AN INTRODUCTION TO QUASIFUCHSIAN MANIFOLDS IN HYPERBOLIC SPACE

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Abstract. In this paper, we introduce the idea of a quasifuchsian manifold in hyperbolic space. We assume little to no background in hyperbolic geometry, instead starting from the ground up, providing all the necessary concepts along the way. We end with the definition of a quasifuchsian group and a few examples. For the sake of brevity, we do not provide proofs for most results. However, for the interested reader, we provide references at the end of each section that go into further depth on each topic.

Contents

1. Hyperbolic Geometry

We will begin with a summary of hyperbolic geometry in two dimensions. There are several spaces in which hyperbolic geometry can be modeled. One such space is called the upper half-plane.

Definition 1.1. The **upper half-plane** is defined as follows:

$$\mathbb{H}^2 = \{x + iy \in \mathbb{C} \mid y > 0\}.$$

As the name suggests, the upper half-plane is the set of all points above the x-axis.

Next, we define a metric on this space.

Definition 1.2. The **hyperbolic metric** is defined in the following way:

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

The **hyperbolic length** of a path is computed by integrating the path against the hyperbolic metric, and the **hyperbolic distance** between two points is the minimum of the lengths of these paths (the infimum is always attained).

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We see that the hyperbolic metric is defined similarly to the Euclidean one–in fact, the term in the numerator is precisely the Euclidean metric. The term in the denominator implies that the hyperbolic distance between points tends to infinity as the points get close (in the Euclidean sense) to the x-axis, provided that the x-coordinates stay a bounded distance apart. This remark motivates the following definition:

Definition 1.3. The **Euclidean boundary**, or **boundary at infinity**, is defined as $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$.

We now turn our attention to the isometries (distance-preserving maps) of $\mathbb{H}^2$.

Definition 1.4. Denote points of $\mathbb{H}^2$ by $z = x + iy$. A **Möbius transformation**, or **fractional linear transformation**, is a function of the form:

$$z \mapsto \frac{az+b}{cz+d}, \;\; a, b, c, d \in \mathbb{R}, \; ad - bc = 1$$

We can extend the action of the Möbius transformation to the boundary at infinity in the following way: for ordinary points on the real line, we simply apply the function normally. When $c \neq 0$, we define $-\frac{d}{c} \mapsto \infty$, and $\infty \mapsto \frac{a}{c}$. If $c = 0$, $\infty \mapsto \infty$.

We note that the Möbius transformations form a group under the operation of function composition. This is easily verified by identifying Möbius transformations with 2×2 matrices in a way we describe now.

Recall that we denote the set of real-valued matrices with determinant 1 as SL(2,$\mathbb{R}$). As such, we can represent every Möbius transformation by a matrix in SL(2,$\mathbb{R}$), namely, the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is associated with the map $z \mapsto \frac{az+b}{cz+d}$. Here, we note that the reason we may consider only SL(2,$\mathbb{R}$) and not GL(2,$\mathbb{R}$) is that if we multiply each entry of a matrix by a constant k, then the determinant is multiplied by k^2, but the Möbius transformation remains the same due to cancellation in the numerator and denominator. Thus, it suffices only to consider matrices with determinant 1. However, there is still not a one-to-one correspondence between SL(2,$\mathbb{R}$) and the Möbius transformations: for each matrix A, both A and $-A$ correspond to the same transformation. As such, we instead define the group PSL(2,$\mathbb{R}$) as SL(2,$\mathbb{R}$)$/\pm I$. In plain English, we treat both A and $-A$ as the same object in $\mathrm{PSL}(2, \mathbb{R})$. By the above remarks, it follows that we have a bijection between the Möbius transformations and PSL(2,$\mathbb{R}$). We note that a composition of two Möbius transformations corresponds to the product of the two corresponding matrices.

In fact, we have an even stronger result.

Theorem 1.5. $\mathrm{Isom}^+(\mathbb{H}^2) \cong \mathrm{PSL}(2, \mathbb{R})$*, where* $\mathrm{Isom}^+(\mathbb{H}^2)$ *represents the orientation-preserving isometries of* $\mathbb{H}^2$*.*

A consequence of this result provides a more intuitive grasp of the nature of hyperbolic geometry.

Theorem 1.6. *The geodesics (straight lines) in* $\mathbb{H}^2$ *are semicircles and straight lines orthogonal to the real axis* $\mathbb{R}$*.*

Proof. For proofs of these two theorems, see [1], chapter 1.1-1.3. $\square$

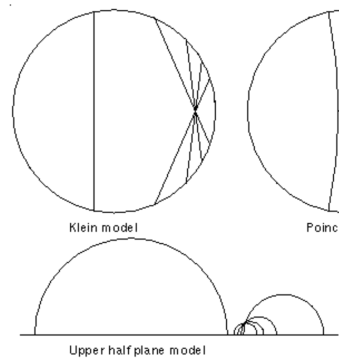

FIGURE 1. The same five lines visualized in different models of hyperbolic space (we will not use the Klein model for our purposes). Figure borrowed from [2]

It will often be useful to work with another model of hyperbolic space. We present it here.

Definition 1.7. The Poincaré Disk Model
Let

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\},$$

or simply the open unit disk. Then, the metric on $\mathbb{D}$ is

$$ds^2 = \frac{4dz^2}{(1 - |z|^2)^2},$$

and the boundary at infinity is $\partial\mathbb{D}$, or S^1.

The straight lines in the Poincaré disk model are arcs of circles which are orthogonal to the boundary at both intersection points, as displayed in Figure 1.

We note that the upper half-space model and the Poincaré disk model are isometric via the function $f : \mathbb{H}^2 \to \mathbb{D}$,

$$f(z) = \frac{zi + 1}{z + i}.$$

Since the models are equivalent, we will use them interchangeably and denote both simply as $\mathbb{H}^2$ for ease of notation. It should be clear from context which model we are working with.

This section has provided a brief overview of hyperbolic geometry which is needed for the rest of the paper. However, we have omitted many interesting and important results. If the reader is interested in learning more and gaining better intuition, they should consult [3], [4], and [5].

2. Discrete Groups of Isometries and Limit Sets

In this section, we discuss a special type of subgroup of $\mathrm{Isom}^+(\mathbb{H}^2)$.

Definition 2.1. A subgroup Γ of $\mathrm{Isom}(\mathbb{H}^2)$ is a **discrete group** if the induced subspace topology on Γ is the discrete topology.

When we discuss the topology on $\mathrm{Isom}^+(\mathbb{H}^2)$, we view the elements as points inside $\mathbb{R}^4$ (the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is identified with the point (a,b,c,d)) and consider the normal subspace topology there. When discussing $\mathrm{Isom}(\mathbb{H}^2)$ as a whole, we follow a similar process.

An alternative characterization of a discrete group is that for any sequence A_n which converges to the identity I, there exists $N > 0$ such that for all $n \geq N$, $A_n = I$. It follows that a discrete group has no accumulation points.

Definition 2.2. A **Fuchsian group** is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^2)$.

Definition 2.3. The **orbit** of a point x under a Fuchsian group Γ is denoted by

$$\Gamma x = \{\gamma x \mid \gamma \in \Gamma\}$$

Definition 2.4. Let Γ be a Fuchsian group. Then, the **limit set** Λ_Γ is defined as the set of limit points of Γx for some $x \in \mathbb{H}^2$.

Since Γ is discrete, there are no accumulation points in $\mathbb{H}^2$, as remarked above. This implies that $\Lambda_\Gamma \subset \mathbb{R} \cup \{\infty\}$.

Remark 2.5. The limit set is invariant under the choice of reference point x. To see this, consider the Poincaré disk model. Suppose that $\gamma_i x \to z \in S^1$. Then, consider any other point $y \in \mathbb{D}$. Since γ_i are all isometries, $d_{\mathbb{H}^2}(\gamma_i x, \gamma_i y) = d_{\mathbb{H}^2}(x, y) < \infty$. However, we know that any point on the boundary S^1 is infinitely far away from any other point in $\overline{\mathbb{D}}$ by the definition of the hyperbolic metric. Therefore, since $\gamma_i x$ is tending to z on the boundary, then both $\gamma_i x$ and $\gamma_i y$ must be converging to the same point.

The complement of Λ_Γ is also a useful concept.

Definition 2.6. Let Γ be a Fuchsian group. Then, $\Omega_\Gamma = S^1 \setminus \Lambda_\Gamma$ is referred to as the **domain of discontinuity**.

We now discuss another important feature of Fuchsian groups, the fundamental region.

Definition 2.7. Let Γ be a Fuchsian group. Then, a region F is called a **fundamental region** for Γ if

(i) $\bigcup_{\gamma \in \Gamma} \gamma(F) = \mathbb{H}^2$;
(ii) $F^\circ \cap \gamma(F^\circ) = \varnothing$ for all $\gamma \neq \mathrm{id}$,

where F° is the interior of F.

Let's look at a couple of examples. First, let's examine one in the complex plane for easier visualization.

Example 2.8. Consider $\Gamma = \langle z+1, z+i \rangle$, where $z+1$ represents the transformation which takes the complex number z to $z+1$, and $\langle\cdot\rangle$ represents the group generated by the elements inside the brackets. Then, Γ consists of all transformations which move a point to another point on the integer lattice centered at the point. One fundamental region for this group (there are infinitely many) is the unit square formed by the vertices (0,0), (1,0), (1,1), and (0,1). Since the square has side length one, we see that any point in the interior will be moved outside by any non-identity transformation in Γ. Additionally, since elements in Γ move up/down and left/right in increments of one unit, we see that the union of all of them will

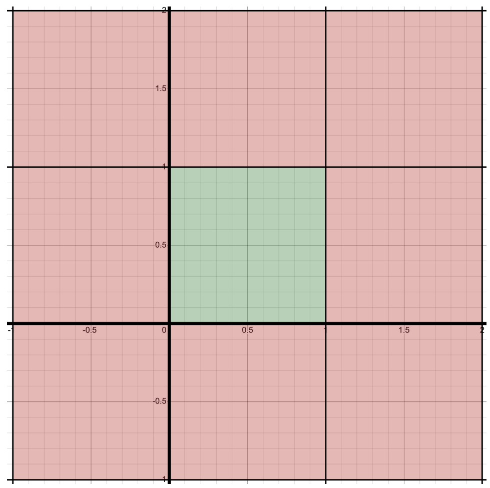

FIGURE 2. A tessellation of the Euclidean plane by squares. The fundamental region is marked in green, and the translates under Γ are marked in red (figure made with Desmos Online Graphing Calculator).

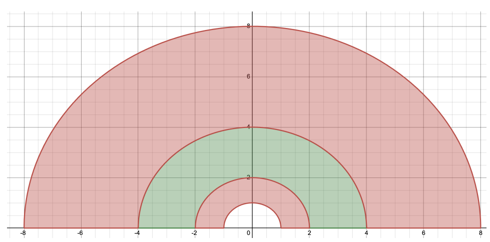

FIGURE 3. We can construct a tessellation of the hyperbolic plane using half-annuli as depicted in the figure. As above, the green section is the fundamental region, and the red regions are the shifted and scaled copies, doubling or halving the size. We could continue the process to cover the whole upper half-plane (figure made with Desmos Online Graphing Calculator; idea taken from [1]).

form a perfect tessellation of $\mathbb{C}$, where the only overlap is on the boundary of each square. See figure 2.

Example 2.9. Consider now the hyperbolic plane with the group $\Gamma = \langle 2z \rangle$, where $2z$ represents the map which sends the complex number z to $2z$. Then, a fundamental region for this group is a half-annulus whose inner radius is half the length of the outer radius. See Figure 3.

A natural question to ask is whether all Fuchsian groups have a fundamental region. The answer turns out to be yes.

Definition 2.10. Let Γ be a Fuchsian group, and $p \in \mathbb{H}^2$ be not fixed by any element $\gamma \neq \mathrm{id}$. Then we define the **Dirichlet region for Γ centered at p** as

follows:
$$D_p(\Gamma) = \{z \in \mathbb{H}^2 \mid d(z,p) \le d(z,\gamma p) \text{ for all } \gamma \in \Gamma\}.$$

Theorem 2.11. *If Γ is a Fuchsian group and p is not fixed by any non-identity element, then the Dirichlet region $D_p(\Gamma)$ is a connected fundamental region for Γ.*

Proof. See [1], theorem 3.2.2. □

We will explore fundamental regions in a little more depth. By rewriting the definition of the Dirichlet region slightly differently:
$$D_p(\Gamma) = \bigcap_{\gamma \in \Gamma} \{z \in \mathbb{H}^2 \mid d(z,p) \le d(z,\gamma p)\},$$
we see that $D_p(\Gamma)$ is the intersection of closed half-planes, and so it is itself a closed, convex hyperbolic region whose boundary contains at most one segment from the boundary line of each half-plane. In the two examples we saw above, the fundamental region was a nice simple polygon. However, this is not always the case. We present a sufficient criterion for this to hold:

Theorem 2.12. *Suppose that Γ is finitely generated. Then, there exists a (polygonal) fundamental region for Γ which has finitely many sides.*

Proof. See [1], theorem 4.6.1. □

For a more careful and detailed exploration of Fuchsian groups and fundamental regions, see [1].

3. Covering Spaces and Hyperbolic Surfaces

We ultimately wish to discuss surfaces and manifolds in hyperbolic space. To do this, we introduce the idea of a covering space.

Definition 3.1. Let X be a topological space. A **covering space** of X is a space $\tilde{X}$ along with a map $p : \tilde{X} \to X$ which satisfies the following property: for each $x \in X$, there exists an open neighborhood $U_x \subset X$ such that $p^{-1}(U_x)$ is the union of disjoint open neighborhoods in $\tilde{X}$. Each of these open neighborhoods must be mapped homeomorphically onto U_x.

X is called the **base space**, p is called the **covering map** or **projection**, and $p^{-1}(x)$ is called the **fiber over** x.

This is a bit of an unintuitive definition at first. The easiest way to grasp the notion of a covering space is through some simple examples.

Example 3.2. Let $X = S^1$, $\tilde{X} = \mathbb{R}$, and $p(x) = (\cos x, \sin x)$. We see that $p(x + 2n\pi) = p(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. In this way, we can imagine the real line as if we took a very long circular coil of wire and unwound it while keeping track of where each point originated on the coil. In order to go back to the base space S^1, we simply need to roll up $\mathbb{R}$ again.

Example 3.3. Let $X = [0,1) \times [0,1) \subset \mathbb{R}^2$ and $\tilde{X} = \mathbb{R}^2$, and let p be the map which takes a point and maps it into X via a translation in the integer lattice. In other words, $(3.5, 2.7) \mapsto (0.5, 0.7)$ and $(-2, -0.6) \mapsto (0, 0.4)$. In this way, we can imagine $\mathbb{R}^2$ as being a tessellation of unit squares, and the corresponding points of each square are identified via the covering map. Note that this is in a sense the inverse of the operation that we discussed in Example 2.8 above.

We now turn our attention to hyperbolic surfaces.

Definition 3.4. A **hyperbolic surface** is a set of points S and a metric d_S such that for each $x \in S$, there exists $\varepsilon > 0$ small so that $B(x, \varepsilon)$ is isometric to a disk in $\mathbb{H}^2$.

A way to summarize this definition is that locally, a hyperbolic surface "looks like the hyperbolic plane". We can also define Euclidean and spherical surfaces analogously by replacing $\mathbb{H}^2$ with $\mathbb{R}^2$ and S^2, respectively.

Remark 3.5. Often, the term "manifold" is also used to describe a similar concept. A manifold is essentially the same idea, except it is not restricted to 2 dimensions. With this in mind, we could also describe a surface as a 2-manifold. However, we will continue to use "surface" to denote a 2-dimensional manifold to avoid any confusion. In this paper, the term "manifold" will always refer to an object of dimension at least 3.

Another note worth mentioning is that there are several different types of manifolds (differential, real analytic, complex), each with its own properties, but we will stick to the above formulation for our purposes.

We now present a deep theorem in geometry:

Theorem 3.6. (Killing-Hopf) *Let S be a complete and connected hyperbolic surface. Then, there exists a Fuchsian group $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^2)$ such that $S \cong \mathbb{H}^2/\Gamma$.*

As above, this result also holds in the Euclidean and spherical case.

Proof. For a great development of this proof, see [6], Chapters 2 and 5. □

Example 3.7. Recalling Examples 2.8 and 3.3 once again, we give one more way to describe the relationship between $\mathbb{R}^2$ and the unit square: $\mathbb{R}^2$ is a covering space for $\mathbb{R}^2/\Gamma$, where Γ is as in Example 2.8. Note that in the quotient operation $\mathbb{R}^2/\Gamma$, two points p, q are identified if there is an element $\gamma \in \Gamma$ such that $\gamma p = q$. Equivalently, two points are "the same" under the quotient if the covering map p maps them to the same point inside $[0, 1) \times [0, 1)$.

Remark 3.8. The converse to the theorem is not true in general. That is, there are Fuchsian groups Γ for which $\mathbb{H}^2/\Gamma$ is not a surface, but instead is a less nice object called an orbifold, which may have cone points.

4. Hyperbolic Geometry in 3 Dimensions

In order to discuss quasifuchsian manifolds, which are 3-dimensional objects, we must first discuss the properties of 3-dimensional hyperbolic space. Fortunately, most of the concepts carry over quite naturally.

Definition 4.1. Upper Half-Space Model

The **upper half-space** is defined as

$$\mathbb{H}^n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid x_n > 0\}.$$

Note that this is simply a generalization of the 2-dimensional case.

The metric on the upper half-space is given by:

$$ds^2 = \frac{dx_1^2 + dx_2^2 + \cdots + dx_n^2}{x_n^2}$$

Once again, this is an easy generalization: take the Euclidean metric and divide it by the last coordinate.

The **boundary at infinity** $\partial\mathbb{H}^n$ is given by the plane $x_n = 0$ and the point ∞.

The geodesics are, as one might expect, semicircles and vertical lines orthogonal to the $x_n = 0$ plane.

The Poincaré disk model also easily generalizes.

Definition 4.2. Poincaré Ball Model

The **Poincaré ball** is defined as:

$$\mathbb{D}^n = \{x \in \mathbb{R}^n \mid |x|_E < 1\},$$

where $|\cdot|_E$ represents the Euclidean norm. Then, the metric on $\mathbb{D}^n$ is given by:

$$ds^2 = \frac{4dx^2}{(1 - |x|^2)^2},$$

and the boundary at infinity is S^{n-1}.

The geodesics are again arcs of circles which are orthogonal to the boundary at both intersection points.

As before, we will use $\mathbb{H}^n$ to refer to both models, and the usage should be clear from context.

We now turn our focus to the isometries of $\mathbb{H}^3$.

Theorem 4.3. $\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2,\mathbb{C})$

Proof. See [7], section 2.6. □

This is an important result although we will not prove it here. Accepting that $\mathrm{Isom}^+(\mathbb{H}^3)$ is isomorphic to $\mathrm{PSL}(2,\mathbb{C})$, let us examine how exactly we can understand the action of these isometries.

Theorem 4.4. *In the Poincaré ball model, the action of an orientation-preserving isometry of $\mathbb{H}^3$ is determined solely by its action on the sphere at infinity.*

Proof. Suppose that two such isometries A, B have the same action on the boundary at infinity. Then we must show that $A = B$ on $\mathbb{H}^3$. Since A, B act the same way on S^2, we have that $AB^{-1} = \mathrm{id}$ on the boundary. We must extend this to all of $\mathbb{H}^3$. Fix any $z \in \mathbb{H}^3$. Then, fix any geodesic ℓ that contains z. Let x, y be the endpoints on the boundary at infinity. Then, we know $AB^{-1}x = x, AB^{-1}y = y$. Thus, since isometries send geodesics to geodesics, we know that $AB^{-1}\ell = \ell$. We have shown that all geodesics are mapped onto themselves. We must now show that the geodesics are preserved pointwise. Suppose that AB^{-1} acts on ℓ via a translation, i.e. points are shifted away from one endpoint and toward the other. Now, choose any other geodesic ℓ' which crosses ℓ only at some point p. Then, since AB^{-1} acts as a translation on ℓ, we have $AB^{-1}p \neq p$ and hence $AB^{-1}p \notin AB^{-1}\ell' = \ell'$ since the two geodesics only intersect once. This is a contradiction because AB^{-1} preserves geodesics, and $p \in \ell'$, but $AB^{-1}p \notin \ell'$. □

This insight is extremely valuable to us as it is much easier to consider the action on the boundary since it is of lower dimension and since we already have a more intuitive grasp of conformal geometry in two dimensions.

Next, it is also worth noting that concepts of limit set and domain of discontinuity transfer over unchanged (noting, of course, that the boundary of $\mathbb{H}^3$ is S^2, not S^1). We also adapt the term "Fuchsian group" into three dimensions.

Definition 4.5. A **Fuchsian group** is a subset of $\mathrm{Isom}^+(\mathbb{H}^3)$ whose limit set is a round circle (with respect to the Poincaré model).

The motivation for this definition is that we can imagine a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ acting on the Poincaré disk. The limit set will be contained inside S^1. If we then embed this disk into the 3-dimensional Poincaré ball, i.e. by considering the disk as the plane $\{z=0\}$ within the ball, and consider the group's action as a subset of $\mathrm{PSL}(2,\mathbb{C})$ in 3 dimensions (we take the same real matrix coefficients but view them as acting on $\mathbb{H}^3$), then the limit set of our Fuchsian group will still lie inside that circle.

We add one more term to the descriptors of discrete groups of isometries:

Definition 4.6. A **Kleinian group** is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$ whose domain of discontinuity is nonempty.

Remark 4.7. The term "Kleinian group" is often used in two different ways. Sometimes it is defined as any discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$, and other times it is used in the context we have just given. For ease of terminology, we will stick to Thurston's convention in [9] by adding the stipulation that the domain of discontinuity is nonempty.

Lastly, we generalize the idea of a surface to higher dimensions. Roughly speaking, a **hyperbolic n-manifold** M is an object that locally looks like $\mathbb{H}^n$, just as a hyperbolic surface locally looks like $\mathbb{H}^2$ (there is a more formal way to define this, but this pseudo-definition will suffice for our purposes). We can express any hyperbolic n-manifold M as $\mathbb{H}^n/\Gamma$ for a Kleinian group Γ.

For more details on 3-dimensional hyperbolic space, see [7], chapter 2.

5. The Fundamental Group and Quasiconformal Maps

We are close to being able to define a quasifuchsian manifold. There are a few more preliminary definitions we must establish before we do so. The first is that of a homotopy. Informally, homotopy is a continuous deformation of one path into another. We will now put this formally.

Definition 5.1. Let $p,q:[0,1]\to S$ be two (continuous) paths with the same starting and ending points. Then, a **homotopy between p and q** is a continuous function $h:[0,1]\times[0,1]\to S$ such that

(i) h(x,0)=p(x) and h(x,1)=q(x)
(ii) h(0,t)=p(0)=q(0) and h(1,t)=p(1)=q(1)

If such a homotopy exists between p and q, then they are said to be **homotopic** to one another.

We can think of the t-coordinate of h as being the "time" parameter which increases as the path $h(x,t)$ is deformed closer to q from p. See Figure 4.

It is relatively straightforward to see that the relation of paths being homotopic to one another is an equivalence relation, so we may categorize paths with the same beginning and ending points up to homotopy. We will denote such an equivalence class as $[p]$. We now present a few important properties of homotopies:

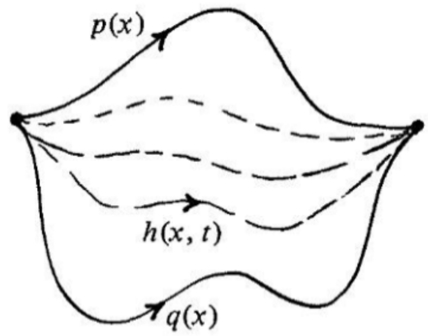

FIGURE 4. A visualization of a homotopy (borrowed from [6]).

Proposition 5.2. Path Lifting Property
Fix a hyperbolic surface $S = \mathbb{H}^2/\Gamma$. Then, for each path q in S with origin O, there exists a unique path $\tilde{q}$ with origin $\tilde{O}$ (lying over O) such that $q = \Gamma\tilde{q}$. We call $\tilde{q}$ the ***lift of*** q ***with origin*** $\tilde{O}$.

Note: when we say that $\tilde{O}$ lies over O, we mean that $\tilde{O} \in p^{-1}(O)$, where p here represents the covering map. Sometimes, mathematicians will also describe the covering space as "upstairs" from the base space.

Proposition 5.3. Homotopy Lifting Property
Let paths p_1 and p_2 have origin $O \in S$. Then, they are homotopic if and only if their lifts $\tilde{p_1}, \tilde{p_2}$ with origin $\tilde{O}$ are homotopic in $\mathbb{H}^2$.

Example 5.4. Recall Example 3.2, in which we discussed the covering of a circle by the real line. We now discuss a similar example in 2 dimensions. Consider a cylinder of some height. We can use the same type of formulation to cover this cylinder by a flat sheet. A visualization of this would be unrolling a roll of tin foil. Two points on the plane fiber over the same point if they are in the same location after rolling up the flat sheet. However, we can distinguish the two points based on their homotopy class, as described in Proposition 5.3 and as shown in Figure 5. If we imagine point A as being at the start of the roll, then, the path p_1 is a straight line across the top of the roll. We can imagine the path p_2 as traversing all the way around the roll and coming to rest at the same point B, but on the second-highest layer of foil.

In fact, we could continue this process indefinitely (assuming the sheet is infinitely long). For each extra trip around the roll, we define a distinct homotopy class for the paths which all begin at point A and end at point B. Intuitively, what Proposition 5.3 tells us in this case is that two paths which begin and end at points A, B, respectively, are homotopic if and only if they traverse around the roll in the same direction and for the same number of times.

Hopefully the previous example has shed some light on the use and importance of homotopies and covering spaces. Building on the theory of homotopies that we've developed so far, we now introduce the idea of the fundamental group.

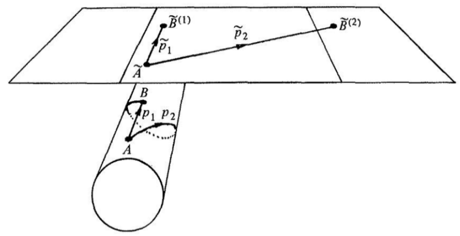

FIGURE 5. A visualization of the differences in homotopy classes of paths with the same origin and terminus (borrowed from [6]).

Definition 5.5. Let S be a hyperbolic surface. Then, we define the **fundamental group of** S $\pi_1(S)$ to be the group of homotopy classes of closed paths (i.e. paths where the origin and terminus are the same) with origin O. The group operation is the concatenation $[p_1][p_2] = [p_1p_2]$, where $[p_1p_2]$ represents the path p_1 followed by p_2, namely:

$$(p_1p_2)(x) = \begin{cases} p_1(2x) & 0 \leq x \leq 1/2, \\ p_2(2x-1) & 1/2 \leq x \leq 1. \end{cases}$$

Remark 5.6. More generally, the fundamental group may be defined in the same way for any topological space and origin (in general, the origin is referred to as the base point).

It is relatively straightforward to show that this does actually define a group. We now present an important theorem:

Theorem 5.7. *If* $S = \mathbb{H}^2/\Gamma$, *then* $\pi_1(S) \cong \Gamma$.

This theorem, and the study of fundamental groups and homotopies, is quite important in the study of algebraic topology. Given that the fundamental group is isomorphic to the Fuchsian group, it would not have been necessary for us to even introduce fundamental groups–we could have simply spoken in terms of the Fuchsian group. However, given the importance of the fundamental group in topology, I felt it pedagogically important to introduce the concept here.

Next, we turn our attention to a special kind of map: the quasiconformal map.

Definition 5.8. A **conformal** map is a homeomorphism that preserves angles.

A **quasiconformal** map is a homeomorphism that somewhat preserves the angle structure while allowing some distortion in a controlled way. That is, similarly to how conformal maps send infinitesimally small circles to circles, quasiconformal maps send infinitesimally small circles to ellipses in which the ratio between the major and minor axes is not too large

There are more precise ways to formulate these definitions, but they are a bit too technical to be useful for our purposes. The intuitive definitions will suffice.

Lastly, we introduce the concept of conjugacy between groups. Recall that in a group G, elements $g_1, g_2 \in G$ are said to be conjugate if there exists $h \in G$ such that $g_2 = hg_1h^{-1}$. When dealing with group actions, the definition takes a similar form.

Definition 5.9. Suppose that we have two Fuchsian groups, Γ, Δ acting on the hyperbolic plane to produce surfaces $S_\Gamma = \mathbb{H}^2/\Gamma$, $S_\Delta = \mathbb{H}^2/\Delta$. Then, we say that Γ, Δ are **quasiconformally conjugate** if there exists a quasiconformal map $f : \mathbb{H}^2 \to \mathbb{H}^2$ such that for all $\gamma \in \Gamma$, we have

$$f \circ \gamma \circ f^{-1} = \delta$$

for some $\delta \in \Delta$.

Expanding this notion to talk about 3-manifolds, suppose we have two Kleinian groups, Γ, Δ acting on 3-D hyperbolic space to produce hyperbolic 3-manifolds $M_\Gamma = \mathbb{H}^3/\Gamma$, $M_\Delta = \mathbb{H}^3/\Delta$. Then, we say that Γ, Δ are **quasiconformally conjugate** if there exists a quasiconformal map $f : S^2 \to S^2$ such that for all $\gamma \in \Gamma$, we have

$$f \circ \gamma|_{S^2} \circ f^{-1} = \delta|_{S^2}$$

for some $\delta \in \Delta$.

Here, we use our above characterization of the action $\mathrm{PSL}(2, \mathbb{C})$ so that we may only consider the action on the boundary at infinity.

The usefulness of this concept is that we have a convenient way to describe Fuchsian and Kleinian groups that are similar to one another in a particular sense. That is, the surfaces which they produce are quasiconformally equivalent (quasiconformal equivalence is an equivalence relation), and the groups themselves have a similar structure in that the quasiconformal map induces a conjugation between the elements of both.

For more information on homotopies and the fundamental group, see [6]. For more details on quasiconformal maps, see [8].

6. Quasifuchsian Groups

After much discussion, we are finally ready to discuss quasifuchsian groups.

Definition 6.1. A **quasifuchsian group** is a Kleinian group that is quasiconformally conjugate to a Fuchsian group.

We note that "Fuchsian group" in this case refers to the 3-dimensional generalization of the term. Recognizing that we have introduced a large number of definitions in this paper, we will break down term by term what, precisely, this definition means. First, a quasifuchsian group is a Kleinian group, meaning that it is a discrete group of orientation-preserving isometries in $\mathbb{H}^3$ with non-empty domain of discontinuity. Second, we recall that a Fuchsian group has a limit set which is a circle. Lastly, a quasifuchsian group is quasiconformally conjugate to a Fuchsian group. As we discussed at the end of section 5, this means that the actions of the quasifuchsian and Fuchsian groups on $\mathbb{H}^3$ are related by a homeomorphism on the boundary at infinity which somewhat preserves angle structure, and that this same homeomorphism relates the elements of the two groups to one another in a specific way.

This definition is a little tricky to wrap one's head around. We now present a simpler, yet equivalent, formulation.

Definition 6.2. A **quasifuchsian group** is a Kleinian group whose limit set is topologically S^1.

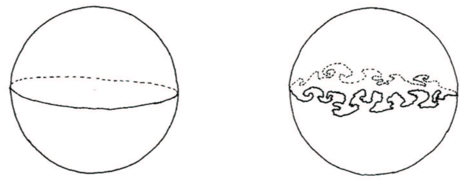

FIGURE 6. A deformation of S^1 under a quasiconformal map (borrowed from [9]).

It is not immediately obvious why these definitions are equivalent, although one implication is simple enough. A Fuchsian group has a circle as its limit set. Therefore, when we apply a quasiconformal map (which is a reasonably well-behaved homeomorphism) to the surface, the limit set still retains the same structure. For a visual interpretation of this, see Figure 6. When the homeomorphism has many points where it is not differentiable as illustrated in the figure, we end up with a fractal circle. For the other direction of implication, the author needs to study the topic deeper to find a rigorous explanation.

We conclude with two examples of quasifuchsian groups that open the door to a rich discussion and exploration.

Example 6.3. Fix a hyperbolic surface $S = \mathbb{H}^2/\Gamma$. Then, consider the inclusion map $\iota : \mathrm{PSL}(2,\mathbb{R}) \hookrightarrow \mathrm{PSL}(2,\mathbb{C})$. This map takes 2 x 2 matrices with real coefficients and considers them inside $\mathrm{PSL}(2,\mathbb{C})$, still as 2 x 2 matrices with real coefficients. Then, $\Gamma' = \iota(\Gamma)$ is a quasifuchsian group with a very special property. Consider the 3-manifold $\overline{M} = (\mathbb{H}^3 \cup \Omega_{\Gamma'})/\Gamma'$. There exists a homeomorphism between $\overline{M}$ and $\mathrm{int}(S) \times [0,1]$.

For emphasis, we can imagine $\mathrm{int}(S) \times [0,1]$ as being the result if we took the hyperbolic surface S and stacked it on itself like a stack of pancakes. This object is deeply connected with the 3-manifold generated by the quasifuchsian group given by the inclusion of Γ into $\mathrm{PSL}(2,\mathbb{C})$.

We can take the example one step deeper. Fix a hyperbolic surface $S = \mathbb{H}^2/\Gamma$. Now, take any representation $\rho : \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$ (recall that $\pi_1(S) \cong \Gamma$) such that $\Gamma' = \rho(\pi_1(S))$ is quasifuchsian. We denote the set of all such representations by $QF(S)$. For any $\rho \in QF(S)$, the corresponding Γ' satisfies the same property as above, namely that there exists a homeomorphism between $\overline{M} = (\mathbb{H}^3 \cup \Omega_{\Gamma'})/\Gamma'$ and $\mathrm{int}(S) \times [0,1]$.

We could go even further in this example, but we would need to discuss Teichmüller theory in order to do so, and that is beyond the scope of this paper. See [8] for more details.

Example 6.4. Consider a 2-holed hyperbolic surface such as the one in Figure 7. We will deform the corresponding Fuchsian group by bending the surface along a closed path γ by an angle $\pi/2$. If the length of γ is sufficiently short, then this process will give rise to a quasifuchsian group. The interesting result occurs when we consider the limit set of this quasifuchsian group. We can visualize the limit set by considering the lifts of γ into the hyperbolic plane (viewed as the plane $\{z = 0\}$ in $\mathbb{H}^3$) and then bending the plane in $\mathbb{H}^3$ by an angle of $\pi/2$ along each one of the lifts, like creases on a paper. The result is a limit set which Thurston described

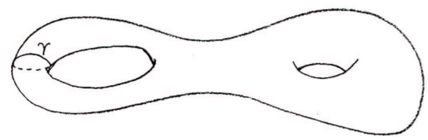

FIGURE 7. A 2-holed surface with a short closed geodesic γ (borrowed from [9]).

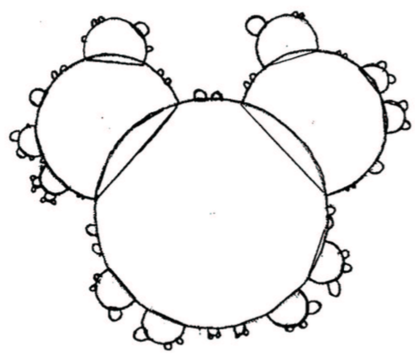

FIGURE 8. The limit set of the quasifuchsian group described in Example 6.4 is visualized by the boundary of the image above. It looks like an outline of Mickey Mouse! (borrowed from [9]).

in [9] as looking like Mickey Mouse (see Figure 8). It is worth noting that we can see that the limit set is topologically the same as S^1: if we trace the outside of the limit set, we would get a deformed circle with many bulges.

As we can see from this example, the study of quasifuchsian groups is closely connected to the study of fractal sets in the plane. There is a rich area of exploration to be done and discoveries to be made, and we have only scratched the surface in this paper. I hope that this paper has helped to explain some applications of hyperbolic geometry and interest readers in the subject.

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References

[1] Svetlana Katok, Pennsylvania State University. Fuchsian Groups. The University of Chicago Press. 1992.

[2] Evelyn Sander, University of Minnesota. Diagram of Hyperbolic Lines. 1994. http://www.geom.uiuc.edu/docs/forum/hype/model.html

[3] Zhiren Wang. Two Models of Hyperbolic Geometry. The University of Chicago Math REU. 2023. https://math.uchicago.edu/~may/REU2023/REUPapers/Wang,Zhiren(Su).pdf

[4] Will Adkisson. Geodesics of Hyperbolic Space. The University of Chicago Math REU. 2014. https://math.uchicago.edu/~may/REU2014/REUPapers/Adkisson.pdf

[5] Sarah Zhang. A Roundabout Introduction to Hyperbolic Area. The University of Chicago Math REU. 2019. https://math.uchicago.edu/~may/REU2019/REUPapers/Zhang,Sarah.pdf

[6] John Stillwell, Monash University. Geometry of Surfaces. Springer. 1992.

[7] William P. Thurston, Princeton University. Three-Dimensional Geometry and Topology. Princeton University Press. 1991.

[8] Curtis T. McMullen, Harvard University. Renormalization and 3-Manifolds which Fiber over the Circle. Princeton University Press. 1996.

[9] William P. Thurston, Princeton University. The Geometry and Topology of Three-Manifolds. 1980.

[10] Anne Thomas, The University of Chicago. Covering Spaces. 2004. https://math.uchicago.edu/~womp/2004/athomas04.pdf