BOUNDS ON SUMS OF BETTI NUMBERS

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Abstract. In this note we survey three different iterations of bounds on sum of Betti numbers of algebraic varieties. The first is Milnor's approach using Morse theory for singular cohomology, then Katz's approach using the affine Lefschetz hyperplane theorem in the ℓ-adic setting. Finally we discuss quantitative sheaf theory, a theory which "functorially" bounds the sum of Betti numbers of arbitrary objects $A \in D_c^b(X)$, . We discuss an application of quantitative sheaf theory to horizontal equidistribution of exponential sums.

Contents

1. Introduction

Why should one care about bounds on sums of Betti numbers? The perspective we take is that the Lefschetz trace formula lets one translate bounds on sums of Betti numbers into useful information on fixed points of endomorphisms. Let us state two versions of the Lefschetz trace formula.

Theorem 1.1 (Lefschetz-Hopf fixed point theorem)**.** *Let X be a compact oriented manifold and $\phi : X \to X$ an endomorphism with isolated singularities. Then*

$$\#\textit{Fixed pts of } \phi = \sum_i (-1)^i Tr(\phi^*, H^*(X, \mathbb{R})).$$

Note that the left side counts fixed points with multiplicities, more precisely it should be the intersection number $\Delta_X \cdot \Gamma_\phi$ in $X \times X$. In positive characteristic there is an analagous theorem. Let X_0 be a variety over a finite field $k = \mathbb{F}_q$ and denote $X := X_0 \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$. Let $\mathcal{F}_0$ be an ℓ-adic sheaf on $\mathcal{F}_0$ and denote $\mathcal{F}$ the

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base change to X. We have the geometric Frobenius on k which induces maps $F : H^i(X, \mathcal{F}) \to H^i(X, \mathcal{F})$ and $F_x : \mathcal{F}_{\overline{x}} \to \mathcal{F}_{\overline{x}}$ for any closed point $x \in |X_0|$.

Theorem 1.2 (Grothendieck-Lefschetz trace formula)**.** *Let X_0 and F_0 be as above. Then*

$$\sum_{x \in X(\mathbb{F}_q)} Tr(F_x, \mathcal{F}_{\overline{x}}) = \sum_i (-1)^i Tr(F, H^i_c(X, \mathcal{F}))$$

Note that if $\mathcal{F}$ is the constant sheaf, then the left side is $\#X(\mathbb{F}_q)$, which are the fixed points of Frobenius. If $\mathcal{F}$ is an Artin-Schreier sheaf (or some variant of it), then the left side is an exponential sum. So in order to bound the left side, it suffices to bound the size of the eigenvalues of Frobenius, and control the number of eigenvalues, i.e the Betti numbers. For ℓ-adic cohomology, the eigenvalues of Frobenius have been studied by Deligne.

Theorem 1.3 ([5], 3.3.6)**.** *Let X_0 be a smooth and proper variety over a finite field $k = \mathbb{F}_q$. If $\mathcal{F}_0$ is pure of weight n, then $H^i(X, \mathcal{F})$ is pure of weight $n + i$*

We recall that pure of weight i means for all field isomorphisms $\tau : \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$, all eigenvalues of geometric Frobenius have complex norm $q^{i/2}$ under τ. In general, without smoothness or properness, we still have the following theorem.

Theorem 1.4 ([5], 3.3.1)**.** *Let X be a variety over $\mathbb{F}_q$ and $\mathcal{F}$ an ℓ-adic sheaf on X, mixed of weight $\leq n$. Then $H^i_c(X, \mathcal{F})$ is mixed of weights $\leq n + i$.*

Recall that mixed means there exists a filtration whose successive quotients are pure of some weight. For mixed sheaves we no longer have precise control of the size of eigenvalues, but the bound is often still useful. We now give an example of how the Grothendieck-Lefschetz trace formula can be applied to give estimates on the number of rational points. The following appears in Kat's appendix to [8].

Theorem 1.5 ([8] Appendix Thm 1)**.** *Let $N \geq 1, r \geq 1$ be integers, and $d_1 \dots d_r$ an r-tuple of positive integers. Then there exists a constant $C(N, d_1, \dots d_r)$ so that for all finite fields $k = \mathbb{F}_q$ and complete intersections $X \subset \mathbb{P}^{N+r}_k$ of dimension N and multidegree $(d_1, \dots d_r)$, with singular locus dimension D (set $D = -1$ for X smooth over k), we have*

$$|\#X(k) - \#\mathbb{P}^N(k)| \leq C(N, d_1, \dots d_r)\sqrt{q}^{N+1+D}.$$

To prove this, Katz first shows that there exists a bound $\sum_i h^i(X \otimes_k \overline{k}, \mathbb{Q}_\ell) \leq S(N, d_1 \dots d_r)$ for all X as above (no assumption on dimension of the singular locus). Then Katz computes for $i > N + 1 + D$, we have $H^i(X \otimes_k \overline{k}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-i)$ for i even and 0 for i odd. Combining these two facts along with the Grothendieck trace formula and the bound from Theorem 1.4, we obtain the result using $C(N, d_1 \dots d_r) = N + 1 + S(N, d_1 \dots d_r)$.

1.1. **Outline of paper.** In section 2 we discuss Milnor's approach to bounding the cohomology of real affine varieties using Morse theory. In section 3 we discuss Kat's approach in the ℓ-adic setting using a Lefschetz hyperplane theorem and highlight an independence of ℓ corollary. In section 4 we discuss singular support and the characteristic cycle, and their application to the proof of continuity under the 6 operations in quantitative sheaf theory. In section 5 we discuss an application of quantitative sheaf theory to horizontal equidistribution (i.e. taking $p \to +\infty$) of exponential sums.

1.2. **Notation.** $p \neq \ell$ will always be distinct primes. A variety is a reduced, separated scheme of finite type over a field. In the context of ℓ-adic cohomology, all functors will be derived unless otherwise noted, e.g. $f_* = Rf_*$. For an object $F \in D^b_c(X)$, we denote $D(F)$ for the Verdier dual of F, and $t_F : X(\mathbb{F}_q) \to \overline{Q}_\ell$ for the trace function. In quantitative sheaf theory, we fix once and for all an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$. By pure or mixed sheaves we will mean ι-pure (respectively mixed).

2. Milnor's bound via Morse theory

In this section, Betti numbers are with respect to singular cohomology. We outline the proof of Milnor's bound in [14] on the sum of Betti numbers of real affine algebraic varieties (in terms of the degrees of defining polynomials).

Lemma 2.1. *Let $X \subset \mathbb{R}^n$ be a 0 dimensional variety cut out by polynomials $f_1 \dots f_n$ of degree $d_1 \dots d_n$. Suppose that the gradient vectors df_i are linearly independent at each point of X. Then $|X| \leq \prod_i d_i$.*

Proof. Perturb the coefficients of the polynomials f_i so that they are all algebraically independent. If X' denotes the variety cut out by the new polynomials, then $X'_\mathbb{C}$ has exactly $\prod_i d_i$ points. Then the points of X must be close to a point of $X'_\mathbb{C}$, hence the bound we desired. A direct proof using intersection theory can be found for example in [7, 8.4]. □

Theorem 2.2. *Let $X \subset \mathbb{R}^n$ be a compact nonsingular hypersurface defined a polynomial f of degree $2k$. Then the sum of the Betti numbers of X is $\leq 2k(k-1)^{m-1}$.*

Proof. Consider the map $n : X \to S^{n-1}$ given by $n = \text{grad} f / |\text{grad} f|$. By Sard's theorem, the critical values of n have measure 0 so we may rotate the coordinates on $\mathbb{R}^n$ (without changing the Betti numbers) and assume that $(0, \dots 0, \pm 1)$ are not values of n. Given local coordinates $u_1 \dots u_{m-1}$ on X, this means that $det(\partial n_i / \partial u_j) \neq 0$ on $n^{-1}(0, \dots 0, \pm 1)$.

Define the height function $h : X \to \mathbb{R}$ by $(x_1, \dots x_m) \mapsto x_m$. We claim that h has no degenerate critical points (i.e. h is a Morse function). Indeed, if $u_1 \dots u_{m-1}$ are local coordinates at a critical point, then

$$n(u_1 \dots u_{m-1}) = \pm(\partial h/\partial u_1 \dots \partial h/\partial u_{m-1}, -1) / \sqrt{\sum_j \partial (h/\partial u_j)^2 + 1}.$$

At a critical point of h, $\partial h/\partial u_j = 0$ for all j so $\partial n_i / \partial u_j = \pm \partial^2 h / \partial u_i \partial u_j$, hence the matrix $(\partial^2 h/\partial u_i \partial u_j)$ is nonsingular. By the weak Morse inequality [13, Thm 5.2], the sum of the Betti numbers of X (compact manifold by assumption) is bounded by the number of critical points of the Morse function h. The critical points of h are the zeros of the polynomials

$$\partial h/\partial x_1, \dots \partial h/\partial x_{m-1}, f.$$

So it suffices to show that the m vectors $d(\partial h/\partial x_1), \dots \partial d(h/\partial x_{m-1}), df$ are linearly independent at the critical points of h and apply Lemma 2.1. Indeed, note that differentiating the equality

$$f(u_1, \dots u_{m-1}, h(u_1, \dots u_{m-1})) = 0$$

twice gives

$$\frac{\partial^2 f}{\partial x_i \partial x_j} + \frac{\partial f}{\partial x_m}\frac{\partial^2 h}{\partial u_i \partial u_j} + \frac{\partial^2 f}{\partial x_j \partial x_m}\frac{\partial h}{\partial u_i}.$$

The third term is zero since we are evaluating at a critical point of h, so we have the matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j}$ is nonsingular. Again since we are at a critical point, df is nonzero except in the last coordinate, hence the vectors are linearly independent. □

Theorem 2.3. *Let $X \subset \mathbb{R}^m$ be defined by the zeros of polynomials $f_1 \dots f_n$ all of degree $\leq k$. Then the sum of Betti numbers of X is $\leq k(2k-1)^{n-1}$.*

Proof. Let B_r be the ball of radius r centered at the origin in $\mathbb{R}^m$. For $\epsilon, \delta > 0$, denote $K(\epsilon, \delta)$ by the set of points where

$$f_1^2 + \dots f_n^2 + \epsilon^2 |x|^2 \leq \delta^2$$

where $|x|$ is the Euclidean norm. We have $K(\epsilon, \delta) \subseteq B_{\delta/\epsilon}$ hence is compact. For $r \leq \delta/\epsilon$, we have $X \cap B_r \subset K(\epsilon, \delta)$. Note that

$$\partial K(\epsilon, \delta) = \left\{ f_1^2 + \dots f_n^2 + \epsilon^2 |x|^2 = \delta^2 \right\}$$

is a compact algebraic hypersurface of degree $\leq 2k$. It is nonsingular if δ^2 is a noncritical value of $f_1^2 + \dots f_n^2 + \epsilon^2|x|^2$. By Sard's theorem, for a fixed ϵ, $\partial K(\epsilon, \delta)$ is nonsingular for fixed almost all δ. From now on assume that ϵ, δ are chosen so that this is the case.

By Theorem 2.2, $H^*(\partial K)$ has dimension $\leq 2k(2k-1)^{m-1}$. By Alexander duality and the collar neighborhood theorem, we have dim $H^*(K) = \frac{1}{2}$dim $H^*(\partial K) \leq k(2k-1)^{m-1}$. Choose sequences ϵ_i, δ_i so that ϵ_i decrease monotonically to 0 and δ_i/ϵ_i decreases monotonically to r, and that each $K(\epsilon_i, \delta_i)$ is nonsingular. We have

$$\dots \subset K(\epsilon_3, \delta_3) \subset K(\epsilon_2, \delta_2) \subset K(\epsilon_1, \delta_1)$$

with intersection $X \cap B_r$. By the continuity of Cech cohomology [6, X 3.1], we have dim $H^*(X \cap B_r) \leq k(2k-1)^{m-1}$. It thus suffices to take the limit as $r \to \infty$. One can utilize the fact that X as as a smooth variety has a triangulation (we refer to [14] for details). □

3. Katz's bounds in ℓ-adic cohomology

In this section we prove an analagous result on bounds of sums of Betti numbers in ℓ-adic cohomology due to Katz. One remarkable feature is the independence of ℓ. In the setting of manifolds, one uses Morse theory to get off the ground. Here the main tool is the affine (weak) Lefschetz theorem.

Theorem 3.1. *Affine Weak Lefschetz Let k be an algebraically closed field, and $X \subset \mathbb{A}_k^N$ a closed local complete intersection of dimension n. Then there exists a dense open set of hyperplanes $H \subset \mathbb{A}^n$ so that for every lisse $\mathbb{Q}_\ell$-sheaf $\mathcal{F}$ on X, the restriction maps*

$$H^i(X, \mathcal{F}) \to H^i(X \cap H, i^*F)$$

are isomorphisms for $i < d-1$ and injective for $i = d-1$.

Proof. Apply [9] Corollary 3.4.1 with $V = X$, $f : X \to \mathbb{A}^1$ the 0 map, and $\pi : X \to \mathbb{A}^N$ the inclusion. □

Thus in order to bound the sum of Betti numbers of X (which is affine so cohomology goes up to $dim(X)$), we can inductively control the sum of Betti numbers of $X \cap H$, and separately bound h^{d-1} and h^d. For this last step one uses bounds on the Euler characteristic. In [3], Bombieri combining p-adic methods due to Dwork and results from Deligne's Weil II proved that for $X \subset \mathbb{A}^N$ a closed subscheme cut out by $f_1, \dots f_r$ all of degree $\leq d$, then

$$|\chi_c(X, \mathbb{Q}_\ell)| \leq (4(1+d)+5)^{N+r}.$$

Note that by [12], there is no difference between compactly supported Euler characteristic and the usual one. Fix an universal bound $|\chi_c(X, \mathbb{Q}_\ell)| \leq E(N, r, d)$, and define

$$A(N, r, d) := E(N, r, d) + 2 + 2\sum_{n=1}^{N-1} E(n, r, d)$$

$$B(N, r, d) := 1 + \sum_{\substack{S \subset \{0,1\dots r\} \\ S \neq \varnothing}} A\left(N+1, 1, d \cdot (\#S) + 1\right).$$

We fix an algebraically closed field k and $X \subset \mathbb{A}^N$ a closed subscheme defined by $f_1 \dots f_r$ all of degree $\leq d$. Fix a prime $\ell \neq p$. Denote

$$\sigma(X, \mathbb{Q}_\ell) := \sum_i h^i(X, \mathbb{Q}_\ell)$$

$$\sigma_c(X, \mathbb{Q}_\ell) := \sum_i h^i_c(X, \mathbb{Q}_\ell).$$

We sketch proofs of the following two theorems.

Theorem 3.2. [10, Thm 2] *In the above notation, assume either $dim(X) = 0$ or X is smooth of dimension $n \geq 1$ and connected. Then $\sigma_c(X, \mathbb{Q}_\ell) \leq A(N, r, d)$.*

Theorem 3.3. [10, Thm 1] *We have for X as above (no assumption on smoothness), $\sigma_c(X, \mathbb{Q}_\ell) \leq B(N, r, d)$*

Proof sketch of Theorem 3.2. Since X is smooth we have $\sigma = \sigma_c$ by Poincare duality. For $dim(X) = 0$ we only have cohomology in degree 0 so $\sigma_c(X, \mathbb{Q}_\ell) = \chi_c(X, \mathbb{Q}_\ell) \leq E(N, r, d) \leq A(N, r, d)$ as desired. The $n = 1$ case is also immediate. Induct on n, and use the affine Lefschetz hyperplane theorem, noting that $X \cap H$ is smooth and connected in A^{N-1}, again cut out by r equations of degree $\leq d$. We get the inequality

$$\sigma(X, \mathbb{Q}_\ell) \leq (-1)^n \chi(X, \mathbb{Q}_\ell) + (-1)^{n-1}\chi(X \cap H, \mathbb{Q}_\ell) + \sigma(X \cap H, \mathbb{Q}_\ell)$$

which complete the inductive step based on our definition of $A(N, r, d)$. □

Proof. of Theorem 3.3

Using the excision long exact sequence for compactly supported cohomology on $\mathbb{A}^N \backslash X \subset \mathbb{A}^N$, we get

$$\sigma_c(X, \mathbb{Q}_\ell) \leq \sigma_c(\mathbb{A}^N, \mathbb{Q}_\ell) + \sigma_c(\mathbb{A}^N \backslash X, \mathbb{Q}_\ell) = 1 + \sigma(\mathbb{A}^N \backslash X, \mathbb{Q}_\ell)$$

since $\mathbb{A}^N \backslash X$ is open in $\mathbb{A}^N$ hence smooth (so $\sigma = \sigma_c$). We have $\mathbb{A}^N \backslash X$ is covered by $D(f_i)$ for $i = 1 \dots r$, so the Mayer-Vietoris spectral sequence implies that

$$\sigma(\mathbb{A}^N \backslash X, \mathbb{Q}_\ell) \leq \sum_{p=1}^{r} \sum_{1 \leq i_1 \dots \leq i_p \leq r} \sigma(U_{i_1} \cap \dots U_{i_p}, \mathbb{Q}_\ell).$$

The intersection $U_{i_1} \cap \dots U_{i_p}$ is the non-vanishing locus in $\mathbb{A}^N$ of $f_{i_1} \dots f_{i_p}$, or equivalently the smooth hypersurface in $\mathbb{A}^{N+1}$ cut out by the single equation $x_{n+1} \prod_{j=1}^{r} f_{i_j}$ which is of degree $\leq 1 + dp$. So by Theorem 3.2 we have $\sigma(U_{i_1} \cap \dots U_{i_p}, \mathbb{Q}_\ell) \leq A(N+1, 1, dp+1)$.

□

We now give one application to independence of ℓ.

Theorem 3.4. [10, Thm 5] *Let k be an algebraically closed field of characteristic p, and X/k a variety. Then there exists a constant $M(X/k)$ so that for all $\ell \neq p$, we have*

$$\sigma(X, \mathbb{Q}_\ell) \leq M(X/k).$$

Proof. By the Mayer-Vietoris spectral sequence we may assume X is affine, and furthermore reduced and irreducible. By de Jong's theorem on alterations [4, Thm 4.1], we get a hypercover $X. \to X$ where X_n is affine and smooth over k for all n. We have an associated spectral sequence

$$E_1^{p,q} = H^q(X_p, \mathbb{Q}_\ell) \Rightarrow H^{p+q}(X, \mathbb{Q}_\ell).$$

Since X is affine, $h^i(X, \mathbb{Q}_\ell) = 0$ for $i > dim(X)$. So

$$\sigma(X, \mathbb{Q}_\ell) \leq \sum_{i=0}^{dim(X)} \sigma(X_i, \mathbb{Q}_\ell) = \sum_{i=0}^{dim(X)} \sigma_c(X_i, \mathbb{Q}_\ell)$$

where the last equality is by Poincare duality since X_i is smooth. Each term $\sigma_c(X_i, \mathbb{Q}_\ell)$ by Theorem 3.3 is bounded by some constant depending on the degree of equations cutting out X_i (in particular independent of ℓ). So such a constant $M(X/k)$ independent of ℓ exists.

□

4. Quantitative sheaf theory

In this section we outline the general constructions and properties of quantitative sheaf theory, developed in [16]. The main technical input is the theory of Beilinson's singular support and Saito's characteristic cycle for ℓ-adic sheaves. In this note we make a beeline to one of their main results on horizontal equidistribution. We will blackbox some proofs and refer the interested reader to the original paper [16], which we found very readable. In the previous sections we obtained bounds on sums of Betti numbers of constant sheaves. However, often the trace sums we care about use more complicated sheaves, and various combinations of the 6 functors applied to them. The goal of quantitative sheaf theory is to control the sum of Betti numbers of sheaves under the 6 functors.

The main construction of [16] is to assign a number $c(F)$ to each complex of sheaves F, which bounds the sum of Betti numbers of F, and satisfies various continuity properties under the 6 operations. By continuity, we mean, for example, the property that $c(F \otimes G) \ll c(F)c(G)$ for some constant not depending on F, G. Unfortunately, a definition with the various continuity properties cannot be intrinsic to a variety X (see [16, Example 6.1]).

The definition of complexity will depend on a variety X, an object $F \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$, and a locally closed embedding $u : X \hookrightarrow \mathbb{P}^n$. The definition will turn out to be $c_u(F) := c(u_! F)$ where $c(-)$ is an intrinsic notion of complexity for projective space. An embedding into projective space can be thought of as analogous to how in earlier sections we bounded the sum of Betti numbers of varieties which were presented as vanishing loci of polynomials with bounded degree. We now give the precise definition of complexity.

Notation: Throughout the rest of this paper, k will denote an algebraically closed field unless specified otherwise. Coefficients for the constructible derived category $D^b_c(X)$ will be $\overline{\mathbb{Q}}_\ell$ for some prime ℓ relatively prime to the characteristic of k.

4.1. **Complexity.** In this section we introduce the notion of complexity, which is the fundamental construction of QuantitativeSheafTheory. As the name suggests, to any sheaf we assign a number, measuring how “complex” it is.

Definition 4.1. Let k be a field, and $0 \leq m \leq n$ be integers. Let $M_k^{n+1,m+1}$ denote the variety over k of $(n+1) \times (m+1)$ matrices of maximal rank. For an extension k'/k and a point $\mathbf{a} \in M_k^{n+1,m+1}(k')$, denote $l_{\mathbf{a}} : \mathbb{P}^m_{k'} \to \mathbb{P}^n_{k'}$ the associated linear map.

We can now define complexity for sheaves on projective space.

Definition 4.2. Let k be a field and $n \geq 0$ an integer. For each $0 \leq m \leq n$, let $\mathbf{a}_m$ be a geometric generic point of $M_k^{n+1,m+1}$ defined over an algebraically closed field k'. For an object $F \in D^b_c(\mathbb{P}^n_k)$, define the complexity $c(F)$ to be

$$c(F) := \max_{1 \leq m \leq n} \sum_{i \in \mathbb{Z}} h^i(\mathbb{P}'^m_k, l^*_{\mathbf{a}_m} F).$$

The notion of complexity interpolates between the sum of Betti numbers ($m = n$) and the sum of the generic ranks of the cohomology sheaves ($m = 0$). Note that Betti numbers are invariant under base change of algebraically closed fields, so the choice of k' does not matter. As mentioned above, we can now define complexity for arbitrary quasi-projective varieties with a given embedding into projective space.

Definition 4.3. Let X be a variety over a field k and $u : X \to \mathbb{P}^n_k$ a locally closed embedding for some $n \geq 0$. Define the complexity of an object $F \in D^b_c(X)$ relative to u to be

$$c_u(F) := c(u_! F).$$

We also define the complexity of the locally closed embedding u to be

$$c(u) := c_u(\overline{\mathbb{Q}}_\ell).$$

One can also extend the notion of complexity to morphisms between quasi-projective varieties.

Definition 4.4 ([16] Def 6.6)**.** Let $f : X \to Y$ be a morphism of quasi-projective varieties over k, and $u : X \to \mathbb{P}^n_k$, $v : Y \to \mathbb{P}^m_k$ locally closed embeddings. For integers $0 \leq p \leq n$ and $0 \leq q \leq m$, let $\mathbf{a}_p$, $\mathbf{b}_q$ be geometric generic points of $M_k^{n+1,p+1}$ and $M_k^{m+1,q+1}$ respectively, defined over an algebraically closed field k'. Then define the complexity of f relative to u, v to be

$$c_{u,v}(f) := \max_{0\leq p\leq n} \max_{0\leq q\leq n} \sum_{i\in\mathbb{Z}} h^i_c(X_{k'}, u^* l_{\mathbf{a}_p *}\overline{\mathbb{Q}}_\ell \otimes f^* v^* l_{\mathbf{b}_q *}\overline{\mathbb{Q}}_\ell).$$

This notion of complexity of a morphism will play a role in the fundamental result of [16] on continuity of complexity under the 6 operations (see Theorem 4.12).

In order to prove the continuity of the 6 operations, the key one is the tensor product property.

$$c_u(A \otimes B) \ll c_u(A) c_u(B).$$

All other parts of the continuity theorem follow from formal arguments. The proof of this result takes up the bulk of the first half of [16] and we only briefly give an idea of the proof. As mentioned earlier, the key technical input is the characteristic cycle.

4.2. **Singular support and characteristic cycles.** In this section we recall the basic properties of singular support due to Beilinson [2] and characteristic cycle due to Saito [15]. We omit the somewhat technical definitions.

Remark 4.5. Beilinson and Saito work with finite coefficients. The characteristic cycle was extended to $\overline{\mathbb{Q}}_\ell$ coefficients in [17], while singular support for $\overline{\mathbb{Q}}_\ell$ coefficients was defined recently in [1]. In particular, we have an intrinsic definition of singular support for rational coefficients, and we don't need to choose an integral model of our sheaves. There is always a good integral model so that the singular support of its reduction in the sense of Beilinson agrees with the singular support for ℓ-adic sheaves as defined by Barrett.

Let k be a field of characteristic $p > 0$, and let X be a smooth k-scheme of dimension n. The *singular support* $SS(F)$ of an object $F \in D^b_c(X)$ is a closed conical subset $SS(F) \subseteq T^*X$ where T^*X is the cotangent bundle of X viewed as a scheme. Conical means invariant under the $\mathbb{G}_m$ scaling action on T^*X. The *characteristic cycle* $CC(F)$ is an algebraic cycle

$$CC(F) = \sum_i m_i [C_i]$$

where the C_i are the irreducible components of $SS(F)$, and $m_i \in \mathbb{Z}$ are integers. Intuitively, one can think of the singular support $SS(F)$ as measuring where F fails to be a local system. The characteristic cycle $CC(F)$ will be useful to us in controlling the Euler characteristic of F. We have the following formula for the

Euler characteristic as the intersection number of the characteristic cycle with the 0-section:

$$\chi(X, F) = CC(F) \cdot [T^*_X X].$$

Example 4.6. Let X be a curve, and $D \subset X$ a divisor. Let G be a local system on $X\backslash D$. Let $j : X\backslash D \hookrightarrow X$ be the inclusion, and set $F := j_! G$. Then $SS(F)$ is the union of the 0 section $T^*_X X$ and the vertical fibers $T^*_x X$ for each $x \in D$ (note that $x \in D$ are exactly the points where F fails to be a local system). The characteristic cycle in this case is

$$CC(F) = -\left(\text{rank } \cdot G[T^*_X X] + \sum_{x\in D} (\text{rank } G + \text{swan}_x G) \cdot [T^*_x X]\right).$$

For this example, the above formula for the Euler characteristic recovers the Grothendieck-Ogg-Shafarevich formula. In general one has $CC(F[n]) = (-1)^n CC(F)$. Here, $F[1]$ is perverse and $CC(F[1])$ is effective. In general, for F perverse, $CC(F)$ is effective.

Recall that for proving things about complexity, we are primarily interested in the case $X = \mathbb{P}^n$. If V is a vector bundle on a scheme X, let $\overline{V}$ denote the projective bundle $\mathbb{P}(V + \mathcal{O}_X)$ over X. For $F \in D^b_c(X)$, let $\overline{CC(F)}$ denote the closure of $CC(F)$ in $\overline{T^*X}$.

Definition 4.7 ([16] Definition 2.4)**.** There is an isomorphism $CH_*(\mathbb{P}^n) \to CH_n(\overline{T^*\mathbb{P}^n})$ given by

$$(a_i) \mapsto \sum_i p^* a_i h^i$$

where h is the first Chern class of the dual of the universal line bundle, and $p : \overline{T^*\mathbb{P}^n} \to \mathbb{P}^n$ is the projection map. For an object $F \in D^b_c(X)$, we define the *characteristic class* $cc(F) \in CH_*(\mathbb{P}^n) \simeq \mathbb{Z}^{n+1}$ to be the inverse image of $\overline{CC(F)}$ under the above isomorphism.

Note that the isomorphism $CH_*(\mathbb{P}^n) \simeq \mathbb{Z}^{n+1}$ comes from the basis of $CH_*(\mathbb{P}^n)$ given by linear subspaces of dimensions 0 to n. The group $\mathbb{Z}^{n+1}$ inherits the intersection pairing on $CH_n(\overline{T^*\mathbb{P}^n})$. Now, recall that we want to control the complexity of a tensor product of sheaves, and the first step is to control the Euler characteristic of a tensor product. We can do this under sufficient transversality, which is why a modification by a generic automorphism will show up. One can later work to remove the generic automorphism via a "double induction" argument (see [16] §5).

Theorem 4.8 ([16] Corollary 3.14)**.** *Let A, B be objects of $D^b_c(\mathbb{P}^n_k)$. For every geometric generic point* ð$of GL_{n+1,k}$ *over an algebraically closed field k', we have*

$$\chi(\mathbb{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g}} B) = (-1)^n cc(A) \cdot cc(B).$$

We will also need the following lemma, which shows that restriction of sheaves to a generic linear codimension 1 subspace gives a linear function on the characteristic class.

Lemma 4.9 ([16] Lemma 3.15)**.** *There exists a linear function $f_n : \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ so that for every perverse sheaf $A \in D^b_c(\mathbb{P}^n_k)$, and every geometric point $\boldsymbol{b}$ of $M^{n+1,n}_k$, we have*

$$cc(l_{\boldsymbol{b}}^* A) = f_n(cc(A)).$$

Strictly speaking I don't believe the perverse condition is necessary. However we will only be applying this lemma to perverse sheaves. We defer the proofs of the above two results to [16]. We can now prove the main result of [16, §3] which will be useful in controlling the complexity of a tensor product. We include the proof as it is similar to arguments in Section 3, where we used the Lefschetz hyperplane argument to inductively control cohomology in degree $< n-1$ and the Euler characteristic to control the remaining cohomology. Here, we use perversity to inductively control cohomology in all degrees except $-n$, and Euler characteristic to control h^{-n}.

Proposition 4.10 ([16] Proposition 3.17)**.** *Let $f_n : \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ be the linear function as above. We define a bilinear form*

$$b_n : \mathbb{Z}^{n+1} \times \mathbb{Z}^{n+1} \to \mathbb{Z}$$

inductively using the formula $b_0(x, y) = xy$ for $x, y \in \mathbb{Z}$, and

$$b_n(x, y) = x \cdot y + 4b_{n-1}(f_n(x), f_n(y)).$$

Recall that $\cdot$ is the intersection pairing on $CH_n(\overline{T^*X}) \simeq \mathbb{Z}^{n+1}$. Then for any perverse sheaves $A, B \in D_c^b(\mathbb{P}_k^n)$, and any geometric generic point $\boldsymbol{g}$ of $GL_{n+1,k}$ over an algebraically closed field k', we have

$$\sum_{i\in\mathbb{Z}} h^i(\mathbb{P}_{k'}^n, A \otimes l_{\boldsymbol{g}}^* B) \leq b_n(cc(A), cc(B)).$$

Proof. We induct on n. For $n = 0$, A, B are just vector spaces (since they are perverse), the characteristic class is their dimension, and the Euler characteristic is the product of the dimensions.

Now let $n > 0$. By [16] Lem 3.11, $A \otimes l_{\mathbf{g}}^* B[-n]$ is perverse. Let

$$\sigma := \sum_{i\in\mathbb{Z}} h^i(\mathbb{P}_{k'}^n, A \otimes l_{\mathbf{g}}^* B)$$

be the quantity we are estimating. By definition of Euler characteristic, we have

$$\sigma \leq 2\sum_{i<-n} h^i(\mathbb{P}_{k'}^n, A \otimes l_{\mathbf{g}}^* B) + (-1)^n \chi(\mathbb{P}_{k'}^n, A \otimes l_{\mathbf{g}}^* B) + 2\sum_{i>-n} h^i(\mathbb{P}_{k'}^n, A \otimes l_{\mathbf{g}}^* B).$$

Let $\mathbf{a} \in M_k^{n+1,n}$ be a geometric generic point defined over k'. Since affine morphisms are perverse t-exact, the compactly supported cohomology of perverse sheaves on $\mathbb{A}^n$ vanish in degrees < 0. Since $A \otimes l_{\mathbf{g}}^* B[-n]$ is perverse, excision shows that the canonical map

$$H^i(\mathbb{P}_{k'}^n, A \otimes l_{\mathbf{g}}^* B) \to H^i(\mathbb{P}_{k'}^{n-1}, l_{\mathbf{a}}^*(A \otimes l_{\mathbf{g}}^* B))$$

is an isomorphism for $i < -n$. Since duality exchanges $l_{\mathbf{a}}^*$ and $l_{\mathbf{a}}^!$ and preserves perversity, the map

$$H^i(\mathbb{P}_{k'}^{n-1}, l_{\mathbf{a}}^!(A \otimes l_{\mathbf{g}}^* B)) \to H^i(\mathbb{P}_{k'}^n, A \otimes l_{\mathbf{g}}^* B)$$

is an isomorphism for $i > -n$. So we have the estimate

$$\sigma \leq 2\sum_{i\in\mathbb{Z}} H^i(\mathbb{P}^{n-1}_{k'}, l^*_{\mathbf{a}}(A\otimes l^*_{\mathbf{g}}B)) + (-1)^n\chi(\mathbb{P}^n_{k'}, A\otimes l^*_{\mathbf{g}}B) + 2\sum_{i\in\mathbb{Z}} H^i(\mathbb{P}^{n-1}_{k'}, l^!_{\mathbf{a}}(A\otimes l^*_{\mathbf{g}}B)).$$

By [16] Lemma 3.12, the functors $l^*_{\mathbf{a}}$ and $l^!_{\mathbf{a}}$ agree up to shift and Tate twist, so

$$\sigma \leq 4\sum_{i\in\mathbb{Z}} H^i(\mathbb{P}^{n-1}_{k'}, l^*_{\mathbf{a}}(A\otimes l^*_{\mathbf{g}}B)) + (-1)^n\chi(\mathbb{P}^n_{k'}, A\otimes l^*_{\mathbf{g}}B).$$

Let $\mathbf{h}$ be a geometric generic point of $GL_{n,k(\mathbf{a},\mathbf{g})}$, and set $\mathbf{b} = \mathbf{g}\mathbf{a}\mathbf{h}^{-1}$. By [16] Lemma 3.7, $\mathbf{h}$ is generic over $k(\mathbf{a},\mathbf{b})$. We can write $l^*_{\mathbf{a}}(A\otimes l^*_{\mathbf{g}}B) = l^*_{\mathbf{a}}A\otimes l^*_{\mathbf{h}}l^*_{\mathbf{b}}$. So by the inductive hypothesis and Proposition 4.10,

$$\sum_{i\in\mathbb{Z}} H^i(\mathbb{P}^{n-1}_{k'}, l^*_{\mathbf{a}}(A\otimes l^*_{\mathbf{g}}B)) \leq b_{n-1}(cc(l^*_{\mathbf{a}}A), cc(l^*_{\mathbf{b}})) = b_{n-1}(f_n(cc(A)), f_n(cc(B))).$$

By 4.9, we are done. □

We now sketch the proof of the following result. One of the key ingredients which we have omitted is the theory of test sheaves in [16] §4. They allow us to bound how "big" $cc(A)$ is in terms of the complexity $c(A)$.

Theorem 4.11 ([16] Corollary 5.3)**.** *Let A, B be objects of $D^b_c(\mathbb{P}^n_k)$. Then*

$$c(A\otimes B) \ll c(A)c(B)$$

where the implicit constant only depends on n.

Proof. We sketch the proof. One first notes that by unraveling the definition of complexity, and using the fact that the intersection of a generic linear subspace of dimension m and dimension m' is a generic linear subspace of dimension $m+m'-n$, the proof reduces to showing

$$\sum_{i\in\mathbb{Z}} h^i(\mathbb{P}^n_k, A\otimes B) \ll c(A)c(B).$$

By a double induction argument, one further reduces to replacing $A\otimes B$ with $A\otimes l^*_{\mathbf{g}}B$ for a geometric generic point $\mathbf{g}$ of $GL_{n+1,k}$. Using the spectral sequence associated to the perverse filtrations on A and B, we can apply Proposition 4.10 to bound the $\sum_{i\in\mathbb{Z}} h^i(\mathbb{P}^n_k, A\otimes l^*_{\mathbf{g}}B)$ in terms of the characteristic classes (and the bilinear form b_n which only depends on n). Then the theory of test sheaves allows us to get to the complexity. We refer to [16] §4, 5 for more details. □

We can now state the main theorem of complexity, which is the continuity of the 6 operations. We recall the following notation: $D(-)$ denotes the Verdier dual $Hom(-,\omega_X)$, and $\otimes_!$ is defined to be $A\otimes_! B := \Delta^!(A\boxtimes B)$, which satisfies $D(A\otimes_! B) = D(A)\otimes D(B)$.

Theorem 4.12 ([16] Theorem 6.8)**.** *Let X, Y be quasiprojective varieties over k, and $u : X\to\mathbb{P}^n_k$, $v : Y\to\mathbb{P}^m_k$ locally closed embeddings, and $f : X\to Y$ a morphism. For objects $A, B\in D^b_c(X)$ and $C\in D^b_c(Y)$, we have:*

(1) $c_u(D(A)) \ll c(u)c_u(A)$
(2) $c_u(A \otimes B) \ll c_u(A)c_u(B)$
(3) $c_u(A \otimes_! B) \ll c_u(A)^3 c_u(A)c_u(B)$
(4) $c_u(\mathscr{H}om(A,B)) \ll c(u)c_u(A)c_u(B)$
(5) $c_u(f^*C) \ll c_{u,v}(f)c_v(C)$
(6) $c_v(f_!A) \ll c_{u,v}(f)c_v(A)$
(7) $c_u(f^!C) \ll c(u)c(v)c_{u,v}(f)c_v(C)$
(8) $c_u(f_*A) \ll c(u)c(v)c_{u,v}(f)c_v(A)$

where constants only depend on n, m.

The proof of the above theorem follows from Theorem 4.11 using formal properties of ℓ-adic cohomology (e.g. projection formula, Verdier duality, etc.) and the definition of complexity. The non-formal fact that $l_{\mathbf{a}}^* = l_{\mathbf{a}}^!$ up to shift and Tate twist for $\mathbf{a}$ a geometric generic point of $M^{n+1,m+1}$ is again used.

5. Horizontal equidistribution

In this section we highlight an application of complexity to a horizontal equidistribution result. The meaning of "horizontal equidistribution" is that the result will be about distributions of exponential sums as $p \to \infty$, instead of studying $t_n : X(\mathbb{F}_p^n) \to \mathbb{C}$ at a fixed prime p and taking $n \to \infty$ which one could call "vertical". We first record a preliminary result.

Proposition 5.1 ([16] Proposition 6.33)**.** *Let $G \subset GL_n$ be a reductive algebraic group with finite center, and let Std denote the standard n-dimensional representation. If $\rho : G \to GL(V)$ is an irreducible representation, then ρ is a subrepresentation of $Std^{\otimes a}$ for some $a \geq 0$.*

By applying (2) of the continuity theorem, if $u : X \to \mathbb{P}_k^n$ is an embedding of a quasiprojective variety X/k, then any $A \in D_c^b(X)$ satisfies $c_u(\rho(A)) \ll c_u(A)^a$ for a constant a only depending on ρ.

Now we introduce some notation for the equidistribution theorem. Let $n \geq 1$ be an integer, and $d \geq 1$ an odd integer. Let $P(n,d)$ denote the space of polynomials of degree d in n variables, and $P(n,d,\text{odd})$ the polynomials only containing monomials of odd degree. Let $D(n,d) \subset P(n,d)$ denote the dense open of Deligne polynomials, i.e. where the homogeneous part of largest degree defines a smooth hypersurface in $\mathbb{P}^{n-1}$, and similarly $D(n,d,\text{odd}$. The schemes $D(n,d), D(n,d,\text{odd})$ are smooth over $Spec(\mathbb{Z})$. For any prime number p and $f \in P(n,d)(\mathbb{F}_p)$, set

$$S(f,p) := \frac{1}{w(f)} \sum_{x \in \mathbb{A}^n(\mathbb{F}_p)} exp\left(\frac{f(x)}{p}\right)$$

where $w(f)$ is the smallest integer so that $H_c^i(\mathbb{A}^n_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\psi(f)}) = 0$ for all $i > w(f)$. Here, we have fixed a nontrivial additive character $\psi : \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^\times \simeq \mathbb{C}^\times$, and set $\mathcal{L}_{\psi(f)}$ to be the pullback of the Artin-Schreier sheaf along the map $f : \mathbb{A}^n \to \mathbb{A}^1$. More generally, if $\mathbb{A}^n \times D(n,d,) \to \mathbb{A}^1$ is the evaluation map sending $x, f \mapsto f(x)$, then define $\mathcal{L}_{\psi(f(x))}$ to be the pullback of the Artin-Schreier sheaf.

Theorem 5.2. *Let $n \geq 1$ be an integer, and $d \geq 1$ an odd integer. For n odd, set $K_{n,odd} = U_{(d-1)^n}(\mathbb{C}) \cap Sp_{(d-1)^n}(\mathbb{C})$, and for n even, set $K_n = O_{(d-1)^n}(\mathbb{C})$. Then the families $(S(f,p))_{f \in P(n,d,odd)(\mathbb{F}_p}$ are equidistributed as $p \to \infty$ under the trace of the probability Haar measure on $K_{n,odd}$.*

Remark 5.3. This theorem is interesting because, as explained in [11, Sections 12.5, 12.6], there exist analogs of Kummer sheaves over $Spec(\mathbb{Z})$ which allow one to prove horizontal equidistribution results, but there is no analogue for Artin-Schreier sheaves over $Spec(\mathbb{Z})$.

Proof. The case of $d = 1$ is immediate, since exponential sums of linear polynomials vanish, and $(d-1)^n = 0$, so $K_{n,\text{odd}}$ is trivial. For $d \geq 3$, since the complexity of Artin-Schreier sheaves are constant as p varies, the Riemann hypothesis implies that $|S(f,p)|$ is bounded as f,p varies. Since $D(n,d,\text{odd}) \subset P(n,d,\text{odd})$ is dense, it suffices to prove equidistribution for $D(n,d,\text{odd})$. Now, consider the projection to the second factor

$$\pi_2 : \mathbb{A}^n \times D(n,d,\text{odd}) \to D(n,d,\text{odd}).$$

Define $\mathcal{D}_p := \pi_{2,!}\mathcal{L}_{\psi(f(x))}[n](n/2)$, so that by construction,

$$S(f,p) = (-1)^n t_{\mathcal{D}_p}(f)$$

since shriek pushforward on the level of trace functions implements summing over the fiber. By [11, Cor 3.5.11, 3.5.12] this object of $D_c^b(D(n,d,\text{odd}))$ is actually a local system of rank $(d-1)^n$ and pure of weight 0 for each prime $p \nmid d$. For $p \geq 7$ and $p \nmid d$, the geometric and arithmetic monodromy groups agree, and are $Sp_{(d-1)^n}$ for n odd and $O_{(d-1)^n}$ for n even [11, Theorem 12.6.3]. So by definition, $K_{n,\text{odd}}$ is a maximal compact subgroup of the monodromy group of $\mathcal{D}_p$.

Using (5) and (6) along with the facts that the Artin-Schreier sheafs have bounded complexity as p varies, and that complexities of morphisms are bounded in families (by constructibility), we deduce that $c(\mathcal{D}_p)$ is bounded as $p \to \infty$. To show equidistribution, we now need to check the Weyl equidistribution theorem, namely that

$$\lim_{p\to\infty} \frac{1}{|D(n,d,\text{odd})|} \sum_{x\in\mathbb{F}_p^n} t_{\rho(\mathcal{D}_p)}(x) = 0$$

for every nontrivial irreducible representation ρ of $K_{n,\text{odd}}$. By the remark following Proposition 5.1, since we showed $c(\mathcal{D}_p)$ is bounded, so is $c(\rho(\mathcal{D}_p))$. Again by Proposition 5.1, the sheaf $\rho(\mathcal{D}_p)$ is pure of weight 0 since it is a subhseaf of $\mathcal{D}_p^{\otimes a}$. Finally, applying the Lang-Weil bound, the Riemann hypothesis, and the geometric irreducibility and non-triviality of $\rho(\mathcal{D}_p)$, we deduce that

$$\frac{1}{|D(n,d,\text{odd})|} \sum_{x\in\mathbb{F}_p^n} t_{\rho(\mathcal{D}_p)}(x) \ll c_u(\rho(\mathcal{D}_p))p^{-1/2}$$

hence the limit as $p \to \infty$ is zero. □

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