N-RADIAL PARTITION FUNCTION OF LOOP-ERASED RANDOM WALK

ZONGJIN (TRACY) WU

Abstract. We consider n-radial loop-erased random walk (LERW) on the square grid, each running from a boundary point to an interior point of the square. We compute the probability that the n-radial LERW meet at a single point in the scaling limit as lattice size goes to 0. For $n = 3$, we prove that the scaling exponent of this asymptotic probability coincides with that of $3-$crossings of disjoint uniform spanning trees in an annulus.

Contents

1. Introduction

The loop-erased random walk (LERW) is a random walk with loops erased according to their chronological order. It is a natural object that produces a self-avoiding path. The LERW path is known to have a scaling limit as a curve up to reparameterization to the chordal SLE_2 path as the lattice size goes to zero [LSW04]. In this paper, we are interested in the $n-$radial LERW, namely n loop-erased random walks running from boundary points to interior points of a square grid. We are interested in the asymptotic

Date: September, 2024.

behavior of these LERWs. Specifically, we want to compute the probability that the $n-$radial LERWs meet at a single point in the scaling limit as lattice mesh size goes to 0. This leads us to consider n non-intersecting LERWs running from the outer boundary to the inner boundary of an annulus. By taking the radius of the inner boundary to 0, we can approximate this probability. In Section 1.5, we introduce Fomin identity, a combinatorial fact that relates the non-intersecting LERWs with the regular LERWs, which allows us to compute the probability using the LERW Green's function. By comparing the discrete Green's function of LERW with the continuous Green's function, we obtain the exponent of the scaling limit of $n-$radial LERWs meeting at a single point.

1.1. Notation.

Let $\mathbb{Z}^2 = \mathbb{Z} + i\mathbb{Z}$ denote the 2D discrete integer lattice. For $z \in \mathbb{Z}^2$, let $\mathcal{S}_z$ denote the square of size 1 centered at z. In particular, let $\mathcal{S} = \mathcal{S}_0$ be the unit square. Suppose $A \subset \mathbb{Z}^2$ is a connected subset of the integer lattice. Let D_A be the interior of $\bigcup_{z\in A} \mathcal{S}_z$. Then $\partial A = \{z \in \mathbb{Z}^2 : \mathrm{dist}(z, A) = 1\}$ is the boundary of A. A path on $\mathbb{Z}^2$ is denoted by $\omega = [\omega_0, \cdots, \omega_k] \subset \mathbb{Z}^2$, and $|\omega| = k$ is the length of ω.

The weight of a path of a simple random walk is given by

$$p(\omega) = 4^{-|\omega|}.$$

Note that a trivial path $[\omega_0]$ has length 0 and is assigned weight 1.

1.2. Green's Function and Poisson kernel.

Let $z, w \in A$. We write $\mathcal{P}^A_{z,w}$ as the collection of paths from z to w in A. The random walk *Green's function* counts the sum of the weights of all such paths in A:

$$G_A(z, w) = \sum_{\omega\in P^A_{z,w}} p(\omega).$$

The random walk Poisson kernel is the analogous quantity, where one of the points is on the boundary of A. If $z \in A, w \in \partial A$, we define the Poisson kernel as follows:

$$H_A(z, w) = \sum_{\omega\in \mathcal{P}^A_{z,w}} p(\omega).$$

If both points are on the boundary, that is, if $z, w \in \partial A$, we call the following the boundary Poisson kernel:

$$H_{\partial A}(z, w) = \sum_{\omega\in \mathcal{P}^A_{z,w}} p(\omega).$$

1.3. Loop-erased Random Walk.

A rooted loop is a path that starts and ends at the same point. The loop measure $\tilde{m}$ assigns the following measure to loop l :

$$\tilde{m}(l) = |l|^{-1}p(l).$$

An unrooted loop is an equivalence class of rooted loops with the following equivalence relation that identifies loops up to cyclic shifts:

$$[\omega_0, \omega_1, \cdots, \omega_k = \omega_0] \sim [\omega_1, \cdots, \omega_k = \omega_0, \omega_1] \sim \cdots$$

The unrooted loop measure m associated to weight p is defined as the sum of measures of all the unrooted loops in this equivalence class. Let $[\ell]$ denote the unrooted loop equivalence class and $\#[\ell]$ denote the number of representatives in this equivalence class. Since any rooted loop $l \in [\ell]$ has the same weight and length, let us denote them by $p(\ell) = p([\ell]) = p(l)$.

Denote $|\ell|$ similarly. Then the unrooted loop measure m associated to weight p is defined as follows:

$$m(\ell) = m^p(\ell) = \sum_{l \in \ell} \tilde{m}(l) = \frac{\#[\ell]}{|\ell|} p(\ell).$$

If $V \subset A \subset \mathbb{Z}^2$, we denote

$$F_V(A) = \exp\left\{ \sum_{\ell \subset A, \ell \cap V \neq \emptyset} m(\ell) \right\} = \exp\left\{ \sum_{l \subset A, l \cap V \neq \emptyset} \tilde{m}(\ell) \right\}.$$

A *self-avoiding walk (SAW)* is a path $\eta = [\eta_0, \ldots, \eta_k]$ such that $\eta_0, \ldots, \eta_k$ are all distinct. The *(chronological) loop erasure* is an operation on paths ω that removes nontrivial loops that are encountered when tracing paths in chronological order and produces a SAW denoted by $\mathrm{LE}(\omega)$. Let $\omega = [\omega_0, \cdots, \omega_\tau]$ be a simple random walk with stopping time $\tau < \infty$. The *(chronological) loop-erasure* $\mathrm{LE}(\omega)$ is defined as follows:

- If ω is self-avoiding, i.e. $\omega_i \neq \omega_j$ for any $i \neq j$, then set $\mathrm{LE}(\omega) = \omega$.
- Otherwise, let $s_0 = \max\{t \leq \tau : \omega_t = \omega_0\}$. Let $\mathrm{LE}(\omega)_0 = \omega_{s_0}$.
- For $i \geq 0$, if $s_i < \tau$, let $s_{i+1} = \max\{t : \omega_t = \omega_{s_i+1}\}$. Set $\mathrm{LE}(\omega)_{i+1} = \omega_{s_{i+1}}$.
- If $i = \min\{i : \mathrm{LE}(\omega)_i = \omega_\tau\}$ then $\mathrm{LE}(\omega) = [\mathrm{LE}(\omega)_0, \cdots, \mathrm{LE}(\omega)_i]$.

A loop-erased random walk (LERW) weight $\hat{p}$ of a self-avoiding path η is defined as the sum of weights of paths whose loop-erasure is η :

$$\hat{p}(\eta; A) = \sum_{w \subset A, \mathrm{LE}(\omega) = \eta} p(\omega).$$

Proposition 1.1. *Let $\eta = [\eta_0, \cdots, \eta_k]$ be a self-avoiding path. For $1 \leq j \leq k$, denote $A_j = A \setminus \{\eta_0, \cdots, \eta_{j-1}\}$ and let $A_0 = A$. Then*

$$\hat{p}(\eta; A) = p(\eta) \prod_{j=0}^{k} G_{A_j}(\eta_j, \eta_j) = p(\eta) F_\eta(A) = p(\eta) \det[G_A(\eta_i, \eta_j)]_{0 \leq i,j \leq k}.$$

Proof. According to the definition of loop-erasure, the loops that intersect η are erased chronologically. For any loop l such that $l \cap \eta \neq \emptyset$, let $i = \min\{0 \leq i \leq k : \eta_i \in l\}$ be the first index point where l and η meet. Then l is erased at step i. Namely, by setting $s_{i+1} = \max\{t : \omega_t = \omega_{s_i}\}$ from the loop-erasing procedure, l is erased from $\mathrm{LE}(\omega)_{i+1}$. Therefore, to avoid over-counting, each loop is counted only once as a (rooted) loop with root i (the first index point). For any j, $G_{A_j}(\eta_j, \eta_j)$ counts the number of loops that start and end at η_j and do not intersect $\{\eta_0, \cdots, \eta_{j-1}\}$. Hence $\prod_{j=0}^{k} G_{A_j}(\eta_j, \eta_j)$ counts the total number of paths in A whose loop-erasure is η, which gives the first equality.

Recall that $F_\eta(A) = \exp\{\sum_{l \subset A, l \cap \eta \neq \emptyset} \tilde{m}(l)\}$. By the loop-erasure procedure, we can decompose the sum in the previous exponent by the smallest index i such that l intersects η_i. For any i, we count the loops that intersect η_i and do not intersect $\{\eta_0, \cdots, \eta_{i-1}\}$. Hence, $F_\eta(A)$ can be written as the following product:

$$F_\eta(A) = \prod_{j=0}^{k} F_{\eta_j}(A_j). \tag{1.2}$$

To prove the second equality, let us consider $G_A(\eta_0, \eta_0)$ first. Let $\mathcal{L}_1$ denote the paths from η_0 to η_0 that do not return to η_0 more than once. Let $H_{\partial A_1}(\eta_0, \eta_0)$ denote the boundary

Poisson kernel of $A_1 = A \setminus \{\eta_0\}$. Then

$$H_{\partial A_1}(\eta_0, \eta_0) = \sum_{l \in \mathcal{L}_1} p(l) = p(\mathcal{L}_1).$$

Let $\mathcal{L}_2$ denote the paths from η_0 to η_0 that consist of two loops from $\mathcal{L}_1$. Namely, suppose $l_1, l_2 \in \mathcal{L}_2$, then $l_1 \oplus l_2 \in \mathcal{L}_2$. Note that $p(\mathcal{L}_2) = p(\mathcal{L}_1)^2$ due to the concatenation process. Define $\mathcal{L}_k$ analogously. Suppose $|p(\mathcal{L}_1)| < 1$, then

$$G_A(\eta_0, \eta_0) = 1 + \sum_{k=1}^{\infty} p(\mathcal{L}_1)^k = \frac{1}{1 - p(\mathcal{L}_1)}, \tag{1.3}$$

where the extra term 1 comes from the measure of trivial loops. On the other hand, let us consider the unrooted loop measure on $\mathcal{L}_1$:

$$m(\mathcal{L}_1) = \sum_{l \in \mathcal{L}_1} \frac{p(l)}{|l|} = \sum_{[\ell] \in \mathcal{L}_1} m([\ell]) = \sum_{[\ell] \in \mathcal{L}_1} p(\ell) = p(\mathcal{L}_1).$$

Since loops in $\mathcal{L}_1$ do not return to η_0 until the last step, there are exactly $|l|$ representatives in the same equivalence class $[\ell]$ that each $l \in \mathcal{L}_1$ belongs to. Hence $m([\ell]) = \frac{\#[\ell]}{|\ell|} p(\ell) = p(\ell)$. Suppose $l_1, l_2 \in \mathcal{L}_1$. Then $l_1 \oplus l_2$ and $l_2 \oplus l_1$ belong to the same unrooted loop equivalence class. Then the unrooted measure of $\mathcal{L}_2$ is obtained from $p(\mathcal{L}_1)^2$, with half of the representatives:

$$m(\mathcal{L}_2) = \sum_{[\ell_1], [\ell_2] \in \mathcal{L}_1} m([\ell_1 \oplus \ell_2]) = \frac{p(\mathcal{L}_1)^2}{2}.$$

For any $k \in \mathbb{N}$, let $l_i \in \mathcal{L}_1$ for $1 \le i \le k$. Let $\sigma \in S_k$ be a counterclockwise cyclic permutation (cyclic shifts of the identity). Then the following concatenations are in the same unrooted loop equivalence class:

$$l_1 \oplus \cdots \oplus l_k \sim l_{\sigma(1)} \oplus \cdots \oplus l_{\sigma(k)}.$$

Hence $m(\mathcal{L}_k) = \frac{p(\mathcal{L}_1)^k}{k}$. Let $\mathcal{L} = \bigcup_{k=1}^{\infty} \mathcal{L}_k$, we have

$$m(\mathcal{L}) = \sum_{k=1}^{\infty} \frac{p(\mathcal{L}_1)^k}{k} = \log\left(\frac{1}{1 - p(\mathcal{L}_1)}\right). \tag{1.4}$$

Combining (1.3) and (1.4), we have

$$G_A(\eta_0, \eta_0) = e^{m(\mathcal{L})} = F_{\eta_0}(A).$$

Together with (1.2), we have the second equality.

The last equality is proved using induction. See Proposition 2.9 of [Law22] for a proof using Cramer's rule. □

Proposition 1.5. *Let $A \subset \mathbb{Z}^2$ be a bounded domain. Let $\mathbf{z} = (z_1, \cdots, z_n) \subset A$ and $\mathbf{w} = (w_1, \cdots, w_n) \subset \partial A$. Denote $A \setminus \mathbf{z} := D_A \setminus \{z_1, \cdots, z_n\}$. Let $H_{\partial(A\setminus \mathbf{z})}(\mathbf{z}, \mathbf{w}) := [H_{\partial(A\setminus \mathbf{z})}(z_i, w_j)]_{i,j=1}^n$ denote the $n \times n$ matrix of boundary Poisson kernels from points in $\mathbf{z}$ and $\mathbf{w}$. Define $G_A(\mathbf{z}, \mathbf{z})$ as the $n \times n$ matrix of Green's function among points of $\mathbf{z}$ in A. Then*

$$\det H_{\partial(A\setminus \mathbf{z})}(\mathbf{z}, \mathbf{w}) = \frac{\det H_A(\mathbf{z}, \mathbf{w})}{\det G_A(\mathbf{z}, \mathbf{z})}.$$

Note that we can also write the statement as

$$\det H_{\partial(A\setminus \mathbf{z})}(\mathbf{z}, \mathbf{w}) F_{\mathbf{z}}(A) = \det H_A(\mathbf{z}, \mathbf{w}).$$

Proof. For a given i, j,

$$H_{\partial(A\setminus\mathbf{z})}(z_i, w_j) = \sum_{\substack{\omega=[\omega_0=z_i,\omega_1,\cdots,\omega_{k+1}=w_j]\\ \omega_j\cap\mathbf{z}=\emptyset, 1\le j\le k}} p(\omega)$$

is the sum of paths from z_i to w_j that otherwise do not intersect $\mathbf{z}$. Note that the Poisson kernel from z_k to w_j can be decomposed into paths from z_k to z_i for some $1 \le i \le n$ and paths from z_i to w_j that do not intersect $\mathbf{z}$:

$$H_A(z_m, w_j) = \sum_{i=1}^{n} G_A(z_k, z_i) H_{\partial(A\setminus\mathbf{z})}(z_i, w_j),$$

which is the row k, column j entry of the matrix product $G_A(\mathbf{z},\mathbf{z})H_{\partial(A\setminus\mathbf{z})}(\mathbf{z},\mathbf{w})$. The result follows from taking the determinant. □

1.4. **Multiple LERW.** A *boundary excursion* in A is a path $\omega = [\omega_0, \cdots, \omega_n]$ such that $\omega_0, \omega_n \in \partial A$ and $\omega_i \in A$ for $1 \le i < n$. Let $p(x, y)$ denote the transition probability of a simple random walk from x to y. Let $p(\omega) := \prod_{i=1}^{n} p(\omega_{i-1}, \omega_i)$ denote the excursion measure of ω. Let $\mathcal{E}_A(x, y)$ denote the set of excursions in A that start at x and end at y. Let $\mathcal{E}_A = \bigcup_{x,y\in\partial A} \mathcal{E}_A(x, y)$ denote the union of all excursions in A. Let $\hat{\mathcal{E}}_A(x, y)$ denote the subset of $\mathcal{E}_A(x, y)$ that are self-avoiding. The *loop-erased excursion measure* of a self-avoiding path η is given by

$$\hat{p}(\eta) := \sum_{\omega\in\mathcal{E}_A, \mathrm{LE}(\omega)=\eta} p(\omega).$$

Let $A \subset \mathbb{Z}^2$. The boundary Poisson kernel $H_{\partial A} : \partial A \times \partial A \to [0, \infty)$ is defined as the total mass of excursion measure between two boundary points:

$$H_{\partial A}(x, y) = \sum_{\omega\in\mathcal{E}_A(x,y)} p(\omega).$$

If $x \ne y$, we can also write

$$H_{\partial A}(x, y) = \sum_{\eta\in\hat{\mathcal{E}}_A(x,y)} \hat{p}(\eta).$$

Let $n \in \mathbb{N}$. Let $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ be distinct points on ∂A. We define the excursion measure of n pairs of points $(x_1, y_1), \ldots, (x_n, y_n)$ as the product measure

$$p(\omega^1, \ldots, \omega^n) = p(\omega^1) \times \cdots \times p(\omega^n)$$

on the product space

$$\mathcal{E}_A(\mathbf{x}, \mathbf{y}) := \mathcal{E}_A(x_1, y_1) \times \cdots \times \mathcal{E}_A(x_n, y_n).$$

On the analogous space $\hat{\mathcal{E}}_A(x_1, y_1) \times \cdots \times \hat{\mathcal{E}}_A(x_n, y_n)$ with paths that are individually self-avoiding, there may be paths that still intersect each other. Let us define $\hat{\mathcal{E}}_A(\mathbf{x}, \mathbf{y})$ to be the restriction of this product space to the case that all paths are mutually non-intersecting: i.e., if $\eta^i \in \hat{\mathcal{E}}_A(x_i, y_i)$ is a self-avoiding path for each i, then $\boldsymbol{\eta} = (\eta^1, \cdots, \eta^n) \in \hat{\mathcal{E}}_A(\mathbf{x}, \mathbf{y})$ if and only if $\eta_i \cap \eta_j = \emptyset$ for $i \ne j$. The *non-intersecting self-avoiding excursion measure* at $(\mathbf{x}, \mathbf{y})$ is the product measure $\hat{p} \times \cdots \times \hat{p}$ restricted to the mutually non-intersecting space $\hat{\mathcal{E}}_A(\mathbf{x}, \mathbf{y})$. That is,

$$\hat{p}(\boldsymbol{\eta}) = \hat{p}(\eta^1) \times \cdots \times \hat{p}(\eta^n).$$

if $\boldsymbol{\eta} = (\eta^1, \cdots, \eta^n) \in \hat{\mathcal{E}}_A(\mathbf{x}, \mathbf{y})$ and zero otherwise. Let $\hat{H}_{\partial A}$ denote the the total mass of non-intersecting loop-erased excursion measure:

$$\hat{H}_{\partial A}(\mathbf{x}, \mathbf{y}) := \sum_{\boldsymbol{\eta} \in \hat{\mathcal{E}}_A(\mathbf{x}, \mathbf{y})} \hat{p}(\boldsymbol{\eta}).$$

1.5. **Fomin's Identity.** A key lemma that allows us to compute the Poisson kernel of the LERW using that of a simple random walk is Fomin's identity. For $n = 2$, there exists a bijection between the set of paths from b to 1 that intersect the LERW from a to 0 and paths from b to 0 that intersect the LERW from a to 1, leading to the following identity. Let us denote the condition that a path ω starts at x and ends at y as $\omega : x \to y$.

Lemma 1.6. *Let $A \subset \mathbb{Z}^2$. Let $a, b \in \partial_e A$ be two points on the boundary of A. Let $K = A \setminus [0, 1]$. Let ω^1, ω^2 be simple random walks in K that start and end on the boundary of K. Then*

$$\sum_{\omega^1 : a \to 0} \sum_{\substack{\omega^2 : b \to 1 \\ \mathrm{LE}[\omega^1] \cap \omega^2 = \emptyset}} p(\omega^1) p(\omega^2) - \sum_{\omega^1 : a \to 1} \sum_{\substack{\omega^2 : b \to 0 \\ \mathrm{LE}[\omega^1] \cap \omega^2 = \emptyset}} p(\omega^1) p(\omega^2)$$
$$= \sum_{\omega^1 : a \to 0} \sum_{\omega^2 : b \to 1} p(\omega^1) p(\omega^2) - \sum_{\omega^1 : a \to 1} \sum_{\omega^2 : b \to 0} p(\omega^1) p(\omega^2).$$

For a general n, Fomin's identity states that the determinant of the corresponding quantities are the same regardless of the non-intersecting criterion of the walks.

Lemma 1.7. *Let $A \subset \mathbb{Z}^2$. Let $\mathbf{x} = (x_1, \cdots, x_n), \mathbf{y} = (y_1, \cdots, y_n)$ be two $n-$tuples of disjoint points of A. Then*

$$\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \hat{H}_{\partial A}(\mathbf{x}, \sigma(\mathbf{y})) = \det[H_{\partial A}(x_i, y_j)]_{i,j=1}^n.$$

See [LL10] for additional references on Fomin's identity. In particular, if D_A is simply connected, by planarity, there exists only one element of the permutation group S_n such that the n paths have the non-intersection property. Similarly, for a discrete annulus A (i.e., $D_A \subset \mathbb{C}$ is an annulus), let $\partial^i A$ (resp. $\partial^o A$) be the part of ∂A contained in the bounded (resp. unbounded) component of $\mathbb{Z}^2 \setminus A$. Let $\mathbf{z} = (z_1, \cdots, z_n)$ be an n-tuple of distinct points in $\partial^i A$ and $\mathbf{w} = (w_1, \cdots, w_n)$ an n-tuple of distinct points in $\partial^o A$. Then the only non-zero terms are those that have a n-cyclic permutation. Suppose n is odd, then the signs of the n-cyclic permutations are all positive, and we have the following:

$$\sum_{\sigma \text{ is } n-\text{cyclic}} \hat{H}_{\partial A}(\mathbf{z}, \sigma(\mathbf{w})) = \det[H_{\partial A}(z_i, w_j)]_{i,j=1}^n.$$

We would like to approximate the determinant of the discrete boundary Poisson kernel using the continuous boundary Poisson kernel. By Donsker's Theorem, under appropriate normalization, the Poisson kernel of the simple random walk converges to the continuous Poisson kernel. To extend the convergence to the determinant, we need a further bound on the error terms.

1.6. **Complex Weights.** Note that when n is even, the sign of an $n-$cycle is negative, and Fomin's identity becomes

$$\sum_{\sigma \text{ is } n\text{-cyclic}} \mathrm{sgn}(\sigma) \hat{H}_{\partial A}(\mathbf{x}, \sigma(\mathbf{y})) = \det[H_{\partial A}(x_i, y_j)]_{i,j=1}^k \neq \sum_{\sigma \text{ is } n-\text{cyclic}} \hat{H}_{\partial A}(\mathbf{z}, \sigma(\mathbf{w})).$$

To account for this sign misalignment, we can consider a complex weight on paths. Let $z, w \in \mathbb{Z}^2$ be two adjacent points in the integer lattice. Let $\arg(z), \arg(w)$ be the argument of the two points with certain branch cut such that $\arg(w) - \arg(z) \in [-\frac{\pi}{4}, \frac{\pi}{4}]$. Denote the directed edge between them as $e = [z, w]$. Then the q weight on e is defined as

$$q(e) = 4^{-1} \exp\left(\frac{\arg(e)}{2} i\right),$$

where $\arg(e) = \arg(w) - \arg(z) \in [-\frac{\pi}{4}, \frac{\pi}{4}]$ is the continuous change of argument. Note that if e^R is the reverse path $[w, z]$, then $q(e^R) = \overline{q(e)}$.

The q measure on paths is defined as follows. For $\omega = [\omega_0, \cdots, \omega_k]$, define $e_j = [\omega_i, \omega_{i-1}]$ as the edge between two adjacent points in path ω. We have

$$q(\omega) = 4^{-|w|} \exp\left(\frac{i}{2} \sum_{i=1}^{k} \arg(e_j)\right).$$

The LERW Green's function and Poisson kernel are defined analogously as in section 1.2. For $n = 2$, the complex q weights account for the even and odd winding numbers by assigning them with positive and negative weights, respectively. See [BLV16] for an example of negative weights. If $\omega^R = [\omega_k, \omega_0]$ is the reverse path, then again we have $q(\omega^R) = \overline{q(\omega)}$. If l is a rooted loop, and $N(l)$ is the winding number, then

$$q(l) = (-1)^{N(l)} p(l).$$

Let $\tilde{m}^q = |l|^{-1} q(l)$ denote the rooted loop measure using q weights, and $m^q(\ell) = \sum_{l \in \ell} \tilde{m}^q(l)$ denote the unrooted loop measure. Then

$$m^q(\ell) = (-1)^{N(\ell)} m^p(\ell).$$

Note that Fomin's identity is independent of which weights we use on paths, hence

$$\sum_{\sigma \ n\text{-cycle}} \operatorname{sgn}(\sigma) \hat{H}^q_{\partial A}(\mathbf{x}, \sigma(\mathbf{y})) = \det[H^q_{\partial A}(x_i, y_j)]_{i,j=1}^n.$$

We use Fomin's identity to compute the Poisson kernel of n-radial LERW from n boundary points to n interior points in the unit square. In this paper, we compute explicitly the scaling exponent of lattice size $\frac{1}{N}$ in the Poisson kernel of 3-radial LERW in the unit square; see Proposition 3.17. The exponent is given by the formula $\frac{n^2-1}{4}$, which yields 2 when $n = 3$, times $\frac{1}{N^3}$, which comes from the scaling of the interior points. The exponent $\frac{n^2-1}{4}$ coincides with scaling exponent of $n-$crossings of disjoint uniform spanning tree branches from one boundary of the annulus to the other [Ken00].

2. Continuous Poisson kernel

In this section, we will compute the scaling exponent of the determinant of the continuous boundary Poisson kernel. The following *Poisson summation formula* will be useful for our calculation.

Lemma 2.1 (Poisson summation formula)**.** *For a Schwartz function $f : \mathbb{R} \to \mathbb{C}$, let*

$$\hat{f}(m) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i m x} dx.$$

Then

$$\sum_{k \in \mathbb{Z}} f(k) = \sum_{m \in \mathbb{Z}} \hat{f}(m).$$

To obtain the n-radial LERW boundary Poisson kernel of the unit square, we can consider end points on an annulus first. Let $A \subset \mathbb{C}$ be an annulus with outer radius 1 and inner radius r. Let $\mathbf{z} \subset \partial^o A$ and $\mathbf{w} \subset \partial^i A$ be n-tuples of points on the outer and inner bounaries of A, respectively. By sending r to the appropriate scale, the continuous Poisson kernel of the annulus can be used to approximate the discrete Poisson kernel of the refined lattice of size $\frac{1}{N}$ as $N \to \infty$.

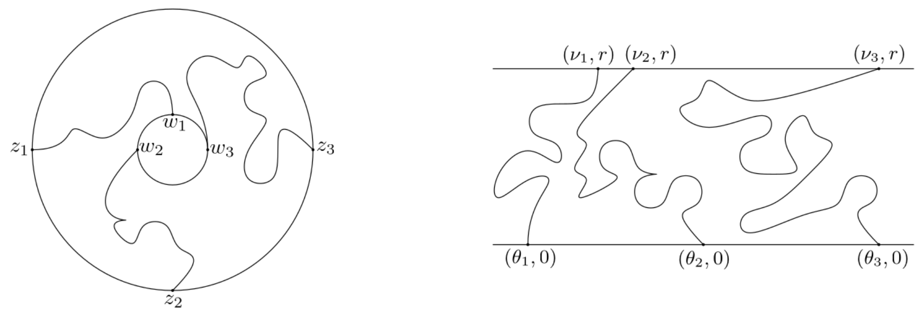

FIGURE 1. 3-radial LERWs on an annulus and a cylinder

2.1. **Cylinder Poisson kernel.** Since an appropriate conformal map sends the unit disk to the upper half plane, we can compute the continuous Poisson kernel on a cylinder (infinite strip) first. The boundary Poisson kernel of the following infinite strip can be computed explicitly as follows:

Proposition 2.2. *Let $C_r = \{x + iy \in \mathbb{H} : 0 < y < r\}$ denote the infinite strip of width r in the upper half plane. Then the continuous boundary Poisson kernel is the following:*

$$h_{\partial C_r}(0, x + ir) = \frac{\pi^2}{4r^2}[\cosh(\frac{\pi}{2r}x)]^{-2}.$$

Proof. Recall that the standard Poisson kernel of the upper half plane $\mathbb{H}$ is given by

$$h_{\mathbb{H}}(x, y) = \frac{1}{|x - y|^2},$$

for $x, y \in \mathbb{R}$. Note that the map $f(z) = e^{\frac{\pi}{r}z}$ maps the infinite strip C_r to $\mathbb{H}$. By the conformal invariance of the two-dimensional Brownian motion, the boundary Poisson kernel of C_r is conformally covariant [KL05]:

$$\begin{aligned} h_{\partial C_r}(0, x + ir) &= |f'(0)||f'(x + ir)|h_{\mathbb{H}}(f(0), f(x + ir)) \\ &= \frac{\pi^2}{r^2} e^{\frac{\pi}{r}x} h_{\mathbb{H}}(1, -e^{\frac{\pi}{r}x}) \\ &= \frac{\pi^2}{r^2} \frac{e^{\frac{\pi}{r}x}}{(1 + e^{\frac{\pi}{r}x})^2} \\ &= \frac{\pi^2}{r^2} \frac{1}{(e^{\frac{\pi}{2r}x} + e^{-\frac{\pi}{2r}x})^2} \\ &= \frac{\pi^2}{4r^2}[\cosh(\frac{\pi}{2r}x)]^{-2}. \end{aligned}$$

□

Consider n loop-erased random walk paths from the bottom to the top boundary of C_r. Denote the initial points of these paths by $\{(\theta_j,0)\}_{j=1}^n$ and the terminal points by $\{(\nu_k,r)\}_{k=1}^n$, where $0\le\theta_1<\cdots<\theta_n<2\pi$ and $0\le\nu_1<\cdots<\nu_n<2\pi$. If n is odd, define

$$h_c(j,k):=h_{\partial C_r}((\theta_j,0),(\nu_k,r))=\sum_{m\in\mathbb{Z}}\frac{\pi^2}{4r^2}[\cosh(\frac{\pi}{2r}(2\pi m+\nu_k-\theta_j))]^{-2}.$$

If n is even, define

$$\begin{aligned}h_c(j,k):=&h^q_{\partial C_r}((\theta_j,0),(\nu_k,r))\\=&\sum_{m\in\mathbb{Z}}\frac{\pi^2}{4r^2}[\cosh(\frac{\pi}{2r}(2\pi m+\nu_k-\theta_j))]^{-2}\exp\left(\frac{2\pi m+\nu_k-\theta_j}{2}i\right),\end{aligned}$$

where the weight of paths is given by measure q, which accounts for odd and even winding numbers.

Motivated by a similar result by [AO19], we obtain the following lemma using the Poisson summation formula:

Lemma 2.3. *If n is odd, then*

$$h_c(j,k)=\frac{1}{2r}+\sum_{m\in\mathbb{Z}\setminus\{0\}}\frac{m}{2\sinh(mr)}e^{im(\nu_k-\theta_j)}. \tag{2.4}$$

If n is even, then

$$h^q_c(j,k)=\sum_{m\in\mathbb{Z}+\frac12}\frac{m}{2\sinh(mr)}e^{i(m+\frac12)(\nu_k-\theta_j)}. \tag{2.5}$$

Proof. Let

$$f(x)=\frac{\pi^2}{4r^2}[\cosh(\frac{\pi}{2r}(2\pi x+\nu_k-\theta_j))]^{-2}\exp\left(\xi(2\pi x+\nu_k-\theta_j)i\right),$$

where $\xi=0$ when n is odd and $\xi=\frac12$ when n is even. By the Poisson summation formula, we have

$$\begin{aligned}h_c(j,k)&=\sum_{m\in\mathbb{Z}}\int_{-\infty}^{\infty}f(x)e^{-2\pi mxi}dx\\&=\sum_{m\in\mathbb{Z}}\int_{-\infty}^{\infty}\frac{\pi^2}{4r^2}[\cosh(\frac{\pi}{2r}(2\pi x+\nu_k-\theta_j))]^{-2}\exp\left((2\pi x+\nu_k-\theta_j)\xi i-2\pi mxi\right)\end{aligned}$$

Substitute $u=2\pi x+\nu_k-\theta_j$, we have

$$\begin{aligned}h_c(j,k)&=\sum_{m\in\mathbb{Z}}\frac{\pi}{8r}e^{(\nu_k-\theta_j)mi}\int_{-\infty}^{\infty}[\cosh(\frac{\pi}{2r}u)]^{-2}\exp\left(u(\xi-m)i\right)du\\&=\sum_{m\in\mathbb{Z}}\frac{m-\xi}{2\sinh(r(m-\xi))}e^{im(\nu_k-\theta_j)}.\end{aligned}$$

The only case that $2\sinh(r(m-\xi))=0$ is when $m=\xi=0$, which only occurs when n is odd. Since $\lim_{x\to0}\frac{x}{2\sinh(rx)}=\frac{1}{2r}$, we can replace $\frac{m-\xi}{2\sinh(r(m-\xi))}$ with $\frac{1}{2r}$ for $m=0$. Then

$$h_c(j,k)=\begin{cases}\frac{1}{2r}+\sum_{m\in\mathbb{Z}\setminus\{0\}}\frac{m}{2\sinh(rm)}e^{im(\nu_k-\theta_j)} & n\text{ is odd,}\\ \sum_{m\in\mathbb{Z}+\frac12}\frac{m}{2\sinh(rm)}e^{i(m+\frac12)(\nu_k-\theta_j)} & n\text{ is even.}\end{cases}$$

□

2.2. **Punctured disk Poisson kernel.** Alternatively, we can compute the continuous Poisson kernel on a punctured disk from a boundary point to an interior point:

$$h_{\mathbb{D}}(e^{-s+i\theta}, e^{i\nu}) = \sum_{k=-\infty}^{\infty} \frac{s}{s^2 + (2\pi k + (\nu - \theta))^2}. \tag{2.6}$$

Consider n pairs of points on $\partial\mathbb{D}$ and the annulus of radius e^{-s}. Denote the angle term of the n boundary points by $\{\nu_k\}_{k=1}^n$, where $0 \leq \nu_k < 2\pi$. Denote the angle term of n interior points by $\{\theta_j\}_{j=1}^n$. Define

$$h_{\mathbb{D}}(j,k) := h^q_{\partial\mathbb{D}}(e^{-s+i\theta_j}, e^{i\nu_k}) = \sum_{m\in\mathbb{Z}} \frac{s}{s^2 + (2\pi m + (\nu_k - \theta_j))^2} \exp[i\xi(2\pi m + (\nu_k - \theta_j))],$$

where $\xi = 0$ for n is odd and $\xi = \frac{1}{2}$ for n is even. For odd n, the above Poisson kernel is defined using the regular p weight, while for even n it is defined using the q weight of the loop-erased paths. For a fixed pair j, k, let

$$f_{j,k}(x) := \frac{s}{s^2 + (2\pi x + (\nu_k - \theta_j))^2} \exp[i\xi(2\pi x + (\nu_k - \theta_j))].$$

Then

$$\hat{f}_{j,k}(m) = \int_{-\infty}^{\infty} \frac{s}{s^2 + (2\pi x + (\nu_k - \theta_j))^2} \exp[i\xi(2\pi x + (\nu_k - \theta_j))] \exp(-2\pi i m x) dx.$$

Let $u = 2\pi x + (\nu_k - \theta_j)$, we have

$$\begin{aligned} \hat{f}_j(m) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{s}{s^2 + u^2} \exp(i\xi u + ((\nu_k - \theta_j) - u) i m) du \\ &= \frac{1}{2} e^{im(\nu_k - \theta_j)} \int_{-\infty}^{\infty} \frac{s}{\pi(s^2 + u^2)} \exp(i(\xi - m)u) du. \end{aligned}$$

Note that $\frac{s}{\pi(s^2+u^2)}$ is the probability density function of the Cauchy distribution. We then have

$$\mathbf{E}[e^{iX(\xi - m)}] = \int_{-\infty}^{\infty} \frac{s}{\pi(s^2 + u^2)} \exp(i(\xi - m)u) du = e^{-s|\xi - m|},$$

where X is the random variable with Cauchy distribution. Hence,

$$\hat{f}_j(m) = \frac{1}{2} e^{im(\nu_k - \theta_j)} \mathbf{E}[e^{iX(\xi - m)}] = \frac{1}{2} e^{im(\nu_k - \theta_j)} e^{-s|\xi - m|}.$$

By the Poisson summation formula, we have

$$h_{\mathbb{D}}(j,i) = \sum_{m\in\mathbb{Z}} \hat{f}_j(m) = \begin{cases} \frac{1}{2} \sum_{m\in\mathbb{Z}} e^{im(\nu_k - \theta_j)} e^{-s|m|} & n \text{ is odd,} \\ \frac{1}{2} \sum_{m\in\mathbb{Z}+\frac{1}{2}} e^{i(m+\frac{1}{2})(\nu_k - \theta_j)} e^{-s|m|} & n \text{ is even.} \end{cases}$$

Substitute $s = -\log r$,

$$h_{\mathbb{D}}(j,k) = \begin{cases} \frac{1}{2} \sum_{m\in\mathbb{Z}} e^{im(\nu_k - \theta_j)} r^{|m|} & n \text{ is odd,} \\ \frac{1}{2} \sum_{m\in\mathbb{Z}+\frac{1}{2}} e^{i(m+\frac{1}{2})(\nu_k - \theta_j)} r^{|m|} & n \text{ is even.} \end{cases}$$

2.3. **Determinant of the continuous Poisson kernel in an annulus.** In this section, we compute the determinant of the continuous Poisson kernel of an annulus as a function of the inner radius. By setting the inner radius to $\frac{1}{N}$, we obtain the scaling exponent for the continuous case, which we later use for an approximation to the discrete Poisson kernel.

We need the following lemma for our preliminary calculation of the determinant. This result is standard in the discussion of unitary ensembles in the random matrix theory. See [Meh04] for more on this topic.

Lemma 2.7. *Let $\{m_j\}_{j=1}^n$ take the values $(-\frac{n-1}{2}, \cdots, \frac{n-1}{2})$ in ascending order. That is, let $m_j = j - \frac{n+1}{2}$. Let $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_n) \in [0, 2\pi)^n$. Then,*

$$\Phi(\boldsymbol{\theta}) := \det(e^{im_j\theta_k})_{j,k=1}^n = i^{\frac{n(n-1)}{2}} \prod_{1\le j<k\le n} |e^{i\theta_k} - e^{i\theta_j}| = (2i)^{\frac{n(n-1)}{2}} \prod_{1\le j<k\le n} \sin\left(\frac{\theta_k - \theta_j}{2}\right).$$

Proof. Suppose $\theta_1 < \cdots < \theta_n$. Note that if $\theta_j < \theta_k$, a geometric property (fig. 2) yields the following:

$$i \cdot e^{\frac{i}{2}(\theta_j+\theta_k)} = \frac{e^{i\theta_k} - e^{i\theta_j}}{|e^{i\theta_j} - e^{i\theta_k}|}.$$

Then we can write

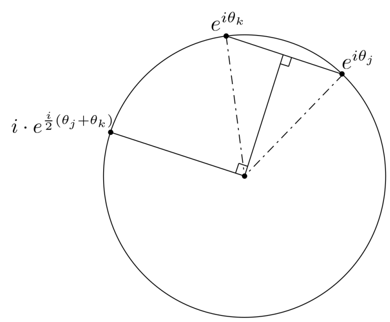

FIGURE 2. A geometric property of the unit circle.

$$\prod_{1\le j<k\le n} |e^{i\theta_k} - e^{i\theta_j}| = i^{\frac{-n(n-1)}{2}} e^{-\frac{i(n-1)}{2}\sum_{j=1}^n \theta_j} \prod_{1\le j<k\le n} (e^{i\theta_k} - e^{i\theta_j}).$$

The Vandermonde determinant is defined as the following:

$$\Delta(e^{i\theta_1}, \cdots, e^{i\theta_n}) := \det[e^{i(j-1)\theta_k}]_{j,k=1}^n = \prod_{1\le j<k\le n} (e^{i\theta_k} - e^{i\theta_j}).$$

Then we have

$$\begin{aligned}\prod_{1\le j<k\le n} |e^{i\theta_k} - e^{i\theta_j}| &= i^{\frac{-n(n-1)}{2}} e^{-\frac{i(n-1)}{2}\sum_{j=1}^n \theta_j} \Delta(e^{i\theta_1}, \cdots, e^{i\theta_n}) \\ &= i^{\frac{-n(n-1)}{2}} \det[e^{im_j\theta_k}]_{j,k=1}^n.\end{aligned}$$

Rearranging the terms, we have

$$i^{\frac{n(n-1)}{2}} \prod_{1\le j<k\le n} |e^{i\theta_k} - e^{i\theta_j}| = \det[e^{im_j\theta_k}]_{j,k=1}^n = \Phi(\boldsymbol{\theta}).$$

Note that

$$|e^{i\theta_k} - e^{i\theta_j}| = 2|\sin\left(\frac{\theta_k - \theta_j}{2}\right)|.$$

Since $\theta_k - \theta_j \in (0, \pi)$, we can remove the absolute value on the right hand side. Hence

$$\Phi(\boldsymbol{\theta}) = (2i)^{\frac{n(n-1)}{2}} \prod_{1\le j<k\le n} \sin\left(\frac{\theta_k - \theta_j}{2}\right)$$

as claimed. □

Similarly, denote $-\boldsymbol{\theta} = (-\theta_1, \cdots, -\theta_n) \in (-2\pi, 0]^n$. Then we have

$$\Phi(-\boldsymbol{\theta}) = (2i)^{\frac{n(n-1)}{2}} \prod_{1\le j<k\le n} \sin\left(\frac{\theta_j - \theta_k}{2}\right) = (-2i)^{\frac{n(n-1)}{2}} \prod_{1\le j<k\le n} \sin\left(\frac{\theta_k - \theta_j}{2}\right). \tag{2.8}$$

2.3.1. *Odd n.* Now we compute the determinant of the continuous Poisson kernel of n paths for odd n :

$$\begin{aligned} \det(h_{\mathbb{D}}(j,k))_{j,k=1}^n &= \frac{1}{2}\sum_{\sigma\in S(n)} \operatorname{sgn}(\sigma) \prod_{j=1}^n \sum_{m_j\in\mathbb{Z}} r^{|m_j|} e^{im_j(\nu_{\sigma(j)}-\theta_j)} \\ &= \frac{1}{2}\sum_{m_1,\cdots,m_n\in\mathbb{Z}} r^{\sum_{j=1}^n |m_j|} \sum_{\sigma\in S(n)} \operatorname{sgn}(\sigma) e^{i\sum_{j=1}^n m_j(\nu_{\sigma(j)}-\theta_j)}. \end{aligned} \tag{2.9}$$

If $m_j = m_k$ for $j \neq k$, then each term from the above is canceled by its counterpart with a transposition of j and k. Hence we can rewrite the sum in the following form:

$$\begin{aligned} &\det(h_{\mathbb{D}}(j,k))_{j,k=1}^n \\ &= \frac{1}{2}\sum_{m_1<\cdots<m_n\in\mathbb{Z}} r^{\sum_{j=1}^n |m_j|} \sum_{\sigma,\tau\in S(n)} \frac{1}{n!}(\operatorname{sgn}(\sigma\circ\tau) e^{i\sum_{j=1}^n m_j\nu_{\sigma\circ\tau(j)}})(\operatorname{sgn}(\tau)e^{-i\sum_{j=1}^n m_j\theta_{\tau(j)}}) \\ &= \frac{1}{2n!}\sum_{m_1<\cdots<m_n\in\mathbb{Z}} r^{\sum_{j=1}^n |m_j|} \sum_{\tau\in S(n)} (\operatorname{sgn}(\tau)e^{-i\sum_{j=1}^n m_j\theta_{\tau(j)}})(\sum_{\sigma\in S(n)} \operatorname{sgn}(\sigma\circ\tau) e^{i\sum_{j=1}^n m_j\nu_{\sigma\circ\tau(j)}}) \\ &= \frac{1}{2n!}\sum_{m_1<\cdots<m_n\in\mathbb{Z}} r^{\sum_{j=1}^n |m_j|} \det(e^{im_j\nu_k})_{j,k=1}^n \det(e^{-im_j\theta_k})_{j,k=1}^n. \end{aligned}$$

Since $r < 1$, the sum $\sum_{m_1<\cdots<m_n\in\mathbb{Z}} r^{\sum_{j=1}^n |m_j|}$ is maximized by the following values:

$$(m_1, \cdots, m_n) = \left(-\frac{n-1}{2}, \cdots, 0, \cdots, \frac{n-1}{2}\right). \tag{2.10}$$

In fact, we will show that this set of values for m_j is the dominating term. With the above values, we have

$$\sum_{j=1}^n |m_j| = \frac{n^2-1}{4}.$$

Note that the set of values of m_j can be rearranged into a set of m'_j, where $m'_1 = \inf_{1\le j\le n} |m_j|$, and $m'_l = \inf_{|m_j|\ge m'_{l-1}} |m_j|$ is the l^{th} smallest absolute value. By symmetry, we have $|m'_2| \ge$

$|m_1'|, |m_3'| \geq 1 + |m_1'|$. In general, $|m_j'| \geq \lceil \frac{j-2}{2} \rceil + |m_1'|$. Then

$$\begin{aligned}
&\sum_{m_1 < \cdots < m_n \in \mathbb{Z}} r^{\sum_{j=1}^n |m_j|} \\
&= \sum_{m_1' \in \mathbb{Z}} \sum_{|m_2'| \geq 1 + |m_1'|} \cdots \sum_{|m_n'| \geq \frac{n-1}{2} + |m_1'|} r^{n|m_1'| + \sum_{j=2}^n (|m_j'| - |m_1'|)} \\
&= r^{\frac{n^2-1}{4}} [(1 + \sum_{k \in \mathbb{N}} r^k) + r^n (\sum_{k \in \mathbb{N}} r^k + \cdots)] \\
&= r^{\frac{n^2-1}{4}} (1 + O(r)),
\end{aligned}$$

as $r \to 0$. Since the set of values in (2.10) contributes to the dominating term $r^{\frac{n^2-1}{4}}$, it suffices to consider $\det(e^{im_j\nu_k})_{j,k=1}^n \det(e^{-im_j\theta_k})_{j,k=1}^n$ for this set of values. From the definition in Lemma 2.7, this is $\Phi(\boldsymbol{\nu})\Phi(-\boldsymbol{\theta})$. As $r \to 0$,

$$\det(h_{\mathbb{D}}(j,k))_{j,k=1}^n = \frac{1}{2n!} r^{\frac{n^2-1}{4}} \Phi(\boldsymbol{\nu})\Phi(-\boldsymbol{\theta})(1 + O(r|\Phi(\boldsymbol{\nu})\Phi(-\boldsymbol{\theta})|^{-1})). \tag{2.11}$$

Example 2.12 ($n = 3$)**.** *Let us consider the case for $n = 3$. From Lemma 2.7, we have*

$$\Phi(\boldsymbol{\nu}) = (2i)^3 \sin\left(\frac{\nu_2 - \nu_1}{2}\right) \sin\left(\frac{\nu_3 - \nu_2}{2}\right) \sin\left(\frac{\nu_3 - \nu_1}{2}\right).$$

Note that since $\boldsymbol{\nu} \in [0, 2\pi)^3$ and $\nu_1 < \nu_2 < \nu_3$, the product of the sine functions is always positive. Using the product-to-sum formula for trigonometric functions, we have

$$\begin{aligned}
\Phi(\boldsymbol{\nu}) =& \frac{(2i)^3}{2} [\cos\left(\frac{2\nu_2 - \nu_1 - \nu_3}{2}\right) - \cos\left(\frac{\nu_3 - \nu_1}{2}\right)] \sin\left(\frac{\nu_3 - \nu_1}{2}\right) \\
=& \frac{(2i)^3}{4} [\sin(\nu_2 - \nu_1) + \sin(\nu_3 - \nu_2) - \sin(\nu_3 - \nu_1) - \sin(0)] \\
=& -2i[\sin(\nu_2 - \nu_1) + \sin(\nu_3 - \nu_2) - \sin(\nu_3 - \nu_1)].
\end{aligned}$$

Similarly, from (2.8), we have

$$\Phi(-\boldsymbol{\theta}) = 2i[\sin(\theta_2 - \theta_1) + \sin(\theta_3 - \theta_2) - \sin(\theta_3 - \theta_1)].$$

As $r \to 0$,

$$\begin{aligned}
\det(h_{\mathbb{D}}(j,k))_{j,k=1}^3 =& \frac{1}{3} r^2 [\sin(\nu_2 - \nu_1) + \sin(\nu_3 - \nu_2) - \sin(\nu_3 - \nu_1)] \\
& \times [\sin(\theta_2 - \theta_1) + \sin(\theta_3 - \theta_2) - \sin(\theta_3 - \theta_1)] + O(r^3).
\end{aligned} \tag{2.13}$$

3. Comparison between the continuous and discrete Poisson kernels

3.1. **Discrete Poisson kernel in the unit square.** Let $\mathcal{K}_{N,d} = \{(x^1, \cdots, x^d) \in \mathbb{Z}^d | 1 \leq x^i \leq N - 1\}$ be a $d-$dimensional cube with positive coordinates. We have the following proposition for the discrete Poisson kernel in $K_{2,d}$ [LL10].

Proposition 3.1. *Suppose $(x, y) \in \mathcal{K}_{N,2}$ and $(N, \tilde{y}) \in \partial\mathcal{K}_{N,2}$. The discrete Poisson kernel of $\mathcal{K}_{N,2}$ is given by*

$$H_N(x + iy, N + i\tilde{y}) = \frac{2}{N} \sum_{z=1}^{N-1} \frac{\sinh(\frac{\alpha_z \pi x}{N})}{\sinh(\alpha_z \pi)} \sin(\frac{zy\pi}{N}) \sin(\frac{z\tilde{y}\pi}{N}),$$

where α_z is the unique positive solution to

$$\cosh(\frac{\alpha_z\pi}{N}) + \cosh(\frac{z\pi}{N}) = 2.$$

Proof. Define

$$f(x,y) = \sinh(\frac{\alpha_z\pi x}{N})\sin(\frac{z\pi y}{N}).$$

Then we can check that f is a discrete harmonic function:

$$\begin{aligned}\sum_{d((x',y'),(x,y))=1} f(x',y') &= \sinh(\frac{\alpha_z\pi}{N}(x+1))\sin(\frac{z\pi y}{N}) + \sinh(\frac{\alpha_z\pi}{N}(x-1))\sin(\frac{z\pi y}{N}) \\ &+ \sinh(\frac{\alpha_z\pi}{N}x)\sin(\frac{z\pi}{N}(y+1)) + \sinh(\frac{\alpha_z\pi}{N}x)\sin(\frac{z\pi}{N}(y-1)) \\ &= 2\sinh(\frac{\alpha_z\pi}{N}x)\sin(\frac{\alpha_z\pi}{N}y)[\cosh(\frac{\alpha_z\pi}{N}) + \cos(\frac{\alpha_z\pi}{N})] \\ &= 4\sinh(\frac{\alpha_z\pi}{N}x)\sin(\frac{\alpha_z\pi}{N}y) \\ &= 4f(x,y).\end{aligned}$$

For a fixed z, we can define a new function $\hat{f}_z(y)$ on $K_{N,1}$:

$$\hat{f}_z(y) = \frac{(\frac{2}{N})^{\frac{1}{2}}}{\sinh(\alpha_z\pi)} f(N,y) = \sqrt{\frac{2}{N}}\sin(\frac{z\pi y}{N}).$$

Then we have

$$\sum_{z=1}^{N-1} \hat{f}_z(y)\hat{f}_z(\tilde{y}) = \begin{cases} 0 & y \neq \tilde{y}, \\ 1 & y = \tilde{y}. \end{cases}$$

This forms an orthonormal basis for functions on $K_{N,1}$. For any function $g(x)$ on $K_{N,1}$, we have

$$g(x) = \sum_{y=1}^{N-1} c_z(y)\hat{f}_z(x),$$

where

$$c_z(y) = \sum_{z=1}^{N-1} \hat{f}_z(y)g(y)$$

Then the delta function at y is given by

$$\delta_y(x) = \sum_{z=1}^{N-1} \hat{f}_z(y)\hat{f}_z(x).$$

Then for each $\tilde{y} \in \mathcal{K}_{N,1}$,

$$(x,y) \to \sum_{z=1}^{N-1} \frac{\sinh(\frac{\alpha_z\pi x}{N})}{\sinh(\alpha_z\pi)}\hat{f}_z(y)\hat{f}_z(\tilde{y}) = \frac{2}{N}\sum_{z=1}^{N-1} \frac{\sinh(\frac{\alpha_z\pi x}{N})}{\sinh(\alpha_z\pi)}\sin(\frac{zy\pi}{N})\sin(\frac{z\tilde{y}\pi}{N})$$

is a harmonic function and takes the value $\delta_{(N,\tilde{y})}$ when restricted to $x = N$. By the uniqueness of the solution to the Dirichlet problem on a bounded, simply connected set $A \subset \mathbb{Z}^2$, the right-hand side above is the Poisson kernel. □

The above result is set in a square in the first quadrant. We now want to map it to the unit square. By first scaling $\mathcal{K}_{N,2}$ to length $2N-1$ and translating it to the center, we obtain the discrete Poisson kernel for the discrete unit square $S_N = \mathcal{S} \cap (\frac{\mathbb{Z}}{N} + \frac{i\mathbb{Z}}{N})$ of lattice size $\frac{1}{N}$.

Corollary 3.2. *Let $(x,y) \in S_N$, $(1,\tilde{y}) \in \partial S_N$. Then the Poisson kernel of S_N between these two points is given by:*

$$H_{S_N}(x+iy, 1+i\tilde{y}) = \frac{1}{N}\sum_{m=1}^{2N-1} \frac{\sinh(\frac{\beta_m \pi(x+1)}{2})}{\sinh(\beta_m \pi)} \sin(\frac{m\pi(y+1)}{2}) \sin(\frac{m\pi(\tilde{y}+1)}{2}), \tag{3.3}$$

where β_m is the unique positive solution to

$$\cosh(\frac{\beta_m \pi}{2N}) + \cos(\frac{m\pi}{2N}) = 2.$$

3.2. **Error Estimate Between Discrete and Continuous Poisson kernels.** Recall that $\mathcal{S}$ is the unit square. Let $(x,y) \in S$ and $(1,\tilde{y}) \in \partial\mathcal{S}$. Then the continuous Poisson kernel between these two points in $\mathcal{S}$ is given by the following:

$$h_{\mathcal{S}}(x,y,\tilde{y}) := h_{\mathcal{S}}((x,y),(1,\tilde{y})) = \sum_{m=1}^{\infty} \frac{\sinh(m\pi\frac{1+x}{2})}{\sinh(m\pi)} \sinh(\frac{1+y}{2}) \sin(\frac{1+\tilde{y}}{2}), \tag{3.4}$$

whose derivation is similar to the above using separation of variables. To compare $h_{\mathcal{S}}(x,y,\tilde{y})$ with $H_{S_N}(x,y,\tilde{y})$, first note that

$$e^{\cosh^{-1}(2-\cos(t))} = 2 - \cos(t) + \sqrt{(2-\cos(t))^2 - 1}.$$

Hence

$$\begin{aligned} \beta_m = \frac{2N}{\pi}\cosh^{-1}(2-\cos(\frac{m\pi}{2N})) &= \frac{2N}{\pi}\log(2-\cosh(\frac{m\pi}{2N}) + \sqrt{(2-\cosh(\frac{m\pi}{2N}))^2 - 1}) \\ &= \frac{2N}{\pi}[\frac{m\pi}{2N} - \frac{1}{12}(\frac{m\pi}{2N})^3 + \frac{1}{96}(\frac{m\pi}{2N})^5 + O(\frac{m^7}{N^7})] \\ &= m[1 - \frac{\pi^2}{48}(\frac{m}{N})^2 + \frac{\pi^4}{1536}(\frac{m}{N})^4 + O(\frac{m}{N})^6], \end{aligned} \tag{3.5}$$

as $\frac{m}{N} \to 0$.

The following lemma and its corollary relate β_m and m when plugged inside hyperbolic functions, which will be useful in comparing the discrete and the continuous Poisson kernel.

Lemma 3.6. *Let $\alpha_2 = -\frac{\pi^2}{48}, \alpha_4 = \frac{\pi^4}{1536}$. As $\frac{m^3}{N^2} \to 0$, the following holds with error term uniform over $u \in (0,\pi]$:*

$$\sinh(\beta_m u) = \sinh(mu)[1 + \alpha_2 u \coth(mu)(\frac{m^3}{N^2}) + (\frac{\alpha_2^2}{2}u^2 + \frac{\alpha_4}{m}u\coth(mu))(\frac{m^3}{N^2})^2 + O((\frac{m^3}{N^2})^3)].$$

Proof. Substitute α_2, α_4 for the constants in (3.5), we have

$$(\beta_m - m) = m[\alpha_2(\frac{m}{N})^2 + \alpha_4(\frac{m}{N})^4 + O(\frac{m}{N})^6].$$

The Taylor series of $\sinh(x)$ at $x = mu$ is

$$\sinh(x) = \sinh(mu) + \cosh(mu)(x-mu) + \frac{\sinh(mu)}{2}(x-mu)^2 + O(\cosh(mu)(x-mu)^3).$$

Now substitute $\beta_m u$ for x, we have

$$\begin{aligned} \frac{\sinh(\beta_m u)}{\sinh(mu)} =& 1 + u\coth(mu)[\alpha_2\frac{m^3}{N^2} + \alpha_4\frac{m^5}{N^4} + O(\frac{m^7}{N^6})] \\ &+ \frac{u^2}{2}[\alpha_2\frac{m^3}{N^2} + \alpha_4\frac{m^5}{N^4} + O(\frac{m^7}{N^6})]^2 + O(\coth(mu)u^3(\frac{m^3}{N^2})^3). \end{aligned}$$

Note that $\coth(mu)$ is uniformly bounded over positive integers m and $u \in (0, \pi]$. As $\frac{m^3}{N^2} \to \infty$, we have

$$\frac{\sinh(\beta_m u)}{\sinh(mu)} = 1 + u\coth(mu)\alpha_2(\frac{m^3}{N^2}) + [\frac{\alpha_2^2 u^2}{2} + \frac{\alpha_4 u}{m}\coth(mu)](\frac{m^3}{N^2})^2 + O((\frac{m^3}{N^2})^3).$$

Rearranging the terms, we have the result in the statement. □

Corollary 3.7. *As* $\frac{m^3}{N^2} \to 0$, *for* $x \in (-1, 1)$,

$$\begin{aligned}\frac{\sinh(\beta_m \pi \frac{1+x}{2})}{\sinh(\beta_m \pi)} =& \frac{\sinh(m\pi\frac{1+x}{2})}{\sinh(m\pi)}\{1 + (\frac{m^3}{N^2})\alpha_2\pi[\frac{1+x}{2}\coth(m\pi\frac{1+x}{2}) - \coth(m\pi)]\\ &+ (\frac{m^3}{N^4})^2[\frac{\alpha_2^2\pi^2}{2}(\frac{(1+x)^2}{4} - 1 - (1+x)\coth(\frac{1+x}{2}m)\coth(m\pi) + 2\coth^2(m\pi))\\ &+ \frac{\alpha_4\pi}{m}(\frac{1+x}{2}\coth(m\frac{1+x}{2}) - \coth(m\pi)] + O((\frac{m^3}{N^2})^3\}.\end{aligned}$$

Proof. As $x \to 0$,

$$\begin{aligned}\frac{1 + c_1 x + c_2 x^2 + O(x^3)}{1 + d_1 x + d_2 x^2 + O(x^3)} =& 1 + \frac{(c_1 - d_1)x + (c_2 - d_2)x^2 + O(x^3)}{1 + d_1 x + d_2 x^2 + O(x^3)}\\ =& 1 + (c_1 - d_1)x + (c_2 - d_2 - c_1 d_1 + d_1^2)x^2 + O(x^3).\end{aligned}$$

Let

$$x = \frac{m^3}{N^2}, c_1 = \alpha_2\pi\frac{1+x}{2}\coth(m\pi\frac{1+x}{2}), c_2 = (\frac{\alpha_2^2}{2}(\pi\frac{1+x}{2})^2 + \frac{\alpha_4}{m}\pi\frac{1+x}{2}\coth(m\pi\frac{1+x}{2}))$$

$$d_1 = \pi\coth(m\pi)\alpha_2, d_2 = \frac{\alpha_2^2\pi^2}{2} + \frac{\alpha_4\pi}{m}\coth(m\pi).$$

The statement follows from substitution. □

Now we want to write the discrete Poisson kernel of the unit square of lattice size $\frac{1}{N}$ in terms of the continuous Poisson kernel. Recall that $h_{\mathcal{S}}(x, y, \tilde{y}) := h_{\mathcal{S}}(x + iy, 1 + i\tilde{y})$ is the continuous Poisson kernel from an interior point to the right vertical boundary of the unit square.

Proposition 3.8. *As* $N \to \infty$,

$$H_{S_N}(x, y, \tilde{y}) = \frac{1}{N}[h_{\mathcal{S}}(x, y, \tilde{y}) + \frac{1}{N^2}h_2(x, y, \tilde{y}) + \frac{1}{N^4}h_4(x, y, \tilde{y}) + O(\frac{1}{N^6})],$$

where

$$h_2(x, y, \tilde{y}) = \sum_{m\in\mathbb{N}} f_m(x, y, \tilde{y})\alpha_2 m^3[\frac{1+x}{2}\coth(\frac{1+x}{2}m\pi) - \coth(m\pi)],$$

$$\begin{aligned}h_4(x, y, \tilde{y}) =& \sum_{m\in\mathbb{N}} f_m(x, y, \tilde{y})m^6\\ &\times\{\frac{\alpha_2^2\pi^2}{2}[\frac{x^2 + 2x - 3}{4} - (1+x)\coth(\frac{1+x}{2}m)\coth(m\pi) + 2\coth^2(m\pi)]\\ &+ \frac{\alpha_4}{m}[\frac{1+x}{2}\coth(m\frac{1+x}{2}) - \coth(m\pi)]\},\end{aligned}$$

and

$$f_m(x, y, \tilde{y}) = \frac{\sinh(m\pi\frac{1+x}{2})}{\sinh(m\pi)}\sin(m\pi\frac{1+y}{2})\sin(m\pi\frac{1+\tilde{y}}{2}).$$

The error term is uniform over $x \in (-1 + \epsilon, 1 - \epsilon)$.

Proof. Let $x, \tilde{x} \in (-1+\epsilon, 1-\epsilon)$. As $m \to \infty$,

$$\frac{\sinh(m\pi\frac{1+x}{2})}{\sinh(m\pi)} \sim e^{m\pi\frac{x-1}{2}}.$$

Then $f_m(x, y, \tilde{y})$ has exponential decay in m. Then there exists $c_1 > 0$ such that

$$\sum_{m\in\mathbb{N}} f_m(x, y, \tilde{y}) = \sum_{m=1}^{c_1 \log N} f_m(x, y, \tilde{y}) + O(\frac{1}{N^6}). \tag{3.9}$$

Note that β_m is strictly increasing in m when $\frac{m}{N} \in (0, 2)$, and $\beta_m < \cosh^{-1}(3)$. Let

$$\tilde{f}_m(x, y, \tilde{y}) = \frac{\sinh(\beta_m\pi\frac{1+x}{2})}{\sinh(\beta_m\pi)} \sin(m\pi\frac{1+y}{2}) \sin(m\pi\frac{1+\tilde{y}}{2}).$$

From (3.5), we have that as $\frac{m}{N} \to 0$, $\frac{\beta_m}{m} \to 1$. Then there exists $c_2 > 0$ such that

$$\sum_{m\in\mathbb{N}} \tilde{f}_m(x, y, \tilde{y}) = \sum_{m=1}^{c_2 \log N} \tilde{f}_m(x, y, \tilde{y}) + O(\frac{1}{N^6}), \tag{3.10}$$

as $\frac{m}{N} \to 0$.

Choose c large such that both (3.9) and (3.10) hold. For $m \le c \log N$, $\frac{m^3}{N^2} \to 0$ as $N \to \infty$. From Corollary 3.7, we have

$$\begin{aligned}\tilde{f}_m(x, y, \tilde{y}) = f_m(x, y, \tilde{y})\{1 &+ (\frac{m^3}{N^2})\alpha_2\pi[\frac{1+x}{2}\coth(m\pi\frac{1+x}{2}) - \coth(m\pi)] \\ &+ (\frac{m^3}{N^2})^2\frac{\alpha_2^2\pi^2}{2}[\frac{x^2+2x-3}{4} - (1+x)\coth(\frac{1+x}{2}m)\coth(m\pi) + 2\coth^2(m\pi)] \\ &+ O((\frac{m^3}{N^2})^3)\}.\end{aligned}$$

Then

$$\sum_{m=1}^{c \log N} \tilde{f}_m(x, y, \tilde{y}) = \sum_{m=1}^{c \log N} f_m(x, y, \tilde{y}) + \frac{1}{N^2}h_2 f_m(x, y, \tilde{y}) + \frac{1}{N^4}h_4 f_m(x, y, \tilde{y}) + O(\frac{1}{N^6}).$$

Recalling the explicit formula for the discrete and continuous Poisson kernel from (3.3) and (3.4), we have

$$H_{S_N}(x, y, \tilde{y}) = \frac{1}{N}[h_{\mathcal{S}}(x, y, \tilde{y}) + \frac{1}{N^2}h_2(x, y, \tilde{y}) + \frac{1}{N^4}h_4(x, y, \tilde{y}) + O(\frac{1}{N^6})]$$

as claimed. □

3.3. **Conformal mapping between the unit square and the unit disk.** We have obtained the estimate for the discrete Poisson kernel using the continuous Poisson kernel in the unit square. To relate that to the result of the scaling exponent of the determinant in Section 2.3, we need to compare the continuous Poisson kernel in the unit square $\mathcal{S}$ and the unit disk $\mathbb{D}$. The continuous Poisson kernel is dependent on the radius of the interior points. It is useful to consider where the interior points in $\mathcal{S}$ are mapped to in $\mathbb{D}$. The following lemma is a consequence of standard complex analysis results [Rud87].

Lemma 3.11. *Let $f : \mathbb{D} \to \mathcal{S}$ be a conformal map with $f(0) = 0$, $f'(0) > 0$, as provided by the Riemann mapping theorem. Let $f(z) = \sum_{n=1}^{\infty} c_n z^n$ be its power series expansion. Then for any $z \in \mathbb{D}$,*

$$|f(z)| = |c_1 z| + O(|z|^5) \tag{3.12}$$

as $z \to 0$.

Proof. Let $g : \mathbb{C} \to \mathbb{C}$ be the map such that $g(z) = iz$. Since g is a conformal map, $h := g^{-1} \circ f \circ g$ is a conformal map from $\mathbb{D} \to \mathcal{S}$ with $h(0) = f(0) = 0$ and $h'(0) = f'(0) > 0$. By uniqueness of the map designated by the Riemann mapping theorem, we must have $h = f$. Hence $f(iz) = if(z)$, for any $z \in \mathbb{D}$. Let

$$u(z) := \sum_{m=1}^{\infty}(i+1)c_{4m-2}z^{4m-2} + \sum_{m=1}^{\infty} 2ic_{4m-1}z^{4m-1} + \sum_{m=1}^{\infty}(i-1)c_{4m}z^{4m},$$

which is holomorphic in an open disk Ω at centered at 0. Note that $u(z) = f(iz) - if(z) = 0$ on Ω, which contains an accumulation point. Hence $u(z) \equiv 0$ is constant. Thus we conclude that $c_n \neq 0$ only if $n \equiv 1 \mod 4$. Then

$$f(z) = \sum_{m=0}^{\infty} c_{4m+1}z^{4m+1} = c_1 z + c_5 z^5 + c_9 z^9 + \cdots$$

Letting $z \to 0$, we have (3.12). $\square$

3.4. **Example:** $n = 3$**.** The following lemma is useful when computing the determinant of the discrete Poisson kernel in the unit square.

Lemma 3.13. *Let $v_1 = (\frac{1}{N}, 0), v_2 = (0, \frac{1}{N}), v_3 = (0, -\frac{1}{N})$ be three points on S_N. Let $\phi : \mathcal{S} \to \mathbb{D}$ be a conformal map with $\phi(0) = 0$. Then*

$$|\phi(v_1)| = |\phi(v_2)| = |\phi(v_3)| = a_1\frac{1}{N} + O(\frac{1}{N^5}),$$

where $a_1 \in \mathbb{R}$ is a constant independent on the choice of ϕ, and

$$|\phi'(v_i)| = a_1 + O(\frac{1}{N^4}),$$

for $i = 1, 2, 3$.

Proof. Suppose $\phi(z) = \sum_{n=1}^{\infty} c_n z^n$. Similar to result in Section 3.3, we have $\phi(iz) = i\phi(z)$ and $\phi(z) = c_1 z + O(|z|^5)$. Note that we have $iv_1 = v_2$ and $iv_3 = v_1$. Hence $\phi(v_2) = i\phi(v_1) = -\phi(v_3)$. In other words, these three points are mapped to the same circle. Moreover, for $i = 1, 2, 3$,

$$\begin{aligned}|\phi(v_i)| &= |c_1 v_i| + O(|v_i|^5) \\ &= |c_1|\frac{1}{N} + O(\frac{1}{N^5}),\end{aligned}$$

as $N \to \infty$.

Let $a_1 = |c_1|$. Note that a_1 does not depend on the choice of ϕ since all admissible choices differ only by rotations. We have

$$\phi'(v_i) = c_1 + 5c_5 v_i^4,$$

which gives the lemma. $\square$

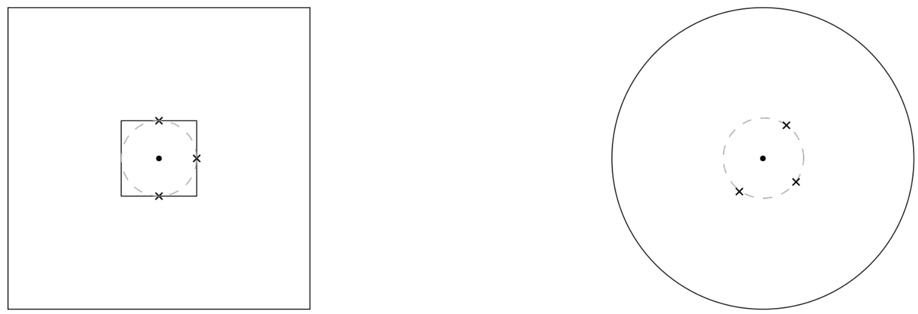

FIGURE 3. Three points in $\mathcal{S}$ and their image in $\mathbb{D}$ under a conformal map

Remark 3.14. See figure 3 for an illustration of a conformal map. Note that the dashed grey circle in the two graphs do not have the same radius. The constant a_1 can be computed explicitly by an exact formula given by the Schwarz–Christoffel transformation, which involves Jacobi elliptic equations. The power $\frac{1}{N^5}$ is also not optimal, but it suffices for our purpose.

Proposition 3.15. *Fix $z_1 = 1, z_2 = i, z_3 = -i$ and let $w_1, w_2, w_3 \in \partial S$. As $N \to \infty$, we have*

$$\det h_{\mathcal{S}}(\frac{z_i}{N}, w_{\sigma(i)}) = O(\frac{1}{N^2}).$$

Proof. From Lemma 3.13, we have

$$|\phi(\frac{z_i}{N})| = a_1 \frac{1}{N} + O(\frac{1}{N^5}),$$

$$|\phi'(\frac{z_i}{N})| = a_1 + O(\frac{1}{N^4}).$$

Let $r = |\phi(\frac{z_i}{N})|$, $\arg(z_i) = \theta_i$, and $\arg(w_i) = \nu_i$. From (2.13), the determinant of the continuous Poisson kernel in $\mathbb{D}$ is given by

$$\begin{aligned}
&\det(h_{\mathbb{D}}(\frac{z_i}{N}, w_{\sigma(i)}))_{i=1}^3 \\
&= \frac{1}{3} r^2 [\sin(\nu_2 - \nu_1) + \sin(\nu_3 - \nu_2) - \sin(\nu_3 - \nu_1)][\sin(\theta_2 - \theta_1) + \sin(\theta_3 - \theta_2) - \sin(\theta_3 - \theta_1)] + O(r^3) \\
&= \frac{2}{3} r^2 [\sin(\nu_2 - \nu_1) + \sin(\nu_3 - \nu_2) - \sin(\nu_3 - \nu_1)] \\
&= \frac{2}{3} [\sin(\nu_2 - \nu_1) + \sin(\nu_3 - \nu_2) - \sin(\nu_3 - \nu_1)] a_1^2 \frac{1}{N^2} + O(\frac{1}{N^6}).
\end{aligned}$$

By conformal covariance of the Poisson kernel, we have

$$\begin{aligned}
\det h_{\mathcal{S}}(\frac{z_i}{N}, w_{\sigma(i)})_{i=1}^3 &= \prod_{i=1}^3 |\phi'(\frac{z_i}{N})| \det h_{\mathbb{D}}(\frac{z_i}{N}, w_{\sigma(i)})_{i=1}^3 \\
&= \frac{2a_1^5}{3} [\sin(\nu_2 - \nu_1) + \sin(\nu_3 - \nu_2) - \sin(\nu_3 - \nu_1)] \frac{1}{N^2} + O(\frac{1}{N^6}).
\end{aligned}$$

□

Remark 3.16. The exponent $\frac{n^2-1}{4}$, which yields 2 when $n = 3$, coincides with the scaling exponent of 3−crossings (3 disjoint tree branches running from one boundary to the other) of a uniform spanning tree on an annulus [Ken00].

Proposition 3.17. *Let $z_1, z_2, z_3 \in S$ and $w_1, w_2, w_3 \in \partial S$. As $N \to \infty$,*

$$\det(H_{S_N}(\frac{z_j}{N}, w_k))_{j,k=1}^3 = O(\frac{1}{N^5}).$$

Proof. Note that

$$\det(H_{S_N}(\frac{z_j}{N}, w_k))_1^n = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^n H_{S_N}(\frac{z_i}{N}, w_{\sigma(i)}).$$

From Proposition 3.8 and the formula of the determinant, we have

$$\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^n H_{S_N}(\frac{z_i}{N}, w_{\sigma(i)}) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \frac{1}{N^n} \prod_{i=1}^n [h_{\mathcal{S}}(\frac{z_i}{N}, w_{\sigma(i)}) + \frac{1}{N^2} h_2(\frac{z_i}{N}, w_{\sigma(i)}) + O(\frac{1}{N^4})]$$

$$= \frac{1}{N^n}[\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^n h_{\mathcal{S}}(\frac{z_i}{N}, w_{\sigma(i)}) + \frac{1}{N^2} \sum_{k=1}^n \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \frac{h_2(\frac{z_k}{N}, w_{\sigma(k)})}{h_{\mathcal{S}}(\frac{z_k}{N}, w_{\sigma(k)})} \prod_{i=1}^n h_{\mathcal{S}}(\frac{z_i}{N}, w_{\sigma(i)}) + O(\frac{1}{N^4})].$$

From Corollary 3.15, we have

$$\sum_{\sigma \in S_3} \mathrm{sgn}(\sigma) \prod_{i=1}^3 h_{\mathcal{S}}(\frac{z_i}{N}, w_{\sigma(i)}) = \det h_{\mathcal{S}}(\frac{z_i}{N}, w_{\sigma(i)})_{i=1}^3 = O(\frac{1}{N^2}).$$

Thus, it suffices to show that

$$\sum_{k=1}^n \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \frac{h_2(\frac{z_k}{N}, w_{\sigma(k)})}{h_{\mathcal{S}}(\frac{z_k}{N}, w_{\sigma(k)})} \prod_{i=1}^n h_{\mathcal{S}}(\frac{z_i}{N}, w_{\sigma(i)}) = O(1). \tag{3.18}$$

Let $z_k = x + iy, w_{\sigma(k)} = 1 + i\tilde{y}$, we have

$$\frac{h_2(\frac{z_k}{N}, w_{\sigma(k)})}{h_{\mathcal{S}}(\frac{z_k}{N}, w_{\sigma(k)})} = \frac{\sum_{m \in \mathbb{N}} f_m(\frac{x}{N}, \frac{y}{N}, \tilde{y}) \alpha_2 \pi m^3 [\frac{1+\frac{x}{N}}{2} \coth(\frac{1+\frac{x}{N}}{2} m\pi) - \coth(m\pi)]}{\sum_{m \in \mathbb{N}} f_m(\frac{x}{N}, \frac{y}{N}, \tilde{y})}.$$

Note that

$$\alpha_2 \pi m^3 [\frac{1+\frac{x}{N}}{2} \coth(\frac{1+\frac{x}{N}}{2} m\pi) - \coth(m\pi)]$$

is uniformly bounded over m and N, and $f_m(\frac{x}{N}, \frac{y}{N}, \tilde{y})$ decreases exponentially in N. Therefore, $\frac{h_2(\frac{z_k}{N}, w_{\sigma(k)})}{h_{\mathcal{S}}(\frac{z_k}{N}, w_{\sigma(k)})}$ is uniformly bounded over k and σ. Since $\prod_{i=1}^n h_{\mathcal{S}}(\frac{z_i}{N}, w_{\sigma(i)})$ is uniformly bounded as well, we have (3.18). For $n = 3$, we have

$$\det(H_{S_N}(\frac{z_j}{N}, w_k))_1^3 = O(\frac{1}{N^5}).$$

as claimed. □

Acknowledgments

I would like to thank my mentor and collaborator, Jinwoo Sung, for his support and guidance throughout this project. I am very grateful to Professor Greg Lawler for his great suggestions and helpful discussions. I would like to thank Professor Peter May for his dedication to this program.

References

[AO19] Jonas Arista and Neil O'Connell. Loop-erased walks and random matrices. *J. Stat. Phys.*, 177(3):528–567, 2019.

[BLV16] Christian Beneš, Gregory F. Lawler, and Fredrik Viklund. Scaling limit of the loop-erased random walk Green's function. *Probab. Theory Related Fields*, 166(1-2):271–319, 2016.

[Ken00] Richard Kenyon. Long-range properties of spanning trees. *J. Math. Phys.*, 41(3):1338–1363, 2000. Probabilistic techniques in equilibrium and nonequilibrium statistical physics.

[KL05] Michael J. Kozdron and Gregory F. Lawler. Estimates of random walk exit probabilities and application to loop-erased random walk. *Electron. J. Probab.*, 10:1442–1467, 2005.

[Law22] Gregory F. Lawler. *Random explorations*, volume 98 of *Student Mathematical Library*. American Mathematical Society, Providence, RI, 2022.

[LL10] Gregory F. Lawler and Vlada Limic. *Random walk: a modern introduction*, volume 123 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.

[LSW04] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. Conformal invariance of planar loop-erased random walks and uniform spanning trees. *Ann. Probab.*, 32(1B):939–995, 2004.

[Meh04] Madan Lal Mehta. *Random matrices*, volume 142 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, third edition, 2004.

[Rud87] Walter Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.