VISCOSITY SOLUTIONS IN OPTIMAL CONTROL THEORY AND ZERO-SUM DIFFERENTIAL GAMES

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Abstract. Viscosity solutions, a notion first formulated by Crandall and Lions, presents a way of preserving the well-posedness of first-order partial differential equations (PDEs) to which the solutions need not be differentiable. This, in turn, allows this concept to be used as natural generalized solutions in many subfields of PDEs. In this paper, we devise the framework for applying viscosity solutions to a broad class of time-evolution Hamilton-Jacobi equations by establishing results for existence, stability, and uniqueness. We also take an in-depth look at two important applications, namely optimal control theory and zero-sum differential games, for which the cost value functions can be modeled as viscosity solutions to specified Hamilton-Jacobi systems. Along the way, we also highlight important identities, such as the comparison principle and the dynamic programming principle.

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1. Introduction

In the theory of partial differential equations, it is not unusual to encounter problems without classical solutions. A canonical example of this is the two-dimensional eikonal equation:

$$\begin{cases} |u'(x)| = 1 & \text{for all } x \in (-1,1) \\ u(-1) = u(1) = 0 \end{cases}$$

Despite having no C^1 solutions, this equation has infinitely many almost everywhere Lipschitz solutions, making it ill-posed in the classical sense. The notion of viscosity solutions, therefore, aims to recover the well-posedness for these type of equations by identifying generalized solutions that obey the three principles of existence, uniqueness, and stability under uniform limits. A large body of research has been done on this topic since the pioneering works of Crandall, Lions, and Evans in the 1980s, and developments have been achieved for both fully nonlinear second-order equations and a small class of higher-order PDEs. In

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the scope of this paper, however, we are mainly concerned with the first-order time-evolution initial value problem:

$$\begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{on } \mathbb{R}^n \end{cases} \tag{1.1}$$

The framework for this class of equation is laid out in Section 2. In particular, with the vanishing viscosity method, the existence of the solution to (1.1) is devised in Theorem 2.10. Furthermore, the comparison principle, an important identity stating that subsolutions stay below supersolutions on the whole space, is given in Theorem 2.12, along with uniqueness of viscosity solutions as an immediate consequence. These characterizations are derived under sufficient regularity conditions as presented in Assumption 2.1, 2.2, and 2.3, but there are many other sets of conditions available in the literature that one can use to arrive at these results.

In Section 3, we investigate the application of viscosity solutions in optimal control theory. Specifically, we consider a finite horizon problem as in Definition 3.1, in which we seek to optimize the cost incurred by the evolution of the state process in finite time. Here, an emphasis is placed on the dynamic programming principle in Theorem 3.11, which is a mathematical formulation of the recursive process of optimizing subparts of the control. With this identity, we identify, in Theorem 3.13, a Hamilton-Jacobi equation that the cost value function solves. Additionally, for the case of spatially-independent Hamiltonians, we present the Hopf-Lax formula as in 3.24, a neat representation formula using the Hamiltonian's convex conjugate, the Lagrangian.

Finally, in Section 4, we explore the application of viscosity solutions in zero-sum differential games, where the two players' objectives are to maximize and minimize the cost respectively. Here, we derive curated dynamic programming principles in Theorem 4.11, where despite having similar ideas as the previous section, the opposing goals of the players create a somewhat more complicated dynamic. This allows us to point out the value functions' corresponding Hamilton-Jacobi equations in Theorem 4.15, culminating in Isaac's condition for the game to attain a value.

2. Viscosity solutions

Following Tran [5], we denote

- $\mathrm{BUC}(\mathbb{R}^n)$ the space of bounded, uniformly continuous functions on $\mathbb{R}^n$, and
- $\mathrm{Lip}(\mathbb{R}^n)$ the space of Lipschitz functions on $\mathbb{R}^n$.

In this section, many results requires appropriate regularity conditions. In particular, for the existence of the viscosity solution to (1.1), we want the Hamiltonian $H(x, p)$ to be sufficiently smooth.

Assumption 2.1. The function $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies

$$\begin{cases} H \in C^2(\mathbb{R}^n \times \mathbb{R}^n) \\ H, D_pH \in \mathrm{BUC}(\mathbb{R}^n \times B(0, R)) \text{ for each } R > 0 \\ \lim_{|p|\to\infty} \inf_{x\in\mathbb{R}^n} \left(\frac{1}{2}H^2 - D_pH \cdot p\right) = +\infty \end{cases}$$

Additionally, for the uniqueness of the viscosity solution to (1.1), we require that H is Lipschitz in both variables.

Assumption 2.2. There exists a $C > 0$ such that

$$\begin{cases} |H(x, p) - H(y, p)| \le C(1 + |p|)|x - y|, \\ |H(x, p) - H(x, q)| \le C|p - q|. \end{cases}$$

The reason for these assumptions shall be made apparent in the upcoming sections. Finally, we also want a bounded, globally Lipschitz, and sufficiently smooth initial condition.

Assumption 2.3. The initial condition $u_0 : \mathbb{R}^n \to \mathbb{R}$ satisfies

$$\begin{cases} u_0 \in C^2(\mathbb{R}^n) \\ ||u_0||_{L^\infty(\mathbb{R}^n)} < \infty \\ ||Du_0||_{L^\infty(\mathbb{R}^n)} < \infty \end{cases}$$

2.1. **Definition of viscosity solutions.** In the context of the time-evolution problem (1.1), we shall provide the following definition courtesy of Crandall, Evans and Lions [2], given that the initial data is bounded and uniformly continuous, as well as Lipschitz.

Definition 2.4 (Viscosity solutions)**.** Given a terminal time T and an initial data $u_0 \in \mathrm{BUC}(\mathbb{R}^n) \cap \mathrm{Lip}(\mathbb{R}^n)$, we call a function $u \in \mathrm{BUC}(\mathbb{R}^n \times [0,T))$ as

(1) a *viscosity subsolution* of (1.1) if for any $\varphi \in C^1(\mathbb{R}^n \times (0,T))$ such that $u - \varphi$ has a strict maximum at $(x_0, t_0) \in \mathbb{R}^n \times (0,T)$, we have

$$\begin{cases} \varphi_t(x_0,t_0) + H(x_0, D\varphi(x_0,t_0)) \leq 0 \\ u(\cdot, 0) \leq u_0 \end{cases}$$

(2) a *viscosity supersolution* of (1.1) if for any $\varphi \in C^1(\mathbb{R}^n \times (0,T))$ such that $u - \varphi$ has a strict minimum at $(x_0, t_0) \in \mathbb{R}^n \times (0,T)$, we have

$$\begin{cases} \varphi_t(x_0,t_0) + H(x_0, D\varphi(x_0,t_0)) \leq 0 \\ u(\cdot, 0) \leq u_0 \end{cases}$$

(3) a *viscosity solution* of (1.1) if it is both a viscosity subsolution and a viscosity supersolution of (1.1).

Remark 2.5. In the definition above, note that the solution u is flexible up to addition of constants. Therefore, throughout this paper, we shall further require the condition

$$u(x_0, t_0) = \varphi(x_0, t_0)$$

on the *test functions* φ. This addition helps from a geometric perspective, as subsolutions are now entirely below the test function (and vice versa for supersolutions), only touching at the point (x_0, t_0).

Finally, note that in Definition 2.4, the subsolution test only applies on the time interval $(0,T)$ for a predetermined T. However, in the context of the time-evolution problem (1.1), which are defined on the entirety of $\mathbb{R}^n \times (0,\infty)$, we are also interested on what happens at the terminal time T. This is formulated in the proposition below, following Tran [5].

Proposition 2.6 (Extrema at terminal time)**.** *Let u be a viscosity subsolution to (1.1) and $\varphi \in C^1(\mathbb{R}^n \times [0,\infty))$ such that $u(x_0,t_0) = \varphi(x_0,t_0)$ and $u - \varphi$ has a strict max at $(x_0,t_0) \in \mathbb{R}^n \times (0,T]$. Then, the subsolution test still holds, that is:*

$$\varphi_t(x_0,t_0) + H(x_0, D\varphi(x_0,t_0)) \leq 0$$

2.2. **Existence of viscosity solutions.** In this section, we shall prove the existence of viscosity solutions for (1.1) using the vanishing viscosity procedure. The main idea is to consider the solutions to the following equation for each $\varepsilon > 0$:

$$\begin{cases} u_t^\varepsilon + H(x, Du^\varepsilon) = \varepsilon \Delta u^\varepsilon & \text{in } \mathbb{R}^n \times (0,\infty) \\ u^\varepsilon(x,0) = u_0(x) & \text{on } \mathbb{R}^n \end{cases} \tag{2.7}$$

Under Assumptions 2.1 and Assumption 2.3, the equation admits a unique solution thanks to the classical results presented by Friedman [3] - furthermore, following Tran [5], the unique solution u_ε has a gradient bound $C > 0$ that is independent of $\varepsilon \in (0,1)$:

$$|u_t^\varepsilon(x,t)| + |Du^\varepsilon(x,t)| \leq C \text{ for all } (x,t) \in \mathbb{R}^n \times [0,\infty) \tag{2.8}$$

The final tool we need is the following stability lemma from Tran [5].

Lemma 2.9 (Stability of extremum points). *Let $u \in C(\mathbb{R}^n)$ and $\varphi \in C^1(\mathbb{R}^n)$ such that for some $x_0 \in \mathbb{R}^n$:*

$$\begin{cases} u(x_0) = \varphi(x_0) \\ u - \varphi \text{ has a strict local extremum at } x_0 \end{cases}$$

Furthermore, provide a sequence of functions $\{u^\varepsilon\} \in C(\mathbb{R}^n)$ that converges to u locally uniformly on $\mathbb{R}^n$ as $\varepsilon \to 0^+$. Then, for $\varepsilon > 0$ small enough, $u^\varepsilon - \varphi$ has a local extremum at x_ε close to x_0. Additionally, there exists a subsequence $\{\varepsilon_j\} \to 0$ such that the corresponding sequence of extremum points $\{x_{\varepsilon_j}\}$ approaches x_0 as $j \to \infty$.

We are now equipped to prove the existence of the viscosity solution to (1.1).

Theorem 2.10 (Existence of viscosity solutions). *Take a Hamiltonian H and an initial data u_0 satisfying Assumptions 2.1 and 2.3. That is, as remarked above, for each $\varepsilon > 0$, the perturbed equation (2.7) has a unique solution u^ε, and there exists a constant $C > 0$ independent of $\varepsilon \in (0,1)$ such that, for each $\varepsilon \in (0,1)$,*

$$|u_t^\varepsilon(x,t)| + |Du^\varepsilon(x,t)| \leq C \text{ on } \mathbb{R}^n \times [0,\infty). \tag{2.11}$$

Then, there exists a subsequence $\{\varepsilon_j\} \to 0$ such that $u^{\varepsilon_j} \to u$ locally uniformly on $\mathbb{R}^n \times [0,\infty)$ for some continuous function $u \in C(\mathbb{R}^n \times [0,\infty))$. This limit function u is a viscosity solution of (1.1).

Proof. Using the Arzelà–Ascoli theorem, the gradient bound (2.11) means there exists a subsequence $\{\varepsilon_j\} \to 0$ such that $u^{\varepsilon_j} \to u$ locally uniformly on $\mathbb{R}^n \times [0,\infty)$ for some continuous function $u \in C(\mathbb{R}^n \times [0,\infty))$.

We now show that u is a viscosity subsolution of (1.1). Choose a test function $\varphi \in C^2(\mathbb{R}^n \times (0,T))$ such that at a point $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$,

$$\begin{cases} u(x_0) = \varphi(x_0) \\ u - \varphi \text{ has a strict local maximum at } x_0. \end{cases}$$

By Lemma 2.9, we can obtain a further subsequence $\{\varepsilon_i\} \to 0$ such that

$$\begin{cases} u^{\varepsilon_i} - \varphi \text{ has a strict local maximum at } (x_i,t_i), \text{ and} \\ \lim_{i\to\infty}(x_i,t_i) = (x_0,t_0). \end{cases}$$

Since (x_i,t_i) is a local maximum of $u^{\varepsilon_i} - \varphi$, we have:

$$\begin{cases} D(u^{\varepsilon_i} - \varphi)(x_i,t_i) = 0 \\ (u^{\varepsilon_i} - \varphi)_t(x_i,t_i) = 0 \\ \Delta(u^{\varepsilon_i} - \varphi)(x_i,t_i) \leq 0 \end{cases}$$

Substituting these relations into (2.7) yields:

$$\varphi_t(x_i,t_i) + H(D\varphi(x_i,t_i)) = \varepsilon_i \Delta(e^{\varepsilon_i})(x_i,t_i) \leq \varepsilon_i \Delta\varphi(x_i,t_i)$$

Letting $i \to \infty$ gives

$$\varphi_t(x_0,t_0) + H(D\varphi(x_0,t_0)) \leq 0,$$

which satisfies the subsolution test. The supersolution case is done similarly, and by combining the two tests, we have that u has to be a viscosity solution of (1.1). ∎

2.3. **The comparison principle and uniqueness of viscosity solutions.** One of the key features of viscosity solutions is the comparison principle, which states that the subsolution has to lie below the supersolution everywhere at any time. In a sense, this acts as a maximum principle for the difference of the subsolution and supersolution. We formalize this concept below following Tran [5], utilizing only the additional Lipschitz constraint.

Theorem 2.12 (Comparison principle for time-evolution problem). *Let the Hamiltonian $H(x,p)$ satisfy Assumption 2.2. For a fixed time $T > 0$, take $u, v \in \mathrm{BUC}(\mathbb{R}^n \times [0,T])$ to be a subsolution and supersolution to (1.1). Then, everywhere on $\mathbb{R}^n \times [0,T]$, we have:*

$$u(x,t) \leq v(x,t)$$

Proof. Assume by contradiction that

$$\sup_{(x,t)\in\mathbb{R}^n\times(0,T]} (u(x,t)-v(x,t)) = \sigma > 0$$

Then, there exists $(x',t') \in \mathbb{R}^n \times (0,T]$ such that $u(x',t')-v(x',t') > \frac{3\sigma}{4}$. Take positive numbers ε and λ such that

$$\varepsilon < \frac{\sigma}{16(|x'|^2+1)} \qquad \text{and} \qquad \lambda < \frac{\sigma}{16(t'+1)}$$

With these constants, consider the auxiliary function $\Phi : \mathbb{R}^n \times \mathbb{R}^n \times [0,T] \times [0,T] \to \mathbb{R}$:

$$\Phi(x,y,t,s) = u(x,t) - v(y,s) - \frac{|x-y|^2+|t-s|^2}{\varepsilon^2} - \varepsilon(|x|^2+|y|^2) - \lambda(t+s)$$

Since Φ is continuous and bounded above, it must achieve its maximum at some point $(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon)$. We have:

$$u(x_\varepsilon,t_\varepsilon) - v(y_\varepsilon,s_\varepsilon) \geq \Phi(x_\varepsilon,y_\varepsilon,t_\varepsilon,s_\varepsilon) \geq \Phi(x',x',t',t') > \frac{3\sigma}{4} - 2\varepsilon|x'|^2 - 2\lambda t' > \frac{\sigma}{2}$$

Furthermore, since $\Phi(x_\varepsilon,y_\varepsilon,t_\varepsilon,s_\varepsilon) \geq \Phi(0,0,0,0)$, we get

$$\begin{aligned}
&u(x_\varepsilon,t_\varepsilon) - v(y_\varepsilon,s_\varepsilon) - (u_0(0)-v_0(0)) \\
&= u(x_\varepsilon,t_\varepsilon) - v(y_\varepsilon,s_\varepsilon) - (u(0,0)-v(0,0)) \\
&\geq \frac{|x_\varepsilon-y_\varepsilon|^2+|t_\varepsilon-s_\varepsilon|^2}{\varepsilon^2} + \varepsilon(|x_\varepsilon|^2+|y_\varepsilon|^2) + \lambda(t_\varepsilon+s_\varepsilon)
\end{aligned}$$

This gives us two bounds:

$$|x_\varepsilon-y_\varepsilon| + |t_\varepsilon-s_\varepsilon| \leq C\varepsilon \qquad \text{and} \qquad |x_\varepsilon|+|y_\varepsilon| \leq \frac{C}{\sqrt{\varepsilon}}$$

These two bounds in conjunction with $\Phi(x_\varepsilon,y_\varepsilon,t_\varepsilon,s_\varepsilon) \geq \Phi(x_\varepsilon,x_\varepsilon,t_\varepsilon,t_\varepsilon)$ give

$$\begin{aligned}
\frac{|x_\varepsilon-y_\varepsilon|^2+|t_\varepsilon-s_\varepsilon|^2}{\varepsilon^2} &\leq v(x_\varepsilon,t_\varepsilon) - v(y_\varepsilon-s_\varepsilon) + \varepsilon(|x_\varepsilon|^2-|y_\varepsilon|^2) + \lambda(t_\varepsilon-s_\varepsilon) \\
&\leq v(x_\varepsilon,t_\varepsilon) - v(y_\varepsilon-s_\varepsilon) + \varepsilon\cdot\frac{C}{\sqrt{\varepsilon}}\cdot C\varepsilon + C\varepsilon
\end{aligned}$$

which, together with the uniform continuity of v, means that

$$\lim_{\varepsilon\to 0} \frac{|x_\varepsilon-y_\varepsilon|^2+|t_\varepsilon-s_\varepsilon|^2}{\varepsilon^2} = 0.$$

Next, we shall prove that t_ε and s_ε are bounded away from 0. Observe that

$$\begin{aligned}
\frac{\sigma}{2} &< u(x_\varepsilon,t_\varepsilon) - v(y_\varepsilon,s_\varepsilon) \\
&= u(x_\varepsilon,t_\varepsilon) - u(x_\varepsilon,0) + u(x_\varepsilon,0) - v(x_\varepsilon,0) + v(x_\varepsilon,0) - v(x_\varepsilon,t_\varepsilon) + v(x_\varepsilon,t_\varepsilon) - v(y_\varepsilon,s_\varepsilon) \\
&\leq \omega(t_\varepsilon) + \omega(|x_\varepsilon-y_\varepsilon| + |t_\varepsilon-s_\varepsilon|)
\end{aligned}$$

where $\omega(\cdot)$ is a modulus of continuity. Since the argument of the second term goes to zero, there must be a $\mu > 0$ independent of ε such that $t_\varepsilon > \mu > 0$. By a similar argument, we also get $s_\varepsilon > \mu > 0$.

Finally, we have to apply the viscosity solution tests. On the one hand, note that the map $(x,t) \to \Phi(x,y_\varepsilon,t,s_\varepsilon)$ has a maximum at $(x_\varepsilon,t_\varepsilon)$, which means the following map has a maximum at $(x_\varepsilon,t_\varepsilon)$ as well:

$$(x,t) \to u(x,t) - \left[\frac{|x-y_\varepsilon|^2+|t-s_\varepsilon|^2}{\varepsilon^2} + \varepsilon|x|^2 + \lambda t\right]$$

Since u is a viscosity subsolution to (1.1), we can use the test to get

$$\frac{2(t_\varepsilon-s_\varepsilon)}{\varepsilon^2} + \lambda + H\left(x_\varepsilon, \frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2} + 2\varepsilon x_\varepsilon\right) \leq 0.$$

On the other hand, the map $(y, s) \to \Phi(x_\varepsilon, y, t_\varepsilon, s)$ has a maximum at $(y_\varepsilon, s_\varepsilon)$. That means the following map has a minimum at $(y_\varepsilon, s_\varepsilon)$:

$$(y, s) \to v(y, s) + \left[\frac{|x_\varepsilon - y|^2 + |t_\varepsilon - s|^2}{\varepsilon^2} + \varepsilon|y|^2 + \lambda s\right]$$

Here, with v being a supersolution to (1.1), we get

$$\frac{2(t_\varepsilon - s_\varepsilon)}{\varepsilon^2} - \lambda + H\left(y_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} - 2\varepsilon y_\varepsilon\right) \geq 0.$$

By combining the two inequalities from the viscosity solution tests, we get

$$2\lambda \leq H\left(y_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} - 2\varepsilon y_\varepsilon\right) - H\left(x_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} + 2\varepsilon x_\varepsilon\right)$$

Furthermore, with the Lipschitz condition in Assumption 2.2, we get:

$$H\left(y_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} - 2\varepsilon y_\varepsilon\right) - H\left(y_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2}\right) \leq 2C\varepsilon|y_\varepsilon|$$
$$H\left(y_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2}\right) - H\left(x_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2}\right) \leq C|x_\varepsilon - y_\varepsilon|\left(1 + \frac{2|x_\varepsilon - y_\varepsilon|}{\varepsilon^2}\right)$$
$$H\left(x_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2}\right) - H\left(x_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} + 2\varepsilon x_\varepsilon\right) \leq 2C\varepsilon|x_\varepsilon|$$

This means that for every $\varepsilon > 0$:

$$\begin{aligned} 2\lambda &\leq 2C\varepsilon(|x_\varepsilon| + |y_\varepsilon|) + C|x_\varepsilon - y_\varepsilon| + \frac{2C|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} \\ &\leq 2C\varepsilon \cdot \frac{C}{\sqrt{\varepsilon}} + C \cdot C\varepsilon + \frac{2C|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} \end{aligned}$$

Letting $\varepsilon \to 0^+$ gives us $2\varepsilon \leq 0$, a contradiction. This finalizes the proof. ☐

The uniqueness of the viscosity solution comes immediately as a corollary of this comparison principle.

Theorem 2.13 (Uniqueness of viscosity solutions)**.** *Assume the Lipschitz conditions on the Hamiltonian as in Assumption 2.2. If* $u, v \in \mathrm{BUC}(\mathbb{R}^n \times [0, T])$ *are viscosity solutions of the time-evolution problem (1.1), then* $u \equiv v$ *on* $\mathbb{R}^n \times (0, \infty)$.

Proof. Since u is a viscosity subsolution and v is a supersolution of (1.1), by the comparison principle in Theorem 2.12, we have that $u(x, t) \leq v(x, t)$ everywhere on $\mathbb{R}^n \times [0, T]$. On the other hand, u is a supersolution and v is a subsolution, so $u \geq v$ everywhere. Thus, $u \equiv v$ on $\mathbb{R}^n \times [0, T]$, which results in uniqueness of the viscosity solution to (1.1). ☐

3. Optimal control theory

In this section, we shall explore optimal control theory under the lens of viscosity solutions for first-order Hamilton-Jacobi equations with convex Hamiltonians. Following Bardi and Calpuzzo-Dolcetta [1], we mainly discuss the finite horizon problem.

Definition 3.1 (Finite horizon problem)**.** Consider the ODE representing the time-dependent evolution of the state process $y : [0, T] \to \mathbb{R}^n$ as follows for a given starting time t_0 and terminal time T:

$$\begin{cases} y'(t) = b(y(t), v(t)) & \text{for all } t \in [t_0, T] \\ y(t_0) = x_0 \end{cases} \tag{3.2}$$

Here, we shall briefly address the definitions of each component of this equation:

- V is a given compact metric space acting as the control set.
- $b : \mathbb{R}^n \times V \to \mathbb{R}^n$ is a vector field.

- $v : [t_0, T] \to V$ is the control function we choose.

Here, we restrict to vector fields which are continuous, bounded, and spatially Lipschitz.

Assumption 3.3. There exists a constant $C > 0$ such that:

$$\begin{cases} b \in C(\mathbb{R}^n \times V) \\ |b(x,v)| \leq C & \text{for all } (x,v) \in \mathbb{R}^n \times V \\ |b(x_1,v) - b(x_2,v)| \leq C|x_1 - x_2| & \text{for all } x_1, x_2 \in \mathbb{R}^n,\ v \in V \end{cases}$$

We denote the unique solution to the ODE (3.2), under these assumptions, as $y_{x_0,t_0,v}$ to emphasize that the process depends exclusively on these three parameters. We shall also abbreviate the solution to y_{x_0,t_0} or y if no confusion arises. Clearly, the solution can be written in the following integral form:

$$y(t) = x_0 + \int_{t_0}^{t} b(y(s), v(s))\, ds \quad \text{for all } t \in [t_0, T] \tag{3.4}$$

The following lemma shows that the solution is Lipschitz with regards to both the time variable and the starting position.

Lemma 3.5 (Lipschitz properties of the state process)**.** *The following claims hold.*

(1) Given a solution y to the ODE (3.2), for $t, s \geq 0$, we have:

$$|y(t) - y(s)| \leq C|t - s|$$

(2) Fix a control v and consider two processes y_{x_0,t_0} and y_{z_0,t_0} starting at two points x_0 and z_0, respectively. Then:

$$|y_{x_0,t_0}(t) - y_{z_0,t_0}(t)| \leq e^{Ct}|x_0 - z_0| \qquad \text{for all } t > 0$$

Proof. For the first claim, assume without loss of generality that $t \geq s$. Using the Lipschitz property of the vector field in Assumption 3.3 and the fundamental theorem of calculus, we have:

$$|y(t) - y(s)| = \left| \int_t^s b(y(\tau), v(\tau))\, d\tau \right| \leq \left| \int_t^s C\, d\tau \right| = C|t - s|$$

For the second claim, define $\varphi(s) = y_{x_0,t_0}(s) - y_{z_0,t_0}(s)$ for all $s \geq 0$. Then, the Lipschitz property in the first variable of b in Assumption 3.3 gives:

$$\begin{aligned} |\varphi'(s)| &= |b(y_{x_0,t_0}(s), v(s)) - b(y_{z_0,t_0}(s), v(s))| \\ &\leq C|y_{x_0,t_0}(s) - y_{z_0,t_0}(s)| \\ &= C|\varphi(s)| \end{aligned}$$

By Gronwall's inequality, we obtain:

$$|y_{x_0,t_0}(s) - y_{z_0,t_0}(s)| = |\varphi(t)| \leq e^{Ct}|\varphi(0)| = e^{Ct}|x_0 - z_0|$$

This finishes the proof. □

Associated with the vector field b above is a function $f : \mathbb{R}^n \times V \to \mathbb{R}$ depicting the running cost of the process. We shall also place assumptions of continuity, boundedness, and Lipschitz regularity on this function.

Assumption 3.6. There exists a constant $C > 0$ such that:

$$\begin{cases} f \in C(\mathbb{R}^n \times V) \\ |f(x,v)| \leq C & \text{for all } (x,v) \in \mathbb{R}^n \times V \\ |f(x_1,v) - f(x_2,v)| \leq C|x_1 - x_2| & \text{for all } x_1, x_2 \in \mathbb{R}^n,\ v \in V \end{cases}$$

Our goal for the optimization problem shall be to find the control that minimizes the cost functional among all possible controls. To that end, we shall set our cost functional and value function as follows.

Definition 3.7 (Cost functional and value function)**.** The cost functional representing the total cost of the process is defined as:

$$J(x_0, t_0, v) = \int_{t_0}^{T} f(y(s), v(s))\, ds + g(y(T))$$

Here, $g \in \mathrm{BUC}(\mathbb{R}^n)$ is the terminal cost of the process. The value function $u : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ is then set to be the minimum cost for a given starting position x:

$$u(x_0, t_0) = \inf_v J(x_0, t_0, v) = \inf_v \left(\int_{t_0}^{T} f(y(s), v(s))\, ds + g(y(T)) \right) \tag{3.8}$$

With the given setting, the upcoming sections shall be dedicated to inspecting properties of this value function. On the one hand, u satisfies a recursive identity, the dynamic programming principle, as in Theorem 3.11. On the other hand, u is a viscosity solution of the time-evolution Hamilton-Jacobi problem

$$\begin{cases} -u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^n \times [0, T] \\ u(x, T) = g(x) & \text{on } \mathbb{R}^n \end{cases}$$

where H is

$$H(x, p) = \sup_{v \in V} \left(-b(x, v) \cdot p - f(x, v) \right) \tag{3.9}$$

as specified in Theorem 3.13. Both of these theorems shall help in reformulating the finite horizon problem to a PDE viewpoint, while allowing us to derive uniqueness and regularity results for the optimal control. Before that, however, we would like to leave some last remarks on the regularity of the Hamiltonian H defined in (3.9) and examine whether it fits into our assumptions in the previous sections, such as Assumption 2.2. Following Tran [5], we have the following lemma.

Lemma 3.10. *Let H be defined as in (3.9). Then:*

(1) $H \in C(\mathbb{R}^n \times \mathbb{R}^n)$, and $p \to H(x, p)$ is convex for each $x \in \mathbb{R}^n$.
(2) There exists a $C > 0$ such that for all $x, y, p, q \in \mathbb{R}^n$:

$$\begin{cases} |H(x, p) - H(y, p)| \le C(1 + |p|)|x - y|, \\ |H(x, p) - H(x, q)| \le C|p - q|. \end{cases}$$

3.1. **Dynamic programming principle.** In the setting of optimal control, the dynamic programming principle aims to optimize subparts of the time-evolution process. Here, the value function has to unite the optimal partial cost up to a predefined time with the optimal cost for all following times, as detailed in the next theorem.

Theorem 3.11 (Dynamic programming principle)**.** *For every initial position $x \in \mathbb{R}^n$ and starting time $0 \le t_0 < T$, the value function u specified in (3.8) satisfies the following identity for all $0 < h \le T - t_0$:*

$$u(x_0, t_0) = \inf_v \left(\int_{t_0}^{t_0 + h} f(y(s), v(s))\, ds + u(y(t_0 + h), t_0 + h) \right) \tag{3.12}$$

Proof. We first prove that LHS $\ge$ RHS in (3.12). Let $\tilde{v}$ be the restriction of the control v for time $t_0 + h$ onwards. Using the fact that u is the infimum of cost functionals, we have:

$$\begin{aligned} J(x_0, t_0, v) &= \int_{t_0}^{t_0 + h} f(y(s), v(s))\, ds + J(y(t_0 + h), t_0 + h, \tilde{v}) \\ &\ge \int_{t_0}^{t_0 + h} f(y(s), v(s))\, ds + u(y(t_0 + h), t_0 + h) \end{aligned}$$

Taking the infimum of both sides over all controls v yields:

$$u(x_0, t_0) = \inf_v J(x_0, t_0, v) \ge \inf_v \left(\int_{t_0}^{t_0 + h} f(y(s), v(s))\, ds + u(y(t_0 + h), t_0 + h) \right)$$

Now, we turn to proving that RHS $\geq$ LHS in (3.12). Start by letting $z = y_{x_0,t_0}(t_0+h)$. Since $u(z, t_0+h)$ is the minimum cost on $[t_0+h, T]$, for any $\varepsilon > 0$, there exists a control w such that:

$$u(z, t_0+h) > J(z, t_0+h, w) - \varepsilon$$

Here, we form a new control by connecting our original control v on $[t_0, t_0+h]$ with the control w on $[t_0+h, T]$. Define this new control v^* as:

$$\begin{cases} v^*(s) = v(s) & \text{if } s \in [t_0, t_0+h] \\ v^*(s) = w(s) & \text{if } s \in [t_0+h, T] \end{cases}$$

By the uniqueness of the solution to the ODE (3.2), we must have that $y_{x_0,t_0,v^*}(\cdot) \equiv y_{z,t_0+h,w}(\cdot)$ on $[t_0+h, T]$. Thus, we get the following inequality:

$$\begin{aligned} u(z, t_0+h) &\geq \int_{t_0+h}^{T} f(y_{z,t_0+h,w}(s), w(s))\, ds + g(y_{z,t_0+h,w}(T)) - \varepsilon \\ &= \int_{t_0+h}^{T} f(y_{x_0,t_0,v^*}(s), v^*(s))\, ds + g(y_{x_0,t_0,v^*}(T)) - \varepsilon \end{aligned}$$

On the other hand, since the controls overlap up to time t_0+h, we must also have:

$$y_{x_0,t_0,v} \equiv y_{x_0,t_0,v^*} \quad \text{on } [t_0, t_0+h]$$

Along with the inequality above, we obtain:

$$\begin{aligned} \int_{t_0}^{t_0+h} f(y_{x_0,t_0,v}(s), v(s))\, ds + u(z, t_0+h) &\geq \int_{t_0}^{T} f(y_{x_0,t_0,v^*}(s), v^*(s))\, ds + g(y_{x_0,t_0,v^*}(T)) - \varepsilon \\ &\geq u(x_0, t_0) - \varepsilon \end{aligned}$$

Taking the infimum over all controls v yields:

$$\inf_v \left(\int_{t_0}^{t_0+h} f(y_{x_0,t_0,v}(s), v(s))\, ds + u(z, t_0+h) \right) \geq u(x_0, t_0) - \varepsilon$$

Because this is true for all $\varepsilon > 0$, we deduce that RHS $\geq$ LHS and complete the proof. □

3.2. **Hamilton-Jacobi equation of the value function.** With the dynamic programming principle, we are now ready to put the value function u in the perspective of the theory of viscosity solutions.

Theorem 3.13. *The value function u as defined by (3.8) is a unique viscosity solution of the following time-evolution Hamilton-Jacobi equation:*

$$\begin{cases} -u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^n \times (0, T) \\ u(x, T) = g(x) & \text{on } \mathbb{R}^n \end{cases} \tag{3.14}$$

Here, the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is

$$H(x, p) = \sup_{v \in V} \left(-b(x, v) \cdot p - f(x, v) \right)$$

Remark 3.15. In Theorem 3.13, the Hamilton-Jacobi equation are presented in a slightly different manner compared to equation (1.1) that we considered in the previous section. Specifically, the equation is given in the form of a terminal-value problem as opposed to an initial-value problems. Fortunately, we can flexibly interchange between the two types of problems by defining, for a viscosity solution u of (3.14), a function $\tilde{u}$ satisfying:

$$\tilde{u}(x, t) = u(x, T - t)$$

This function shall now be the unique viscosity solution of the initial-value problem

$$\begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^n \times (0, T) \\ u(x, 0) = g(x) & \text{on } \mathbb{R}^n \end{cases}$$

and we can apply all derived identities to this new solution accordingly.

Proof of Theorem 3.13. We begin with the subsolution test. Take a test function $\varphi \in C^1(\mathbb{R}^n \times (0,T))$ such that at a point $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$, we have:

$$\begin{cases} u(x_0,t_0) = \varphi(x_0,t_0) \\ u - \varphi \text{ has a strict local maximum at } (x_0,t_0) \end{cases}$$

The goal is to show that:

$$-\varphi_t(x_0,t_0) + H(x_0, D\varphi(x_0,t_0)) \leq 0$$

We start by picking any control v, and take the unique solution y to the ODE:

$$\begin{cases} y'(s) = b(y(s),v(s)) \qquad \text{for all } s > 0 \\ y(t_0) = x_0 \end{cases}$$

Using the dynamic programming principle in Theorem 3.11, we have that for all $0 < h \leq T - t_0$:

$$\begin{aligned} \varphi(x_0,t_0) = u(x_0,t_0) &\leq \int_{t_0}^{t_0+h} f(y(s),v(s))\,ds + u(y(t_0+h),t_0+h) \\ &\leq \int_{t_0}^{t_0+h} f(y(s),v(s))\,ds + \varphi(y(t_0+h),t_0+h) \end{aligned}$$

We rewrite this expression, using the fundamental theorem of calculus, as:

$$\begin{aligned} -\int_{t_0}^{t_0+h} \frac{d}{ds}\left(\varphi(y(s),s)\right) ds &= \varphi(y(t_0),t_0) - \varphi(y(t_0+h),t_0+h) \\ &\leq \int_{t_0}^{t_0+h} f(y(s),v(s))\,ds \end{aligned}$$

However, the left-hand side can be further expanded:

$$\begin{aligned} \int_{t_0}^{t_0+h} \frac{d}{ds}\left(\varphi(y(s),s)\right) ds &= \int_{t_0}^{t_0+h} \varphi_t(y_{x_0}(s),s) + D\varphi(y(s),s)\cdot y'_{x_0}(s)\,ds \\ &= \int_{t_0}^{t_0+h} \varphi_t(y_{x_0}(s),s) + D\varphi(y(s),s)\cdot b(y(s),v(s))\,ds \end{aligned}$$

Combining this with the inequality above gives:

$$\int_{t_0}^{t_0+h} -\varphi_t(y(s),s) - D\varphi(y(s),s)\cdot b(y(s),v(s)) - f(y(s),v(s))\,ds \leq 0$$

Since this is valid for all controls v and every $0 < h \leq T - t_0$, we can pick a control v that is constant at all times and average out the integral to get:

$$\frac{1}{h}\left(\int_{t_0}^{t_0+h} -\varphi_t(y(s),s) - D\varphi(y(s),s)\cdot b(y(s),v) - f(y(s),v)\,ds\right) \leq 0$$

Letting $\tau \to 0^+$ yields:

$$-\varphi_t(x_0,t_0) - D\varphi(x_0,t_0)\cdot b(x_0,v) - f(x_0,v) \leq 0$$

Here, taking the supremum over all controls v concludes the subsolution test. We now turn to the supersolution test - let $\psi \in C^2(\mathbb{R}^n \times (0,T))$ such that at a point $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$, we have:

$$\begin{cases} u(x_0,t_0) = \psi(x_0,t_0) \\ u - \psi \text{ has a strict local minimum at } (x_0,t_0) \end{cases}$$

We will aim to show the inequality:

$$-\psi_t(x_0,t_0) + H(x_0, D\psi(x_0,t_0)) \geq 0$$

By using the dynamic programming principle in Theorem 3.11, we have for any $0 < h \leq T - t_0$:

$$\begin{aligned}\psi(x_0, t_0) = u(x_0, t_0) &= \inf_v \left(\int_{t_0}^{t_0+h} f(y(s), v(s))\, ds + u(y(t_0+h), t_0+h) \right) \\ &\geq \inf_v \left(\int_{t_0}^{t_0+h} f(y(s), v(s))\, ds + \psi(y(t_0+h), t_0+h) \right)\end{aligned}$$

This directly leads to:

$$0 \geq \inf_v \left(\int_{t_0}^{t_0+h} f(y(s), v(s))\, ds + \psi(y(t_0+h), t_0+h) - \psi(x_0, t_0) \right)$$

We shall inspect the inner expression further. With the fundamental theorem of calculus, we have:

$$\begin{aligned}&\int_{t_0}^{t_0+h} f(y(s), v(s))\, ds + \psi(y(t_0+h), t_0+h) - \psi(x_0, t_0) \\ &= \int_{t_0}^{t_0+h} f(y(s), v(s))\, ds + \int_{t_0}^{t_0+h} \frac{d}{ds}\left(\psi(y(s), s)\right) ds \\ &= \int_{t_0}^{t_0+h} \psi_t(y(s), s) + D\psi(y(s), s) \cdot b(y(s), v(s)) + f(y(s), v(s))\, ds \\ &\geq \int_{t_0}^{t_0+h} \psi_t(y(s), s) - H(y(s), D\psi(y(s), s))\, ds\end{aligned}$$

However, by the Lipschitz property of the process as in Lemma 3.5, as well as the uniform continuity of ψ, we have that for any $t_0 \leq s \leq t_0 + h$:

$$\begin{cases} |\psi(y(s), s) - \psi(x_0, t_0)| \leq Cs \leq Ch \\ |\psi_t(y(s), s) - \psi_t(x_0, t_0)| \leq Cs \leq Ch \end{cases}$$

Similarly, by the Lipschitz property of H as in Lemma 3.10, we have that for any $t_0 \leq s \leq t_0 + h$:

$$|H(y(s), D\psi(y(s), s)) - H(x_0, D\psi(x_0, t_0))| \leq Cs \leq Ch$$

This means:

$$\begin{aligned}&\int_{t_0}^{t_0+h} f(y(s), v(s))\, ds + \psi(y(t_0+h), t_0+h) - \psi(x_0, t_0) \\ &\geq \int_{t_0}^{t_0+h} \psi_t(y(s), s) - H(y(s), D\psi(y(s), s))\, ds \\ &\geq \int_{t_0}^{t_0+h} \psi_t(x_0, t_0) - H(x_0, D\psi(x_0, t_0))\, ds - Ch^2\end{aligned}$$

Finally, replacing this expression into the infimum and taking the limit gives:

$$\begin{aligned}0 &\geq \lim_{h \to 0^+} \frac{1}{h} \left(\inf_v \left(\int_{t_0}^{t_0+h} f(y(s), v(s))\, ds + \psi(y(t_0+h), t_0+h) - \psi(x_0, t_0) \right) \right) \\ &\geq \lim_{h \to 0^+} \frac{1}{h} \left(\inf_v \left(\int_{t_0}^{t_0+h} \psi_t(x_0, t_0) - H(x_0, D\psi(x_0, t_0))\, ds - Ch^2 \right) \right) \\ &= \lim_{h \to 0^+} \left(\frac{1}{h} \int_{t_0}^{t_0+h} \psi_t(x_0, t_0) - H(x_0, D\psi(x_0, t_0))\, ds - Ch \right) \\ &= \psi_t(x_0, t_0) - H(x_0, D\psi(x_0, t_0))\end{aligned}$$

Flipping the signs gives us the desired inequality for the supersolution test. This establishes that u is a viscosity solution, and from Remark 3.15 as well as Theorem 2.13, we get that u has to be unique. This finalizes the proof completely. □

3.3. The Legendre's transform and the Hopf-Lax formula. In this final subsection, we shall aim to arrive at the celebrated Hopf-Lax formula, an elegant representation of the viscosity solution for the time-evolution terminal value problem with spatially homogeneous Hamiltonian H - that is:

$$\begin{cases} -u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times [0,T) \\ u(x,T) = g(x) & \text{on } \mathbb{R}^n \end{cases} \tag{3.16}$$

Before that, however, we have to establish some properties about the Legendre's transform, one that takes the Hamiltonian H to its dual operator, the Lagrangian L. For the upcoming theorems, we would like to place on H assumptions of boundedness, smoothness, convexity, and superlinearity.

Assumption 3.17. The Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies:

$$\begin{cases} H \in C^2(\mathbb{R}^n \times \mathbb{R}^n) \\ H \in \text{BUC}(\mathbb{R}^n \times B(0,R)) \text{ for each } R > 0 \\ p \to H(x,p) \text{ is convex for all } x \in \mathbb{R}^n \\ \lim_{|p|\to\infty} \left(\inf_{x\in\mathbb{R}^n} \frac{H(x,p)}{|p|} \right) = +\infty \end{cases}$$

We can now proceed to defining the Legendre's transform.

Definition 3.18. Given a Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, we define its Legendre's transform $H^* = L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as:

$$L(x,v) = \sup_{p\in\mathbb{R}^n} (-p \cdot v - H(x,p)) \quad \text{for all } (x,v) \in \mathbb{R}^n \times \mathbb{R}^n$$

It turns out that the Lagrangian retains many regularity properties that the Hamiltonian has. Specifically, following Tran [5], one can observe that if we place on H the conditions as in Assumption 3.17, then L shall have the exact same premises.

Theorem 3.19 (Duality of the Lagrangian)**.** *Assume that the Hamiltonian H satisfies Assumption 3.17. Then, the following statements hold.*

(1) $L^* = H^{**} = H$.
(2) *L is well-defined (finite), and $v \to L(x,v)$ is convex.*
(3) $L \in C^2(\mathbb{R}^n \times \mathbb{R}^n)$*, and* $L \in \text{BUC}(\mathbb{R}^n \times B(0,R))$ for each $R > 0$.
(4) *L is superlinear, that is:*

$$\lim_{|v|\to\infty} \left(\inf_{x\in\mathbb{R}^n} \frac{L(x,v)}{|v|} \right) = +\infty$$

Counterparts of Theorem 3.11 and 3.13 can then be defined with a value function u defined as follows for all $(x,t) \in \mathbb{R}^n \times [0,T]$:

$$u(x,t) = \inf_v \left\{ \int_t^T L(v(s))\, ds + g(y(T)) : y(t) = x, y' = v \in L^1([t,T]) \right\} \tag{3.20}$$

We shall illustrate this with a pair of theorems below, following Tran [5]. Specifically, Theorem 3.21 shall give a dynamic programming principle for u, while Theorem 3.23 shall place u as the unique viscosity solution to problem (1.1).

Theorem 3.21 (Dynamic programming principle)**.** *The value function (3.20) satisfies, for all $(x,t) \in \mathbb{R}^n \times [0,T)$ and all $s \in [t,T)$, the following identity:*

$$u(x,t) = \inf \left\{ \int_t^s L(v(r))\, dr + u(v(s),s) : y(t) = x, y' = v \in L^1([t,s]) \right\} \tag{3.22}$$

Theorem 3.23. *Under Assumption 3.17, the value function u defined in (3.20) is the unique viscosity solution to (1.1).*

Both theorems have very similar proofs to their counterparts, hence we shall omit them here. Using Theorem 3.23, we shall get to our final goal, the Hopf-Lax formula.

Theorem 3.24. *Given a spatially homogeneous Hamiltonian H satisfying Assumption 3.17 and a terminal data $g \in \mathrm{BUC}(\mathbb{R}^n)$. Then, the viscosity solution u to the time-evolution problem (3.16) shall have the following representation formula for all $(x,t) \in \mathbb{R}^n \times [0,T]$:*

$$u(x,t) = \inf_{z\in\mathbb{R}^n} \left\{ (T-t)L\left(\frac{z-x}{T-t}\right) + g(z) \right\} = \min_{z\in\mathbb{R}^n} \left\{ (T-t)L\left(\frac{z-x}{T-t}\right) + g(z) \right\} \tag{3.25}$$

Proof. First, fix a $(x,t) \in \mathbb{R}^n \times [0,T]$ and consider the straight path y connecting (x,t) to (z,T) - we have:

$$y'(s) = \frac{z-x}{T-t} \qquad \text{for all } s \in [t,T]$$

By the definition of the value function in (3.20), we have:

$$u(x,t) \le \int_t^T L(y'(s))\,ds + g(y(T)) = (T-t)L\left(\frac{z-x}{T-t}\right) + g(z)$$

Since this is for all $z \in \mathbb{R}^n$, we have one side of the desired equality:

$$u(x,t) \le \inf_{z\in\mathbb{R}^n} \left\{ (T-t)L\left(\frac{z-x}{T-t}\right) + g(z) \right\}$$

On the other hand, using Jensen's inequality on any admissible path y connecting (x,t) to (z,T) gives:

$$L\left(\frac{z-x}{T-t}\right) = L\left(\frac{1}{T-t}\int_t^T y'(s)\,ds\right) \le \frac{1}{T-t}\int_t^T L(y'(s))\,ds$$

This directly leads to:

$$(T-t)L\left(\frac{z-x}{T-t}\right) + g(z) \le \int_t^T L(y'(s))\,ds + g(z) = \int_t^T L(y'(s))\,ds + g(y(T))$$

Taking the infimum of both sides yields us:

$$\inf_{z\in\mathbb{R}^n} \left\{ (T-t)L\left(\frac{z-x}{T-t}\right) + g(z) \right\} \le u(x,t)$$

Combining with the inequality above, we get the identity:

$$u(x,t) = \inf_{z\in\mathbb{R}^n} \left\{ (T-t)L\left(\frac{z-x}{T-t}\right) + g(z) \right\}$$

Finally, note that since $g \in \mathrm{BUC}(\mathbb{R}^n)$ and L is superlinear, the infimum must occur at a point $z \in \mathbb{R}^n$. This fully justifies (3.25). ☐

4. Zero-sum differential games

In this section, we shall take a look at the theory of zero-sum differential games as an application of viscosity solutions. Here, we consider a two-player game, whose objective is to minimize and maximize the payoff respectively. Following Pierre [4], we impose the following dynamics to the so-called Bolza problem.

Definition 4.1 (Bolza problem)**.** Consider the following ODE, which represents the evolution of the state y of a differential game from a initial position (x_0, t_0) up to a terminal time T:

$$\begin{cases} y'(t) = b(y(t), t, u(t), v(t)) & \text{for all } t \in [t_0, T] \\ y(t_0) = x_0 \end{cases} \tag{4.2}$$

We shall define the components of this equation:

- U and V are compact metric spaces representing the controls by Player 1 and Player 2, respectively.
- $b : [0, T] \times \mathbb{R}^n \times U \times V \to \mathbb{R}^n$ is a map representing the rules of the game.
- $u : [t_0, T] \to U$ and $v : [t_0, T] \to V$ are the chosen control functions by the two players.
- Additionally, we denote $\mathcal{U}(t_0)$ and $\mathcal{V}(t_0)$ to be the set of all control functions u and v starting at t_0.

Similar to the optimal control section, the differential game is predicated on a cost functional called the payoff specified below.

Definition 4.3 (Payoffs)**.** The payoff for each player depends on a running payoff $f : \mathbb{R}^n \times [0, T] \times U \times V \to \mathbb{R}$ and a terminal payoff $g : \mathbb{R}^n \to \mathbb{R}$. Using these two functions, the cost functional J is defined as:

$$J(x_0, t_0, u, v) = \int_{t_0}^{T} f(v(s), s, u(s), v(s))\, ds + g(v(T))$$

For differential games, we also introduce the notion of *strategies*, a way of specifying players' responses to the opponents' control. Following Pierre [4], we shall focus on non-anticipative delay strategies, which are based on two premises. First, the response depends only on the past actions by the opponent, hence *non-anticipative*. Second, this response is unique up to a specified *delay* d after the opponent's move. These notions are formulated as follows.

Definition 4.4 (Non-anticipative delay strategies)**.** Given a delay $d > 0$, a non-anticipative strategy with delay for Player 1 is a map $\alpha : \mathcal{V}(t_0) \to \mathcal{U}(t_0)$ such that for any two controls $v_1, v_2 \in \mathcal{V}(t_0)$ and any time $t \geq t_0$, if $v_1 \equiv v_2$ on the interval $[t_0, t]$, then:

$$\alpha(v_1) \equiv \alpha(v_2) \qquad \text{on } [t_0, t + d]$$

Additionally, we let $\mathcal{A}(t_0)$ be the set of such strategies α for Player 1. These definitions are replicated for Player 2, with $\beta : \mathcal{U}(t_0) \to \mathcal{V}(t_0)$ being the strategy map and $\mathcal{B}(t_0)$ being the set of such maps. Furthermore, we utilize the notation:

$$\alpha_v := \alpha(v)$$

The following lemma from Pierre [4] establishes the unique pairings of strategies and controls.

Lemma 4.5. *Given a pair of non-anticipative delay strategies $(\alpha, \beta) \in \mathcal{A}(t_0) \times \mathcal{B}(t_0)$, there exists a unique pair of controls $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ such that on $[t_0, T]$, we have:*

$$\begin{cases} \alpha_v \equiv u \\ \beta_u \equiv v \end{cases}$$

For every choice of a starting position and a control pair, the ODE (4.2) has an unique solution. With Lemma 4.5, we can instead represent the control pair with the corresponding strategy pair and, consequently, denote the solution by $y_{x_0,t_0,\alpha,\beta}$ while abbreviating to y_{x_0,t_0} or y if no confusion arises. In each game, we set the goals for Player 1 and 2 to minimize and maximize the payoff, respectively. This gives us two value functions.

Definition 4.6 (Value functions)**.** The upper value function, which is exercised by Player 1 and aims to minimize the highest payoff achievable by Player 2, is defined as:

$$V^+(x_0, t_0) = \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{\beta \in \mathcal{B}(t_0)} J(x_0, t_0, \alpha, \beta) \tag{4.7}$$

On the other hand, the lower value function, which is exercised by Player 2 and aims to maximize the lowest payoff achievable by Player 1, is defined as:

$$V^-(x_0, t_0) = \sup_{\beta \in \mathcal{B}(t_0)} \inf_{\alpha \in \mathcal{A}(t_0)} J(x_0, t_0, \alpha, \beta) \tag{4.8}$$

The game shall have a value if and only if $V^+ = V^-$.

We also have the following lemma that provides an alternative definition of the value functions.

Lemma 4.9. *The value functions satisfy the following alternative identities:*

$$\begin{cases} V^+(x_0,t_0) = \inf_{\alpha\in\mathcal{A}(t_0)} \sup_{v\in\mathcal{V}(t_0)} J(x_0,t_0,\alpha_v,v) \\ V^-(x_0,t_0) = \sup_{\beta\in\mathcal{B}(t_0)} \inf_{u\in\mathcal{U}(t_0)} J(x_0,t_0,u,\beta_u) \end{cases}$$

Proof. We shall do the first statement - the second one can be proven similarly. On the one hand, note that

$$V^+(x_0,t_0) \geq \inf_{\alpha\in\mathcal{A}(t_0)} \sup_{v\in\mathcal{V}(t_0)} J(x_0,t_0,\alpha_v,v)$$

since the set of strategies $\mathcal{B}(t_0)$ is a superset of $\mathcal{V}(t_0)$ (for example, a supremum achievable by a $v^* \in \mathcal{V}(t_0)$ can also be achieved by a strategy β that maps every $u \in \mathcal{U}(t_0)$ to v^*). On the other hand, by Lemma 4.5, for any strategy pair $(\alpha,\beta) \in \mathcal{A}(t_0)\times\mathcal{B}(t_0)$, there is a unique control pair $(u_0,v_0) \in \mathcal{U}(t_0)\times\mathcal{V}(t_0)$ such that $\alpha_{u_0} = v_0$ and $\beta_{v_0} = u_0$ on (t_0,T). Then, we have:

$$J(x_0,t_0,\alpha,\beta) = J(x_0,t_0,\alpha_{v_0},v_0) \leq \sup_{v\in\mathcal{V}(t_0)} J(x_0,t_0,\alpha_v,v)$$

Since this is for any strategy $\beta \in \mathcal{B}(t_0)$, then:

$$\sup_{\beta\in\mathcal{B}(t_0} J(x_0,t_0,\alpha,\beta) \leq \sup_{v\in\mathcal{V}(t_0)} J(x_0,t_0,\alpha_v,v)$$

Taking the infimum over all strategies in $\mathcal{A}(t_0)$ finalizes the proof. □

Throughout this section, we shall derive the necessary steps to establish these value functions as viscosity solutions. In particular, much like the optimal control section, both value functions satisfy their own dynamic programming principle, outlined in Theorem 4.11. This, in turn, allow us to find their respective Hamilton-Jacobi equations in Theorem 4.15. Finally, Corollary 4.20, also known as Isaac's condition, shall tell us when a game can attain a value. Before that, we conclude our setup of the zero-sum differential game by stating the assumptions that we shall utilize throughout this section.

Assumption 4.10. The following statement holds.

(1) The map b is continuous, bounded, and Lipschitz in the space variable. That is, there exists a constant $C > 0$ such that:

$$\begin{cases} b \in C(\mathbb{R}^n\times[0,T]\times U\times V) & \\ |b(x,t,u,v)| \leq C & \text{for all } (t,x,u,v) \in \mathbb{R}^n\times[0,T]\times U\times V \\ |b(x_1,t,u,v) - b(x_2,t,u,v)| \leq C|x_1-x_2| & \text{for all } x_1,x_2\in\mathbb{R}^n,\ t\in[0,T],\ u\in U,\ v\in V \end{cases}$$

(2) The running payoff f is also continuous, bounded, and Lipschitz in the space variable, similar to b.
(3) The terminal payoff g is bounded and Lipschitz continuous, that is, there exists a constant $C > 0$ such that:

$$\begin{cases} g \in C(\mathbb{R}^n) & \\ |g(x_1) - g(x_2)| \leq C|x_1-x_2| & \text{for all } x_1,x_2\in\mathbb{R}^n \end{cases}$$

4.1. **Dynamic programming principle.** The main result for this subsection is the following identity regarding the two value functions.

Theorem 4.11 (Dynamic programming principle)**.** *Take an initial position $(x_0,t_0) \in \mathbb{R}^n\times[0,T)$ and $h \in (0,T-t_0)$. Then, the upper value function (4.7) satifies:*

$$V^+(x_0,t_0) = \inf_{\alpha\in\mathcal{A}(t_0)} \sup_{v\in\mathcal{V}(t_0)} \left\{ \int_{t_0}^{t_0+h} f(y(s),s,\alpha_v(s),v(s))\,ds + V^+(y(t_0+h),t_0+h) \right\} \tag{4.12}$$

To prove this theorem, we want to have the following lemma, which states that nearly optimal strategies remain nearly optimal in a neighborhood of the initial position.

Lemma 4.13 (Stability of optimal strategies)**.** *Take any initial position $(x_0, t_0) \in \mathbb{R}^n \times [0, T]$ and any $\varepsilon > 0$. Also, take any ε-optimal strategy $\alpha \in \mathcal{A}(t_0)$ for the game $V^+(x_0, t_0)$, that is:*

$$\sup_{v \in \mathcal{V}(t_0)} J(x_0, t_0, \alpha_v, v) \leq V^+(x_0, t_0) + \varepsilon$$

Then, there must exist a $\eta > 0$ such that the strategy remains (2ε)-optimal for the game $V^+(z, t_0)$ for any $z \in B(x_0, \eta)$. In other words:

$$\sup_{v \in \mathcal{V}(t_0)} J(z, t_0, \alpha_v, v) \leq V^+(z, t_0) + 2\varepsilon \qquad \text{for all } z \in B(x_0, \eta)$$

Proof. First, we want to prove that both the cost functional and the value function are uniformly Lipschitz in the space variable. Take $t_0 \in [0, T]$ and two starting points $x, z \in \mathbb{R}^n$. Then, for any pair of controls $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$, the paths y_{x,t_0} and y_{z,t_0} satisfy the following bound thanks to Gronwall's inequality:

$$|y_{x,t_0}(t) - y_{z,t_0}(t)| \leq |x - z| e^{C|t-t_0|} \qquad \text{for all } t \in [t_0, T]$$

This, along with the Lipschitz property of the running cost f and the terminal cost g, gives the following:

$$\begin{aligned} |J(x, t_0, u, v) - J(z, t_0, u, v)| &\leq \int_{t_0}^{T} |f(y_{x,t_0}(s), s, u, v) - f(y_{z,t_0}(s), s, u, v)|\, ds + |g(y_{x,t_0}(T)) - g(y_{z,t_0}(T))| \\ &\leq C \int_{t_0}^{T} |y_{x,t_0}(s) - y_{z,t_0}(s)|\, ds + C|y_{x,t_0}(T) - y_{z,t_0}(T)| \\ &\leq C|x - z| \end{aligned}$$

The Lipschitz property of the value function V^+ follows immediately since this applies to all control pairs. Thus, we have that for any $z \in \mathbb{R}^n$, there are Lipschitz constants C and C' such that:

$$\begin{aligned} \sup_{v \in \mathcal{V}(t_0)} J(z, t_0, \alpha_v, v) &\leq \sup_{v \in \mathcal{V}(t_0)} J(x_0, t_0, \alpha_v, v) + C|z - x_0| \\ &\leq V^+(x_0, t_0) + \varepsilon + C|z - x_0| \\ &\leq V^+(z, t_0) + \varepsilon + (C + C')|z - x_0| \end{aligned}$$

By setting $\eta = \varepsilon/(C + C')$, the strategy α becomes (2ε)-optimal for $V^+(z, t_0)$, and the proof is done. □

Proof of Theorem 4.11. We divide the proof into two steps. For the first step, we want to prove that LHS $\geq$ RHS in (4.12). First, fix any $\varepsilon > 0$ and by the property of the infimum, we can select an ε-optimal strategy $\alpha \in \mathcal{A}(t_0)$, that is:

$$\sup_{v \in \mathcal{V}(t_0)} J(x_0, t_0, \alpha_v, v) \leq V^+(x_0, t_0) + \varepsilon$$

Now, we fix a control $v_0 \in \mathcal{V}(t_0)$ and take $\tilde{\alpha} \in \mathcal{A}(t_0 + h)$ to be the restriction of the original strategy to times after $t_0 + h$. In other words, for all $v \in \mathcal{V}(t_0 + h)$, if we denote v^* to be the mixed control

$$\begin{cases} v^*(s) = v_0(s) & \text{on } s \in [t_0, t_0 + h] \\ v^*(s) = v(s) & \text{otherwise} \end{cases}$$

then we must have, for all $t \in [t_0 + h, T]$:

$$\tilde{\alpha}_v(t) = \alpha_{v^*}(t)$$

Let $y = y_{x_0,t_0,\alpha_{v^*},v^*}$ be the state corresponding to the original strategy and mixed control. First, this yields:

$$V^+(y(t_0 + h), t_0 + h) \leq \sup_{v \in V(t_0+h)} J(y(t_0 + h), t_0 + h, \tilde{\alpha}_v, v)$$

However, for all $v \in \mathcal{V}(t_0 + h)$, we have:

$$J(x_0, t_0, \alpha_{v^*}, v^*) = \int_{t_0}^{t_0+h} f(y(s), s, \alpha_{v_0}(s), v_0(s))\, ds + J(y(t_0 + h), t_0 + h, \tilde{\alpha}_v, v)$$

This means that if we let S be the set of mixed controls as defined above, then:

$$\begin{aligned}&\int_{t_0}^{t_0+h} f(y(s),s,\alpha_{v_0}(s),v_0(s))\,ds + V^+(y(t_0+h),t_0+h)\\ &\leq \int_{t_0}^{t_0+h} f(y(s),s,\alpha_{v_0}(s),v_0(s))\,ds + \sup_{v\in\mathcal{V}(t_0+h)} J(y(t_0+h),t_0+h,\tilde{\alpha}_v,v)\\ &= \sup_{v^*\in S} J(x_0,t_0,\alpha_{v^*},v^*)\\ &\leq V^+(x_0,t_0)+\varepsilon\end{aligned}$$

where the last line is because S is a subset of the set of controls $\mathcal{V}(t_0)$. Because this applies to all controls $v_0 \in \mathcal{V}(t_0)$, we can take the supremum of both sides to get:

$$\sup_{v\in\mathcal{V}(t_0)} \left\{ \int_{t_0}^{t_0+h} f(y(s),s,\alpha_v(s),v(s))\,ds + V^+(y(t_0+h),t_0+h) \right\} \leq V^+(x_0,t_0)+\varepsilon$$

Finally, since ε was selected arbitrarily, letting it go to 0 and taking the infimum over all strategies gives us LHS $\geq$ RHS. We now turn to the other step, which is proving RHS $\geq$ LHS. Again, we fix a $\varepsilon > 0$ and select a strategy $\alpha^0 \in \mathcal{A}(t_0)$ such that:

$$\int_{t_0}^{t_0+h} f(y(s),s,\alpha_v^0(s),v(s))\,ds + V^+(y(t_0+h),t_0+h) \leq \text{RHS}+\varepsilon$$

Furthermore, for every $x \in \mathbb{R}^n$, select a strategy α^x that is ε-optimal for the game $V^+(x,t_0+h)$. By Lemma 4.13, the strategy α^x is (2ε)-optimal for the game $V^+(z,t_0+h)$ for all $z \in B(x,\eta)$ with some $\eta > 0$. Here, since $\mathbb{R}^n$ is locally compact, then we can find a countable family of points $\{x_i\}$ such that $\{B(x_i,\eta_i)\}$ is a locally finite covering of $\mathbb{R}^n$ - that is, every point x belongs to a finite number of balls $B(x_i,\eta_i)$. We make this covering disjoint by defining the family $\{S_i\}$ such that:

$$\begin{cases} S_1 = B(x_1,\eta_1)\\ S_i = B(x_i,\eta_i)\setminus\bigcup_{j<i} B(x_j,\eta_j)\end{cases}$$

Also, we fix a common delay $\tau \in (0,\tau/||b||_\infty)$ with properties defined later. With this setup, we define the hybrid strategy α as follows for all $v \in \mathcal{V}(t_0)$:

$$\begin{cases} \alpha_v(t) = \alpha_v^0(t) & \text{for all } t\in[t_0,t_0+h]\\ \alpha_v(t) = \alpha_v^{x_i}(t) & \text{for all } t\in[t_0+h,T] \text{ and } y_{x_0,t_0,\alpha_v^0,v}(t_0+h-\tau)\in S_i\end{cases}$$

We have to first prove that α is a delay strategy. To do this, we first establish that the definition of α only involves a finite number of strategies. Indeed, since b is bounded, then the state y is also bounded with:

$$|y(t_0+h)| \leq |x_0| + ||b||_\infty \cdot T$$

Therefore, there exists a collection of sets that contains $y(t_0+h)$, as with some $n \geq 1$:

$$B(0,|x_0|+||b||_\infty\cdot T) \subseteq \bigcup_{1\leq i\leq n} B(x_i,\eta_i) = \bigcup_{1\leq i\leq n} S_i$$

This means α only involves α^0 and α^{x_i} with $1 \leq i \leq n$. We can then take $\tau \in (0,\tau/||b||_\infty)$ to be the minimum common delay of these strategies and prove that τ is the delay of α. Let $v_1, v_2 \in \mathcal{V}(t_0)$ be two controls overlapping on $[t_0,t]$. If $t \leq t_0+h-\tau$, then:

$$\alpha_{v_1} = \alpha_{v_1}^0 = \alpha_{v_2}^0 = \alpha_{v_2} \qquad \text{for all } t\in[t_0,t+\tau]$$

On the other hand, if $t \geq t_0+h-\tau$, then we must have that $y(t_0+h)$ belongs to some S_i, which means that barring the times before t_0+h which still overlaps under α^0, we have, for all times after t_0+h till $t+\tau$:

$$\alpha_{v_1} = \alpha_{v_1}^{x_i} = \alpha_{v_2}^{x_i} = \alpha_{v_2}$$

This proves that α is a delay strategy. Finally, note that since the strategy a^{x_i} has to be (2ε)-optimal for the game $V^+(y(t_0+h), t_0+h)$, then we have:

$$\int_{t_0+h}^{T} f(y(s), s, \alpha_v^{x_i}(s), v(s))\,ds + g(v(T)) \le V^+(y(t_0+h), t_0+h) + 2\varepsilon$$

Combining this with the ε-optimality of α^0 gives us the following:

$$J(x_0, t_0, \alpha_v, v) \le \int_{t_0}^{t_0+h} f(y(s), s, \alpha_v^0(s), v(s))\,ds + V^+(y(t_0+h), t_0+h) + 2\varepsilon \le \text{RHS} + 3\varepsilon$$

Because this applies to all controls $v \in \mathcal{V}(t_0)$, then applying the supremum yields:

$$\sup_{v\in\mathcal{V}(t_0)} J(x_0, t_0, \alpha_v, v) \le \text{RHS} + 3\varepsilon$$

However, since ε was selected arbitrarily, letting $\varepsilon \to 0$ and taking the infimum over all strategies in $\mathcal{A}(t_0)$ gives us RHS $\ge$ LHS. This finalizes the proof of the dynamic programming principle. ☐

Remark 4.14. In the proof above, we only focused on the upper value function. This is because results regarding the upper value function are directly translatable to the lower value function, knowing that

$$-V^-(x_0, t_0) = \inf_{\beta\in\mathcal{B}(t_0)} \sup_{\alpha\in\mathcal{A}(t_0)} (-J(x_0, t_0, \alpha, \beta))$$

is the upper value function of a game with running payoff $-f$, terminal payoff $-g$, and player roles swapped (so that now Player 1 is maximizing and Player 2 is minimizing).

4.2. **Isaac's condition and Hamilton-Jacobi equations of the value functions.** Using the dynamic programming principle, we now try to find Hamilton-Jacobi equations which the value functions are viscosity solutions to. It turns out that they solve two different, yet closely related equations, as formulated in the theorem below.

Theorem 4.15. *The upper value function V^+ as defined by (4.7) is a viscosity solution of the time-evolution Hamilton-Jacobi equation:*

$$\begin{cases} -u_t + H^+(x, t, Du) = 0 & \text{in } \mathbb{R}^n \times (0, T) \\ u(x, T) = g(x) & \text{on } \mathbb{R}^n \end{cases} \tag{4.16}$$

where the Hamiltonian $H^+ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is

$$H^+(x, t, p) = \sup_{u\in U} \inf_{v\in V} (-b(x, t, u, v)\cdot p - f(x, t, u, v)) \tag{4.17}$$

On the other hand, the lower value function V^- as defined by (4.8) is a viscosity solution of the equation:

$$\begin{cases} -u_t + H^-(x, t, Du) = 0 & \text{in } \mathbb{R}^n \times (0, T) \\ u(x, T) = g(x) & \text{on } \mathbb{R}^n \end{cases} \tag{4.18}$$

where the Hamiltonian $H^- : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is

$$H^-(x, t, p) = \inf_{v\in V} \sup_{u\in U} (-b(x, t, u, v)\cdot p - f(x, t, u, v)) \tag{4.19}$$

Proof. Here, once again, we only need to prove for the upper value function V^+, since Remark 4.14 allows us to obtain versions of the subsolution and supersolution tests for the lower value function. We start with the subsolution test. Take a test function $\varphi \in C^1(\mathbb{R}^n \times (0, T))$ such that at a point $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$, we have:

$$\begin{cases} V^+(x_0, t_0) = \varphi(x_0, t_0) \\ V^+ - \varphi \text{ has a strict local maximum at } (x_0, t_0) \end{cases}$$

The goal is to show that:

$$-\varphi_t(x_0, t_0) + H^+(x_0, t_0, D\varphi(x_0, t_0)) \le 0$$

First, since $V^+ - \varphi$ has a strict local maximum at (x_0, t_0), there exists a ball $B((x_0, t_0), r)$ such that for any point (x, t) in the ball, we have:

$$V^+(x_0, t_0) - \varphi(x_0, t_0) - (V^+(x, t) - \varphi(x, t)) \geq 0$$

Furthermore, the state process y is Lipschitz, so for h sufficiently small, we must have that $y(t_0 + h) \in B((x_0, t_0), r)$. Applying the dynamic programming principle in Theorem 4.11 then gives us:

$$\begin{aligned} 0 &\leq (V^+(x_0, t_0) - V^+(x, t)) + \varphi(y(t_0 + h), t_0 + h) - \varphi(x_0, t_0) \\ &= \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{v \in \mathcal{V}(t_0)} \left\{ \int_{t_0}^{t_0+h} f(y(s), s, \alpha_v(s), v(s))\, ds + \varphi(y(t_0 + h), t_0 + h) - \varphi(x_0, t_0) \right\} \end{aligned}$$

Now, fix a control $u^* \in \mathcal{U}(t_0)$ and take the strategy α that maps every control $v \in \mathcal{V}(t_0)$ to u^*. Then, for every $\varepsilon > 0$ and every time interval $h > 0$, there exists a control v^h such that:

$$-\varepsilon h \leq \int_{t_0}^{t_0+h} f(y(s), s, u^*(s), v^h(s))\, ds + \varphi(y(t_0 + h), t_0 + h) - \varphi(x_0, t_0)$$

Here, keeping in mind the uniform continuity of φ_t, $d\varphi$, and b in the time and space variables, by the fundamental theorem of calculus, we get the following:

$$\begin{aligned} \varphi(y(t_0 + h), t_0 + h) - \varphi(x_0, t_0) &= \int_{t_0}^{t_0+h} \varphi_t(y(s), s) + D\varphi(y(s), s) \cdot b(y(s), s, u^*(s), v^h(s))\, ds \\ &\leq Ch + h\varphi_t(x_0, t_0) + \int_{t_0}^{t_0+h} D\varphi(x_0, t_0) \cdot b(x_0, t_0, u^*(s), v^h(s))\, ds \end{aligned}$$

On the other hand, using the Lipschitz continuity in the time and space variables of f, we have:

$$\int_{t_0}^{t_0+h} f(y(s), s, u^*(s), v^h(s))\, ds \leq Ch + \int_{t_0}^{t_0+h} f(x_0, t_0, u^*(s), v^h(s))\, ds$$

Combining these two inequalities gives us:

$$\begin{aligned} -\varepsilon h - Ch &\leq h\varphi_t(x_0, t_0) + \int_{t_0}^{t_0+h} D\varphi(x_0, t_0) \cdot b(x_0, t_0, u^*(s), v^h(s)) + f(x_0, t_0, u^*(s), v^h(s))\, ds \\ &= h\varphi_t(x_0, t_0) + h \sup_{v \in V} \{D\varphi(x_0, t_0) \cdot b(x_0, t_0, u^*(s), v(s)) + f(x_0, t_0, u^*(s), v(s))\} \end{aligned}$$

Dividing both sides by h and then letting $h \to 0$ and $\varepsilon \to 0$ gives:

$$0 \leq \varphi_t(x_0, t_0) + \sup_{v \in V} \{D\varphi(x_0, t_0) \cdot b(x_0, t_0, u^*(s), v(s)) + f(x_0, t_0, u^*(s), v(s))\}$$

Because this applies to all controls u^*, we can take the infimum across all controls in $\mathcal{U}(t_0)$ and then flip the signs to get the desired inequality. Next, we do the supersolution test. Let $\psi \in C^1(\mathbb{R}^n \times (0, \infty))$ such that at a point $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$, we have:

$$\begin{cases} V^+(x_0, t_0) = \psi(x_0, t_0) \\ V^+ - \psi \text{ has a strict local minimum at } (x_0, t_0) \end{cases}$$

We will aim to show the inequality:

$$-\psi_t(x_0, t_0) + H^+(x_0, t_0, D\psi(x_0, t_0)) \geq 0$$

The proof for this part shall largely resemble that of the subsolution test. Using the dynamic programming principle in Theorem 4.11, we have the following inequality for h sufficiently small:

$$\psi(x_0, t_0) = V^+(x_0, t_0) \geq \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{v \in \mathcal{V}(t_0)} \left\{ \int_{t_0}^{t_0+h} f(y(s), s, \alpha_v(s), v(s))\, ds + \psi(y(t_0 + h), t_0 + h) \right\}$$

Fix a control $v^* \in \mathcal{V}(t_0)$. Then, for every $\varepsilon > 0$ and every time interval $h > 0$, there exists a strategy α^h such that:

$$\begin{aligned}\varepsilon h &\geq \int_{t_0}^{t_0+h} f(y(s), s, \alpha^h_{v^*}(s), v^*(s))\,ds + \psi(y(t_0+h), t_0+h) - \psi(x_0, t_0)\\ &= \int_{t_0}^{t_0+h} f(y(s), s, \alpha^h_{v^*}(s), v^*(s)) + \psi_t(y(s), s) + D\psi(y(s), s)\cdot b(y(s), s, \alpha^h_{v^*}(s), v^*(s))\,ds\end{aligned}$$

With the uniform continuity of φ_t, $d\varphi$, and b as well as the Lipschitz continuity of f in the time and space variables, we get:

$$\begin{aligned}\varepsilon h + Ch &\geq h\psi_t(x_0, t_0) + \int_{t_0}^{t_0+h} f(x_0, t_0, \alpha^h_{v^*}(s), v^*(s)) + D\psi(x_0, t_0)\cdot b(x_0, t_0, \alpha^h_{v^*}(s), v^*(s))\,ds\\ &\geq h\psi_t(x_0, t_0) + h \inf_{\alpha\in\mathcal{A}(t_0)} \{f(x_0, t_0, \alpha_{v^*}(s), v^*(s)) + D\psi(x_0, t_0)\cdot b(x_0, t_0, \alpha_{v^*}(s), v^*(s))\}\end{aligned}$$

Note that since the set of strategies $\mathcal{A}(t_0)$ is a superset of the set of controls $\mathcal{U}(t_0)$, then we can replace the infimum to be over all $u \in \mathcal{U}(t_0)$ without changing the inequality. Dividing both sides by h and then letting $h \to 0$ and $\varepsilon \to 0$ gives:

$$0 \geq \psi_t(x_0, t_0) + \inf_{u\in\mathcal{U}(t_0)} \{f(x_0, t_0, u(s), v^*(s)) + D\psi(x_0, t_0)\cdot b(x_0, t_0, u(s), v^*(s))\}$$

Since this applies to all controls $v^* \in \mathcal{V}(t_0)$, taking the infimum over $\mathcal{V}(t_0)$ and flipping the signs gives us the desired inequality. This finalizes the proof that V^+ is a viscosity solution of (4.16). □

The upcoming corollary, known as Isaac's condition, follows immediately from the theorem above.

Corollary 4.20 (Isaac's condition)**.** *The game has a value, that is*

$$V^+(x,t) = V^-(x,t) \qquad \text{for all } (x,t) \in \mathbb{R}^n \times [0,T]$$

if Isaac's condition holds:

$$H^+(x,t,p) = H^-(x,t,p) \qquad \text{for all } (x,t) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n$$

Moreover, $V^+ = V^-$ is the unique viscosity solution of the Hamilton-Jacobi equation (4.16)/(4.18).

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