SMOOTH MANIFOLDS AND DE RHAM COHOMOLOGY

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Abstract. This paper gives a quick introduction to smooth manifolds. Then we introduce de Rham Cohomology and compute some examples.

Contents

1. Manifolds and Smooth Maps

A manifold is a higher-dimensional analogue of a smooth curve or surface. Its prototype is the Euclidean space $\mathbb{R}^n$, with coordinates $r_1, \dots, r_n$. Let U be an open subset of $\mathbb{R}^n$. A function

$$f = (f_1, \dots, f_m) : U \to \mathbb{R}^m$$

is smooth on U if the partial derivatives

$$\frac{\partial^k f}{\partial r^{j_1} \cdots \partial r^{j_k}}$$

exist on U for all integers $k \geq 1$ and for all choices of indices $j_1, \dots, j_k$. In this paper we use the words "smooth" and C^∞ interchangeably.

Definition 1.1 (Locally Euclidean of Dimension n)**.** A topological space M is *locally Euclidean of dimension* n if, for every point $p \in M$, there is a homeomorphism

$$\varphi \colon U \longrightarrow V \subset \mathbb{R}^n$$

where U is a neighborhood of p in M, and is open in $\mathbb{R}^n$. Such a pair $\big(U, \varphi\big)$ is called a *coordinate chart* (or simply a *chart*). If $p \in U$, then we say that (U, φ) is a chart about p.

Definition 1.2 (Compatibility of Charts, C^∞ Atlas)**.** A collection of charts $\{(U_\alpha, \varphi_\alpha)\}$ is C^∞*-compatible* if, for every α, β, the transition map

$$\varphi_\alpha \circ \varphi_\beta^{-1} \; : \; \varphi_\beta(U_\alpha \cap U_\beta) \longrightarrow \varphi_\alpha(U_\alpha \cap U_\beta)$$

is a C^∞ function. A C^∞ *atlas* on M is a collection of C^∞-compatible charts that covers M. Such a C^∞ atlas is *maximal* if it contains every chart that is C^∞-compatible with all the charts in the atlas.

Definition 1.3 (Topological Manifold)**.** A *topological manifold* is a Hausdorff, second countable, locally Euclidean topological space. By "second countable," we mean that the space has a countable basis of open sets.

Definition 1.4 (Smooth (C^∞) Manifold)**.** A *smooth* or C^∞ *manifold* is a pair $(M, \{(U_\alpha, \varphi_\alpha)\})$ where M is a topological manifold and $\{(U_\alpha, \varphi_\alpha)\}$ is a *maximal* C^∞ atlas on M. In this paper, all manifolds are assumed smooth.

The Hausdorff condition ensures that the topology is not too small (we can separate points), while second countability ensures it is not too large (it admits a countable basis). With these two conditions, the topology of a manifold strikes a useful balance. If one shows a manifold has any C^∞ atlas, then by Zorn's lemma that atlas is contained in a unique maximal one. Hence it is sufficient in practice to exhibit *some* C^∞ atlas.

Example 1.1 ($S^1 \subset \mathbb{R}^2$)**.** Consider the circle $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Define open sets:

$$U_{x^+} = \{(x, y) \in S^1 : x > 0\}, \quad U_{x^-} = \{(x, y) \in S^1 : x < 0\},$$
$$U_{y^+} = \{(x, y) \in S^1 : y > 0\}, \quad U_{y^-} = \{(x, y) \in S^1 : y < 0\}.$$

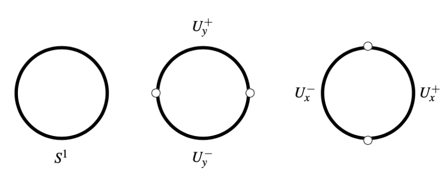

FIGURE 1. A C^∞ atlas on S^1

One checks that the set of charts

$$\{(U_{x^+}, y), (U_{x^-}, y), (U_{y^+}, x), (U_{y^-}, x)\}$$

forms a C^∞ atlas on S^1. For instance, the transition map from $(0, 1) \subset \mathbb{R}$ (the image under x) to $(-1, 0) \subset \mathbb{R}$ (the image under y) can be given by $y = -\sqrt{1 - x^2}$, which is C^∞ on $(0, 1)$.

Definition 1.5 (Smooth Map into $\mathbb{R}^n$)**.** A map $F : M \to \mathbb{R}^n$ on a manifold M is *smooth* or C^∞ at $p \in M$ if there is a chart (U, φ) of M about p such that

$$F \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

is C^∞. The map F is *smooth on* M if it is smooth at every point of M.

Definition 1.6 (Algebra over $\mathbb{R}$)**.** An *algebra* over $\mathbb{R}$ is a vector space A with a bilinear map $\mu : A \times A \to A$ (called multiplication) such that A forms a ring under this multiplication and addition.

Remark 1.1. The set of all smooth real-valued functions on M, denoted $C^\infty(M)$, is naturally an algebra over $\mathbb{R}$ via pointwise addition and multiplication.

Definition 1.7 (Smooth Map between Manifolds)**.** A map $F : N \to M$ between two manifolds is *smooth* or C^∞ at $p \in N$ if there exist charts (U, ϕ) about $p \in N$ and (V, ψ) about $F(p) \in M$ such that $V \supset F(U)$ and

$$\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$$

is C^∞ at $\phi(p)$. If F is invertible and both F and its inverse are smooth, then F is a *diffeomorphism.* i.e., a smooth map $G : M \to N$ such that

$$F \circ G = 1_M \quad \text{and} \quad G \circ F = 1_N.$$

In linear algebra, an $m \times n$ matrix typically represents a linear map $\mathbb{R}^n \to \mathbb{R}^m$. By analogy, one often writes $F : N \to M$ to highlight source and target.

1.1. **Tangent Vectors.** The derivatives of a function f at a point p in Rn depend only on the values of f in a small neighborhood of p. To make precise what is meant by a "small" neighborhood, we introduce the concept of the germ of a function.

Definition 1.8 (Germ of a Function)**.** If $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ are smooth functions defined on neighborhoods U and V of a point p in M, we say $f \sim g$ at p if there is a neighborhood $W \subset U \cap V$ of p on which f and g agree. The equivalence class of such f is called the *germ* of f at p. Denote by $C_p^\infty(M)$ the set of all germs at p.

Remark 1.2. Addition, multiplication, and scalar multiplication of germs are well-defined, so $C_p^\infty(M)$ is an algebra over $\mathbb{R}$.

Definition 1.9 (Derivation, Tangent Space)**.** A *derivation at* p is a linear map $D : C_p^\infty(M) \to \mathbb{R}$ satisfying the Leibniz rule:

$$D(fg) \;=\; (Df)\, g(p) \;+\; f(p)\, (Dg) \quad \text{for all } f, g \in C_p^\infty(M).$$

Such a derivation is also called a *tangent vector* at p. The set of all tangent vectors at p forms a vector space T_pM, called the *tangent space* of M at p.

Example 1.2 (Partial Derivatives in $\mathbb{R}^n$)**.** For $M = \mathbb{R}^n$, the familiar partial derivatives

$$\left.\frac{\partial}{\partial r_1}\right|_p, \ldots, \left.\frac{\partial}{\partial r_n}\right|_p$$

are elements of $T_p(\mathbb{R}^n)$. They form a basis of the tangent space.

Definition 1.10 (Coordinate Vectors)**.** At a point p in a coordinate chart $(U, \varphi) = (U, x_1, \ldots, x_n)$, where

$$x_i = r_i \circ \varphi,$$

we define the coordinate vectors

$$\left.\frac{\partial}{\partial x_i}\right|_p \in T_pM$$

by

$$\left.\frac{\partial f}{\partial x_i}\right|_p = \left.\frac{\partial (f \circ \varphi^{-1})}{\partial r_i}\right|_{\varphi(p)}$$

for any $f \in C_p^\infty(M)$.

Definition 1.11 (Differential of a Smooth Map)**.** If $F : N \to M$ is a smooth map and $p \in N$, its *differential* at p

is the linear map defined by

$$F_{*,p} : T_pN \longrightarrow T_{F(p)}M$$

$$(1.2) \qquad F_p^* : T_pN \to T_{F(p)}M, \qquad (F_p^* X_p)(h) = X_p(h \circ F)$$

for $X_p \in T_pN$ and $h \in C^\infty(M)$. Usually the point p is clear from the context and we write F^* instead of F_p^*. We can verify that if

$$F : N \to M \quad \text{and} \quad G : M \to P$$

are C^∞ maps, then for any $p \in N$

$$(G \circ F)_*, p = G_*, F(p) \circ F_p^*, (G \circ F)^* = G^* \circ F^*.$$

Or, suppressing the points,

$$(G \circ F)_* = G_* \circ F_*.$$

Remark 1.3. If $F : N \to M$ and $G : M \to P$ are smooth, then $(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}$. In coordinates, this is the chain rule in multivariable calculus.

Definition 1.12 (Vector Field)**.** A *vector field* on M is the assignment of a tangent vector $X_p \in T_pM$ to each point $p \in M$. At every p in a chart $(U, x^1, \ldots, x^n)$, since the coordinate vectors

$$\left.\frac{\partial}{\partial x^i}\right|_p$$

form a basis of the tangent space T_pM, the vector X_p can be written as a linear combination

$$X_p = \sum_i a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p,$$

with $a^i(p) \in \mathbb{R}$. As p varies over U, the coefficients $a^i(p)$ become functions on U.

Definition 1.13. A frame of vector fields on an open set $U \subset M$ is a collection of vector fields

$$X_1, \ldots, X_n$$

on U such that at each point $p \in U$, the vectors

$$(X_1)_p, \ldots, (X_n)_p$$

form a basis for the tangent space T_pM.

For example, in a coordinate chart $(U, x_1, \ldots, x_n)$, the coordinate vector fields

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$$

form a frame of vector fields on U.

If $f : N \to M$ is a C^∞ map, its differential

$$f_{,p}^* : T_pN \to T_{f(p)}M$$

pushes forward a tangent vector at a point in N to a tangent vector in M. It should be noted, however, that in general there is no push-forward map

$$f^* : \mathfrak{X}(N) \to \mathfrak{X}(M)$$

for vector fields. For example, when f is not one-to-one, say $f(p) = f(q)$ for $p \neq q$ in N, it may happen that for some

$$X \in \mathfrak{X}(N),$$

we have

$$f^*_{,p} X_p \neq f^*_{,q} X_q;$$

in this case, there is no way to define f^*X so that

$$(f^*X)(f(p)) = f^*_{,p} X_p$$

for all $p \in N$. Similarly, if $f : N \to M$ is not onto, then there is no natural way to define f^*X at a point of M not in the image of f. Of course, if $f : N \to M$ is a diffeomorphism, then

$$f^* : \mathfrak{X}(N) \to \mathfrak{X}(M)$$

is well defined.

1.2. **Differential Forms.**

Definition 1.14. For $k \geq 1$, a *k-form* or a form of degree k on M is the assignment to each $p \in M$ of an alternating k-linear function

$$\omega_p : T_pM \times \cdots \times T_pM \to \mathbb{R}.$$

Here “alternating” means that for every permutation σ of $\{1, 2, \ldots, k\}$ and for all

$$v_1, \ldots, v_k \in T_pM,$$

we have

$$\omega_p(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = (\operatorname{sgn} \sigma)\, \omega_p(v_1, \ldots, v_k),$$

where $\operatorname{sgn} \sigma$, the sign of the permutation σ, is ± 1 depending on whether σ is even or odd. We define a *0-form* to be the assignment of a real number to each $p \in M$; in other words, a 0-form on M is simply a real-valued function on M. When $k = 1$, the condition of being alternating is vacuous. Thus, a 1-form on M is the assignment of a linear function

$$\omega_p : T_pM \to \mathbb{R}$$

to each $p \in M$. For $k < 0$, a k-form is defined to be 0 by convention.

A k-linear function on a vector space V is also called a *k-tensor* on V. As above, a 0-tensor is a constant and a 1-tensor on V is a linear function $f : V \to \mathbb{R}$. Let $A^k(V)$ be the vector space of all alternating k-tensors on V. Then $A^0(V) = \mathbb{R}$ and $A^1(V) = V^* := \operatorname{Hom}(V, \mathbb{R})$, the dual vector space of V. In this language a k-form on M is the assignment of an alternating k-tensor

$$\omega_p \in A^k(T_pM)$$

to each point $p \in M$.

Definition 1.15. Let S_k be the group of all permutations of $\{1, 2, \ldots, k\}$. A (k, l)-*shuffle* is a permutation $\sigma \in S_{k+l}$ such that

$$\sigma(1) < \cdots < \sigma(k) \quad \text{and} \quad \sigma(k+1) < \cdots < \sigma(k+l).$$

Definition 1.16. The *wedge product* of an alternating k-tensor α and an alternating l-tensor β on a vector space V is by definition the alternating $(k+l)$-tensor

$$(\alpha \wedge \beta)(v_1, \ldots, v_{k+l}) = \sum (\operatorname{sgn} \sigma)\, \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\, \beta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}),$$

where the sum is over all (k, l)-shuffles.

For example, if α and β are alternating 1-tensors, then

$$(\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1).$$

The wedge of an alternating 0-tensor, i.e., a constant c, with another alternating tensor β is simply scalar multiplication. In this case, in keeping with the traditional notation for scalar multiplication, we often replace the wedge by a dot or even by nothing:

$$c \wedge \beta = c \cdot \beta = c\beta.$$

Proposition 1.1. *The wedge product $\wedge$ is bilinear, associative, and graded-commutative in its two arguments. Graded-commutativity means that for two alternating tensors α, β on a vector space V,*

$$\alpha \wedge \beta = (-1)^{\deg \alpha \, \deg \beta}\, \beta \wedge \alpha.$$

Proposition 1.2. *If $\alpha_1, \ldots, \alpha_n$ is a basis for the 1-covectors on a vector space V, then a basis for the k-covectors on V is the set*

$$\{\, \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k} \mid 1 \le i_1 < \cdots < i_k \le n \,\}.$$

A k-tuple of integers $I = (i_1, \ldots, i_k)$ is called a multi-index. *If $i_1 \le \cdots \le i_k$, we call I an* ascending multi-index, *and if $i_1 < \cdots < i_k$, we call I a* strictly ascending multi-index. *We will write*

$$\alpha_I = \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k}.$$

As noted earlier, for a point p in a coordinate chart $(U, x_1, \ldots, x_n)$, a basis for the tangent space T_pM is

$$\frac{\partial}{\partial x_1}\Big|_p, \ldots, \frac{\partial}{\partial x_n}\Big|_p.$$

Let $(dx_1)_p, \ldots, (dx_n)_p$ be the dual basis for the cotangent space $A^1(T_pM) = T_p^*M$, i.e.,

$$(dx_i)_p\Big(\frac{\partial}{\partial x_j}\Big|_p\Big) = \delta_{ij}.$$

By Proposition 1.2, if ω is a k-form on M, then at each $p \in U$, ω_p is a linear combination:

$$\omega_p = \sum a_I(p)\, (dx_I)_p = \sum a_I(p)\, (dx_{i_1})_p \wedge \cdots \wedge (dx_{i_k})_p.$$

Definition 1.17. A k-form ω is *smooth* if for every point $p \in M$, there is a chart $(U, x_1, \ldots, x_n)$ about p such that on U the coefficients

$$a_I : U \to \mathbb{R}$$

of

$$\omega = \sum a_I\, dx_I$$

are smooth. By *differential k-forms*, we will mean smooth k-forms on a manifold.

Definition 1.18. A frame of differential k-forms on an open set $U \subset M$ is a collection of differential k-forms

$$\omega_1, \dots, \omega_r$$

on U such that at each point $p \in U$, the alternating k-tensors

$$(\omega_1)_p, \dots, (\omega_r)_p$$

form a basis for the vector space $A^k(T_pM)$ of alternating k-tensors on the tangent space at p.

For example, on a coordinate chart $(U, x_1, \dots, x_n)$, the k-forms

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \quad 1 \le i_1 < \cdots < i_k \le n,$$

constitute a frame of differential k-forms on U.

Definition 1.19. A subset B of a left R-module V is called a *basis* if every element of V can be written uniquely as a finite linear combination

$$\sum r_i b_i,$$

where $r_i \in R$ and $b_i \in B$. An R-module is said to be *free* if it has a basis, and if the basis is finite with n elements, then the free R-module is said to be of rank n.

It can be shown that if a free R-module has a finite basis, then any two bases have the same number of elements, so that the rank is well defined. We denote the rank of V by $\operatorname{rk} V$.

Let $\Omega^k(M)$ denote the vector space of C^∞ k-forms on M and let

$$\Omega^*(M) = \bigoplus_{k=0}^{n} \Omega^k(M).$$

If $(U, x_1, \dots, x_n)$ is a coordinate chart on M, then $\Omega^k(U)$ is a free module over $C^\infty(U)$ of rank

$$\binom{n}{k},$$

with basis $\{dx^I\}$ as above.

Definition 1.20. An algebra A is said to be *graded* if it can be written as a direct sum

$$A = \bigoplus_{k=0}^{\infty} A_k$$

of vector spaces such that under multiplication

$$A_k \cdot A_l \subset A_{k+l}.$$

A graded algebra $A = \bigoplus_{k=0}^{\infty} A_k$ is said to be a *graded commutative algebra* if for all $x \in A_k$ and $y \in A_l$,

$$x \cdot y = (-1)^{kl}\, y \cdot x.$$

The wedge product $\wedge$ makes $\Omega^*(M)$ into a graded commutative algebra over $\mathbb{R}$.

1.3. **Exterior Differentiation.** On any manifold M there is a linear operator

$$d : \Omega^*(M) \to \Omega^*(M),$$

called *exterior differentiation*, uniquely characterized by three properties:

(1) d is an antiderivation of degree 1, i.e., d increases the degree by 1 and for $\omega \in \Omega^k(M)$ and $\tau \in \Omega^l(M)$,

$$d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^k \, \omega \wedge d\tau;$$

(2) $d^2 = d \circ d = 0$;

(3) On 0-forms, the exterior derivative coincides with the differential: for a 0-form $f \in C^\infty(M)$ and a vector field $X \in \mathfrak{X}(M)$, we have

$$(df)(X) = Xf.$$

By induction the antiderivation property (1) extends to more than two factors; for example,

$$d(\omega \wedge \tau \wedge \eta) = d\omega \wedge \tau \wedge \eta + (-1)^{\deg \omega} \, \omega \wedge d\tau \wedge \eta + (-1)^{\deg \omega + \deg \tau} \, \omega \wedge \tau \wedge d\eta.$$

Proposition 1.3. *Let $(U, x_1, \ldots, x_n)$ be a coordinate chart. Suppose*

$$d : \Omega^*(U) \to \Omega^*(U)$$

is an exterior differentiation. Then:

(i) For any $f \in \Omega^0(U)$,

$$df = \sum \frac{\partial f}{\partial x^i} \, dx^i;$$

(ii) $d(dx^I) = 0$;

(iii) For any $a_I \, dx^I \in \Omega^k(M)$,

$$d(a_I \, dx^I) = da_I \wedge dx^I.$$

Proof. (i) Since $(dx^1)_p, \ldots, (dx^n)_p$ is a basis of 1-covectors at each point $p \in U$, we may write (suppressing p):

$$(df)_p = \sum a_i(p) \, (dx^i)_p.$$

That is,

$$df = \sum a_i \, dx^i.$$

Applying both sides to the vector field $\frac{\partial}{\partial x_i}$ gives

$$(df)\Big(\frac{\partial}{\partial x^j}\Big) = \sum a_i \, dx^i \Big(\frac{\partial}{\partial x^j}\Big) = \sum a_i \, \delta_{ij} = a_j.$$

On the other hand, by property (3) of d,

$$(df)\Big(\frac{\partial}{\partial x^j}\Big) = \frac{\partial f}{\partial x^j}.$$

Hence, $a_j = \frac{\partial f}{\partial x^j}$ and

$$df = \sum \frac{\partial f}{\partial x^j} \, dx^j.$$

(ii) By the antiderivation property of d,

$$d(dx^I) = d(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \sum_{j=1}^{k} (-1)^{j-1} dx^{i_1} \wedge \cdots \wedge d(dx^{i_j}) \wedge \cdots \wedge dx^{i_k} = 0,$$

since $d^2 = 0$.

(iii) By the antiderivation property of d,

$$d\big(a_I\, dx^I\big) = da_I \wedge dx^I + a_I\, d(dx^I) = da_I \wedge dx^I,$$

since $d(dx^I) = 0$. □

Proposition 1.3 proves the uniqueness of exterior differentiation on a coordinate chart $(U, x_1, \ldots, x_n)$. To prove its existence, we define d by two of the formulas of Proposition 1.3:

(i) If $f \in \Omega^0(U)$, then

$$df = \sum \frac{\partial f}{\partial x_i}\, dx_i;$$

(iii) If

$$\omega = \sum a_I\, dx_I \in \Omega^k(U) \quad \text{for } k \geq 1,$$

then

$$d\omega = \sum da_I \wedge dx_I.$$

Next we check that so defined, d satisfies the three properties of exterior differentiation.

(1) For $\omega \in \Omega^k(U)$ and $\tau \in \Omega^l(U)$,

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^k\, \omega \wedge d\tau.$$

Proof. Suppose

$$\omega = \sum a_I\, dx_I \quad \text{and} \quad \tau = \sum b_J\, dx_J.$$

On functions, $d(fg) = (df)g + f\,(dg)$ is simply another manifestation of the ordinary product rule, since

$$\begin{aligned} d(fg) &= \sum_i \frac{\partial}{\partial x^i}(fg)\, dx^i \\ &= \sum_i \Big(\frac{\partial f}{\partial x^i}\, g \;+\; f\, \frac{\partial g}{\partial x^i} \Big) dx^i \\ &= \Big(\sum_i \frac{\partial f}{\partial x^i}\, dx^i \Big) g \;+\; f \sum_i \frac{\partial g}{\partial x^i}\, dx^i \\ &= (df)\, g \;+\; f\, dg. \end{aligned}$$

Next suppose $k \geq 1$. Since the exterior derivative d is linear and the wedge product $\wedge$ is bilinear over $\mathbb{R}$, we may assume that

$$\omega = a_I\, dx_I \quad \text{and} \quad \tau = b_J\, dx_J,$$

each consisting of a single term. Then

$$\begin{aligned} d(\omega \wedge \tau) &= d(a_I b_J\, dx_I \wedge dx_J) = d(a_I b_J) \wedge dx_I \wedge dx_J \quad \text{(by definition of } d\text{)} \\ &= (da_I) b_J \wedge dx_I \wedge dx_J + a_I\, (db_J) \wedge dx_I \wedge dx_J \end{aligned}$$

$$= da_I \wedge \big(b_J\, dx_J\big) \wedge dx_I + (-1)^k\, a_I\, dx_I \wedge (db_J) \wedge dx_J$$

$$= d\omega \wedge \tau + (-1)^k\, \omega \wedge d\tau.$$

□

(2) $d^2 = 0$ on $\Omega^k(U)$.

Proof. This is a consequence of the fact that the mixed partials of a function are equal. For $f \in \Omega^0(U)$,

$$d^2 f = d(df) = d\Big(\sum \frac{\partial f}{\partial x^i}\, dx^i\Big) = \sum d\Big(\frac{\partial f}{\partial x^i}\Big) \wedge dx^i = \sum\sum \frac{\partial^2 f}{\partial x^j \partial x^i}\, dx^j \wedge dx^i.$$

In this double sum, the factors

$$\frac{\partial^2 f}{\partial x^j \partial x^i}$$

are symmetric in i, j, while $dx^j \wedge dx^i$ are skew-symmetric in i, j. Hence, for each pair $i < j$ there are two terms

$$\frac{\partial^2 f}{\partial x^i \partial x^j}\, dx^i \wedge dx^j, \quad \frac{\partial^2 f}{\partial x^j \partial x^i}\, dx^j \wedge dx^i,$$

that add up to zero. It follows that $d^2 f = 0$.

For

$$\omega = \sum a_I\, dx_I \in \Omega^k(U), \quad k \geq 1,$$

we have

$$d^2\omega = d\Big(\sum da_I \wedge dx^I\Big) = \sum \Big[(d^2 a_I) \wedge dx^I + da_I \wedge d(dx^I)\Big] = 0.$$

In this computation, $d^2 a_I = 0$ by the degree 0 case, and $d(dx^I) = 0$ follows as in the proof of Proposition 1.3(ii). □

Proof. (3) Suppose

$$X = \sum a_j\, \frac{\partial}{\partial x_j}.$$

Then

$$(df)(X) = \Big(\sum \frac{\partial f}{\partial x_i}\, dx_i\Big)\Big(\sum a_j\, \frac{\partial}{\partial x_j}\Big) = \sum a_i\, \frac{\partial f}{\partial x_i} = X(f).$$

□

1.4. **Pullback of Differential Forms.** Unlike vector fields, which in general cannot be pushed forward under a C^∞ map, differential forms can always be pulled back. Let

$$F : N \to M$$

be a C^∞ map. The pullback of a C^∞ function on M is the C^∞ function

$$F^* f := f \circ F$$

on N. This defines the pullback on C^∞ 0-forms. For $k > 0$, the pullback of a k-form ω on M is the k-form $F^*\omega$ on N defined by

$$(F^*\omega)_p(v_1, \dots, v_k) = \omega_{F(p)}\Big(F^*_{,p} v_1, \dots, F^*_{,p} v_k\Big),$$

for $p \in N$ and $v_1, \dots, v_k \in T_pN$. From this definition, it is not obvious that the pullback $F^*\omega$ of a C^∞ form ω is C^∞. To show this, we first derive a few basic properties of the pullback.

Proposition 1.4. *Let $F : N \to M$ be a C^∞ map of manifolds. If ω and τ are k-forms and σ is an l-form on M, then*

(i) $F^*(\omega + \tau) = F^*\omega + F^*\tau$;
(ii) For any real number a, $F^*(a\,\omega) = a\,F^*\omega$;
(iii) $F^*(\omega \wedge \tau) = F^*\omega \wedge F^*\tau$;
(iv) For any C^∞ function h, $dF^*h = F^*(dh)$.

Proof. The first three properties (i), (ii), (iii) follow directly from the definitions. To prove (iv), let $p \in N$ and $X_p \in T_pN$. Then

$$(dF^*h)_p(X_p) = X_p(F^*h) = X_p(h \circ F),$$

and

$$(F^*dh)_p(X_p) = (dh)_{F(p)}\Big(F^*_{,p}X_p\Big) = \Big(F^*_{,p}X_p\Big)(h) = X_p(h \circ F).$$

Hence,

$$dF^*h = F^*(dh).$$

□

We now prove that the pullback of a C^∞ form is C^∞. On a coordinate chart $(U, x_1, \dots, x_n)$ in M, a C^∞ k-form ω can be written as a linear combination

$$\omega = \sum a_I\, dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where the coefficients a_I are C^∞ functions on U. By the preceding proposition,

$$F^*\omega = \sum (F^*a_I)\, d\big(F^*x_{i_1}\big) \wedge \cdots \wedge d\big(F^*x_{i_k}\big) = \sum (a_I \circ F)\, d\big(x_{i_1} \circ F\big) \wedge \cdots \wedge d\big(x_{i_k} \circ F\big),$$

which shows that $F^*\omega$ is C^∞, because it is a sum of products of C^∞ functions and C^∞ 1-forms.

Proposition 1.5. *Suppose $F : N \to M$ is a smooth map. On C^∞ k-forms,*

$$dF^* = F^*d.$$

Proof. Let $\omega \in \Omega^k(M)$ and $p \in M$. Choose a chart $(U, x_1, \dots, x_n)$ about p in M. On U,

$$\omega = \sum a_I \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Then

$$F^*\omega = \sum (a_I \circ F) \, d\big(x_{i_1} \circ F\big) \wedge \cdots \wedge d\big(x_{i_k} \circ F\big).$$

Thus,

$$dF^*\omega = \sum d(a_I \circ F) \wedge d\big(x_{i_1} \circ F\big) \wedge \cdots \wedge d\big(x_{i_k} \circ F\big)$$

$$= \sum d\big(F^* a_I\big) \wedge d\big(F^* x_{i_1}\big) \wedge \cdots \wedge d\big(F^* x_{i_k}\big) = \sum F^*(da_I) \wedge F^*(dx_{i_1}) \wedge \cdots \wedge F^*(dx_{i_k})$$

$$= \sum F^*\big(da_I \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}\big) = F^*(d\omega).$$

□

As computed above, in summary, for any C^∞ map $F : N \to M$, the pullback map

$$F^* : \Omega^*(M) \to \Omega^*(N)$$

is an algebra homomorphism that commutes with the exterior derivative d.

Example 1.3. Let N and M be manifolds. A C^∞ map

$$f : N \to M$$

is called an *immersion* if for all $p \in N$, the differential

$$f^*_{,p} : T_pN \to T_{f(p)}M$$

is injective. A subset S of M with a manifold structure such that the inclusion map

$$i : S \to M$$

is an immersion is called an *immersed submanifold* of M. An example is the image of a line with irrational slope in the torus $\mathbb{R}^2/\mathbb{Z}^2$. (Note that an immersed submanifold need not have the subspace topology.)

If $\omega \in \Omega^k(M)$, $p \in S$, and $v_1, \dots, v_k \in T_pS$, then by the definition of the pullback,

$$(i^*\omega)_p(v_1, \dots, v_k) = \omega_{i(p)}\Big((i^*v_1), \dots, (i^*v_k)\Big) = \omega_p(v_1, \dots, v_k).$$

Thus, the pullback of ω under the inclusion map i is simply the restriction of ω to the submanifold S. We also adopt the more suggestive notation

$$\omega|_S$$

for $i^*\omega$.

2. de Rham Cohomology

Definition 2.1. We have a sequence of vector spaces and linear maps

$$0 \longrightarrow \Omega^0(U) \xrightarrow{d^0} \Omega^1(U) \xrightarrow{d^1} \Omega^2(U) \longrightarrow \cdots \longrightarrow \Omega^n(U) \longrightarrow 0,$$

such that $d^p \circ d^{p-1} = 0$. This is called the *de Rham complex*. Since $d^{p+1} \circ d^p = 0$, we have $\operatorname{im}(d^p) \subset \ker(d^{p+1})$.

Definition 2.2. The k-th *de Rham cohomology* of U is defined by

$$H^k_{\mathrm{dR}}(U) \;:=\; \frac{\ker(d^k)}{\operatorname{im}(d^{k-1})} \;=\; \frac{Z^k(U)}{B^k(U)},$$

Example 2.1 (Cohomology of $\mathbb{R}$)**.** Consider $\mathbb{R}^1$. Then

$$\Omega^0(\mathbb{R}) = C^\infty(\mathbb{R}), \quad \Omega^1(\mathbb{R}) = \{\, f'(x)\,dx : f \in C^\infty(\mathbb{R})\}.$$

Zero-forms: $Z^0(\mathbb{R}) = \ker(d^0) = \{\, f \in C^\infty(\mathbb{R}) : f' = 0\} = \{\text{constant functions}\} \cong \mathbb{R}$. Since $B^0(\mathbb{R}) = 0$, we get

$$H^0_{\mathrm{dR}}(\mathbb{R}) = \frac{Z^0(\mathbb{R})}{B^0(\mathbb{R})} = \frac{\mathbb{R}}{0} = \mathbb{R}.$$

One-forms: We have

$$B^1(\mathbb{R}) = \operatorname{im}(d^0) = \{\, f'(x)\,dx : f \in C^\infty(\mathbb{R})\}.$$

Any $\omega \in \Omega^1(\mathbb{R})$ looks like $g(x)\,dx$; by the fundamental theorem of calculus, $g(x) = f'(x)$ for some f. Hence $\omega \in B^1(\mathbb{R})$, showing $Z^1(\mathbb{R}) = B^1(\mathbb{R})$. Therefore

$$H^1_{\mathrm{dR}}(\mathbb{R}) = \frac{Z^1(\mathbb{R})}{B^1(\mathbb{R})} = 0.$$

2.1. **Compact Supports.**

Definition 2.3. A C^∞ form ω is *closed* if $d\omega = 0$; it is *exact* if

$$\omega = d\tau \quad \text{for some } \tau \in \Omega^{k-1}(M).$$

Let U be an open subset of $\mathbb{R}^n$.

Example 2.2. Suppose a function f is nonzero on the interval $(0, 2)$. Then

$$\operatorname{Supp}(f) \subset \overline{\{x \mid f(x) \neq 0\}} = [0, 2].$$

Definition 2.4. The *zero set* of a p-form ω on U is

$$Z(\omega) \;=\; \{\, p \in U \mid \omega_p = 0\,\}.$$

The *support* of ω is

$$\operatorname{supp}(\omega) \;=\; \operatorname{cl}\bigl\{\, p \in U \mid \omega_p \neq 0\,\bigr\} \;=\; \operatorname{cl}\bigl(U \setminus Z(\omega)\bigr) \;=\; \operatorname{cl}\bigl(Z(\omega)^c\bigr).$$

(Here ^{c} denotes the complement, and $\operatorname{cl}(\,\cdot\,)$ denotes closure.)

Define

$$\Omega^p_c(U) \;:=\; \bigl\{\, C^\infty \ k\text{-forms on } U \text{ with compact support}\bigr\}.$$

Proposition 2.1 (d is support-decreasing)**.** *For $\omega \in \Omega^p_c(U)$, we have*

$$\operatorname{supp}(d\omega) \;\subset\; \operatorname{supp}(\omega).$$

Proof. Suppose $p \notin \operatorname{supp}(\omega)$. Since $\operatorname{supp}(\omega)$ is closed, there exists an open neighborhood V of p which is disjoint from $\operatorname{supp}(\omega)$. On V we have $\omega = 0$, so $d\omega = 0$ on V. Hence $p \notin \operatorname{supp}(d\omega)$. This shows $(\operatorname{supp}(\omega))^c \subset (\operatorname{supp}(d\omega))^c$. Taking complements yields

$$\operatorname{supp}(d\omega) \subset \operatorname{supp}(\omega).$$

□

Corollary 2.1. *If $\omega \in \Omega_c^p(U)$ has compact support, then so does $d\omega$. In fact, $\operatorname{supp}(d\omega)$ is a closed subset of the compact set $\operatorname{supp}(\omega)$.*

Hence we get the differential complex

$$0 \longrightarrow \Omega_c^0(U) \longrightarrow \Omega_c^1(U) \longrightarrow \cdots \longrightarrow \Omega_c^l(U) \longrightarrow 0,$$

the de Rham complex *with compact support* in U.

Definition 2.5. $H_c^*(U)$ is the cohomology of this complex, i.e. the *compactly supported* de Rham cohomology.

Remark 2.1. A k-tensor has k variables. A 0-tensor has no variables.

Definition 2.6. A 0-form on a vector space V is a constant function. Equivalently,

$$A_0(V) = \mathbb{R}.$$

Thus a 0-form on U assigns to each point of U the same constant, so 0-form = function. In general,

$$\Omega^0(U) = C^\infty(U).$$

Example 2.3. Compute $H_c^0(\mathbb{R})$. A 0-form f on $\mathbb{R}$ with compact support must satisfy $df = 0$ (to be in Z_c^0). If $df = 0$, f is constant. But having compact support forces that constant to be 0. Hence $f \equiv 0$. So

$$Z_c^0(\mathbb{R}) = \{0\}.$$

Then

$$H_c^0(\mathbb{R}) = \frac{Z_c^0(\mathbb{R})}{B_c^0(\mathbb{R})} = \frac{0}{0} = 0.$$

Next we compute $H_c^1(\mathbb{R})$

$$Z_c^1(\mathbb{R}) = \Omega_c^1(\mathbb{R}) = \{\, g(x)\,dx : g \in C_c^\infty(\mathbb{R})\},$$
$$B_c^1(\mathbb{R}) = \{\, df = f'(x)\,dx : f \in C_c^\infty(\mathbb{R})\}.$$

If $g(x) = f'(x)$ for some f with compact support, then

$$\int_{-\infty}^{\infty} g(x)\,dx = \int_{-\infty}^{\infty} f'(x)\,dx = f(\infty) - f(-\infty) = 0,$$

since f must vanish at $\pm\infty$ (compact support). So the integral of an exact form with compact support is 0.
Define

$$\int_{-\infty}^{\infty} : Z_c^1(\mathbb{R}) \longrightarrow \mathbb{R}, \quad g(x)\,dx \mapsto \int_{-\infty}^{\infty} g(x)\,dx.$$

We have shown $B_c^1(\mathbb{R}) \subset \ker\Big(\int_{-\infty}^{\infty}\Big)$.

Lemma 2.1. $\ker\Big(\displaystyle\int_{-\infty}^{\infty}\Big) \subset B_c^1(\mathbb{R}).$

Proof. Suppose $\int_{-\infty}^{\infty} g(x)\,dx = 0$. Define

$$f(x) \;=\; \int_0^x g(u)\,du.$$

By the fundamental theorem of calculus, $f'(x) = g(x)$. We must show f has compact support. If $\operatorname{supp}(g)$ is $[a,b]$, then for $x < a$,

$$f(x) = \int_0^x g(u)\,du = \int_0^a g(u)\,du = 0$$

because $g(u) = 0$ outside $[a,b]$. Similarly, for $x > b$,

$$f(x) = \int_0^x g(u)\,du = \int_0^b g(u)\,du = 0$$

by hypothesis. Hence f also vanishes outside $[a,b]$, so f has compact support. Thus $g(x)\,dx = df \in B_c^1(\mathbb{R})$. ☐

By the first isomorphism theorem,

$$H_c^1(\mathbb{R}) = \frac{Z_c^1(\mathbb{R})}{B_c^1(\mathbb{R})} \;\cong\; \operatorname{im}\Big(\int_{-\infty}^{\infty}\Big) = \mathbb{R}.$$

Acknowledgments

I am deeply indebted to my mentor, Catherine Li, for patiently explaining concepts in algebraic topology to me and encouraging me during this experience. I am grateful to Professor Peter May for his hard work organizing the wonderful REU program.

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