \to 0$ and some $(\bar{u}, \bar{v}) \in C \times D$ such that $\bar{u}_{\epsilon_n} \to \bar{u}$ weakly in C and $\bar{v}_{\epsilon_n} \to \bar{v}$ weakly in D. The assumptions on L then imply

$$L(\bar{u}, v) = \liminf_{\epsilon_n \to 0} L(u, v) + \epsilon_n \|u\|_C \leq L(\bar{u}, \bar{v}) \leq \limsup_{\epsilon_n \to 0} L(u, v) + \epsilon_n \|u\|_C = L(u, \bar{v}),$$

so $(\bar{u}, \bar{v})$ is a saddle point of L and equality is attained. The attainment of the extrema within follows from compactness. ◻

Theorem 2.19. Let $L : C \times D$ be a closed concave-convex function, where C and D are closed convex nonempty subsets of reflexive Banach spaces. Suppose that

$$\exists u_0 \in C, \qquad \lim_{\substack{v \in D \\ \|u\| \to \infty}} L(u_0, v) = \infty,$$

$$\exists v_0 \in D, \qquad \lim_{\substack{u \in D \\ \|v\| \to \infty}} L(u, v_0) = -\infty.$$

Then L has at least one saddle point $(\bar{u}, \bar{v})$ such that

$$L(\bar{u}, \bar{v}) = \min_{v \in D} \max_{u \in C} L(u, v) = \max_{u \in C} \min_{v \in D} L(u, v).$$

Proof. For given $\mu > 0$, define $C_\mu = \{u \in C | \|u\| \leq \mu\}$ and $D_\mu = \{v \in D | \|v\| \leq \mu\}$ so that C_μ and D_μ are closed, convex and bounded, satisfying the hypotheses of the preceding theorem. There is then a saddle point $(\bar{u}_\mu, \bar{v}_\mu)$ of $L_\mu : C_\mu \times D_\mu \to \overline{\mathbb{R}}$.

Recall from the proof of Theorem 2.12 that if $u \mapsto L(u, v)$ is coercive, convex and l.s.c., then it is bounded from below by a constant $M > -\infty$. In particular, $-\infty < M \leq L(\bar{u}_\mu, v)$ and similarly, $K(u, \bar{v}) \leq N \leq +\infty$ for some constant N. By the definition of the saddle point, $K(\bar{u}_\mu, v) \leq K(\bar{u}_\mu, \bar{v}_\mu) \leq L(u, \bar{v}_\mu) \leq N$. Boundedness implies that there is a sequence $\mu_j \to \infty$ such that $\bar{u}_{\mu_j} \to \bar{u}$ weakly in C, $\bar{v}_{\mu_j} \to \bar{v}$ weakly in D and $L(\bar{u}_{\mu_j}, \bar{v}_{\mu_j}) = \alpha$. □

2.4. **Karush-Kuhn-Tucker Theorem.** We return to constrained optimization problems to observe how, as in the unconstrained minimization problem, convexity conditions introduce new methods of characterizing the solution set. Specifically, they allow us to practice the transformation to Equation (1.1) without relying on the Inverse Function Theorem for verification and generalize the Lagrange Multiplier Theorem.

The hypotheses in all theorems of this subsection contain the following assumptions. Consider the constrained minimization problem for the convex l.s.c. function $J : C \to \overline{\mathbb{R}}$, is a convex set in the topological vector space X with dual pairing χ^* and $B : C \to Y$ is convex with respect to $\leq$:

$$\forall u, v \in C, \forall \lambda \in (0,1), B(\lambda u + (1-\lambda)v) \leq \lambda B(u) + (1-\lambda)B(v),$$

while the subset of C satisfying $B(u) \leq 0$ is nonempty. Y is similarly a topological vector space with the dual pairing Y^*, and $u \mapsto \langle y^*, B(u)\rangle$ is l.s.c. The partial ordering relation in Y and Y^* are defined such that $\langle p^*, p\rangle \geq 0$ whenever $p^* \geq 0, p \geq 0$ or $p^* \leq 0, p \leq 0$ and $\langle p^*, p\rangle \leq 0$ whenever $p^* \geq 0, p \leq 0$ or $p^* \leq 0, p \geq 0$. It is constructed using the convex cone in [4, Section III.5.], so we denote $\{q \geq 0 | q \in Y\}$ as K and $\{q^* \geq 0 | q^* \in Y^*\}$ as K^*.

Consider the constrained minimization problem $\mathcal{P}$:

$$\inf_{\substack{u \in C \\ Bu \leq 0}} J(u), \text{ or equivalently, } \inf_{\substack{u \in C \\ Bu \leq 0}} F(u), \text{ where } F(u) = \begin{cases} J(u) & u \in C \text{ and } Bu \leq 0 \\ +\infty & \text{otherwise} \end{cases}.$$

We choose the perturbation function to be

$$\Phi(u,p) = \hat{J}(u) + \chi_{E_p}, \text{ where } \hat{J}(u) = \begin{cases} J(u) & u \in C \text{ and } B(u) \leq 0 \\ +\infty & \text{otherwise} \end{cases} \text{ and } E_p = \{u \in C | B(u) \leq p\},$$

$$\chi_{E_p} = \begin{cases} 0 & u \in E_p \\ +\infty & u \notin E_p \end{cases}$$ is called the **indicator function** of E_p.

Proposition 2.20.
(a) The set E_p is closed and convex in Y regardless of the choice for p;
(b) The set $E = \{(u,p) \in X \times Y | u \in C, B(u) \leq p\}$ is closed and convex in $X \times Y$;
(c) $\Phi \in \Gamma_0(X \times Y)$.

Proof. (a) The ordering condition states that for given $u \in C$, $E_p = \{p | \langle p^*, B(u) - p\rangle, \forall p^*\}$. For any fixed p^*, the map $u \mapsto \langle B(u) - p, p^*\rangle$ is convex l.s.c., so its epigraph $\{u \in C | \langle B(u) - p, p^*\rangle\}$ is convex and closed. As the intersection of these sets, $E_p = \cup_{p^* \geq 0}\{u \in C | \langle B(u) - p, p^*\rangle\}$ is convex and closed.

To show (b), apply the same reasoning to the map $(u, p) \mapsto \langle B(u) - p, p^*\rangle$, which is convex and l.s.c. like $u \mapsto \langle B(u) - p, p^*\rangle$.

(c) Note that $E(u,p)$ is represents the same set as $E_p(u)$. Rewrite the perturbation function as $\Phi(u,p) = \hat{J}(u) + \chi_E(u,p)$. $\hat{J}$ is l.s.c. and convex by assumption and so is $\chi_E(u,p)$ as the indicator function of a convex closed set. As their sum, Φ is convex and l.s.c., or equivalently, $\Phi \in \Gamma_0(X \times Y)$. □

Now to construct the dual problem. Writing $p = B(u) + q, q \geq 0$ for $B(u) \leq p$, the conjugate of the perturbation function becomes

$$\Phi^*(0,p^*) = \sup_{\substack{u\in C\\ p\in Y}}\{\langle p^*, p\rangle - \Phi(u,p)\} = \sup_{\substack{u\in C\\ p\in Y\\ Bu\leq p}}\{\langle p^*, p\rangle - J(u)\}$$

$$= \sup_{u\in C}\sup_{\substack{q\in Y\\ q\geq 0}}\{\langle p^*, Bu\rangle + \langle p^*, q\rangle - J(u)\}.$$

Equivalently, because $\langle p^*, q\rangle$ can be inflated to $+\infty$ whenever $p^* \geq 0$,

$$\Phi^*(0,p^*) = \sup_{u\in C}\{\langle p^*, B(u)\rangle - J(u)\} + \chi_{K^*}(-p^*),$$

$$-\Phi^*(0,p^*) = \inf_{u\in C}\{-\langle p^*, B(u)\rangle + J(u)\} - \chi_{K^*}(-p^*).$$

We have the dual problem $\mathcal{P}^*$, defined as

$$\sup_{p^*\leq 0}\inf_{u\in C}\{-\langle p^*, B(u)\rangle + J(u)\} - \chi_{K^*}(-p^*). \tag{2.1}$$

We then have a stability condition is necessary for the finiteness of the solution.

Lemma 2.21. $\mathcal{P}$ is stable if $\inf_{\substack{u\in C\\ Bu\leq 0}} J(u)$ is finite and there is $u \in C$ so that $-B(u) < 0$.

Proof. From the assumption, there is a neighborhood V around 0 in Y such that $-B(u_0)+p \geq 0$, which implies $\Phi(u_0,p) = J(u_0)$, for every $p \in V$ and $h(p)$, so $p \mapsto \Phi(u_0,p)$ is convex, continuous and bounded, say by the constant $M < +\infty$. Then $h(p) = \inf_u \Phi(u,p) \leq \Phi(u_0,p) \leq M$, so h is finite, convex and continuous, which implies it is subdifferentiable in V, in particular at 0, by Proposition 2.8. □

Theorem 2.22. Suppose that $\inf \mathcal{P}$ is finite. Then $\bar{u} \in C$ is a solution to $\mathcal{P}$ if and only if there exists $\bar{p}^* \in Y^*$ such that $(\bar{u}, \bar{p}^*)$ is a saddle point of the Lagrangian. That is,

$$\forall u \in C, \forall p^* \leq 0, \quad J(\bar{u}) - \langle p^*, B(\bar{u})\rangle \leq J(\bar{u}) - \langle \bar{p}^*, B(\bar{u})\rangle \leq J(u) - \langle \bar{p}^*, B(u)\rangle.$$

In this case, $\langle \bar{p}^*, B(\bar{u})\rangle = 0$.

Proof. To compute the Lagrangian in this case,

$$-L(u,p^*) = \sup_{p\in Y}\{\langle p^*, p\rangle - \Phi(u,p)\} = -\hat{J}(u) + \sup_{\substack{p\in Y\\ p\geq B(u)}}\langle p^*, p\rangle$$

$$= -\hat{J}(u) + \langle p^*, B(u)\rangle + \sup_{\substack{q\in Y\\ q\geq 0}}\langle p^*, q\rangle = -\hat{J}(u) + \langle p^*, B(u)\rangle + \chi_{K^*}(-p^*),$$

$$L(u,p^*) = \hat{J}(u) - \langle p^*, B(u)\rangle - \chi_{K^*}(-p^*).$$

The point $(\bar{u}, \bar{p}^*)$ is then a saddle point of the above Lagrangian if

$$\hat{J}(\bar{u}) - \langle p^*, B(\bar{u})\rangle - \chi_{K^*}(-p^*) \leq \hat{J}(\bar{x}) - \langle \bar{p}^*, B(\bar{u})\rangle - \chi_{K^*}(-\bar{p}^*) \leq \hat{J}(\bar{x}) - \langle \bar{p}^*, B(u)\rangle - \chi_{K^*}(-\bar{p}^*).$$

Since is the infimum is finite, it cannot be the case that $\hat{J}(u) = +\infty$ or $\chi_{K^*}(-\bar{p}^*) = +\infty$, so it is necessary and sufficient for $(\bar{u}, \bar{p}^*)$ to satisfy $\bar{u} \in C, \bar{p}^* \le 0$ and

$$\hat{J}(\bar{u}) - \langle p^*, B(\bar{u})\rangle \le \hat{J}(\bar{x}) - \langle \bar{p}^*, B(\bar{u})\rangle \le \hat{J}(\bar{x}) - \langle \bar{p}^*, B(u)\rangle$$

to be a saddle point for L and solutions for $\mathcal{P}$ and $\mathcal{P}^*$.

Setting $p^* = 0$ in the first half of this inequality, we have $\langle \bar{p}^*, B(\bar{u})\rangle \ge 0$, while the assumptions $\bar{p}^* \ge 0$ and $B(u) \le 0$ gives $\langle \bar{p}^*, B(\bar{u})\rangle \le 0$, so $\langle \bar{p}^*, B(\bar{u})\rangle = 0$. The result also follows from the extremal relation:

$$\Phi(\bar{u}, 0) + \Phi^*(0, \bar{p}^*) = J(u) + \langle \bar{p}^*, B(u)\rangle - J(u) = 0.$$

□

We now turn to the special case where $X = \chi^* = \mathbb{R}^n, Y = Y^* = \mathbb{R}^m$ and $C = \{(u_1, \ldots, u_n) | \forall 1 \le i \le n, u_i \ge 0\}$. The function $B : \mathbb{R}^n \to \mathbb{R}^m$ is convex and l.s.c. in every component $B_i : \mathbb{R}^n \to \mathbb{R}$.

Corollary 2.23 (Karush-Kuhn-Tucker). $\bar{u} \in C$ is a solution to $\mathcal{P}$ if and only if there exists $\bar{p} \in \mathbb{R}^m, \bar{p} \le 0$, such that

$$\forall u \in C, \forall p \ge 0, L(\bar{u}, p) \le L(\bar{u}, \bar{p}) \le L(u, \bar{p}).$$

In this case, $\sum_{i=0}^m \bar{p}_i B_i(\bar{u}) = 0$, which implies that for all $1 \le i \le m$, then either $B_i(\bar{u}) < 0$ and $\bar{p}_i = 0$, or $B_i(\bar{u}) = 0$. $\bar{p}^*$ is called the **Kuhn-Tucker vector** corresponding to solution $\bar{u}$ to problem $\mathcal{P}$.

With this result comes a statement of existence conditions analogous to Theorem 2.16:

Corollary 2.24. Assume, for the problem $\mathcal{P} : \inf_{x \in X} F(x)$, that $F : X \times Y \to \overline{\mathbb{R}}$ is a concave-convex function. $\bar{x} \in \mathbb{R}^n$ is a solution to $\mathcal{P}$ if and only if there exists $\bar{x}^* \in \mathbb{R}^m$ such that $(0, 0) \in \partial L(\bar{x}^*, \bar{y})$, where L is the Lagrangian of $\mathcal{P}$. Furthermore, every $\bar{u}^*$ that satisfies this condition is a Kuhn-Tucker vector for $\mathcal{P}$.

Proof. The analytic proof is the application of 2.16. See [6, Theorem 36.6] for a geometric proof using the Minimax Theorem. □

If we impose another inequality condition $-B(u) \le 0$, then the problem $\mathcal{P}$ is subject to the strict equality constraint, $B(u) = 0$. Combining the optimality conditions, we have

(i) $B(\bar{u}) < 0$ and $p_i = 0$, or $B_i(\bar{u}) = 0$,
(ii) $B(\bar{u}) > 0$ and $p_i = 0$, or $B_i(\bar{u}) = 0$,

for every $1 \le i \le m$. This implies that $B_i(\bar{u}) = 0$ is a necessary condition satisfied by the solution, but they also characterize the critical points in **??**, the Lagrangian. The problem $\mathcal{P}$ then becomes equivalent to

$$\max L(u, p) = J(u) - \sum_{i=1}^{n} p_i B_i(u),$$

whose first-order conditions encapsulate the conditions derived using duality:

(i)
$$DL(\bar{u}, \mathbf{p}) = Df(\bar{u}, \mathbf{p}^*) - \sum_{i=1}^{m} p_i DB_i(\bar{u}),$$
such that $J(\bar{u}) - \langle \mathbf{p}^*, B\bar{u}\rangle \le J(u) - \langle \mathbf{p}^*, Bu\rangle$ for every choice of $\mathbf{p}^*$.

(ii)

$$D_i L(u, \bar{\mathbf{p}}^*) = B_i(u) = 0,$$

such that $J(u) - \langle \bar{\mathbf{p}}^*, B(u)\rangle \geq J(u) - \langle \mathbf{p}^*, B(u)\rangle$ and $B(u) = 0$.

The result is identical to the requirements on Equation (1.2), except λ is replaced by $\mathbf{p}$. Hence, Lagrange multipliers are a special case of Kuhn-Tucker vectors.

3. Duality Generalized

3.1. **Conjugation as isomorphism.** We begin by importing two separation theorems from functional analysis:

Theorem 3.1 (Helly-Hahn-Banach, analytic form). Suppose the function $p : X \to \mathbb{R}$ with $p(\lambda x) = \lambda p(x)$ and $p(x + y) \leq p(x) + p(y)$ for every $\lambda > 0$ and $x, y \in X$. Let $g : G \to \mathbb{R}$, where G is a linear subspace of X, be affine with $g(x) \leq p(x)$ for any $x \in X$. Then there exists an affine function $f : X \to \overline{\mathbb{R}}$ that extends g from G to X, satisfying $f = g$ in G and $f \leq p$ in X.

Theorem 3.2 (Hahn-Banach, first geometric form). Let X be a topological vector space with convex subsets A and B which do not intersect. If one of them, say A, is open, then there is a closed hyperplane separating A and B.

Here, we define a hyperplane, or more accurately an affine hyperplane, as $H \subset X$ such that $H = \{f(x) = \alpha\}$ for a given functional $f \in \chi^*$.

Theorem 3.3 (Hahn-Banach, second geometric form). Let X be a topological vector space and A and B its nonempty convex subsets that do not intersect. If A is closed and B compact, then there is a closed hyperplane that strictly separates A and B.

The formulations adapt [3, Theorems 1.1., 1.6. and 1.7.] for convex analysis. Their proofs are also found (with brilliant illustrations) in [5, Theorem 4.17. and Theorem 4.24.]. An important corollary of the separation theorems is that a convex l.s.c. proper function always has an affine minorant:

Corollary 3.4. If $f : X \to \overline{\mathbb{R}}$ is convex, l.s.c. and proper, then $f^* : \chi^* \to \overline{\mathbb{R}}$ cannot be the constant function $+\infty$. In particular, it is bounded below by an affine continuous function.

This result comments on the feasibility to find affine functionals below a convex function and to construct it as their superior envelop, as accomplished in the preceding section. Using this fact, we now show that conjugation is in fact invertible when applied to convex functions. This observation follows intuitively as any $f \in \Gamma(X)$ is constructed as the pointwise supremum of all its affine minorants, which is $\sup_{x^* \in \chi^*} \langle x^*, x\rangle - f^*(x^*)$, so we can restore the original function by conjugating f^*, as formalized below.

Theorem 3.5 (Fenchel-Moreau). If $f : X \to \overline{\mathbb{R}}$ is convex, l.s.c. and proper, then $f^{**} = f$, where $f^{**} : X \to \overline{\mathbb{R}}$ is defined by

$$f^{**}(x) = \sup_{x^* \in \chi^*} \{\langle x^*, x\rangle - f^*(x^*)\}.$$

Proof. We first take the case where $f \geq 0$. By the definition of the conjugate function, $\langle x^*, x\rangle - f^*(x^*) \leq f(x)$ for any $x \in X$ and $x^* \in \chi^*$, so as the pointwise supremum of the left-hand side in χ^*, f^{**} is similarly less than or equal to f on all of X. To show that $f^{**} = x$, assume, for the sake of contradiction, that $f^{**}(x_0) < x_0$ for some $x_0 \in X$. Apply the Hahn-Banach theorem, second geometric form, to the epigraph of $f^{**}(x)$ and the point $(x_0, f^{**}(x_0))$ and there exists a closed hyperplane, in particular, a linear functional strictly separating them. That is, there exists $\bar{x}^* \in \chi^*, k \in \mathbb{R}, \alpha \in \mathbb{R}$ such that

$$\forall (x, y) \in \operatorname{epi} f, \quad \langle \bar{x}^*, x\rangle + ky > \alpha, \tag{3.1}$$

$$\langle \bar{x}^*, x_0\rangle + kf^{**}(x_0) < \alpha. \tag{3.2}$$

Note that $k \geq 0$ since y can be chosen to be arbitrarily large for fixed $x \in \mathcal{D}f$.

We want to construct a contradiction to. From (3.1) and the assumption that $f \geq 0$, we have for $\epsilon > 0$,

$$\forall x \in \mathcal{D}f, \quad \langle \bar{x}^*, x\rangle + (k+\epsilon)f(x) \geq \alpha,$$

$$\left\langle -\frac{\bar{x}^*}{k+\epsilon}, x\right\rangle - f(x) \leq -\frac{\alpha}{k+\epsilon}.$$

As a result,

$$f^*\left(-\frac{\bar{x}^*}{k+\epsilon}\right) = \sup_{x\in X}\left[\left\langle -\frac{\bar{x}^*}{k+\epsilon}, x\right\rangle - f(x)\right] \leq -\frac{\alpha}{k+\epsilon},$$

$$f^{**}(x_0) \geq \left\langle -\frac{\bar{x}^*}{k+\epsilon}, x_0\right\rangle - f^*\left(-\frac{\bar{x}^*}{k+\epsilon}\right) \geq \left\langle -\frac{\bar{x}^*}{k+\epsilon}, x_0\right\rangle + \frac{\alpha}{k+\epsilon}.$$

Upon rearrangement, $\langle \bar{x}^*, x_0\rangle + (k+\epsilon)f^{**}(x_0) \geq \alpha$, contradicting (3.2).

For the general case, fix some $x_0^* \in \mathcal{D}f^*$ and define $\bar{f}(x) = f(x) - \left\langle x_0^*, x\right\rangle + f^*(x_0^*)$, which is convex, proper and l.s.c. as the sum of such functions. Furthermore, $\bar{f} \geq 0$ by the definition of the conjugate function and falls into the previous case such that $(\bar{f})^{**} = \bar{f}$. This then implies $f^{**} = f$, since

$$\begin{aligned}
(\bar{f})^*(x^*) &= \sup_{x\in X}[\langle x^*, x\rangle - f(x) + \left\langle x_0^*, x\right\rangle - f^*(x_0^*)] \\
&= \sup_{x\in X}[\langle x^*, x\rangle - f(x) + \left\langle x_0^*, x\right\rangle] - f^*(x_0^*) = f^*(x^* + x_0^*) - f^*(x_0^*), \\
(\bar{f})^{**}(x) &= \sup_{x^*\in X}[\langle x^*, x\rangle - (\bar{f})^*(x^*)] = \sup_{x^*\in X}[\left\langle x^* + x_0^*, x\right\rangle - (\bar{f})^*(x^* + x_0^*)] + f^*(x_0^*) - \left\langle x_0^*, x\right\rangle \\
&= f^{**}(x) + f^*(x_0^*) - \left\langle x_0^*, x\right\rangle = \bar{f}(x) = f(x) + f^*(x_0^*) - \left\langle x_0^*, x\right\rangle.
\end{aligned}$$

□

The invertibility of conjugation is a special property of convex functions that induces the one-to-one correspondence between the primal and dual problems. Recall the value function of the primal problem $\inf_{x\in X} F(x)$ with perturbation function $\Phi : X \times Y \to \overline{\mathbb{R}}$, defined as $h(y) = \inf_{x\in X}\Phi(x, y) = \inf_{x\in X} F(x)$. The Fenchel-Moreau Theorem states that

$$h^{**}(0) = \sup_{y^*\in Y} -h^*(y^*) = \sup_{y^*\in Y} -\Phi^*(0, y^*),$$

which is consistent with our previous construction of the dual problem.

3.2. **Fenchel Duality Theorem.**

Theorem 3.6 (Fenchel-Rockafellar). Suppose $f, g : X \to \overline{\mathbb{R}}$ are convex functions and $x_0 \in \mathcal{D}f \cap \mathcal{D}gf$. Then

$$\inf_{x\in X}\{f(x)+g(x)\} = \sup_{x^*\in\mathcal{X}^*}\{-f^*(-x^*)-g^*(x^*)\} \tag{3.3}$$
$$= \max_{x^*\in\mathcal{X}^*}\{-f^*(-x^*)-g^*(x^*)\} = -\min_{x^\in\mathcal{X}^*}\{f^*(-x^*)+g^*(x^*)\}$$

Proof. Define $a = \inf_{x\in X}[f(x)+g(x)]$ and $b = \sup_{x^*\in\mathcal{X}^*}[-f^*(-x^*)-g^*(x^*)]$, then $b \le a$, since for every $x^* \in \chi^*$ and $x \in X$,

$$-f^*(x^*) - g^*(x^*) \le -[\langle -x^*, x\rangle - f(x)] - [\langle x^*, x\rangle - g(x)] = f(x) + g(x).$$

Assume $a > -\infty$, otherwise $b = a$ directly follows. Apply the Hahn-Banach Theorem, first geometric form, to $A = \text{epi}\, f^\circ$, which is nonempty because f is continuous at x_0, and $B = \{y \le a - g(x) | [x, y] \in X \times \mathbb{R}\}$. A and B are convex from the convexity of g. Furthermore, $A \cap B = \emptyset$, since for every $(x, y) \in A, y > f(x) \ge a - g(x)$. Then there exists a closed hyperplane that strictly separates$A$ and B. In particular, since $\overline{A} = \text{epi}\, f$ by Corollary 3.4, there is $\bar{x}^* \in \chi^*, k \in \mathbb{R}, \alpha \in \mathbb{R}$ such that the affine hyperplane $\Phi(x, y) = \langle x^*, x\rangle + ky = \alpha$ separates epi f and B. That is,

$$\forall (x, y) \in \text{epi}\, f, \langle \bar{x}^*, x\rangle + ky \ge \alpha, \tag{3.4}$$
$$\forall (x, y) \in B, \langle \bar{x}^*, x\rangle + ky \le \alpha. \tag{3.5}$$

As in the preceding proof, $k \ge 0$ because otherwise ky would not be bounded from below if we choose large enough y for fixed $x_0 \in \mathcal{D}f \cap \mathcal{D}g$. We now show that in fact $k > 0$. Assume, for the sake of contradiction, that $k = 0$, so $\|\bar{x}^*\|_{\chi^*} \ne 0$ and (3.4), (3.5) become

$$\forall x \in \mathcal{D}f, \langle \bar{x}^*, x\rangle \ge \alpha,$$
$$\forall x \in \mathcal{D}g, \langle \bar{x}^*, x\rangle \le \alpha.$$

But for small enough $\epsilon_0 > 0, B(x_0)_{\epsilon_0} \subset \mathcal{D}f$, so $\langle f, x_0 + z\epsilon_0\rangle \ge \alpha$ for any $0 \le z \le 1$. By the definition of functional norm, this implies $\langle f, x_0 + \epsilon_0 z\rangle \ge \alpha + \epsilon_0\|f\|$. At the same time, $\langle f, x_0\rangle \le \alpha$, so $\|f\| = 0$, a contradiction.

From (3.4),

$$f^*\left(-\frac{x^*}{k}\right) = \sup_{x\in X}\left\langle -\frac{x^*}{k}, x\right\rangle - f(x) \le \left\langle -\frac{x^*}{k}, x\right\rangle - \left(\frac{\alpha}{k} - \left\langle \frac{x^*}{k}, x\right\rangle\right) = -\frac{\alpha}{k},$$

and from (3.5),

$$\forall (x, y) \in B, \langle x^*, x\rangle + ka - kg(x) \le \langle x^*, x\rangle + ky \le \alpha,$$
$$g^*\left(\frac{x^*}{k}\right) = \sup_{x\in X}\left\langle \frac{x^*}{k}, x\right\rangle - g(x) \le \frac{\alpha}{k} - a.$$

From this result and from our choice of b, we have

$$-f^*\left(-\frac{x^*}{k}\right) - g^*\left(\frac{x^*}{k}\right) \ge a,$$
$$-f^*\left(-\frac{x^*}{k}\right) - g^*\left(\frac{x^*}{k}\right) \le b.$$

Since $b \leq a$, $a = b = -f^*\left(-\frac{x^*}{k}\right) - g^*\left(\frac{x^*}{k}\right)$, so the relation (3.3) and the supremum in it is attained. □

Another common formulation of Fenchel Duality, useful in maximizing the difference between two functionals, is

$$\inf_{x \in X}\{f(x) - g(x)\} = \sup_{x^* \in X}\{g_*(-x^*) + f^*(-x^*)\},$$

where we define the **concave conjugate** of g as $g_*(x^*) = \inf_{x \in X}\{\langle x^*, x\rangle - g(x)\}$.

3.3. **Special cases.** We now show that results from the previous discussions on the Minimax Theorem and the Karush-Kuhn-Tucker Theorem are special cases of Fenchel-Moreau and Fenchel-Rockafellar duality.

To begin, we introduced a generalized Fenchel duality theorem particular to the reals from [6, Corollary 31.2.1.].

Theorem 3.7. Let f be a closed proper convex function on $\mathbb{R}^n$ and g a closed proper concave function on $\mathbb{R}^m$. Let A be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$. We have

$$\inf_{x}\{f(x) - g(Ax)\} = \sup_{y^*}\{g^*(y^*) - f^*(A^T y^*)\},$$

if either of the following conditions is satisfied:

(a) There exists $x \in \mathrm{ri}\mathcal{D}f$ such that $Ax \in \mathrm{ri}\mathcal{D}g$;
(b) There exists $y^* \in \mathrm{ri}\mathcal{D}g^*$ such that $A^T y^* \in \mathrm{ri}\mathcal{D}f$.

The **relative interior** of the convex set $C \in \mathbb{R}^n$, denoted as ri$C$, consists of all $z \in \mathbb{R}^n$ such that

$$\forall x \in C, \exists \mu > 1, \quad (1-\mu)x + \mu z \in C.$$

Under (a), the supremum on the right-hand side is attained at some y^*, while under (b), the infimum on the left-hand side is attained at some x.

Apply this observation to the Fenchel problem $\mathcal{P}$:

$$\inf_{x \in \mathbb{R}^n}\{\chi_C(x) + \chi_D^*(Ax)\},$$

where C and $D\mathbb{R}^m$ are non-empty convex, and χ_D^* is the conjugate of the indicator function of set D, so $\chi_D^* : \mathbb{R}^m \to \overline{\mathbb{R}}, y^* \mapsto \sup_{y \in D}\langle y^*, y\rangle$. Then,

$$\inf_{x \in \mathbb{R}^N} \mathcal{P} = \inf_{x \in C}\sup_{y \in D}\langle Ax, y\rangle.$$

Furthermore, since D is convex, and so is its indicator function. By Theorem 3.5, $(\chi_D)^{**} = \chi_D$. By Theorem 3.7,

$$\begin{aligned}
\inf_{x \in \mathbb{R}^n} \mathcal{P} &= \inf_{x \in \mathbb{R}^n}\{\chi_C(x) - (-\chi_D^*(Ax))\} \\
&= \sup_{y \in \mathbb{R}^m}\{-\chi_D(y) - \sup_{x \in C}\{-\left\langle A^T y, x\right\rangle\}\} \\
&= \sup_{y \in D}\inf_{x \in C}\langle Ax, y\rangle.
\end{aligned}$$

Hence, Fenchel duality implies the von Neumann Minimax Theorem for linear operators.

Lagrangian duality is a special case of Fenchel-Rockafellar daulity. Consider, again, the constrained problem that induces the Karush-Kuhn-Tucker Theorem. The objective is to choose $\bar{x} \in \mathbb{R}^n$ to minimize $J : \mathbb{R}^n \to \overline{\mathbb{R}}$ subject to $B(\bar{x}) \leq 0$ for $B : \mathbb{R}^n \to \mathbb{R}^m$. Recall that the primal problem is equivalent to

$$\inf_{x\in\mathbb{R}^n} J(x) + \chi_K(-Bx).$$

Apply the Fenchel Duality Theorem with $f(x) = J(x)$ and $g(x) = \chi_K(-Bx)$:

$$\begin{aligned}\inf_{x\in\mathbb{R}^n}[J(x) + \chi_K(-Bx)] &= \sup_{y^*\in Y^*}\left\{-\sup_{x\in X}[\langle y^*, Bx\rangle - J(x)] + -\chi_{K^*}(-y^*)\right\}\\ &= \sup_{y^*\in Y^*}\left\{\inf_{x\in X}[-\langle y^*, Bx\rangle + J(x)] + -\chi_{K^*}(-y^*)\right\}\\ &= \sup_{y^*\leq 0}\inf_{x\in X}\{-\langle y^*, B(x)\rangle + J(x)\},\end{aligned}$$

which is identical to the dual problem described in (2.1).

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