LEBESGUE MEASURE

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Abstract. This paper offers an introduction to the concept of Lebesgue measure and its most important properties. It begins with a development of a more basic measure function, outer measure, that can be applied to all sets of real numbers, and then develops a smaller σ-algebra of Lebesgue measurable sets, applied to which the outer measure function has improved properties. It concludes with a brief exploration of non-measurable sets.

Contents

1. Introduction

In its calculation, the Riemann integral requires partitioning the function of interest's domain into subintervals that are associated with the values of the function over that subinterval. This is limiting in both the types of functions that are integrable and the domains that those functions can have. The Lebesgue integral is able to calculate the integral on more general functions with more general domains by assigning certain sets of the domain a quantity called Lebesgue measure. This quantity creates an integral that is both more widely applicable and has better properties than the Riemann integral. This paper serves to develop the concept of Lebesgue measure.

The Lebesgue measure function, denoted by m, which maps each set in a σ-algebra of sets called the Lebesgue measurable sets to a real number has the following properties:

(1) The Lebesgue measure of any bounded interval is the difference of its endpoints, and the Lebesgue measure of any unbounded interval is ∞.
(2) The Lebesgue measure of any Lebesgue measurable set does not change under a translation. That is, if A is Lebesgue measurable, and x is a real

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number, then the set $A+x = \{a+x \mid a \in A\}$ is measurable and the measure of $A+x$ is equal to the measure of A. This is called translation invariance.

(3) If we have a countable disjoint collection of Lebesgue measurable sets, $\{A_k\}_{k=1}^{\infty}$, then the measure of their union is equal to the sum of the individual measures of each set. That is

$$m(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} m(A_k).$$

This is called countable additivity over countable disjoint unions of sets.

In order to construct the Lebesgue measure function, we will first, in section 2, construct a more general measure function called outer-measure, denoted m^*, that maps every set of real numbers to a real number. It is not possible for this outer-measure function to possess all of the above properties while still being applicable to all sets of real numbers [1]. However, we can construct an outer-measure function that satisfies the first and second properties, as well as satisfying a looser version of the third property, namely that for any disjoint or non-disjoint countable collection of sets, denoted $\{B_k\}_{k=1}^{\infty}$, the outer-measure of their union is less than or equal to the sum of the outer-measures of the individual sets, that is

$$m(\bigcup_{k=1}^{\infty} B_k) \leq \sum_{k=1}^{\infty} m(B_k).$$

After constructing the outer-measure function, in section 3 we will develop the Lebesgue measurable sets, show that they are a σ-algebra, and create a new function called Lebesgue measure by restricting the outer-measure function to these sets. In section 4, we will show that Lebesgue measure satisfies the third property of countable additivity in addition to the first and second properties. This will conclude our development of Lebesgue measure.

In section 5 we will develop a non-measurable set to show that not all sets of real numbers are measurable.

2. Outer Measure

We begin our construction of outer measure by examining the lengths of intervals. For any nonempty interval of real numbers, I, we define its length, $l(I)$ to be the difference of its endpoints if I is bounded, and ∞ if I is unbounded.

Consider a set, A, of real numbers and all of the collections of nonempty, open, bounded intervals that cover this set, denoted $\{I_k\}_{k=1}^{\infty}$. For each such collection, $A \subseteq \bigcup_{k=1}^{\infty} I_k$. For each collection of this type, consider the sum of the lengths of each interval in the collection. We shall define outer measure by the infimum of the collection of all sums of this type. That is,

$$m^*(A) = inf\{\sum_{k=1}^{\infty} l(I_k) \mid A \subseteq \bigcup_{k=1}^{\infty} I_k\}$$

Because the lengths of intervals are always nonnegative, we can see from this definition that outer-measure is always nonnegative. Additionally, the following proposition about the monotonicity of outer measure also follows from the definition:

Proposition 2.1. *For any sets A and B, if $A \subseteq B$, then $m^*(A) \leq m^*(B)$.*

We will now examine three important properties of outer measure that mirror the three properties of Lebesgue measure outlined in the introduction. For each proposition, we will give an informal explanation of how that property follows from the definition, but will forgo a formal proof as the details are more cumbersome than they are helpful.

We begin with the first property of Lebesgue measure which is true for outer measure as well.

Proposition 2.2. *The outer measure of any bounded interval is the difference of its endpoints, and the outer measure of any unbounded interval is* ∞*.*

This follows quite directly from the definition of outer measure as any cover of an interval I by a collection of intervals other than the interval itself is going to have a larger collective length than the interval itself. This makes $l(I)$ the infimum of $\{\sum_{k=1}^{\infty} l(I_k) \mid I \subseteq \bigcup_{k=1}^{\infty} I_k\}$, so that by definition, the outer measure of an interval is its length.

We shall now examine the second property of Lebesgue measure: that it is translation invariant.

Proposition 2.3. *Outer measure is translation invariant, meaning that for any set* A *and for any number* y*,*

$$m^*(A) = m^*(A + y).$$

For any interval I and for any real number y, it is clear that

$$l(I) = l(I + y)$$

and similarly, we can see that if $\{I_k\}_{k=1}^{\infty}$ is a cover of a set A, then $\{I_k + y\}_{k=1}^{\infty}$ is a cover of the set $A + y$. Then, from the definition of outer measure, we can see why this proposition is true.

Finally, we shall establish the countable subadditivity property that outer measure possesses.

Proposition 2.4. *Outer measure is countably subadditive, meaning that for any countable collection of sets,* $\{E_k\}_{k=1}^{\infty}$*,*

$$m^*(\bigcup_{k=1}^{\infty}\{E_k\}) \leq \sum_{k=1}^{\infty} m^*(E_k)$$

This is a similar but distinct property from the countable additivity property possessed by Lebesgue measure. It will be important in our establishment of the collection of Lebesgue measurable sets.

3. Lebesgue Measurable Sets

The one important property of Lebesgue measure that outer-measure does not possess is countable or finite additivity. However, there exists a collection of sets called the Lebesgue measurable sets for which the outer-measure function is countably and finitely additive. Below is one possible definition for a Lebesgue measurable set (referred to simply as a measurable set from now on), and the one that we will be using:

Definition 3.1. A set A is said to be measurable if for any set E,

$$m^*(E) = m^*(E \cap A) + m^*(E \cap A^C)$$

where A^C denotes the complement of A.

In more understandable terms, a set A is measurable if the outer measure of any set E is equal to the sum of the outer measure of the subset of E that overlaps with A and the outer measure of the subset of E that overlaps with A^C.

Consider disjoint sets A and B and assume that A is measurable. By the definition of measurability, we see that

$$m^*(A \cup B) = m^*([A \cup B] \cap A) + m^*([A \cup B] \cap A^C) = m^*(A) + m^*(B)$$

meaning that finite additivity holds for two disjoint sets if at least one of them is measurable. We shall complete the proof of countable additivity in section 4, but for now this observation shows one of the improved properties of outer measure when restricted to measurable sets.

Because for any sets A and E, $E = [E \cap A] \cup [E \cap A^C]$, we can use Proposition 2.4 to conclude that

$$m^*(E) \leq m^*(E \cap A) + m^*(E \cap A^C).$$

Thus, in order to prove measurability, we need only prove that

$$m^*(E) \geq m^*(E \cap A) + m^*(E \cap A^C).$$

This inequality is always true if $m^*(A) = \infty$, so we will only be examining sets with finite outer measure.

We will now define some important terms for the central theorem of this section.

Definition 3.2. A σ-algebra is a collection of subsets of $\mathbb{R}$ that contains $\mathbb{R}$ and is closed with respect to the formation of complements and countable unions. Note that by De Morgan's Identities, a σ-algebra is also closed with respect to the formation of countable intersections.

Definition 3.3. A Borel set is a set that can be formed from countable unions and complements of open sets. The σ-algebra of Borel sets is the smallest σ-algebra of sets of real numbers that contains all of the open sets of real numbers.

Definition 3.4. A G_δ set is a countable intersection of opens sets.

Definition 3.5. An F_δ set is a countable union of closed sets.

Our main objective of this section is to prove the following theorem:

Theorem 3.6. *The collection of measurable sets is a σ-algebra that contains the σ-algebra of Borel sets. Each interval, each open set, each closed set, each G_δ set, and each F_δ set is measurable.*

To begin our establishment of this theorem, we must examine complements of measurable sets and countable unions of measurable sets. These properties will show that the measurable sets are a σ-algebra.

Proposition 3.7. *A set A is measurable if and only if A^C is measurable.*

Proof. If a set A is measurable, then for any set E,

$$m^*(E) = m^*(E \cap A) + m^*(E \cap A^C) = m^*(E \cap A^C) + m^*(E \cap A).$$

Therefore, A^C is measurable.

If for a set A, A^C is measurable, then for any set E,

$$m^*(E) = m^*(E \cap A^C) + m^*(E \cap A) = m^*(E \cap A) + m^*(E \cap A^C).$$

Therefore, A is measurable. □

Proposition 3.8. *The empty set and the set of real numbers are measurable.*

Proof. For any set E,

$$m^*(E) = m^*(E \cap \emptyset) + m^*(E \cap \emptyset^C).$$

Because $\emptyset^C = \mathbb{R}$, by Proposition 3.7, $\mathbb{R}$ is also measurable. □

Now that we have established that the collection of measurable sets contains $\mathbb{R}$ and is closed under complements, we must examine countable unions of measurable sets. To do this, we must first examine finite unions of measurable sets.

Proposition 3.9. *The union of a finite collection of measurable sets is measurable.*

Proof. First, we will prove that the union of two measurable sets is measurable.

Let A_1 and A_2 be measurable sets, and let E be any set. By the measurability of A_1,

$$m^*(E) = m^*(E \cap A_1) + m^*(E \cap A_1^C).$$

By the measurability of A_2,

$$m^*(E \cap A_1^C) = m^*([E \cap A_1^C] \cap A_2) + m^*([E \cap A_1^C] \cap A_2^C).$$

Thus

$$m^*(E) = m^*(E \cap A_1) + m^*([E \cap A_1^C] \cap A_2) + m^*([E \cap A_1^C] \cap A_2^C).$$

We can simplify the following sets like so:

$$[E \cap A_1] \cup [[E \cap A_1^C] \cap A_2] = E \cap [A_1 \cup A_2].$$

$$[E \cap A_1^C] \cap A_2^C = E \cap [A_1 \cup A_2]^C$$

Therefore,

$$\begin{aligned} m^*(E) &= m^*(E \cap A_1) + m^*([E \cap A_1^C] \cap A_2) & +m^*([E \cap A_1^C] \cap A_2^C) \\ &\geq m^*([E \cap A_1] \cup [[E \cap A_1^C] \cap A_2]) & +m^*(E \cap [A_1 \cup A_2]^C) \\ &= m^*(E \cap [A_1 \cup A_2]) & +m^*(E \cap [A_1 \cup A_2]^C) \end{aligned}$$

so that $[A_1 \cup A_2]$ is measurable.

Now consider a finite collection of measurable sets $\{A_k\}_{k=1}^n$. We proceed by means of induction. For $n = 1$ measurable sets, A_1 is measurable so that $\{A_k\}_{k=1}^n$ is measurable.

Now assume that $\bigcup_{k=1}^{n-1} A_k$ is measurable for some natural number n. Then, as we showed above, $[\bigcup_{k=1}^{n-1} A_k] \cup A_n$ is measurable so that $\bigcup_{k=1}^{n} A_k$ is measurable.

By the principle of mathematical induction, a finite union of measurable sets is measurable. □

Proposition 3.10. *For any set E and any finite, disjoint collection of measurable sets $\{A_k\}_{k=1}^n$,*

$$m^*(E \cap [\bigcup_{k=1}^{n} A_k]) = \sum_{k=1}^{n} m^*(E \cap A_k)$$

In particular

$$m(\bigcup_{k=1}^{n} A_k) = \sum_{k=1}^{n} m(A_k).$$

Proof. Let $\{A_k\}_{k=1}^{\infty}$ be a finite, disjoint collection of measurable sets and let E be any set. We proceed by means of induction. Let n be a natural number. For the case of $n = 1$, we see that

$$m^*(E \cap [\bigcup_{k=1}^{n} A_k]) = m^*(E \cap A_1) = \sum_{k=1}^{n} m^*(E \cap A_k).$$

Assume that for a natural number n,

$$m^*(E \cap [\bigcup_{k=1}^{n-1} A_k]) = \sum_{k=1}^{n-1} m^*(E \cap A_k).$$

Then by the measurability of A_n,

$$m^*(E \cap \bigcup_{k=1}^{n} A_k) = m^*([E \cap \bigcup_{k=1}^{n} A_k] \cap A_n) + m^*([E \cap \bigcup_{k=1}^{n} A_k] \cap A_n^C).$$

Because the sets $\{A_k\}_{k=1}^n$ are disjoint,

$$[E \cap \bigcup_{k=1}^{n} A_k] \cap A_n = E \cap A_n$$

$$[E \cap \bigcup_{k=1}^{n} A_k] \cap A_n^C = E \cap \bigcup_{k=1}^{n-1} A_k$$

Thus, by the induction hypothesis,

$$\begin{aligned} m^*(E \cap \bigcup_{k=1}^{n} A_k) &= m^*([E \cap \bigcup_{k=1}^{n} A_k] \cap A_n) &&+ m^*([E \cap \bigcup_{k=1}^{n} A_k] \cap A_n^C). \\ &= m^*(E \cap A_n) &&+ m^*(E \cap \bigcup_{k=1}^{n-1} A_k) \\ &= m^*(E \cap A_n) &&+ \sum_{k=1}^{n-1} m^*(E \cap A_k) \\ &= \sum_{k=1}^{n} m^*(E \cap A_n) \end{aligned}$$

Plugging in $\emptyset$ for E gives us

$$m^*(\bigcup_{k=1}^{n} A_k) = \sum_{k=1}^{n} m^*(A_k).$$

Because A_k is measurable for each natural number k, by Proposition 3.9 $\bigcup_{k=1}^n A_k$ is measurable, so that we can write this as

$$m(\bigcup_{k=1}^n A_k) = \sum_{k=1}^n m(A_k)$$

thus establishing the finite additivity of Lebesgue measure. □

Lemma 3.11. *For any union of a countable collection of measurable sets, there exists a union of a countable collection of disjoint measurable sets that contains all the same elements.*

Proof. Let $\{A_k\}_{k=1}^\infty$ be a countable collection of measurable sets. We define $A_1' = A_1$. For each natural number $k \geq 2$, we define

$$A_k' = A_k \sim \bigcup_{i=1}^{k-1} A_i$$

Because the measurable sets are an algebra, each A_k' is measurable. By construction, the collection $\{A_k'\}_{k=1}^\infty$ is disjoint and $\{A_k'\}_{k=1}^\infty = \{A_k\}_{k=1}^\infty$. □

Now, using Propositions 3.9, 3.10, and Lemma 3.11, we will show that the collection of measurable sets are closed under countable unions, completing the establishment of this collection as a σ-algebra.

Proposition 3.12. *The union of a countable collection of measurable sets is measurable.*

Proof. Let A be the union of a countable collection of measurable sets. By Lemma 3.11, there exists a countable, disjoint collection of measurable sets $\{A_k\}_{k=1}^\infty$ such that $A = \{A_k\}_{k=1}^\infty$. Let E be any set, and let n be a natural number. Then by Proposition 3.9 $\bigcup_{k=1}^n A_k$ is measurable and $\bigcup_{k=1}^n A_k \subseteq A$ so that by Proposition 3.10,

$$\begin{aligned} m^*(E) &= m^*(E \cap \bigcup_{k=1}^n A_k) + m^*(E \cap [\bigcup_{k=1}^n A_k]^C) \\ &\geq m^*(E \cap \bigcup_{k=1}^n A_k) + m^*(E \cap A^C) \\ &= \sum_{k=1}^n m^*(E \cap A_k) + m^*(E \cap A^C). \end{aligned}$$

This inequality does not rely on the value of n. Thus, by the countable subadditivity of outer measure,

$$\begin{aligned} m^*(E) &\geq \sum_{k=1}^\infty m^*(E \cap A_k) + m^*(E \cap A^C) \\ &\geq m^*(\bigcup_{k=1}^\infty (E \cap A_k)) + m^*(E \cap A^C) \\ &= m^*(E \cap \bigcup_{k=1}^\infty A_k) + m^*(E \cap A^C) \\ &= m^*(E \cap A) + m^*(E \cap A^C) \end{aligned}$$

Therefore A is measurable. □

With Propositions 3.7, 3.8, and 3.12, we have established that the collection of measurable sets is a σ-algebra. The next part of Theorem 3.6 states that this σ-algebra contains the σ-algebra of Borel sets. We begin our treatment of this part of the theorem with the following lemma which will help us show that every interval is measurable:

Lemma 3.13. *Let E_1 be any set and let E_2 be a set with outer measure zero. Then*

$$m^*(E_1 \sim E_2) = m^*(E_1).$$

Proof. Because $E_1 \sim E_2 \subseteq E_1$,

$$m^*(E_1 \sim E_2) \leq m^*(E_2).$$

Additionally, by the subadditivity of outer-measure,

$$m^*(E_1) = m^*([E_1 \sim E_2] \cup E_2) \leq m^*(E_1 \sim E_2) + m^*(E_2) = m^*(E_1 \sim E_2)$$

Therefore

$$m^*(E_1 \sim E_2) = m^*(E_1)$$

□

Proposition 3.14. *Every interval is measurable.*

Proof. We will show that intervals of the form (a, ∞) are measurable because all other types intervals can be constructed from this type using finite unions, finite intersections, and complements which all preserve measurability since the set of measurable sets is an algebra.

Let E be any set. If $a \in E$, replace E by $E \sim \{a\}$ which, by Lemma 3.12, does not change the outer-measure since $m^*(\{a\}) = 0$. Otherwise, assume $a \notin E$.

We want to show that

$$m^*(E \cap (a, \infty)) + m^*(E \cap [(a, \infty)]^C) \leq m^*(E)$$

which, because $a \notin E$, is equivalent to showing

$$m^*(E \cap (a, \infty)) + m^*(E \cap (-\infty, a)) \leq m^*(E).$$

By the definition of $m^*(E)$, it suffices to show that for every countable collection of open, bounded intervals $\{I_k\}_{k=1}^{\infty}$ that covers E,

$$m^*(E \cap (a, \infty)) + m^*(E \cap (-\infty, a)) \leq \sum_{k=1}^{\infty} l(I_k). \tag{3.15}$$

For each interval I_k in $\{I_k\}_{k=1}^{\infty}$, define

$$I_k' = I_k \sim (a, \infty) \text{ and } I_k'' = I_k \sim (-\infty, a).$$

Note that

$$l(I_k) = l(I_k') + l(I_k'').$$

$\{I_k'\}_{k=1}^{\infty}$ is a finite collection of open, bounded intervals that covers $E \sim (a, \infty)$ and $\{I_k''\}_{k=1}^{\infty}$ is a finite collection of open, bounded intervals that covers $E \sim (-\infty, a)$ so that by the definition of outer measure,

$$m^*(E \cap (a, \infty)) \leq \sum_{k=1}^{\infty} l(I_k') \text{ and } m^*(E \cap (-\infty, a)) \leq \sum_{k=1}^{\infty} l(I_k'').$$

Thus,

$$m^*(E \cap (a, \infty)) + m^*(E \cap (-\infty, a)) \leq \sum_{k=1}^{\infty} l(I'_k) + \sum_{k=1}^{\infty} l(I''_k) = \sum_{k=1}^{\infty} l(I_k).$$

Therefore, (3.15) holds. □

We can now rigorously support Theorem 3.6. Because every open set is the disjoint union of a countable collection of open intervals [1], by Proposition 3.12 and Proposition 3.14, every open set is measurable.

The Borel σ-algebra is the intersection of every σ-algebra that contains all of the open sets, which means that it is contained in every σ-algebra that contains all of the open sets. Thus, the collection of measurable sets contains the Borel σ-algebra.

Because the measurable sets are closed under complements, and every closed set is the complement of an open set, every closed sets is measurable. From this fact and the fact that the measurable sets are a σ-algebra, we see that every G_δ and F_δ set is measurable.

We have now rigorously defined the σ-algebra of measurable sets and its important properties, which sets us up for the development of the Lebesgue measure function in the next section.

4. Lebesgue Measure & Important Properties

We are now ready to give a formal definition of Lebesgue measure.

Definition 4.1. The Lebesgue measure function is the restriction of the outer-measure function to the collection of measurable sets. We denote Lebesgue measure by m, and for any measurable set A, we define

$$m(A) = m^*(A).$$

With this definition, we are finally ready to establish the three critical properties of Lebesgue measure laid out in the introduction. We begin with the first two properties which are shared with outer measure.

Proposition 4.2. *The measure of any interval is its length.*

This follows from Proposition 3.14 which establishes that every interval is measurable, and Proposition 2.2 which establishes that the outer measure of any interval is its length since the measure of any measurable set is defined as its outer measure.

In order to prove the translation invariance of Lebesgue measure, we must first prove that the translate of a measurable set is, in fact, measurable.

Proposition 4.3. *The translate of a measurable set is measurable.*

Proof. Let A be a measurable set, let E be any set, and let y be a real number. Then by the translation invariance of outer-measure and the measurability of A,

$$\begin{aligned} m^*(E) = m^*(E - y) &= m^*([E - y] \cap A) + m^*([E - y] \cap A^C) \\ &= m^*(E \cap [A + y]) + m^*(E \cap [A + y]^C) \end{aligned}$$

□

With this property established, and Proposition 2.3 which states that outer measure is translation invariant, we have formalized the translation invariance of Lebesgue measure:

Proposition 4.4. *Lebesgue measure is translation invariant.*

We will now establish the last property from the introduction, and the only one that is unique to Lebesgue measure: countable additivity.

Proposition 4.5. *Lebesgue measure is countably additive over countable disjoint unions of sets, meaning that if $\{A_k\}_{k=1}^{\infty}$ is a countable disjoint collection of measurable sets, then $\bigcup_{k=1}^{\infty} A_k$ is also measurable, and*

$$m(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} m(A_k).$$

Proof. Let $\{A_k\}_{k=1}^{\infty}$ be a countable disjoint collection of measurable sets. By Proposition 3.12, $\bigcup_{k=1}^{\infty} A_k$ is measurable.

By Proposition 2.4, we know that outer measure is countably subadditive, so that

$$m(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} m(A_k) \tag{4.6}$$

Thus, to complete the proof, we need only prove that

$$m^*(\bigcup_{k=1}^{\infty} A_k) \geq \sum_{k=1}^{\infty} m(A_k).$$

For each natural number n, $\bigcup_{k=1}^{n} A_k \subseteq \bigcup_{k=1}^{\infty} A_k$, so that by the monotonicity of outer measure,

$$m(\bigcup_{k=1}^{n} A_k) \leq m(\bigcup_{k=1}^{\infty} A_k).$$

By Proposition 3.10, Lebesgue measure is finitely additive so that

$$\sum_{k=1}^{n} m(A_k) = m(\bigcup_{k=1}^{n} A_k) \leq m(\bigcup_{k=1}^{\infty} A_k).$$

Because this inequality holds for any natural number n, we have

$$\sum_{k=1}^{\infty} m(A_k) \leq m(\bigcup_{k=1}^{\infty} A_k). \tag{4.7}$$

Therefore, by (4.6) and (4.7),

$$\sum_{k=1}^{\infty} m(A_k) = m(\bigcup_{k=1}^{\infty} A_k).$$

□

We have now established the following theorem which lays out the most important properties of Lebesgue measure.

Theorem 4.8. *The set function Lebesgue measure, which is defined on the σ-algebra of Lebesgue measurable sets, assigns any interval to its length, is translation invariant, and is countably additive over countable collections of disjoint sets.*

5. Nonmeasurable Sets

Now that we have established the Lebesgue measure function and its domain, the question arises of which sets are not Lebesgue measurable. In this section, we will develop an example of a non-measurable set, and show their pervasiveness among the sets of real numbers.

Lemma 5.1. *Let A be a bounded, measurable set of real numbers. If there exists a bounded, countably infinite set of real numbers Λ for which the collection of translates of A, $(A+\lambda)_{\lambda\in\Lambda}$, is disjoint, then $m(A) = 0$.*

Proof. Because A is measurable, by Proposition 4.3 the set $A + \lambda$ is measurable. Then $\bigcup_{\lambda\in\Lambda}(A+\lambda)$ is a countable union of disjoint measurable sets so that countable additivity applies and,

$$m(\bigcup_{\lambda\in\Lambda}[A+\lambda]) = \sum_{\lambda\in\Lambda} m(A+\lambda).$$

Because A and Λ are both bounded sets, $m(\bigcup_{\lambda\in\Lambda}[A+\lambda])$ is finite so that $\sum_{\lambda\in\Lambda} m(A+\lambda)$ is finite. However, $m(A+\lambda)$ has the same value for all $\lambda \in \Lambda$, so this is only possible if $m(A) = 0$. □

Theorem 5.2. *Any set A with positive outer measure contains a subset that is not measurable.*

Proof. We must first develop some background on the concept of a rational equivalence. Two numbers are rationally equivalent if their difference is a rational number. This relationship is a type of relation called an equivalence relation, which means that every number is rationally equivalent to itself, if a is rationally equivalent to b then b is rationally equivalent to a, and rational equivalence is transitive. Because this is an equivalence relation, there exists a disjoint decomposition of the set A into classes that are rationally equivalent to one another. A choice set is a set that contains exactly one element from each of the equivalence classes. This choice set, denoted C_A, posesses the following characteristics:

(1) The difference between any two points in the choice set C_A is not rational.
(2) For each $x \in A$, there exists $c \in C_A$ and $r \in \mathbb{Q}$ such that $x = c + r$.

We may now begin our proof of the theorem. We assume that A is bounded. By means of contradiction, assume that C_A is measurable. Let Λ_0 be a bounded, countably infinite set of rational numbers. Because of the first characteristic of choice sets, the set $\{C_A + \lambda\}_{\lambda\in\Lambda_0}$ is disjoint. Thus, by Lemma 5.1 and Proposition 4.5,

$$m(\bigcup_{\lambda\in\Lambda_0}[C_A+\lambda]) = \sum_{\lambda\in\Lambda_0} m(C+A+\lambda) = 0.$$

We now choose a specific Λ_0 to obtain a contradiction. Since A is bounded, there exists a real number b such that $A \subseteq [-b, b]$. Define

$$\Lambda_0 = \mathbb{Q} \cap [-2b, 2b]$$

Because of the second characteristic of C_A, for each $x \in A$, there exists $c \in C_A$ and $r \in \mathbb{Q}$ such that $x = c + r$. Because $x, c \in [-b, b]$, $r \in [-2b, 2b]$ so that $r \in \Lambda_0$. Thus,

$$A \subseteq \bigcup_{\lambda\in\Lambda}(C_A+\lambda).$$

By the countable subadditivity of measure,

$$0 < m(A) \leq m(\bigcup_{\lambda \in \Lambda_0} [C_A + \lambda]) = \sum_{\lambda \in \Lambda_0} m(C_A + \lambda) = 0$$

This is a contradiction. Therefore Λ_0 is not measurable. □

We have now successfully shown that there are many non-measurable sets, showing the necessity of the restrictions on the Lebesgue measure function developed in this paper.

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