THE CLOSED SUBGROUP THEOREM

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Abstract. Lie groups are mathematical structures that unify the properties of groups with those of smooth manifolds, enriching them with both algebraic and geometrical traits. Hence, they provide a framework for understanding continuous symmetry, which has numerous applications in mathematics and physics. An important theorem on the structure of Lie groups is the Closed Subgroup Theorem. It states that any closed subgroup of a Lie group is itself a Lie group, and therefore inherits a smooth manifold structure.

Contents

1. Lie Groups

In this section, we will define Lie groups and describe their basic properties. The reader is expected to have a grasp of the concepts of manifolds and diffeomorphisms.

Definition 1.1. A Lie group is a smooth manifold G that is also a group in the algebraic sense, with the additional property that the multiplication map

$$m : G \times G \to G, \quad m(g, h) = gh$$

and the inversion map

$$i : G \to G, \quad i(g) = g^{-1}$$

are both smooth.

One of the most useful properties of Lie groups is that it is possible to map any element of the Lie group to any other element by a diffeomorphism.

Definition 1.2. Let G be a Lie group and let $g \in G$ be an element. The maps

$$L_g, R_g : G \to G$$

called left translation and right translation, respectively, are defined by

$$L_g(h) = gh$$
$$R_g(h) = hg.$$

Theorem 1.3. *Let G be a Lie group and let $g \in G$ be an element. Left translation $L_g : G \to G$ and right translation $R_g : G \to G$ are diffeomorphisms.*

Proof. Let G be a Lie group and let $g \in G$ be an element. The map L_g can be expressed as the composition of smooth maps

$$G \xrightarrow{\iota_g} G \times G \xrightarrow{m} G$$

where $\iota_g(h) = (g, h)$. As a result, L_g is smooth. Moreover, $L_{g^{-1}}$ can be expressed as the composition of smooth maps

$$G \xrightarrow{\iota_{g^{-1}}} G \times G \xrightarrow{m} G.$$

Thus, $L_{g^{-1}}$ is also smooth.

A similar argument shows that right translation is a diffeomorphism. □

2. Some Examples

From the definition of a Lie group, we can prove that certain subsets of $M(n, \mathbb{R})$ are Lie groups.

Proposition 2.1. *The set of invertible $n \times n$ matrices, called the general linear group, is a Lie Group.*

Proof. We can express the general linear group as

$$GL(n, \mathbb{R}) = \{M \in M(n, \mathbb{R}) \mid M \in \det^{-1}(\mathbb{R} \backslash \{0\})\}.$$

The set $GL(n, \mathbb{R})$ is a group under multiplication, since it only consists of invertible matrices.

The determinant is a continuous function, and $\mathbb{R} \backslash \{0\}$ is open, so $\det^{-1}(\mathbb{R}) \backslash \{0\}$ is also open. The set $GL(n, \mathbb{R}) \subset M(n, \mathbb{R})$ is an open set in a Euclidean space, therefore it is also an n^2-dimensional manifold.

For any $A, B \in GL(n, \mathbb{R})$, matrix multiplication is given by

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj},$$

which is a polynomial of the entries, hence smooth.
Cramer's rule tells us that matrix inversion is given by

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A),$$

when $\det(A) \neq 0$. The entries of the adjugate $adj(A)$ are polynomial functions of the entries of A, and the determinant is also a polynomial. Since the reciprocal of a smooth function is smooth on the domain where the function is nonzero, $\frac{1}{\det(A)}$ is smooth, and so is the inversion map. □

Proposition 2.2. *The circle S^1 is a Lie group.*

Proof. The circle S^1 is a one-dimensional manifold. We can embed S^1 into $GL(2, \mathbb{R})$ as

$$S^1 = \left\{ A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \, \theta \in [0, 2\pi) \right\}.$$

Let's prove that S^1 is a group.

(1) If $A(\theta), A(\phi) \in S^1$, matrix multiplication shows that
$$A(\theta)A(\phi) = A(\theta + \phi) \in S^1.$$
(2) For all $\theta \in [0, 2\pi)$,
$$A(\theta)A(0) = A(\theta)I_2 = A(\theta).$$
(3) Additionally,
$$A(\theta)A(2\pi - \theta) = I_2 = A(0).$$

Since we have already seen that matrix multiplication and inversion are smooth, this is sufficient to show that S^1 is a Lie group. □

Theorem 2.3. *The Cartesian product of two Lie groups is a Lie group.*

Proof. Let G, H be two Lie groups. Then, there exist two multiplication maps
$$m_G : G \times G \to G, \quad m_G(g, g') = gg'$$
and
$$m_H : H \times H \to H, \quad m_H(h, h') = hh'$$
that are both smooth. Thus, we can define a smooth multiplication map
$$m : (G \times H) \times (G \times H) \to G \times H,$$
given by
$$m((g, h), (g', h')) = (m_G(g, g'), m_H(h, h')).$$
Similarly, we can define an inversion map
$$i : G \times H \to G \times H, \quad i((g, h)) = (i_G(g), i_H(h))$$
that is the product of the two smooth inversion maps. □

3. Immersions and Embeddings

The fact that Lie groups are smooth manifolds endows them with interesting geometric properties. In this section, we will see that if an embedded submanifold of a Lie group is also a group, then it is automatically a Lie subgroup.

Definition 3.1. An immersion is a map $f : S \to M$ between two manifolds S and M such that, at every point $p \in S$, its differential
$$df_p : T_pS \to T_{f(p)}M$$
is injective. The manifold S is called an immersed submanifold of M.

Definition 3.2. Let G be a Lie group and let $H \subseteq G$ be a subgroup of G. The subgroup H is a Lie subgroup of G if it is itself a Lie group and if it is an immersed submanifold of G.

Definition 3.3. Let M be a k-dimensional manifold, and $S \subset M$ a d-dimensional submanifold of M, where $d \leq k$ necessarily. The manifold S is an embedded submanifold of M if the inclusion map
$$\iota : S \to M$$
is a smooth immersion and a homeomorphism onto its image.

We will now see that, if a manifold H is embedded into a Lie group G and not just immersed, this automatically proves that H is a Lie subgroup without even needing to show that H is itself a Lie group.

Proposition 3.4. *Let G be a Lie group, and suppose $H \subseteq G$ is a subgroup that is also an embedded submanifold. Then H is a Lie subgroup.*

Proof. We only need to check that H is a Lie group, and thus that multiplication $H \times H \to H$ and inversion $H \to H$ are smooth maps. We know that G is a Lie group, thus multiplication is a smooth map from $G \times G$ into G. This is also true if we map $H \times H$ into G. Since H is a subgroup, multiplication maps $H \times H$ into H, and since H is embedded, this is a smooth map into H.

Similarly, inversion is smooth from G to G, and so is it from H into G. Moreover, H contains the multiplicative inverse of all its elements, thus inversion maps H smoothly into H. □

4. One-parameter Subgroups and the Exponential Map

In this section, we will define one-parameter subgroups of $GL(n, \mathbb{R})$, which are just Lie group homomorphisms from $\mathbb{R}$ to G [1]. We will also show that there is a one-to-one correspondence between elements of $M(n, \mathbb{R})$ and one-parameter subgroups of $GL(n, \mathbb{R})$.

While in this section we will define one-parameter subgroups for $GL(n, \mathbb{R})$, they can be defined for any Lie group G. In that case, the one-to-one correspondence will be between the one-parameter subgroups and elements of the tangent space at the identity called the Lie algebra.

Then, we will define the exponential map, a smooth map from the Lie algebra into the Lie group. Fundamentally, the exponential map maps tangent vectors at the identity to one-parameter subgroups of G.

Establishing a local diffeomorphism between the Lie group and one-parameter subgroups at the identity will be essential in proving the Closed Subgroup Theorem as it allows us to show that certain subgroups of Lie groups are manifolds.

Definition 4.1. A one-parameter subgroup of a Lie group G is a smooth map $\gamma : \mathbb{R} \to G$ such that:

(1) $\gamma(0) = e$, where e is the identity element of G,
(2) $\gamma(t + s) = \gamma(t)\gamma(s)$ for all $t, s \in \mathbb{R}$.

We will restrict our focus of interest to Lie groups that are in $GL(n, \mathbb{R})$. The following map will allow us to construct one-parameter subgroups from $\mathbb{R}$ to $GL(n, \mathbb{R})$.

Definition 4.2. For any $A \in M(n, \mathbb{R})$, let:

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I_n + A + \frac{1}{2}A^2 + \dots$$

Proposition 4.3. *For all $A \in M(n, \mathbb{R})$, the series*

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

converges to an invertible matrix.

Proof. Let $A \in M(n, \mathbb{R})$. Matrix multiplication satisfies $|AB| \leq |A||B|$. Thus, $|A^k| \leq |A|^k$ for all $k > 0$. Moreover, the sequence $\sum_{k=0}^{\infty} \frac{|A|^k}{k!}$ converges, and thus

[1] To read about the general case, one can consult Introduction to Smooth Manifolds by Lee.

by the comparison test $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$ also converges. We now want to show that e^A is invertible:

$$\begin{aligned}
e^A e^{-A} &= \sum_{k=0}^{\infty} \frac{A^k}{k!} \sum_{m=0}^{\infty} \frac{(-A)^m}{m!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{A^k(-A)^{n-k}}{k!n-k!} \quad \text{setting } n = k+m \\
&= \sum_{n=0}^{\infty} A^n \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!n-k!} \\
&= \sum_{n=0}^{\infty} \frac{A^n}{n!} \sum_{k=0}^{n} \binom{n}{k} 1^k (-1)^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{A^n}{n!} (1-1)^n.
\end{aligned}$$

When $n \geq 1$, $(1-1)^n = 0$ and when $n = 0$, $(1-1)^n = 1$. Thus, $e^A e^{-A} = \mathrm{Id}$, which proves that e^A is invertible.

□

Proposition 4.4. *Let $A \in M(n, \mathbb{R})$. The function $\gamma : \mathbb{R} \to GL(n, \mathbb{R})$ defined by $\gamma(t) = e^{tA}$ is a one parameter subgroup of $GL(n, \mathbb{R})$.*

Proof. Let $A \in M(n, \mathbb{R})$. We can derive $\gamma(0) = e^{0 \cdot A} = e^0 = I_n$, which is the identity element.

To show that γ is a Lie group homomorphism from $\mathbb{R}$ to G, we can prove that

$$\begin{aligned}
\gamma(t+s) &= e^{(t+s)A} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}(t+s)^k A^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n=0}^{k} \frac{k!}{n!(k-n)!} t^n s^{k-n} A^k \\
&= \sum_{k=0}^{\infty} \sum_{n=0}^{k} \frac{1}{n!(k-n)!} t^n s^{k-n} A^k \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{n!m!} t^n s^m A^{n+m} \quad \text{setting } m = k-n \\
&= e^{tA} e^{sA} \\
&= \gamma(s)\gamma(t)
\end{aligned}$$

for all $s, t \in \mathbb{R}$. □

Therefore, for all $A \in M(n, \mathbb{R})$, we say that $\gamma_A(t) = e^{tA}$ is the one-parameter subgroup generated by A.

Proposition 4.5. *Let $A \in M(n, \mathbb{R})$ and let $\gamma_A(t) = e^{tA}$ be a one parameter subgroup of $GL(n, \mathbb{R})$. Then,*

$$\frac{d}{dt} \gamma_A(t) \bigg|_{t=0} = A$$

Proof. We can calculate directly that

$$\begin{aligned}\frac{d}{dt}\gamma_A(t)\bigg|_{t=0} &= \lim_{h\to 0}\left(\frac{e^{hA}-e^0}{h}\right)\\ &= \lim_{h\to 0}\left(\frac{I_n + Ah + \frac{A^2h^2}{2} + \ldots - I_n}{h}\right)\\ &= A.\end{aligned}$$

□

Definition 4.6. Let G be a Lie group. The Lie algebra $\mathfrak{g}$ of G is defined to be the tangent space of G at the identity.

Remark 4.7. The Lie algebra of $GL(n,\mathbb{R})$ is $M(n,\mathbb{R})$. We know that $M(n,\mathbb{R})$ is a vector space, so the tangent space at any point of $M(n,\mathbb{R})$ is simply $M(n,\mathbb{R})$. Due to $GL(n,\mathbb{R})$ being an open subset of $M(n,\mathbb{R})$, the tangent space of $GL(n,\mathbb{R})$ at every point is also $M(n,\mathbb{R})$.

Definition 4.8. Given a Lie group $G \in GL(n,\mathbb{R})$ with Lie algebra $\mathfrak{g}$, the exponential map

$$\exp : \mathfrak{g} \to G$$

is defined by

$$\exp(A) = \gamma_A(1),$$

where γ_A is the one-parameter subgroup generated by $A \in \mathfrak{g}$. Thus, for all $A \in M(n,\mathbb{R})$,

$$\exp(A) = e^{tA}.$$

Proposition 4.9. *Let G be a Lie group and let $\mathfrak{g} = Lie(G)$. Suppose that $X, Y \in \mathfrak{g}$ are two commuting elements of the Lie algebra. For any $t \in \mathbb{R}$,*

$$(\exp(tX))(\exp(tY)) = \exp(t(X+Y))$$

Proof. Let G be a Lie group and suppose that $X, Y \in \mathrm{Lie}(G)$ commute. Then, for all $t \in \mathbb{R}$,

$$\begin{aligned}(\exp(tX))(\exp(tY)) &= \sum_{k=0}^{\infty}\frac{t^k}{k!}X^k\sum_{n=0}^{\infty}\frac{t^n}{n!}Y^n\\ &= \sum_{m=0}^{\infty}\frac{t^m}{m!}\sum_{k=0}\binom{m}{k}X^kY^{m-k} \qquad \text{setting } m = n+k\\ &= \sum_{m=0}^{\infty}\frac{t^m}{m!}(X+Y)^m\\ &= \exp(t(X+Y)).\end{aligned}$$

□

Using this result, we can prove by induction that

$$\exp(tX)^n = \exp(ntX). \tag{4.10}$$

5. The Closed Subgroup Theorem

In this section, we prove the Closed Subgroup theorem in $GL(n,\mathbb{R})$, which says that every topologically closed subgroup of a Lie group $G \subseteq GL(n,\mathbb{R})$ is actually an embedded Lie subgroup.

Before proving the Closed Subgroup Theorem, we first need to prove a property about the exponential map that will allow us to define the Lie algebra of any closed subgroup.

Proposition 5.1. *Let $G \subseteq GL(n,\mathbb{R})$ be a Lie group and let $\mathfrak{g}$ be its Lie algebra. For any $X, Y \in \mathfrak{g}$, there exists some $\epsilon > 0$ such that for all $t \in (-\epsilon, \epsilon)$,*

$$(\exp(tX))(\exp(tY)) = \exp\left(t(X+Y) + O(t^2)\right).$$

Proof. We saw in Proposition 4.5 that for all $A \in M(n,\mathbb{R})$,

$$\begin{aligned} \frac{d}{dt}\exp(tA)\bigg|_{t=0} &= \frac{d}{dt}\gamma_A(t)\bigg|_{t=0} \\ &= A. \end{aligned}$$

Therefore,

$$d\exp(0) = I_n$$

which is an invertible map. Thus by the inverse function theorem, there exists $\epsilon > 0$ such that the map $\phi : (-\epsilon,\epsilon) \to \mathfrak{g}$ defined by

$$\phi(t) = \exp^{-1}(\exp(tX)\exp(tY))$$

is smooth. We obviously have

$$\phi(0) = 0$$

Observe that we can write φ as the composition:

$$\mathbb{R} \xrightarrow{e^X, e^Y} G \times G \xrightarrow{m} G \xrightarrow{\exp^{-1}} g,$$

where $e^X(t) = \exp(tX)$ and $e^Y(t) = \exp(tY)$.

We now want to show that the derivative $dm(e,e) : T_eG \times T_eG \to T_eG$ satisfies $dm(e,e)(X,Y) = X + Y$ for $X, Y \in T_eG$. We can prove that

$$\begin{aligned} \lim_{t\to\infty} \frac{(I_n + tX)(I_n + tY) - I_n}{t} &= \lim_{t\to\infty} \frac{tX + tY + t^2XY}{t} \\ &= X + Y. \end{aligned}$$

Thus, we have

$$\varphi'(0) = (d\exp)_0^{-1}\left(e_0^X(0) + e_0^Y(0)\right) = X + Y.$$

Therefore, Taylor's theorem yields

$$\varphi(t) = \phi(0) + t\phi'(0) + O(t^2) = t(X+Y) + O(t^2)$$

for $t \in (-\epsilon, \epsilon)$. □

Proposition 5.2. *Let $G \subseteq GL(n,\mathbb{R})$ be a Lie group and let $\mathfrak{g}$ be its Lie algebra. For any $X, Y \in \mathfrak{g}$ and for any $t \in \mathbb{R}$,*

$$\lim_{n\to\infty}\left(\exp\left(\frac{t}{n}X\right)\exp\left(\frac{t}{n}Y\right)\right)^n = \exp(t(X+Y)).$$

Proof. Let G be a Lie group and suppose that $X, Y \in \mathrm{Lie}(G)$ and $t \in \mathbb{R}$. Since $\lim_{n\to\infty} \frac{t}{n} = 0$,

$$\begin{aligned}
\lim_{n\to\infty}\left(\exp\left(\frac{t}{n}X\right)\exp\left(\frac{t}{n}Y\right)\right)^n &= \lim_{n\to\infty}\exp\left(\frac{t}{n}(X+Y)+O\left(\frac{t^2}{n^2}\right)\right)^n. \\
&= \lim_{n\to\infty}\exp\left(n\frac{t}{n}(X+Y)+nO\left(\frac{t^2}{n^2}\right)\right)^n \quad \text{from Identity (4.10)} \\
&= \lim_{n\to\infty}\exp\left(\frac{t}{n}n(X+Y)+O\left(\frac{t^2}{n}\right)\right) \\
&= \exp\left(t(X+Y)\right).
\end{aligned}$$

□

Thus, suppose that G is a closed Lie group whose Lie algebra is $\mathfrak{g}$. This last proposition shows that if we have a closed subgroup such that $X, Y \in G$ are elements of the Lie algebra, and thus correspond to a one-parameter subgroup of G, then $X + Y$ is also an element of the Lie algebra and corresponds to a one-parameter subgroup of G. This is only possible if G is closed. For this reason, as its name indicates, the Closed Subgroup Theorem is only applicable to closed subgroups.

Theorem 5.3. ***(Closed Subgroup Theorem)*** *Suppose $G \subseteq GL(n,\mathbb{R})$ is a Lie group and $H \subseteq G$ is a subgroup that is also a closed subset of G. Then H is an embedded Lie subgroup of G.*

Proof. By Proposition 3.4, it is enough to prove that H is an embedded submanifold.

Let $d = \dim(G)$. Suppose that $\mathfrak{g}$ is the Lie algebra of G. Then, $\mathfrak{g}$ is a d-dimensional vector space. The map

$$\exp : \mathfrak{g} \to G$$

maps elements from the Lie algebra to the Lie group. We saw in Proposition 4.5 that for all $A \in M(n,\mathbb{R})$,

$$\begin{aligned}
\frac{d}{dt}\exp(tA)\bigg|_{t=0} &= \frac{d}{dt}\gamma_A(t)\bigg|_{t=0} \\
&= A.
\end{aligned}$$

Therefore,

$$d\exp(0) = I_n$$

which is an invertible map.

Thus, by the Inverse Function Theorem, there exist two neighborhoods $U \subseteq \mathfrak{g}$ of the 0 element and $V \subseteq G$ of the identity matrix such that

$$\exp : U \to V$$

is a diffeomorphism.

Now let $H \subseteq G$ and define

$$\mathfrak{h} = \{\, X \in \mathfrak{g} |\, e^{tX} \in H \text{ for all } t \in \mathbb{R}\}.$$

We can see from the definition of $\mathfrak{h}$ that if $X \in \mathfrak{h}$, then for any scalar $c \in \mathbb{R}$, cX will also be in X. Thus, $\mathfrak{h}$ is closed under scalar multiplication.

Now suppose X and Y are in $\mathfrak{h}$. Then, for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$, $\exp\left(\frac{t}{n}X\right)$ and $\exp\left(\frac{t}{n}Y\right)$ are in H, and because H is a closed subset,

$$\lim_{n\to\infty}\left(\left(\exp\left(\frac{t}{n}X\right)\right)\left(\exp\left(\frac{t}{n}Y\right)\right)\right)^n$$

is also in H. Let $t \in \mathbb{R}$. Proposition 5.2 tells us that

$$\lim_{n\to\infty}\left(\left(\exp\left(\frac{t}{n}X\right)\right)\left(\exp\left(\frac{t}{n}Y\right)\right)\right)^n = \exp\left(t(X+Y)\right),$$

therefore $X + Y \in \mathfrak{h}$.

This shows that $\mathfrak{h} \subseteq \mathfrak{g}$ is a vector subspace.

Suppose that $m = \dim(\mathfrak{h}) \leq d$. The open set $H \cap \mathfrak{h} \subseteq \mathfrak{g}$ is diffeomorphic to an open set in $\mathbb{R}^m$. At $I_n \in H \subseteq G$, there exists an open neighborhood $V \cap H$ such that

$$\exp : H \cap \mathfrak{h} \to V \cap H$$

is a diffeomorphism.

For all $g \in H$, Theorem 1.3 shows that left translation

$$L_g : H \to H$$

is a diffeomorhpism from H to itself. Thus, for all $g \in H$, there exists an open neighborhood $V_g \cap H \subseteq H$ of g, where

$$V_g = \{X \in G \mid {L_g}^{-1}(X) \in V\}$$

such that

$$(L_g \circ \exp) : U \cap \mathfrak{h} \to V_g \cap H$$

is a diffeomorphism.

$$\begin{tikzcd}
\mathbb{R}^m \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & V_g \cap H \arrow[d, leftrightarrow, "L_g"] \\
U \cap \mathfrak{h} \arrow[r, leftrightarrow, "\exp"'] & V \cap H
\end{tikzcd}$$

FIGURE 1. Commuting diagram illustrating the diffeomorphic relations linking the set $\mathbb{R}^m$ with an open neighborhood of any element of the Lie group G.

As shown in Figure 1, we obtain a diffeomorphism between $\mathbb{R}^m$ and an open neighborhood in H of any element contained in H, showing that H is an m-dimensional manifold.

We now want to show that H is an embedded submanifold. Consider the inclusion map $i : H \to G$. The differential of i at any point $h \in H$ is given by

$$d_h\iota : T_hH \to T_hG.$$

The sets H and G are both smooth manifolds, thus the inclusion map ι is a smooth map. Consider the differential $d\iota_h : T_hH \to T_hG$. Since H is a submanifold of G, thus at every point $h \in H$, T_hH is a subspace of T_hG. Thus, for all $v \in T_hH$, $d\iota_h(v) = v$, shows that $d\iota_h$ is injective, hence ι is a smooth immersion.

Additionally since H is closed in G, the topology induced on H by G matches the manifold topology of H, making ι a homeomorphism onto its image.

This is enough to show that H is a Lie subgroup of G, proving the Closed Subgroup Theorem.

□

6. Applications

We can now use the Closed Subgroup Theorem to prove that the following subsets of $GL(n, \mathbb{R})$ are Lie groups.

Definition 6.1. The orthogonal group $O(n)$ is defined as the set of all matrices in $GL(n, \mathbb{R})$ such that for all $Q \in O(n)$, $Q^T Q = I_n$.

Theorem 6.2. *The orthogonal group is a Lie group.*

Proof. First we can prove closure over multiplication. If $Q_1, Q_2 \in O(n)$, then

$$(Q_1 Q_2)^T (Q_1 Q_2) = {Q_2}^T {Q_1}^T Q_1 Q_2 = {Q_2}^T Q_2 = I_n.$$

We can also prove closure under inversion. Let $Q \in O(n)$, then

$$(Q^{-1})^T Q^{-1} = (Q^T)^{-1} Q^{-1} = (Q^T Q)^{-1} = {I_n}^{-1} = I_n.$$

Hence, $Q^{-1} \in O(n)$.

We can prove that $O(n)$ is closed in $GL(n, \mathbb{R})$. The map

$$f : GL(n, \mathbb{R}) \to GL(n, \mathbb{R}), \quad f(Q) = Q^T Q$$

is continuous because it involves poylnomial entries of Q. We can express $O(n)$ as $f^{-1}(I_n)$, and since I_n is a closed set, $O(n)$ is also a closed set.

We have successfully proved that $O(n)$ is a closed set and that it is a subgroup of $GL(n, \mathbb{R})$. Therefore, by the Closed Subgroup Theorem it is a Lie group. □

Definition 6.3. The special orthogonal group $SO(n)$ is the set of matrices in $O(n)$ whose determinant equals 1.

Theorem 6.4. *The special orthogonal group is a Lie group.*

Proof. We have already shown that $O(n)$ is a Lie group. Let's do it for $SO(n)$.

1. If $Q_1, Q_2 \in SO(n)$, then $Q_1 Q_2 \in O(n)$, and $\det(Q_1) = \det(Q_2) = 1$. Since $\det(Q_1 Q_2) = \det(Q_1)\det(Q_2) = 1$, we can deduce that $Q_1 Q_2 \in SO(n)$.
2. If $Q \in SO(n)$, then $Q^{-1} \in O(n)$. Since $\det(QQ^{-1}) = \det(I_n) = 1$, then $\det(Q^{-1})\det(Q) = \det(Q^{-1}) \cdot 1 = 1$. This shows that $\det(Q^{-1}) = 1$.

Finally, $SO(n) = \det^{-1}(1)$, which shows that $SO(n)$ is closed in $O(n)$. By the Closed Subgroup Theorem, $SO(n)$ is a Lie group. □

Definition 6.5. For any positive integer n, the n-dimensional torus is defined as the product space

$$T^n = S^1 \times \ldots \times S^1.$$

The two-dimensional torus is the Cartesian product of two circles. In the proof to Proposition 2.2, we saw that we can embed S^1 into $GL(n, \mathbb{R}^2)$ as

$$S^1 = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \, \theta \in [0, 2\pi) \right\}$$

Thus, we can embed $T^2 = S^1 \times S^1$ into $GL(4, \mathbb{R})$ as

$$T^2 = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix} \middle| \, \theta, \phi \in [0, 2\pi) \right\}.$$

Theorem 6.6. *The two-dimensional torus T^2 is a Lie group.*

Proof. We first want to show that T^2 is a subgroup. Define

$$A(\theta, \phi) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix}.$$

(1) To prove closure under matrix multiplication, it can be shown that
$$A(\theta, \phi)A(\theta', \phi') = A(\theta + \theta', \phi + \phi') \in T^2$$
(2) The set T^2 contains I_4, which corresponds to $\theta, \phi = 0$, thus I_4 is the multiplicative identity.
(3) Finally, $A(\theta, \phi)A(2\pi - \theta, 2\pi - \phi) = I_4$, so every element of T^2 has a multiplicative inverse in T^2.

The set S^1 is closed, and since the Cartesian product of two closed sets is closed, we can deduce that $T^2 = S^1 \times S^1$ is closed. Hence, by the Closed Subgroup Theorem, T^2 is a Lie group. □

Another way of proving that T^2 is a Lie group would be to have noticed that, since S^1 is a Lie group, Theorem 2.3 tells us that $S^1 \times S^1$ is a Lie group. This theorem also allows us to show that for any positive integer n, the n-dimensional torus T^n is a Lie group.

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