A SURVEY OF TECHNIQUES IN ELLIPTIC REGULARITY THEORY

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Abstract. We introduce partial differential equations and the Laplacian, proving the Harnack inequality for harmonic functions. We then define Holder continuity, proving an equivalent definition of Holder continuous. After this we prove the Schauder estimate (with a preliminary lemma and the maximum principle), which bounds the Holder norm of a weak solution by its L^∞ norm and the L^∞ norm of its Laplacian. We then prove the Holder continuity of a weak solution where $\text{div}(A\nabla u)$ is very close to Δu. Finally, we prove the De Giorgi theorem and Harnack inequality in two dimensions.

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1. Introduction

Partial Differential Equations, or PDEs, are equations which involve functions and their partial derivatives. One of the most important PDEs is asking about a function u where we know its Laplacian,

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = f$$

where f is a given function, and where $u = g$ on the boundary $\partial\Omega$ of the domain. When $\Delta u = 0$, u is called harmonic, and there are many results about harmonic functions, some of which we discuss in the preliminaries. For example, harmonic functions are smooth in the interior of the domain, and they satisfy the maximum principle. They also minimize the Dirichlet integral, which is the functional

$$\int_\Omega |\nabla u|^2.$$

In this paper, however, we also ask to what degree these properties extend to other PDEs. Instead of having a function u where the Laplacian Δu is 0, we may have a function u where

$$\partial_i(a_{ij}\partial_j u) = \sum_{i=1}^{n} \partial_i\Big(\sum_{j=1}^{n} a_{ij}\partial_j u\Big) = 0$$

where each a_{ij} is a function entry of a positive definite matrix A. Note that if A were the identity I, then $\operatorname{div}(A\nabla u) = \operatorname{div}(\nabla u) = \Delta u$. It turns out that functions u which are weak solutions to $\partial_i(a_{ij}\partial_j u) = 0$ are also smooth in the interior of the domain if A is smooth, but this result is much more complicated to prove, and we first need to prove other perturbation and regularity results.

After this, we show De Giorgi and the Harnack Inequality in two dimensions, where the proofs are simpler. The De Giorgi Theorem says that if $\operatorname{div}(A\nabla u) = 0$ for A in L^∞, then $u \in C^\alpha$. This theorem is monumental in the theory of PDEs; in this paper, we prove it in the two-dimensional case. The Harnack Inequality is also an important result. It says that if u is a positive weak solution on B_1, then the ratio of $\sup u$ and $\inf u$ is bounded on $B_{1/2}$.

2. Background on the Laplacian

Before beginning the subject of the paper, we establish some preliminaries.
Firstly, the notion of a weak solution is vital to understanding PDEs. We say that $u \in H^1(\Omega)$ is a weak solution of

$$\begin{cases} \Delta u = 0 & on\ \Omega \\ u = g & on\ \partial\Omega, \end{cases}$$

if $u = g$ on the boundary and

$$\int_\Omega \nabla u \cdot \nabla \phi = 0$$

for any $\phi \in H_0^1(\Omega)$. In general, a weak solution is characterized by its behavior when integrating with a smooth, compactly supported function ϕ. In this paper, saying a function is a solution means it is a weak solution. Also, to denote the ball $B_r(0)$ centered at the origin, we simply write B_r.

The solutions of PDEs can also be characterized as minimizers of certain functionals. For example, harmonic functions minimize the Dirichlet integral:

$$\int_\Omega |\nabla u|^2.$$

First we prove the Harnack inequality for harmonic functions.

Theorem 2.1. *Given a positive harmonic function $u : B_1 \to (0,\infty)$, then there exists some absolute C (independent of u) such that*

$$\frac{1}{C} \le \frac{u(x)}{u(y)} \le C$$

for all $x, y \in B_{1/2}$.

Proof. Let u be a positive harmonic function. Then set $v = \log u$ and $w = |\nabla v|^2$. Then we can see that

$$\Delta v = \sum_{i=1}^{n} \Big[-\frac{1}{u^2}\Big(\frac{\partial u}{\partial x_i}\Big)^2 + \frac{1}{u}\Big(\frac{\partial^2 u}{\partial x_i^2}\Big)\Big] = -\sum_{i=1}^{n} \frac{1}{u^2}\Big(\frac{\partial u}{\partial x_i}\Big)^2 = -w.$$

At this point, we show the equality

$$\begin{aligned}\Delta(|\nabla v|^2) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i^2}\Big(\frac{\partial v}{\partial x_j}\Big)^2 \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} 2\Big(\frac{\partial^2 v}{\partial x_i \partial x_j}\Big)^2 + 2\Big(\frac{\partial v}{\partial x_j}\Big)\frac{\partial v}{\partial x_j}\Big(\frac{\partial^2 v}{\partial x_i^2}\Big) \\ &= 2(|D^2 v|^2 + \nabla \Delta v \cdot \nabla v).\end{aligned}$$

Therefore

$$\Delta w = 2(|D^2 v|^2 + \nabla \Delta v \cdot \nabla v) = 2(|D^2 v|^2 - \nabla w \cdot \nabla v).$$

Because $\nabla w = 2D^2 v \nabla v$, then this means that given some $\phi \in C_c^\infty(B_1)$, we have

$$\begin{aligned}\Delta(\phi w) + 2\nabla v \cdot \nabla(\phi w) &= 2\phi |D^2 v|^2 + 2w\nabla v \cdot \nabla \phi + w\Delta\phi + 2\nabla w \cdot \nabla \phi \\ &= 2\phi |D^2 v|^2 + 2w\nabla v \cdot \nabla \phi + w\Delta\phi + 4D^2 v\nabla v \cdot \nabla \phi.\end{aligned}$$

Now let $\phi = \eta^2$ where η is a cutoff function, so that $\nabla\phi = 2\eta\nabla\eta$. This gives

$$\Delta(\eta^2 w) + 2\nabla v \cdot \nabla(\eta^2 w) = 2\eta^2 |D^2 v|^2 + 4w\eta\nabla v \cdot \nabla\eta + w\Delta\phi + 8\eta\nabla\eta \cdot D^2 v \nabla v.$$

By Cauchy's inequality, for all a, b, we have that given any $\epsilon > 0$, $2ab \geq -\epsilon a^2 - \frac{b^2}{\epsilon}$. Therefore

$$\begin{aligned}8\eta\nabla\eta \cdot D^2 v\nabla v &\geq -C|\eta||\nabla\eta||D^2 v||\nabla v| \\ &\geq -\epsilon\eta^2|D^2 v|^2 - \frac{C'}{\epsilon}|\nabla\eta|^2|\nabla v|^2 = -\epsilon\eta^2|D^2 v|^2 - \frac{C'}{\epsilon}|\nabla\eta|^2 w\end{aligned}$$

(where we have set a as $|\eta||D^2 v|$ and b as $|\nabla\eta||\nabla v|$). By similar logic,

$$\begin{aligned}4\eta w\nabla v \cdot \nabla\eta &\geq -C|\eta||w||\nabla v||\nabla\eta| \\ &\geq -\epsilon\eta^2 w^2 - \frac{C'}{\epsilon}|\nabla v|^2|\nabla\eta|^2 = \epsilon\eta^2 w^2 - \frac{C'}{\epsilon}|\nabla v|^2 w.\end{aligned}$$

We can also bound the $2\phi|D^2 v|^2$ term, noting that

$$\begin{aligned}w^2 = (\Delta v)^2 &= \sum_{i,j}^{n} \frac{\partial^2 v}{\partial x_i^2}\frac{\partial^2 v}{\partial x_j^2} \leq \sum_{i,j}^{n} \frac{1}{2}\Big(\frac{\partial^2 v}{\partial x_i^2}\Big)^2 + \frac{1}{2}\Big(\frac{\partial^2 v}{\partial x_j^2}\Big)^2 \\ &= \sum_{i,j}^{n} \Big(\frac{\partial^2 v}{\partial x_i^2}\Big)^2 = n\sum_{i=1}^{n}\Big(\frac{\partial^2 v}{\partial x_i^2}\Big)^2 \leq n\sum_{i,j}^{n}\Big(\frac{\partial^2 v}{\partial x_i \partial x_j}\Big)^2 = n|D^2 v|^2.\end{aligned}$$

Therefore $w \leq n|D^2 v|^2$, which with the other inequalities means that, taking ϵ small enough that $2/n - \epsilon - \epsilon/n \geq 0$,

$$\begin{aligned}\Delta(\eta^2 w) + 2\nabla v \cdot \nabla(\eta^2 w) &\geq \frac{2}{n}\eta^2 w^2 - \epsilon\eta^2 w^2 - \frac{C}{\epsilon}|\nabla\eta|^2 w - \frac{\epsilon}{n}\epsilon^2 w^2 \\ &\quad - \frac{C}{\epsilon}|\nabla\eta|^2 w - |\Delta(\eta^2)|w \geq \Big(\frac{2}{n} - \epsilon - \frac{\epsilon}{n}\Big)\eta^2 w^2 - C_\eta w.\end{aligned}$$

Then let x_0 be an interior maximum of $n^2 w$; this means that $\Delta(\eta^2 w)(x_0) \leq 0$ and $\nabla(\eta^2 w)(x_0) = 0$. Therefore, using the previous inequality, we have that

$$\Big(\frac{2}{n} - \epsilon - \frac{\epsilon}{n}\Big)(\eta w)^2(x_0) \leq C_\eta w(x_0) \implies (\eta w)^2(x_0) \leq C_{(\eta,n)} w(x_0).$$

Since $\eta^2 w \leq (\eta^2 w)(x_0)$, this means

$$\eta^2 w w(x_0) \leq (\eta w)^2(x_0) \leq C_{(\eta,n)} w(x_0) \implies \eta^2 w \leq C_{(\eta,n)}.$$

Therefore, since $w = |\nabla v|^2$, this means that there exists a constant C depending on η such that

$$\eta^2 |\nabla \log u|^2 \leq C.$$

Choose η to be a cutoff function where $\eta = 1$ on $B_{1/2}$. Thus there exists C independent of u such that $|\nabla \log u| \leq C$ on $B_{1/2}$. As a result, for all $x \neq y \in B_{1/2}$,

$$\frac{|\log u(x) - \log u(y)|}{|x-y|} \leq C \implies e^{|\log u(x) - \log u(y)|} \leq e^C e^{|x-y|}.$$

Assuming without loss of generality that $\log u(x) > \log u(y)$, then this means

$$\frac{e^{\log u(x)}}{e^{\log u(y)}} = \frac{u(x)}{u(y)} \leq e^C e^{|B_{1/2}|} = C'.$$

As a result, there exists C such that $u(x) \leq Cu(y)$ on $B_{1/2}$, which proves the Harnack inequality for harmonic functions. □

3. Holder Preliminaries

Definition 3.1. (Holder continuity.) The space $C^{0,\alpha}(\Omega)$ of functions that are α-Holder on a space Ω consists of functions f such that there exists $M \in \mathbb{R}$ where

$$\sup_{x \neq y \in \Omega} \frac{|f(x) - f(y)|}{|x-y|^\alpha} \leq M.$$

Also, when we write $\operatorname{osc}_\Omega f$, this is the oscillation of the function f over the domain Ω, meaning

$$\operatorname{osc}_\Omega f = \sup_\Omega f - \inf_\Omega f.$$

3.1. Morrey's Characterization of Holder Continuity.

We prove here an equivalent definition of Holder continuity, which is Morrey's characterization. This alternative definition comes in handy when proving the Schauder estimate, and an additional perturbation result, which are both vital to prove De Giorgi's theorem.

Theorem 3.2. *There exists $C > 0$ such that*

$$\frac{1}{|B_r|} \int_{B_r(x)} |u - (u)_{B_r(x)}|^p \leq C r^{\alpha p}$$

for all $B_r(x) \subset B_1$ if and only if $u \in C^\alpha(B_1)$ with $[u]_\alpha \approx C$.

Proof. First, we want to prove that if some $u \in C^\alpha(B_1)$, then u satisfies

$$⨍_{B_r(x_0)} |u - (u)_{B_r(x_0)}|^p \leq C r^{\alpha p}$$

for all $x_0 \in B_{1/2}$, $r \in (0, 1/2)$. We know that if $u \in C^\alpha(B_1)$, then for all $x, y \in B_r(x_0)$ where $x_0 \in B_{1/2}$ and $r \in (0, 1/2)$, there exists M such that

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq M.$$

As a result,

$$|u(x) - u(y)| \leq M|x - y|^\alpha \leq M(2^\alpha)r^\alpha = Cr^\alpha$$

where C depends on α. Then we can see that $\operatorname{osc}_{B_r(x_0)} u \leq Cr^\alpha$. As a result,

$$⨍_{B_r(x_0)} |u - (u)_{B_r(x_0)}|^p \leq \frac{1}{|B_r|}|B_r|(\operatorname{osc}_{B_r(x_0)} u)^p \leq (Cr^\alpha)^p = C' r^{\alpha p}.$$

Here C' is a constant depending on α and p.

For the other direction, we assume that

$$⨍_{B_r(x_0)} |u - (u)_{B_r(x_0)}|^p \leq Cr^{\alpha p}$$

for all $x_0 \in B_1$, $r \in (0, 1/2)$ and want to show that $u \in C^\alpha(B_{1/2})$. Fix some $x, y \in B_1$ such that $|x - y| = r$. Then by the triangle inequality

(3.3)

$$|u(x) - u(y)| \leq |u(x) - (u)_{B_{2r}(x)}| + |(u)_{B_{2r}(y)} - (u)_{B_{2r}(x)}| + |u(y) - (u)_{B_{2r}(y)}|.$$

We proceed to bound each of these terms by some constant times $r^{\alpha p}$. Take the middle term first; we consider the average of u over $B' = B(x, 2r) \cap B(y, 2r)$, or

$$⨍_{B(x,2r)\cap B(y,2r)} u(z)dz = ⨍_{B'} u(z)dz.$$

Then by the triangle inequality, we can bound the middle term thus:

$$\begin{aligned}
|(u)_{B_{2r}(y)} - (u)_{B_{2r}(x)}| &\leq \left|(u)_{B_{2r}(y)} - ⨍_{B'} u(z)dz\right| + \left|⨍_{B'} u(z)dz - (u)_{B_{2r}(x)}\right| \\
&= \left|⨍_{B'} (u)_{B_{2r}(y)} - u(z)dz\right| + \left|⨍_{B'} u(z) - (u)_{B_{2r}(x)}dz\right| \\
&\leq ⨍_{B'} |(u)_{B_{2r}(y)} - u(z)|dz + ⨍_{B'} |u(z) - (u)_{B_{2r}(x)}|dz \\
&\leq C\left(⨍_{B(y,2r)} |(u)_{B_{2r}(y)} - u(z)|dz + ⨍_{B(x,2r)} |u(z) - (u)_{B_{2r}(x)}|dz\right)
\end{aligned}$$

where

$$C = \frac{|B'|}{|B_{2r}(x)|} = \frac{|B'|}{|B_{2r}(y)|}.$$

Then by the assumption, we have

$$\begin{aligned}
|(u)_{B_{2r}(y)} - (u)_{B_{2r}(x)}| &\leq C\left(⨍_{B_{2r}(y)} |(u)_{B_{2r}(y)} - u(z)|dz + ⨍_{B_{2r}(x)} |u(z) - (u)_{B_{2r}(x)}|dz\right) \\
&\leq Cr^{\alpha p}.
\end{aligned}$$

This bounds the middle term of Equation (3.4). Now we want to bound the first and last term. Given some $x_0 \in B_{1/2}$ and $0 < r < \rho$, then

$$\begin{aligned} |(u)_{B_r(x_0)} - (u)_{B_\rho(x_0)}|^p &= \fint_{B_r(x_0)} |(u)_{B_r(x_0)} - (u)_{B_\rho(x_0)}|^p dy \\ &= \fint_{B_r(x_0)} |(u)_{B_r(x_0)} - u(y) + u(y) - (u)_{B_\rho(x_0)}|^p dy. \end{aligned}$$

For $p \geq 1$, $f = x^p$ is a convex equation, so we can see that by Jensen's inequality $|(u)_{B_r(x_0)} - u(y) + u(y) - (u)_{B_\rho(x_0)}|^p \leq 2^{p-1}\Big(|(u)_{B_r(x_0)} - u(y)|^p + |u(y) - (u)_{B_\rho(x_0)}|^p\Big)$.
As a result, by this result and our original assumption, we have

$$\begin{aligned} &\fint_{B_r(x_0)} |(u)_{B_r(x_0)} - u(y) + u(y) - (u)_{B_\rho(x_0)}|^p dy \\ &\quad\leq 2^{p-1}\Big(\fint_{B_r(x_0)} |(u)_{B_r(x_0)} - u(y)|^p + \fint_{B_r(x_0)} |u(y) - (u)_{B_\rho(x_0)}|^p\Big) \\ &\quad\leq Cr^{\alpha p} + C\Big(\frac{\rho}{r}\Big)^n \rho^{\alpha p}. \end{aligned}$$

Therefore, this means

$$|(u)_{B_{r(x_0)}} - (u)_{B_{\rho(x_0)}}| \leq Cr^\alpha + C\Big(\frac{\rho}{r}\Big)^{(n/p)} \rho^\alpha.$$

Then given some $R > 0$, we can set $r = 2^{-k-1}R$ and $\rho = 2^{-k}R$. We thus have

$$\begin{aligned} |(u)_{B_{R/2^N}(x_0)} - (u)_{B_R(x_0)}| &\leq |(u)_{B_{R/2^N}(x_0)} - (u)_{B_{R/2^{N-1}}(x_0)}| + \\ &\quad \ldots + |(u)_{B_{R/2}(x_0)} - (u)_{B_R(x_0)}| \\ &= \sum_{k=0}^{N-1} |(u)_{B_{R/2^{-k-1}}(x_0)} - (u)_{B_{R/2^{-k}}(x_0)}| \\ &\leq \sum_{k=0}^{N-1} C\Big(\frac{R}{2^{k+1}}\Big)^\alpha + 2^{n/p}\Big(\frac{R}{2^k}\Big)^\alpha. \end{aligned}$$

This last term is less than or equal to the limit as N goes to infinity, meaning that

$$\begin{aligned} |(u)_{B_{R/2^N}(x_0)} - (u)_{B_R(x_0)}| &\leq \lim_{N\to\infty} \sum_{k=0}^{N-1} C\Big(\frac{R}{2^{k+1}}\Big)^\alpha + 2^{n/p}\Big(\frac{R}{2^k}\Big)^\alpha \\ &\leq 2C \sum_{k=1}^{\infty} \Big(\frac{R}{2^k}\Big)^\alpha = 2C\Big(\frac{1}{1-2^\alpha}\Big)R^\alpha. \end{aligned}$$

When we take both sides to the power of p, we get that

$$|(u)_{B_{R/2^N}(x_0)} - (u)_{B_R(x_0)}|^p \leq C\Big(\frac{1}{1-2^\alpha}\Big)^p R^{\alpha p} = CR^{\alpha p}.$$

Taking $N \to \infty$, this means

$$|u(x_0) - (u)_{B_{R(x_0)}}|^p \leq CR^{\alpha p} \implies |u(x_0) - (u)_{B_{R(x_0)}}| \leq CR^\alpha.$$

Therefore each term in Equation (3.4) is bounded by some constant times r^α, meaning we have

$$|u(x) - u(y)| \leq Cr^\alpha \implies \operatorname{osc}_{B_{1/2}} u \leq Cr^\alpha.$$

Thus u is C^α on $B_{1/2}$. □

Theorem 3.4. *Suppose $f \in C(\Omega)$, and that for any ball $B_r(x) \subset \Omega$, there exists a linear function $\ell(p) = a \cdot p + b$ where $a \in \mathbb{R}^n$, $b \in \mathbb{R}$, such that*

$$\sup_{B_r(x)} |f(y) - \ell(y)| \leq Cr^{1+\alpha}.$$

Then $f \in C^{1,\alpha}(\Omega)$ with C approximately $||f||_{C^{1,\alpha}}$. Moreover, if there exists a quadratic polynomial p such that

$$\sup_{B_r(x)} |f(y) - p(y)| \leq Cr^{2+\alpha},$$

then $f \in C^{2,\alpha}(\Omega)$.

Proof. First, we show that f is Lipschitz because for all $x, y \in \Omega$, we have

$$\begin{aligned}
\frac{|f(x) - f(y)|}{|x - y|} &\leq \frac{|f(x) - \ell(x)| + |\ell(x) - \ell(y)| + |\ell(y) - f(y)|}{|x - y|} \\
&\leq \frac{|f(x) - \ell(x)|}{|x - y|} + a + \frac{|f(y) - \ell(y)|}{|x - y|} \\
&\leq C|x - y|^\alpha + a + C|x - y|^\alpha \\
&\leq C(\text{diam}(\Omega))^\alpha + a
\end{aligned}$$

which means $|f(x) - f(y)|/|x - y|$ is bounded. To obtain that $f \in C^{1,\alpha}(\Omega)$, we now show that given $x, y \in \Omega$, $\nabla f(x) = \nabla f(y) + O(r^\alpha)$. We define $\ell_x(y) = f(x) + \nabla f(x)(y - x)$ given some $x \in \Omega$. Moreover, given $x, y \in \Omega$, take z such that $r = |y - z| = |x - z|$. Then we have

$$\begin{aligned}
f(z) &= f(x) + \nabla f(x)(z - x) + O(r^{1+\alpha}) \\
f(z) &= f(y) + \nabla f(x)(z - y) + O(r^{1+\alpha}) \\
f(y) &= f(x) + \nabla f(x)(x - y) + O(r^{1+\alpha}).
\end{aligned}$$

As a result we have that

$$\begin{aligned}
&f(x) + \nabla f(x)(z - x) + O(r^{1+\alpha}) = f(y) + \nabla f(y)(z - y) + O(r^{1+\alpha}) \\
&f(x) + \nabla f(x)(z - x) + O(r^{1+\alpha}) = f(x) + \nabla f(x)(x - y) + \nabla f(y)(z - y) + O(r^{1+\alpha}) \\
&\nabla f(x)(z - y) = \nabla f(y)(z - y) + O(r^{1-\alpha}) \\
&\nabla f(x) = \nabla f(y) + \frac{O(r^{1-\alpha})}{z - y} \\
&\nabla f(x) = \nabla f(y) + O(r^\alpha).
\end{aligned}$$

Therefore, this means that

$$|\nabla f(x) - \nabla f(y)| \leq Cr^\alpha.$$

As a result, ∇f is C^α, meaning that $f \in C^{1,\alpha}$.

For the second part of this proof, we assume that given some $B_r(x) \subset \Omega$, there exists a quadratic polynomial p such that $||f - p||_{L^\infty} \leq Cr^{2+\alpha}$. Therefore take a sequence of decreasing radii r_k, and for each $B_{r_k}(x)$ we have a corresponding quadratic polynomial P_k. Then

$$||f - P_k||_{L^\infty(B_{r_k}(x))} \leq Cr_k^{2+\alpha}.$$

Given P_k, P_ℓ such that $\ell \leq k$, by the triangle inequality we have that

$$||P_k - P_\ell||_{L^\infty(B_{r_\ell}(x))} \leq 2Cr_\ell^{2+\alpha}.$$

Suppose that $P_k = a_k + b_k(y-x) + (y-x)M_k(y-x)$, and similarly for P_ℓ. Then as $r_\ell, r_k \to \infty$, we have that $||P_k - P_\ell||_{L^\infty} \leq 0$, meaning that the sequences a_k, b_k and M_k must be Cauchy, since $a_k - a_\ell$, $b_k - b_\ell$, and $M_k - M_\ell$ must converge to 0. Therefore P_k is a Cauchy sequence, so there exists some $P = \lim_{k\to\infty} P_k$ such that

$$||f - P||_{L^\infty(B_r)} \leq Cr^{2+\alpha}.$$

Because the norm $f - P$ is comparable to $r^{2+\alpha}$ on B_r, it follows that P is the second order Taylor polynomial of f (centered at the origin). Thus, we can follow steps similar to the previous proof to reach the conclusion that $f \in C^{2,\alpha}$. □

4. Schauder Theorem

The Schauder theorem states that given some u where $\Delta u = f$ and $u = g$ on $\partial\Omega$, then we can bound all the second derivatives of u by the L^∞ norms of f and g. This means that if f and g are bounded, then $u \in C^{2,\alpha}$. This estimate can help us prove DeGiorgi's theorem.

Before we begin explaining the Schauder theorem, we first introduce harmonic replacements. Given some

$$\begin{cases} u = g & on\ \partial\Omega \\ u = f & on\ \Omega, \end{cases}$$

then the harmonic replacement v of u is a function such that $v = u$ on $\partial\Omega$, and

$$\int_\Omega \nabla\phi \cdot \nabla v = 0$$

for all $\phi \in H_0^1(\Omega)$. Also, before the Schauder theorem, we first need to prove the maximum principle, and then a preliminary lemma. The maximum principle proves that the maxima (and by similar logic, minima) of a weak solution lie on its boundary. We prove this result for the more general uniformly elliptic case, rather than just for harmonic solutions.

Lemma 4.1. *(Maximum Principle.) For an $H^1(B_2)$ weak solution u to $\partial_i(a_{ij}\partial_j u) = 0$ where each a_{ij} is bounded, measurable, and uniformly elliptic,*

$$essup_\Omega u = essup_{\partial\Omega} u$$

for any open set $\Omega \subset B_2$. A similar statement holds for minima.

Proof. Let $v = (u - \text{essup}_{\partial\Omega} u)_+$. Then because v belongs to $H_0^1(\Omega)$ by standard Sobolev Space theory,

$$\int_\Omega a_{ij}\partial_j u \partial_i v = 0.$$

Because this integral is only nonzero on the set $A = \{x \in \Omega \mid u(x) > \text{essup}_{\partial\Omega} u\}$, then

$$\int_\Omega a_{ij}\partial_j u\partial_i v = \int_A a_{ij}\partial_j u\partial_i v = \int_\Omega a_{ij}\partial_j(u - \text{essup}_{\partial\Omega} u)_+ \partial_i v = 0.$$

Since $\partial_j v = \partial_j u$ on A and $\partial_j v = 0$ almost everywhere on $\Omega \setminus A$,

$$0 \geq \lambda \int_\Omega |\nabla(u - \text{essup}_{\partial\Omega} u)_+|^2.$$

Because

$$\lambda \int_{\Omega} |\nabla (u - \text{essup}_{\partial\Omega} u)_+|^2 \geq 0$$

and $\lambda > 0$, this means

$$\int_{\Omega} |\nabla (u - \text{essup}_{\partial\Omega} u)_+|^2 = 0$$

and therefore that $\nabla(u - \text{essup}_{\partial\Omega} u)_+ = 0$ on Ω. Since, however, $(u - \text{essup}_{\partial\Omega} u)_+ = 0$ on $\partial\Omega$, this means $(u - \text{essup}_{\partial\Omega} u)_+ = 0$ on Ω and therefore that $\text{essup}_{\partial\Omega} u = \text{essup}_{\Omega} u$.

By a similar argument using $v = (\text{essinf}_{\partial\Omega} u - u)_+$, we also have that $\text{essinf}_{\partial\Omega} u = \text{essinf}_{\Omega} u$. □

Lemma 4.2. *Let $u \in H^1(B_1)$ be a weak solution of*

$$\begin{cases} \Delta u = f & \text{in } B_1, \\ u = g & \text{on } \partial B_1. \end{cases}$$

Then there exists $C > 0$ such that

$$||u||_{L^\infty(B_1)} \leq C(||f||_{L^\infty(B_1)} + ||g||_{L^\infty(\partial B_1)}).$$

Proof. We define

$$\tilde{u} = \frac{u}{||f||_{L^\infty(B_1)} + ||g||_{L^\infty(\partial B_1)}}.$$

Then $\Delta \tilde{u} = \tilde{f}$ where $\tilde{f} = f/(||f||_{L^\infty(B_1)} + ||g||_{L^\infty(\partial B_1)})$, and $\tilde{u} = \tilde{g}$ on ∂B_1 where $\tilde{g} = g/(||f||_{L^\infty(B_1)} + ||g||_{L^\infty(\partial B_1)})$. Then $||\tilde{f}||_{L^\infty(B_1)} \leq 1$ and $||\tilde{g}||_{L^\infty(B_1)} \leq 1$. Now we define a new function

$$v = \tilde{u} - \frac{|x|^2}{2d} + 1 + \frac{1}{2d}.$$

Recall that $\Delta(|x|^2) = 2d$; therefore we have that $\Delta v = \tilde{f} - 1 \leq 0$ and on ∂B_1 we have $v = \tilde{g} + 1 \geq 0$. Thus by the maximum principle, since $v \geq 0$ on ∂B_1 and $\Delta v \leq 0$, $v \geq 0$ on B_1. Therefore,

$$\tilde{u} - \frac{|x|^2}{2d} + 1 + \frac{1}{2d} \geq 0 \implies \tilde{u} \geq \frac{|x|^2}{2d} - 1 - \frac{1}{2d}$$

on B_1. Therefore, $\tilde{u} \geq -2$. By applying the same argument to $-u$, we also have $\tilde{u} \leq 2$, so $|\tilde{u}| \leq 2$, meaning that $|u| \leq 2(||f||_{L^\infty(B_1)} + ||g||_{L^\infty(\partial B_1)})$, which proves the theorem. □

Now we can move on to the Schauder estimate.

Theorem 4.3. *(Schauder Theorem.) Let $u \in H^1(B_1)$ be a bounded weak solution to $\Delta u = f \in C^\alpha(B_1)$, where $0 < \alpha < 1$. Then there exists some $C > 0$ depending only on α and the dimension d such that*

$$||u||_{C^{2,\alpha}(B_1)} \leq C(||u||_{L^\infty(B_1)} + ||f||_{C^\alpha(B_1)}).$$

Proof. We can suppose without loss of generality that $f(0) = 0$ by replacing a given u' with $u(x) = u'(x) - \frac{f(0)|x|^2}{2d}$, so that $\Delta u = f - f(0)$. We can also reduce to the case that $||u||_{L^\infty(B_1)} \leq 1$ and $||f||_{C^\alpha(B_1)} \leq \epsilon$ for some $\epsilon > 0$; this is because we can multiply u by the constant $\epsilon/(\epsilon||u||_{L^\infty} + ||f||_{C^\alpha})$. Then defining u' and u multiplied by this constant, we have

$$||u'||_{L^\infty} \leq \frac{1}{||u||_{L^\infty}} ||u||_{L^\infty} \leq 1,$$

as well as

$$||\Delta u||_{C^\alpha} \leq \frac{\epsilon}{||f||_{C^\alpha}}||f||_{C^\alpha} \leq \epsilon.$$

Therefore, from now on we assume that our given function u is such that $\Delta u(0) = f(0) = 0$, $||u||_{L^\infty} \leq 1$ and $||\Delta u||_{C^\alpha} = ||f||_{C^\alpha} \leq \epsilon$.

Now let some w be the harmonic replacement of u, meaning that it solves

$$\begin{cases} \Delta w = 0 & \text{on } B_1, \\ w = u & \text{on } \partial B_1. \end{cases}.$$

Then by Lemma 4.1, and because of the assumptions we make on u, we have

$$||u - w||_{L^\infty(B_1)} \leq C||\Delta u||_{L^\infty(B_1)} = C||f||_{L^\infty(B_1)} \leq C||f||_{C^\alpha} \leq C\epsilon.$$

Moreover, again because of the assumptions we made on u, we have

$$||w||_{L^\infty(B_1)} \leq ||w||_{L^\infty(\partial B_1)} = ||u||_{L^\infty(\partial B_1)} \leq 1.$$

Then we define P_1 to be the second order Taylor polynomial of w about 0; in other words,

$$P_1(x) := w(0) + \nabla w(0) \cdot x + \frac{1}{2} x \cdot D^2 w(0) x.$$

Because w is harmonic, the second derivatives are bounded [1]. Then by Theorem 2.10 of [2], and since $||w||_{L^\infty(B_1)} \leq 1$, we have that on a smaller ball $B_{1/2}$ there exists C such that,

$$||P_1||_{L^\infty(B_{1/2})} \leq C||w||_{L^\infty(B_1)} \leq C.$$

Thus $||P_1||_{L^\infty(B_1)} \leq C$ where C depends only on the dimension d. As a result, since P_1 is the second order Taylor polynomial of w, we have that for all $r \leq 1/2e$

$$||w - P_1||_{L^\infty(B_r)} \leq C||D^3 w||_{L^\infty(B_r)} r^3 \leq C||D^3 w||_{L^\infty(B_1)} r^3.$$

Because P_1 is the second order Taylor polynomial of a harmonic function w, one can compute that

$$\Delta P_1(x) = \sum_{i=1}^{d} \frac{\partial^2 w}{\partial x_i^2}(x) = \Delta w(x) = 0.$$

Thus P_1 is harmonic, so $\Delta(w - P_1) = 0$, meaning that

$$||w - P_1||_{L^\infty(B_r)} \leq C||D^3 w||_{L^\infty(B_1)} r^3 \leq C'||w||_{L^\infty} r^3 \leq C' r^3.$$

Since $||w - P_1||_{L^\infty(B_r)} \leq Cr^3$, and $||u - w||_{L^\infty(B_r)} \leq ||u - w||_{L^\infty(B_1)} \leq C\epsilon$, this means that

$$\begin{aligned} ||u - P_1||_{L^\infty(B_r)} &\leq ||w - P_1||_{L^\infty(B_r)} + ||u - w||_{L^\infty(B_r)} \\ &\leq C_1 r^3 + C_2 \epsilon \\ &\leq C(r^3 + \epsilon) \end{aligned}$$

for all $r \leq 1$. Then choose $r = r_0 = (\frac{1}{2C})^{1/(1-\alpha)}$, and pick $\epsilon = r_0^3$. Then

$$C(r^3 + \epsilon) = C\Big(2\Big(\frac{1}{2C}\Big)^{3/(1-\alpha)}\Big) = \Big(\frac{1}{2C}\Big)^{\frac{3}{1-\alpha}-1} = \Big(\frac{1}{2C}\Big)^{\frac{2+\alpha}{1-\alpha}} = r_0^{2+\alpha}$$

which implies that

$$||u - P_1||_{L^\infty(B_{r_0})} \leq r_0^{2+\alpha}.$$

Now we define

$$u_2(x) := \frac{(u - P_1)(r_0 x)}{r_0^{2+\alpha}}$$

for all $x \in B_1$. Then

$$||u_2||_{L^\infty(B_r)} \leq \frac{||u - P_1||_{L^\infty(B_r)}}{r_0^{2+\alpha}} \leq \frac{r_0^{2+\alpha}}{r_0^{2+\alpha}} = 1.$$

Moreover, let f_2 be the Laplacian of u_2; then because P_1 is harmonic we have that

$$\begin{aligned}[\Delta u_2]_{C^\alpha(B_1)} &= \sup_{x \neq y} \frac{r_0^2[(\Delta u - \Delta P_1)(r_0 x) - (\Delta u - \Delta P_1)(r_0 y)]}{r_0^{2+\alpha}|x - y|^\alpha} \\ &= \sup_{x,y} \frac{\Delta u(r_0 x) - \Delta u(r_0 y)}{|r_0 x - r_0 y|^\alpha} \\ &= [f]_{C^\alpha(B_{r_0})} \leq \epsilon.\end{aligned}$$

This means that $||f_2||_{C^\alpha(B_{r_0})} \leq \epsilon$. We can also bound the L^∞ norm of f_2, noting that

$$||f_2||_{L^\infty(B_1)} = r_0^{-\alpha}||f||_{L^\infty(B_{r_0})} \leq \epsilon$$

because for any $|x| < r$, $|f(x)| = |f(x) - f(0)| \leq \epsilon r^\alpha$. Therefore f_2 is C^α since its L^∞ norm and seminorm are bounded. As above, we can again define w_2 to be the harmonic replacement of u_2, and define P_2 to be the second order Taylor polynomial of w_2. We again have w_2 and P_2 such that $||u_2 - P_2||_{L^\infty(B_{r_0})} \leq r_0^{2+\alpha}$. This means that by our definition of u_2,

$$u_2(x) r_0^{2+\alpha} = (u - P_1)(r_0 x).$$

From there we have that

$$u(r_0 x) = P_1(r_0 x) + r_0^{2+\alpha} P_2(r_0 x) + r_0^{2+\alpha}(u_2 - P_2)(r_0 x),$$

which implies that

$$||u - P_1 - r_0^{2+\alpha} P_2||_{L^\infty(B_{r_0})} \leq r_0^{2(2+\alpha)}.$$

Therefore if we define the polynomial $Q_2 = P_1 + r_0^{2+\alpha} P_2$,

$$||u - Q_2||_{L^\infty(B_{r_0})} \leq r_0^{2(2+\alpha)}.$$

We now continue, similarly defining u_3, P_3 and Q_3 such that

$$||u_3 - P_3||_{L^\infty(B_{r_0})} \leq r_0^{2+\alpha} \text{ and } ||u - Q_3||_{L^\infty(B_{r_0^2})} \leq r_0^{3(2+\alpha)}.$$

In other words, we can iterate so that for each k there exists a quadratic Q_k such that

$$||u - Q_k||_{L^\infty(B_{r_0^{k-1}})} \leq r_0^{k(2+\alpha)}.$$

Then by Theorem 3.4, this means that $u \in C^{2,\alpha}(B_1)$. However, we manipulated our original function to get this u. Our original function is

$$u' = u(||u||_{L^\infty} + (||f||_{C^\alpha}/\epsilon)) + \frac{f(0)|x|^2}{2d}.$$

Therefore because $u \in C^{2,\alpha}$ we obtain that

$$||u'||_{C^{2,\alpha}} \leq C(||u||_{L^\infty} + ||f||_{C^\alpha}).$$

□

5. Perturbation Result

In this section, we first look at a perturbation result where $\partial_i(a_{ij}\partial_j u) = 0$ and $||A - I|| \leq \epsilon$, meaning A is very close to the identity. In other words, $\partial_i(a_{ij}\partial_j u)$ is close to the Laplacian, so u is close to being harmonic. We then ask about the regularity of u (whether it is Holder continuous).

Lemma 5.1. *Given some u such that $\fint_{B_r(x_0)} |\nabla u|^p \leq Cr^{(\alpha-1)p}$ for all $x_0 \in B_{1/2}$, $r \in (0, 1/2)$, this implies that u is Holder continuous by Morrey's characterization.*

Proof. Poincaré's Inequality gives us that $||u - (u)_{B_r}||_{L^p(B_r)} \leq C||\nabla u||_{L^p(B_r)}$. Let $u_r(x) = u(rx)$. Poincare's also gives us that $||u_r - (u_r)_{B_1}||_{L^p(B_1)} \leq C||\nabla u_r||_{L^p(B_1)}$. By scaling Poincaré's inequality

$$\int_{B_r} |u(x) - (u)_{B_r}|^p dx \leq C \int_{B_r} |r\nabla u|^p dx,$$

which means by assumption that

$$\int_{B_r} |u(x) - (u)_{B_r}|^p dx \leq Cr^p r^{(\alpha-1)p},$$

and thus $\int_{B_r} |u(x) - (u)_{B_r}|^p dx \leq Cr^{\alpha p}$, which is the condition of Morrey's characterization of Holder continuity. □

Lemma 5.2. *Let ϕ be a nonnegative, increasing function with*

$$\phi(\rho) \leq C\Big(\Big(\frac{\phi}{r}\Big)^{\gamma} + \epsilon\Big)\phi(r)$$

for all $0 < \rho < r$. For any $0 < \beta < \gamma$, there exists $c > 0$ such that $\phi(r) \leq c\phi(1)r^{\beta}$ holds for all r.

Proof. To prove this, we fix $0 < r < 1$. Then we know by assumption that there exists C, γ satisfying the inequality above. Now given some $\beta < \gamma$, we can fix

$$\tau = \Big(\frac{1}{2C}\Big)^{\frac{1}{\gamma-\beta}}.$$

Since $\tau < 1$ this means $\tau r < r$. Then we choose $\epsilon_0 = \tau^{\gamma}$. Setting $\rho = \tau r$, this gives us

$$\phi(\rho) \leq C(\tau^{\gamma} + \tau^{\gamma})\phi(r) = 2C\Big(\frac{1}{2C}\Big)^{\frac{\gamma}{\gamma-\beta}}\phi(r) = \Big(\frac{1}{2C}\Big)^{\frac{\beta}{\gamma-\beta}}\phi(r) = \tau^{\beta}\phi(r).$$

As a result, this means $\phi(\tau) \leq \tau^{\beta}\phi(1)$. Take some $0 < \tau < r < 1$, we can see $\phi(\tau) \leq \tau^{\beta}\phi(1) \leq r^{\beta}\phi(1)$, and therefore

$$\phi(r) \leq r^{\beta}\phi(1)\Big(\frac{\phi(r)}{\phi(\tau)}\Big) \leq r^{\beta}\phi(1)\Big(\frac{\phi(1)}{\phi(\tau)}\Big).$$

Moreover, now suppose $\tau^{k+1} < r < \tau^k$ for some $k \leq 1$. Then we know

$$\phi(\tau^{k+1}) \leq \tau^{\beta}(\phi(\tau^k)) \leq \tau^{\beta}\tau^{\beta}(\phi(\tau^{k-1}))\ldots \leq (\tau^{k+1})^{\beta}(\phi(1)) \leq r^{\beta}(\phi(1)).$$

Therefore, this means

$$\phi(r) \leq r^{\beta}(\phi(1))\Big(\frac{\phi(r)}{\phi(\tau^{k+1})}\Big) \leq r^{\beta}(\phi(1))\Big(\frac{\phi(\tau^k)}{\phi(\tau^{k+1})}\Big) \leq r^{\beta}(\phi(1))\Big(\frac{\phi(1)}{\phi(\tau)}\Big).$$

Thus if we let $c = \phi(1)/\phi(\tau)$, we have some c such that given $0 < \beta < \gamma$, $\phi(r) \leq r^{\beta}\phi(1)c$ for all r. □

We now state and prove the perturbation result.

Theorem 5.3. *Let $\lambda I \leq A = A(x) \leq \Lambda I$ be a uniformly elliptic matrix with $||A - I||_{L^\infty(B_1)} < \epsilon$. For every $\alpha \in (0,1)$, there exists $\epsilon_0(d, \alpha) > 0$ such that if $\epsilon < \epsilon_0$, and $u \in H^1(B_1)$ solves*

$$\partial_i(a_{ij}\partial_j u) = \operatorname{div}(A\nabla u) = 0 \text{ on } B_1,$$

then $u \in C^\alpha(B_{1/2})$.

Proof. Fix some ball $B_r(x_0) \subset B_1$, and suppose without loss of generality that $x_0 = 0$, $0 < r < 1$. Define v as the harmonic replacement of u on $B_r(x_0)$, and set $h = u - v \in H_0^1(B_r)$. Then

$$\Delta h = \Delta u = \operatorname{div}(I\nabla u),$$

and since $\Delta v = 0$, we have

$$\Delta h = \Delta u - \Delta v = \operatorname{div}(I\nabla u) - \operatorname{div}(A\nabla u) = \operatorname{div}(I - A)(\nabla u)$$

by assumption. By the weak formulation of the equation for h, taking the test function to be h, we have

$$\int_{B_r} |\nabla h|^2 = \int_{B_r} ((I - A)\nabla u)\nabla h.$$

Then by Cauchy-Schwartz,

$$\int_{B_r} |\nabla h|^2 = \int_{B_r} ((\delta_{ij} - a_{ij})\partial_j u)\partial_i h \leq \int_{B_r} \epsilon |\nabla u||\nabla h| \leq \epsilon ||\nabla u||_{L^2} ||\nabla h||_{L^2}.$$

As a result,

$$||\nabla h||_{L^2} \leq \epsilon ||\nabla u||_{L^2}.$$

We can also show that $|\nabla v|^2$ is subharmonic, meaning that $\Delta(|\nabla v|^2) \geq 0$. We have that

$$\Delta |\nabla v|^2 = \sum_{i=1}^d \sum_{k=1}^d 2(\partial_i \partial_k v)^2 + \sum_{i=1}^d \sum_{k=1}^d \partial_k v(\partial_k \partial_i^2 v).$$

The first term here is greater than or equal to zero, and the second term is zero because $\Delta v = 0$. Hence $|\nabla v|^2$ is a subharmonic function. A property of subharmonic functions is that their averages are monotone increasing. As a result, this means that given some $0 < \delta < r$,

$$\frac{1}{|B_\delta|} \int_{B_\delta} |\nabla v|^2 \leq \frac{1}{|B_r|} \int_{B_r} |\nabla v|^2$$

which implies

$$\int_{B_\delta} |\nabla v|^2 \leq \frac{\delta^n}{r^n} \int_{B_r} |\nabla v|^2.$$

Because v is the harmonic replacement of u on B_r, we have $\Delta v = 0$ on B_r and $v = u$ on ∂B_r, meaning that because harmonic functions are minimizers of the Dirichlet integral

$$\int_{B_r} |\nabla v|^2 \leq \int_{B_r} |\nabla u|^2.$$

Therefore we have that

$$\begin{aligned}\int_{B_\delta} |\nabla u|^2 &= \int_{B_\delta} |\nabla(v+h)|^2 \\ &\leq \int_{B_\delta} 2(|\nabla v|^2 + |\nabla h|^2) \\ &\leq 2\Big(\frac{\delta^n}{r^n}\Big)\int_{B_r} |\nabla v|^2 + 2\epsilon^2 \int_{B_\delta} |\nabla u|^2 \\ &\leq 2\Big(\Big(\frac{\delta^n}{r^n}\Big) + \epsilon^2\Big)\int_{B_r} |\nabla u|^2.\end{aligned}$$

Returning now to the original proof, we set $I(r) = \int_{B_r} |\nabla u|^2$. This is a nonnegative, increasing function, and we have shown that

$$I(\delta) \leq 2((\frac{\delta^n}{r^n}) + \epsilon^2)I(r)$$

for $\delta < r$. As a result, by Lemma 5.2, there exists a constant c such that given $0 < \rho < 1$, if we take ϵ small enough,

$$I(\rho) \leq C\rho^{n-2+2\alpha} I(1).$$

This is because n is our γ as stated in the lemma, and $n - 2 + 2\alpha < n$ because $0 < \alpha < 1$. This means that

$$\frac{1}{\rho^n}\int_{B_\rho} |\nabla u|^2 \leq C\rho^{2(\alpha-1)} \int_{B_1} |\nabla u|^2 \implies \fint_{B_\rho} |\nabla u|^2 \leq C\rho^{2(\alpha-1)} \fint_{B_1} |\nabla u|^2.$$

Therefore, u satisfies the condition in Lemma 5.1 with constant

$$C' = C\fint_{B_1} |\nabla u|^2.$$

Therefore u is Holder continuous over $B_{1/2}$. □

6. DeGiorgi's Theorem in 2D

We now prove De Giorgi's Theorem in two dimensions. This result is surprising because given only that u is a weak solution and the coefficients a_{ij} are L^∞, this means that $u \in C^\alpha$. Instead of proving the general De Giorgi's Theorem, we only prove it in two dimensions, which is much simpler.

Theorem 6.1. *Let $u \in H^1(B_1)$ solve $\partial_i(a_{ij}\partial_j u) = 0$ in the weak sense. Assume that the dimension is $n = 2$ and the matrix A is uniformly elliptic. Then $u \in C^\alpha(B_{1/2})$ for some $0 < \alpha < 1$.*

Proof. First, we fix $B = B_r(x_0)$ such that $2B = B_{2r}(x_0) \subset B_1$. Then set $\phi = (u-c)\eta^2$ for some constant c to be chosen, and some smooth η such that $\eta = 1$ on B, $\eta = 0$ on $B_1 \setminus 2B$, with $|\nabla \eta| \leq 2/r$. Then because $\phi \in C_c^\infty(B_1)$ and u is a solution, we have

$$\int_{B_1} a_{ij}\partial_j u \partial_i \phi = \int_{B_1} a_{ij}\partial_j u \partial_i((u-c)\eta^2) = 0.$$

We split this into three parts, writing (since the term on $B_1 \setminus B_{2r}$ is zero)

$$\int_{B_r} a_{ij}\partial_j u\partial_i((u-c)\eta^2) + \int_{B_{2r}\setminus B_r} a_{ij}\partial_j u \partial_i((u-c)\eta^2) = 0$$

We also know $\eta = 1$ on B_r, meaning that

$$\int_{B_r} a_{ij}\partial_j u \partial_i u = -\int_{B_{2r}/B_r} a_{ij}\partial_j u \partial_i((u-c)\eta^2).$$

By the uniform ellipticity of A,

$$\lambda \int_{B_r} |\nabla u|^2 \leq \Big| -\int_{B_{2r}\setminus B_r} a_{ij}\partial_j u \partial_i((u-c)\eta^2)\Big| \leq \int_{B_{2r}\setminus B_r} |a_{ij}\partial_j u \partial_i((u-c)\eta^2)|.$$

By the product rule,

$$\nabla((u-c)\eta^2) = \eta^2\nabla(u-c) + (u-c)\nabla(\eta^2) = \eta^2\nabla(u-c) + (u-c)2\eta\nabla\eta.$$

As a result,

$$\lambda \int_{B_r} |\nabla u|^2 \leq \int_{B_{2r}\setminus B_r} A\nabla u \cdot \nabla u + \int_{B_{2r}\setminus B_r} |(u-c)A\nabla u \cdot \nabla \eta|.$$

By the Holder inequality and uniform ellipticity, we have that

$$\int_{B_{2r}/B_r} A\nabla u \nabla \eta (u-c) \leq ||\Lambda |\nabla u||\nabla \eta|||_{L^2(B_{2r}\setminus B_r)} ||u-c||_{L^2(B_{2r}\setminus B_r)}.$$

As a result this gives us

$$\begin{aligned}\lambda \int_{B_r} |\nabla u|^2 &\leq \Lambda \int_{B_{2r}/B_r} |\nabla u|^2 + \Big(\int_{B_{2r}/B_r} (\Lambda|\nabla u||\nabla\eta|)^2\Big)^{1/2}\Big(\int_{B_{2r}/B_r}(u-c)^2\Big)^{1/2} \\ &\leq \Lambda \int_{B_{2r}/B_r} |\nabla u|^2 + \frac{\Lambda}{r}||\nabla u||_{L^2(B_{2r}/B_r)}\Big(\int_{B_{2r}/B_r}(u-c)^2\Big)^{1/2}.\end{aligned}$$

Choosing c to be the average of u on the annulus, we apply the scaled Poincaré' inequality to obtain

$$\begin{aligned}\lambda \int_{B_r} |\nabla u|^2 &\leq \Lambda \int_{B_{2r}/B_r} |\nabla u|^2 + \frac{\Lambda}{r}||\nabla u||_{L^2(B_{2r}/B_r)} cr ||\nabla u||_{L^2(B_{2r}/B_r)} \\ &= (\Lambda + (\Lambda/r)cr)\int_{B_{2r}/B_r} |\nabla u|^2 = (\Lambda + \Lambda c)\int_{B_{2r}/B_r}|\nabla u|^2.\end{aligned}$$

Adding $\lambda \int_{B_r} |\nabla u|^2$ to both sides, we have some constant $C = (\Lambda + \Lambda c)/\lambda$ such that

$$\int_{B_r(x_0)} |\nabla u|^2 \leq C \int_{B_{2r(x_0)}/B_r(x_0)} |\nabla u|^2.$$

Then if we define $I(r) = \int_{B_r(x_0)} |\nabla u|^2$, we can see that $I(r) \leq C(I(2r) - I(r))$, meaning

$$I(r) \leq \frac{C}{C+1} I(2r).$$

Given some $r < 1/2$, choose $k \geq 1$ such that $2^{-(k+1)} \leq r \leq 2^{-k}$. Then by the monotonicity of I, $I(r) \leq I(2^{-k})$. We know

$$I(2^{-k}) \leq \Big(\frac{C}{C+1}\Big)^k I(1).$$

Then if we denote $\zeta = C/(C+1)$, we have

$$\zeta^{-k} = (2^{\log_2 \zeta})^{-k} = (2^{-k})^{\log_2 \zeta} \leq 2r^{\log_2 \zeta}.$$

Therefore $I(r) \leq 2I(1)r^\alpha$ where $\alpha = \log_2 \zeta$. Therefore

$$\int_{B_r(x_0)} |\nabla u|^2 \leq Cr^\alpha$$

for all $r < 1/2$. Then by the scaled Poincaré inequality

$$\frac{1}{|B_r|}\int_{B_r(x_0)} |u - (u)_{B_r(x_0)}|^2 \leq \frac{r^2}{|B_r|}\int_{B_r(x_0)} |\nabla u|^2.$$

Since we are in two dimensions, $r^2/|B_r|$ is just the constant $1/\pi$ (this is why this proof only works in two dimensions). As a result, we have that

$$\frac{1}{|B_r|}\int_{B_r(x_0)} |u - (u)_{B_r(x_0)}|^2 \leq C\int_{B_r(x_0)} |\nabla u|^2 \leq Cr^\alpha.$$

By Morrey's characterization of Holder continuity, this means that $u \in C^{\alpha/2}$, and therefore u is Holder continuous.

□

7. Harnack Inequality in 2D

Harnack's inequality states that for a positive weak solution u, the ratio of the supremum and infimum of u is bounded independent of u. Before we prove this result, we have a preliminary lemma, which only holds in two dimensions.

Lemma 7.1. *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function such that $\omega(r) := osc_{\partial B_r} u$ is increasing. Then it holds that*

$$\omega(r)^2 \leq \frac{\pi}{\log(1/r)}\int_{B_1} |\nabla u|^2.$$

Proof. Given that $u = u(x,y)$, we set $x = r\cos\theta$ and $y = r\sin\theta$. Then because $\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} = \frac{\partial u}{\partial \theta}$, we have $\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta)$ and similarly $\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial y}(r\cos\theta)$. Then $|\frac{\partial u}{\partial \theta}| \leq |r\frac{\partial u}{\partial x}|$ and $|\frac{\partial u}{\partial \theta}| \leq |r\frac{\partial u}{\partial y}|$, meaning that

$$\Big|\frac{\partial u}{\partial \theta}\Big| \leq r\Big(\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial u}{\partial y}\Big)^2\Big)^{1/2}.$$

As a result $|\frac{\partial u}{\partial \theta}| \leq |r\nabla u|$.

Now for a given $r > 0$, assume that angles θ_1 and θ_2 attain the sup and inf on ∂_{B_r}. Then

$$\omega(r) = u(\theta_1) - u(\theta_2) = \int_{\theta_1}^{\theta_2}\Big|\frac{\partial u}{\partial \theta}\Big|\partial\theta \leq r\int_{\theta_1}^{\theta_2} |\nabla u(r,\theta)|d\theta.$$

Then by Cauchy-Schwartz

$$\omega(r) \leq r\int_{\theta_1}^{\theta_2} |\nabla u(r,\theta)|d\theta \leq r\sqrt{\pi}||\nabla u||_{L^2}$$

which implies

$$\frac{\omega(r)^2}{r} \leq r\pi\int_0^{2\pi} |\nabla u(r,\theta)|^2 d\theta.$$

Next we integrate both sides of this equation with respect to r to get

$$\int_{r_0}^{1} \frac{\omega(r)^2}{r} dr \leq \int_{r_0}^{1} r\pi \int_{B_1 \setminus B_{r_0}} |\nabla u|^2.$$

Because ω is increasing, then

$$\omega(r_0)^2 \int_{r_0}^{1} \frac{1}{r} dr \leq \int_{r_0}^{1} \frac{\omega(r)^2}{r}.$$

As a result,

$$\omega(r_0)^2(-\log(r_0)) \leq r\pi \int_{B_1 \setminus B_{r_0}} |\nabla u|^2$$

implying that

$$\omega(r_0)^2 \leq \frac{\pi}{\log(1/r_0)} \int_{B_1} |\nabla u|^2$$

for all $r_0 \in (0, 1)$. This proves Lemma 7.1. □

Theorem 7.2. *(Harnack Inequality in 2D). Let $u \in H^1(B_2)$ be a nonnegative solution to $\partial_i(a_{ij}\partial_j u) = 0$, where the coefficients a_{ij} are assumed only to be uniformly elliptic, i.e., $\lambda I \leq a_{ij}(x) \leq \Lambda I$. Then there exists an absolute constant $C > 0$ such that*

$$\sup_B u \leq C \inf_B u$$

for every ball $B \subset\subset B_1$.

Proof. Suppose we have some weak solution $u \in H_1$ on B_2 in two dimensions. Then Lemma 7.1 holds for u. Since u is a weak solution on B_2, it is a weak solution on every subset B_r of B_2. Therefore, given some $2 > \rho > r > 0$, by the maximum principle (Lemma 5.1)

$$\text{osc}_{\partial B_\rho} u = \text{osc}_{B_\rho} u \geq \text{osc}_{B_r} u = \text{osc}_{\partial B_r}.$$

Thus the oscillation is increasing and we can apply Lemma 7.1.

We assume now that $u > 0$, and set $v = \log u$. Then given some $\phi \in C_c^\infty(B_2)$,

$$\int_{B_2} a_{ij}\partial_i v \partial_j \phi = \int_{B_2} a_{ij}\partial_i u \Big(\frac{\partial_j \phi}{u}\Big).$$

We can see that $(\partial_j\phi/u) = \partial_j(\phi/u) + (\phi\partial_j u)/u^2$. Then we have

$$\int_{B_2} a_{ij}\partial_i v \partial_j \phi = \int_{B_2} a_{ij}\partial_i u \partial_j(\phi/u) + \int_{B_2} a_{ij}\partial_i u \frac{\phi\partial_j u}{u^2}.$$

The first term is zero because u is a weak solution, which means

$$\int_{B_2} a_{ij}\partial_i v \partial_j \phi = \int_{B_2} a_{ij}\partial_i u \frac{\phi\partial_j u}{u^2} = \int_{B_2} (a_{ij}\partial_i v \partial_j v)\phi.$$

Set ϕ to be a cutoff function η^2 which is 1 on B_1, and 0 on $B_2 \setminus B_r$ (where $1 < r < 2$), decreasing on $B_r \setminus B_1$ so that $|\nabla\eta| < 2/r$. Then by the Cauchy-Schwartz inequality

$$\lambda \int_{B_1} |\nabla v|^2 \eta^2 \leq \int_{B_1} a_{ij}\partial_i v \partial_j(\eta^2) = 2\int_{B_1} a_{ij}\partial_i v \partial_j \eta(\eta).$$

This means that
$$\lambda\int_{B_1}|\nabla v|^2\eta^2 \le 2\int_{B_1} a_{ij}\partial_i v\partial_j\eta(\eta) \le 2\Lambda\Big(\int_{B_1}|\nabla v|^2\eta^2\Big)^{1/2}\Big(\int_{B_1}|\nabla\eta|^2\Big)^{1/2}.$$
Since η is fixed, $(\int_{B_1}|\nabla\eta|^2)^{1/2}$ is a constant C. As a result, this means that (squaring both sides)
$$\int_{B_1}|\nabla v|^2\eta^2 \le C\frac{\Lambda^2}{\lambda^2}.$$
Since $\eta = 1$ on B_1, then this means
$$\int_{B_1}|\nabla v|^2 \le \int_{B_1}|\nabla v|^2\eta^2 \le C\frac{\Lambda^2}{\lambda^2} = C'$$
where C' is a constant based on λ, Λ and the dimension.

Now we can apply Lemma 7.1 to the function $v = \log u$, since the function $\omega(r) = \operatorname{osc}_{\partial B_r} v$ is increasing. This means we have that
$$(\sup_{\partial B_r}\log u - \inf_{\partial B_r}\log u)^2 \le \frac{\pi}{\log(1/r)}\int_{B_1}|\nabla v|^2 \le \frac{C\pi}{\log(1/r)}.$$
Therefore
$$\sup_{\partial B_r}\log u - \inf_{\partial B_r}\log u \le \frac{\sqrt{C\pi}}{\sqrt{\log(1/r)}}.$$
We then take both sides as powers of e to get that
$$\frac{e^{\sup_{\partial B_r}\log u}}{e^{\inf_{\partial B_r}\log u}} \le e^{\sqrt{\frac{C\pi}{\log(1/r)}}}.$$
Then we can see that $e^{\sup_{\partial B_r}\log u} = \sup_{\partial B_r} u$, and similarly for $e^{\inf_{\partial B_r}\log u}$, meaning that
$$\frac{\sup_{\partial B_r} u}{\inf_{\partial B_r} u} \le e^{\sqrt{\frac{C\pi}{\log(1/r)}}}.$$
Since we want Harnack to hold for any $B_r \subset\subset B_1$, this means $r < 1$. Thus $\log(1/r) \ge m$ where $m > 0$. This means that $\frac{C\pi}{\log(1/r)}$ is bounded by some C', and thus that
$$\frac{\sup_{\partial B_r} u}{\inf_{\partial B_r} u} \le C',$$
which implies Harnack in 2D. □

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