INTERTWINING CLASSICAL AND QUANTUM DYNAMICS ON HYPERBOLIC SURFACES: A MICROLOCAL PERSPECTIVE

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Abstract. We prove the asymptotic equivalence of two types of phase space distributions associated to eigenfunctions of the Laplacian on a compact hyperbolic surface in the semi-classical limit. We then construct an explicit unitary intertwining operator between the Schrödinger group and geodesic flow on certain Hilbert spaces of symbols on the cotangent bundle of a compact hyperbolic surface. This exposition is primarily based on recent work of Anantharaman and Zelditch [1], [2].

Contents

1. Motivation

1.1. The Semi-Classical Limit. In Hamiltonian mechanics, a formulation of classical mechanics, the motion over time of an electron orbiting a hydrogen atom is modelled by Hamilton's equations

$$\begin{cases} \partial_t x(t) &= \partial_\xi p(x(t), \xi(t)) \\ \partial_t \xi(t) &= -\partial_x p(x(t), \xi(t)) \end{cases} \tag{1.1}$$

with the Hamiltonian

$$p(x, \xi) = \frac{1}{2}|\xi|^2 - V(x) : T^*\mathbf{R}^n \to \mathbf{R}.$$

The orbits of electrons are situated on specific level sets of the Hamiltonian.

In quantum mechanics, the electron orbiting a hydrogen atom is modeled by wave functions ψ_j that have energy levels E_j via Schrödinger's equation

$$\hat{H}\psi_j := \left(-\frac{h^2}{2}\Delta + V\right)\psi_j = E_j\psi_j,$$

where V is the potential and h is the Planck constant, that is, the ratio of the energy of a photon to its frequency. The resemblance between the Schrödinger operator $\hat{H}$ and Hamiltonian p is readily apparent. Bohr's correspondence principle asserts that the behavior of a quantum mechanical system approaches the description of the behavior via classical mechanics in the limit of large quantum numbers (large orbits and high energies), namely, $\lambda \to \infty$. With the parameters $V = 0$, $E = 1$, $h = \lambda^{-1}$, the Schrödinger equation specializes to the following eigenvalue equation for the Laplacian, known as the Helmholtz equation on the domain Ω

$$\begin{cases} (\Delta + \lambda^2)\varphi_\lambda = 0 \\ \varphi_\lambda|_{\partial\Omega} = 0 \end{cases}.$$

Consequently, the semi-classical limit $h \to 0$ can be attained through taking the high energy limit $\lambda \to \infty$. In this limit, there ought to be a relationship between the asymptotics of eigenfunctions and eigenvalues of the Laplacian and the dynamics of the Hamiltonian flow.

1.2. **Global Harmonic Analysis of the Laplacian on Riemannian Manifolds.** The exposition in this part is based on [13]. Let (M, g) be a Riemannian manifold and let Δ_g denote the Laplacian. We are concerned with the eigenvalue problem

$$(\Delta_g + \lambda^2)\varphi_\lambda = 0.$$

When M is compact, it follows from the Spectral Theorem for compact self-adjoint operators applied to Δ^{-1} that there exists an orthonormal basis $\{\varphi_j\}_{j\geq 0}$ of $L^2(M)$ of eigenfunctions,

$$\Delta_g\varphi_j = -\lambda_j^2\varphi_j, \langle\varphi_j, \varphi_k dV_g = \delta_{jk}.$$

Furthermore, when M is compact, the Laplacian has a discrete spectrum with finite multiplicities. Global techniques to study the eigenfunctions rely on the wave equation. To see this, set $\Box = \frac{\partial^2}{\partial t^2} - \Delta_g$. The Cauchy problem for the wave equation on $\mathbf{R} \times M$ is the initial value problem (with Cauchy data f, g) is

$$\begin{cases} \Box u(t, x) = 0 \\ u(0, x) = f, \frac{\partial}{\partial t}u(0, x) = g(x) \end{cases}.$$

The solution operator is the wave group

$$\mathcal{U}(t) = \begin{pmatrix} \cos t\sqrt{\Delta} & \frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}} \\ \sqrt{\Delta}\sin t\sqrt{\Delta} & \cos t\sqrt{\Delta} \end{pmatrix}.$$

To obtain this, we note that this initial value problem can be decomposed into an even part

$$\begin{cases} (\frac{\partial}{\partial t}^2 - \Delta)u = 0 \\ u_{|t=0} = f, \frac{\partial}{\partial t}u|_{t=0} = 0 \end{cases}$$

and an odd part

$$\begin{cases} (\frac{\partial}{\partial t}^2 - \Delta)u = 0 \\ u|_{t=0} = 0, \frac{\partial}{\partial t}u|_{t=0} = g \end{cases}.$$

A solution to the even part is $C(t) := \cos t\sqrt{\Delta}$ and a solution to the odd part is $S(t) := \frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}}$. These functions arise in the solution to the following Cauchy problem

$$\begin{cases} (\frac{1}{i}\frac{\partial}{\partial t} - \sqrt{-\Delta}) = 0 \\ u(0,x) = u_0 \end{cases}.$$

The solution operator to this Cauchy problem is known as the forward half-wave group. The solution is given by $u(t,x) = U(t)u_0(x)$ with $U(t) : L^2(M) \to L^2(M)$ being the half-wave propagator given by

$$U(t) = e^{it\sqrt{-\Delta}} = C(t) + i\sqrt{-\Delta}S(t),$$

which is the unitary group generated by the self-adjoint elliptic operator $\sqrt{-\Delta}$. The half-wave propagator has the eigenfunction expansion

$$U(t,x,y) = \sum_{j=0}^{\infty} e^{it\lambda_j}\varphi_j(x)\varphi_j(y),$$

which is known as the wave kernel. The wave kernel is a simple example of a Fourier integral operator. It is difficult to analyze the eigenfunctions of the half-wave propagator, so the typical approach to study them is to investigate the wave kernel and its singularities. The simplest technique for constructing a Fourier integral representation of the wave kernel is the Hadamard parametrix construction.

1.3. **Quantum Mechanics in Phase Space.** Phase space consists of all possible states of a quantum mechanical system. It keeps track of both position and momentum rather than just one of these. The phase space formulation of quantum mechanics can be viewed in analogy to Hamiltonian mechanics because the setting for Hamiltonian mechanics is phase space. The mathematical description of phase space is the cotangent bundle T^*M, on which the variable x denotes position and the variable ξ denotes momentum. A classical observable is a function on phase space. A quantum observable is a bounded operator on $L^2(M)$. Quantization is a procedure by which we associate a quantum observable to a classical observable. The quantum observables which can be studied in the semi-classical limit are the semi-classical pseudo-differential operators $\mathrm{Op}_h(a) = a(x, hD)$; we refer to such operators as quantizations of classical observables. It is apparent from the following definition that quantum observables are pseudo-differential operators. The semi-classical Weyl quantization on $\mathbf{R}^n$ is denoted by $\mathrm{Op}_h(a) = a^w(x, hD)$ and defined by

$$\mathrm{Op}_h(a)f(x) = (2\pi h)^{-1}\int_{\mathbf{R}^n} e^{\frac{i}{h}(x-y)\xi} a\left(\frac{x+y}{2}, \xi\right) f(y)\, dy\, d\xi,$$

where $0 < h << 1$ is the semi-classical parameter. By piecing together coordinate charts, we can obtain a quantization on an arbitrary manifold M. To the classical observable $a = a(x,\xi) \in C_c^\infty(T^*M)$, we associate the quantum observable

$$\mathrm{Op}_h(a) := a^w\left(x, \frac{h}{i}\partial_x\right) : L^2(M) \to L^2(M).$$

Classical evolution is evolution with respect to the geodesic or, more generally, Hamiltonian flow because the geodesic flow on the punctured cotangent bundle $T^*M \setminus 0$ is the Hamiltonian flow of the metric norm. A quantization procedure provides a mapping from classical Hamiltonian dynamics to quantum dynamics in phase space. To see this, let $p : \mathbf{R}^{2n} \to \mathbf{R}$, $p = p(x, \xi)$ be an arbitrary Hamiltonian. The time-evolution of the position and momentum of the system is given by Hamilton's system of equations 1.1. The solution of this system is

$$\varphi_t = \exp(tH_p),$$

where

$$H_p q := \{p, q\} = \langle \partial_\xi, \partial_x q \rangle - \langle \partial_x p, \partial_\xi q \rangle$$

is the Poisson bracket. We describe how quantization provides the analogy between the time-evolution of a classical observable and the time-evolution of a quantum observable, following [16]. Let a be a symbol and set

$$a_t(x, \xi) := a(\varphi_t(x, \xi)).$$

The classical time-evolution of the symbol is given by

$$\partial_t a_t = \{p, a_t\}.$$

To define the quantum time-evolution, set $P = p^w(x, hD)$ and $A = a^w(x, hD)$, and define

$$A(t) := F^{-1}(t) A F(t),$$

where $F(t) := e^{-\frac{itP}{h}}$. Here, $A(t)$ represents the time evolution of the quantum observable A. The time evolution equation of the quantum observable A is

$$\partial_t A(t) = \frac{i}{h}[P, A(t)].$$

This is obviously an analogue of the classical evolution equation, where the Poisson bracket has been replaced with a commutator. Generally, assertions about Hamiltonian dynamics involving a Poisson bracket will involve a commutator in quantum dynamics.

Another statement that links classical and quantum mechanics is the following theorem. In particular, it relates the time evolution of an observable to the time evolution of its principal symbol.

Theorem 1.2 (Egorov [13])**.** *If $A \in \Psi^0(M)$ (i.e. A is a pseudo-differential operator of order zero) and $U^t = e^{it\sqrt{-\Delta}}$, then $\alpha_t(A) := U^t A (U^t)^* \in \Psi^0(M)$ and the principal symbol of $\alpha_t(A)$ is $a \circ G^t$, where G^t is the Hamiltonian flow. Quantitatively,*

$$U_\hbar^t Op_h(a) U_\hbar^{-t} - Op_h(a \circ G^t) \in \Psi_h^{-1}(M),$$

i.e. the difference is a pseudo-differential operator of order -1. This means that the difference is of order $\mathcal{O}(\hbar)$.

1.4. **Quantum Ergodicity.** Let (M, g) be a closed manifold and let $\{\varphi_j\}$ be an orthonormal basis of eigenfunctions of the Laplacian arranged in increasing order of eigenvalues. The matrix entries of a quantum observable A are $\rho_{jk}(A) := \langle A\varphi_j, \varphi_k \rangle$. The diagonal matrix element $\rho_{jj}(A)$ is interpreted as the expected value of the observable A in the energy state φ_j (of energy λ_j^2). The off-diagonal matrix elements $\rho_{jk}(A)$ with $j \neq k$ are transition amplitudes. Once we fix a quantization map

$a \mapsto \mathrm{Op}_h(a)$, the matrix elements can be represented by *Wigner distributions.* For the diagonal elements, we define $W_k \in \mathcal{D}'(T^*M)$ by

$$\int_{T^*M} a \, dW_k := \langle \mathrm{Op}_h(a)\varphi_k, \varphi_k \rangle.$$

The central question of quantum ergodicity is that of determining the set of weak* limit points of the sequence $\{dW_k\}$. For the off-diagonal elements of a quantum observable, the appropriate question is that of determining the set of weak* limit points of the sequence $\{\rho_{jk}\}$ on the classical phase space T^*M.

We say that a subsequence $\{\varphi_{j_k}\}$ equidistributes in physical space if for all $a \in C^\infty(M)$,

$$\int_M a(x)|\varphi_{j_k}(x)|^2 \, d\mathrm{vol}_g(x) \to \frac{1}{\mathrm{vol}_g(M)} \int_M a(x) \, d\mathrm{vol}_g(x)$$

as $k \to \infty$. We say that $\{\varphi_{j_k}\}$ equidistributes in phase space if for all $a \in C_c^\infty(T^*M)$ we have

$$\langle \mathrm{Op}_{h_{j_k}}(a)\varphi_{j_k}, \varphi_{j_k} \rangle \to \int_{S^*M} a(x,\xi) \, d\mu_L(x,\xi)$$

as $k \to \infty$. The Liouville measure μ_L on the cosphere bundle $S^*M = \{(x,\xi) \in T^*M : |\xi|_g = 1\}$ is $d\mu_L(x,\xi) := c \, d\mathrm{vol}_g(x) \, dS(\xi)$ where the densities $d\mathrm{vol}_g$ on M and dS on the fibers of S^*M are induced by the metric g and c is chosen to be a normalizing constant to ensure that μ_L is a probability measure. Note that equidistribution in phase space is a stronger property than equidistribution in physical space. For example, on $M = \mathbf{R}/2\pi\mathbf{Z}$, the sequence of eigenfunctions $\varphi_j = e^{ijx}$ equidistributes in physical space but not in phase space. The following results on the equidistribution of eigenfunctions are foundational in quantum chaos.

Theorem 1.3 (Shnirelman [4])**.** *Assume that the geodesic flow on M is ergodic with respect with respect to the Liouville measure. Then, there exists a density one subsequence $\{\varphi_{j_k}\}$ of eigenfunctions which equidistributes in physical space.*

Density one means that

$$\frac{\#\{k : \lambda_{j_k} \le R\}}{\#\{j : \lambda_j \le R\}} \to 1$$

as $R \to 1$, where eigenvalues are counted with multiplicity. Physically, Shnirelman's Theorem means that the probability of finding a particle in a set approaches the volume of that set in the high energy limit. The semi-classical version of this result, known as the Quantum Ergodicity Theorem, is below.

Theorem 1.4 ([4])**.** *Assume that the geodesic flow on M is ergodic with respect to the Liouville measure. Then there exists a density one subsequence $\{\varphi_{j_k}\}$ that equidistributes in phase space.*

Both of these theorems have a broad scope because many manifolds have ergodic geodesic flows, including manifolds of negative sectional curvature. A question that naturally arises is whether all eigenfunctions rather than a density one subsequence of eigenfunctions equidistribute in phase space. The property that all eigenfunctions equidistribute is known as quantum unique ergodicity. In other words, quantum unique ergodicity is the problem of determining which geodesic flow invariant probability measures arise as weak* limits of Wigner distributions. Rudnick and Sarnak conjecture in [12] that quantum unique ergodicity holds for any orthonormal basis of eigenfunctions if the geodesic flow is Anosov. Negatively curved manifolds are a setting where the geodesic flow is Anosov. The conjecture of quantum unique ergodicity, in its originally stated form, remains open. Thus, we investigate distributions associated to eigenfunctions of the Laplacian on hyperbolic surfaces, which are compact surfaces of constant curvature -1.

2. Outline

2.1. **Setup and Identifications.** Here, we specify the notation and conventions that will be used in the remainder of this paper. We denote the duality pairing on the space Y between a distribution $E \in \mathcal{D}'(Y)$ and a test function $f \in \mathcal{D}(Y) = C_0^\infty(Y)$ by $\langle f, E\rangle_Y = \int_Y f(y)E(dy)$. For the hyperbolic plane, we utilize two models: the upper half-plane model $\mathbf{H}$ and the Poincaré disk model $\mathbf{D}$. Let $S\mathbf{D}$ denote the unit tangent bundle for $\mathbf{D}$, $S\mathbf{H}$ denote the unit tangent bundle for $\mathbf{H}$, and set $G = PSU(1,1)$. Since G acts freely and transitively on $S\mathbf{D}$, we can identify G and $S\mathbf{D}$. Likewise, since $PSL(2,\mathbf{R})$ acts freely and transitively by Möbius transformations on $S\mathbf{H}$, we can identify $PSL(2,\mathbf{R})$ and $S\mathbf{H}$. Since all models of the hyperbolic plane are isometric, we identify $S\mathbf{D}$, $S\mathbf{H}$, $PSU(1,1)$, and $PSL(2,\mathbf{R})$. Let K be the maximal compact subgroup $PSO(2,\mathbf{R})$ and identify the corresponding symmetric space G/K with $\mathbf{D}$. Let $\Gamma \subset G$ be a cocompact discrete subgroup. Then, $\mathbf{X}_\Gamma = \Gamma\backslash\mathbf{D}$ is a hyperbolic surface.

On $S\mathbf{H}$, we define three flows that preserve the Liouville measure. The geodesic flow $g^t : S\mathbf{H} \to S\mathbf{H}$ is defined by moving distance t along the oriented geodesic tangent to a given vector v. The horocycle flow $h^s : S\mathbf{H} \to S\mathbf{H}$ is defined by moving distance s along the horocycle perpendicular to v with v pointing inward. The elliptic flow $e^r : S\mathbf{H} \to S\mathbf{H}$ is defined by rotating v through angle r in its tangent space. The geodesic, horocyclic, and elliptic flows on $S\mathbf{H}$ correspond to the right action of the one-parameter subgroups

$$A = \left\{a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbf{R}\right\}$$

$$N = \left\{n_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbf{R}\right\}$$

$$K = \left\{k_t = \begin{pmatrix} \cos(t/2) & \sin(t/2) \\ -\sin(t/2) & \cos(t/2) \end{pmatrix} : t \in \mathbf{R}\right\},$$

respectively. In fact, every connected subgroup of G is conjugate to $\{1\}$, K, A, N, or AN.

We utilize two parametrizations of $S\mathbf{D}$. The first is $G \sim G/K \times K \sim \mathbf{D} \times B$, where B is the boundary at infinity of $\mathbf{D}$ identified with S^1 in the Poincaré disk model. We obtain this with the Gram-Schmidt process as G/K is upper triangular and K is orthogonal. Geometrically, $(z, b) \in \mathbf{D} \times B$ is identified with the unit tangent vector (z, v), where $v \in S_z\mathbf{D}$ is the vector tangent to the unique geodesic through z ending at b. Second, $B \times B \setminus \Delta$ can be naturally identified with the set of oriented geodesics on $\mathbf{D}$ since each oriented geodesic $\gamma_{b',b}$ is determined by its unique forward limit point b in B and unique backward limit point $b' \neq b$ in B. Since the space of geodesics on $\mathbf{D}$ is defined as the quotient of $S\mathbf{D}$ by the action of the geodesic flow, this gives rise to the identification $S\mathbf{D} \equiv (B \times B \setminus \Delta) \times \mathbf{R}$. We specify the time parameter by identifying $(b', b, \tau) \in (B \times B \setminus \Delta) \times \mathbf{R}$ with

$(z, b) \in \mathbf{D} \times B$ where z is on the geodesic $\gamma_{b',b}$ situated τ units from the point $z_{b,b'} \in \gamma_{b,b'}$ closest to the origin $o := eK \in \mathbf{D}$.

Without modifications, the geodesic flow acts on functions on $T\mathbf{X}_\Gamma$ and the Schrodinger flow acts on $\mathbf{X}_\Gamma$. The Schwarz kernel is a distribution on $\mathbf{X}_\Gamma \times \mathbf{X}_\Gamma$. By taking the local Fourier transform of the kernel with respect to the second component, we obtain a distribution on the cotangent bundle $T^*\mathbf{X}_\Gamma$ called the symbol of the operator. We let the Schrödinger group act on the space of operators by conjugation. These operators have a Schwarz kernel and taking the Fourier transform in the second variable gives the action on $T^*\mathbf{X}_\Gamma$. We identify $T\mathbf{X}_\Gamma$ and $T^*\mathbf{X}_\Gamma$ with the metric and this results in the Schrödinger flow acting on the same space as the geodesic flow.

The geodesic flow g^t on $S\mathbf{D}$ is given by $g^t(z, v) = (\gamma_v(t), \gamma_v'(t))$ where $\gamma_v(t)$ is the unit speed geodesic with initial value (z, v). Thus, the natural setting on which to define geodesic flow-invariant distributions (namely, Patterson-Sullivan distributions) is $S\mathbf{X}_\Gamma$. The diagonal quantum-invariant distributions (Wigner distributions) are defined on $S\mathbf{X}_\Gamma$.

The off-diagonal Wigner distributions will be defined on $S\mathbf{X}_\Gamma \times \mathbf{R}$. We identify $S\mathbf{X}_\Gamma$ with $S\mathbf{X}_\Gamma \times \{\frac{r_j+r_k}{2}\}$, thereby extending Patterson-Sullivan distributions to $S\mathbf{X}_\Gamma \times \mathbf{R}$. With this identification, the off-diagonal Patterson-Sullivan and Wigner distributions are defined on the same space.

2.2. **Main Results.** Quantum dynamics are intimately related to the automorphic eigenvalue problem for the Laplacian on the compact hyperbolic surface $\mathbf{X}_\Gamma = \Gamma \backslash G/K$.

$$\begin{cases} \Delta\phi & = -\lambda\phi \\ \phi(\gamma z) & = \phi(z) \text{ for all } \gamma \in \Gamma \text{ and for all } z \end{cases}$$

The eigenvalues of the Laplacian can be expressed as $\lambda = \lambda_r = 1/4 + r^2$ and also $\lambda = \lambda_s = s(1-s)$ where $s = 1/2 + ir$. We denote by $\{\lambda_j = 1/4 + r_j^2\}$ the set of eigenvalues repeated according to multiplicity, and a corresponding orthonormal basis of eigenfunctions by $\{\phi_{ir_j}\}$. We define two types of phase space distributions associated with the automorphic eigenfunctions ϕ_{ir_j} of the Laplacian on $\mathbf{X}_\Gamma$. Patterson-Sullivan distributions PS_{ir_j}, which are the residues of the dynamical zeta functions $\mathcal{Z}(s; a) := \sum_\gamma \frac{e^{-sL_\gamma}}{1-e^{-L_\gamma}} \int_{\gamma_0} a$ (where the sum runs over closed geodesics) at the poles $1/2 + ir_j$, are invariant under the geodesic flow (classical evolution). Wigner distributions $\int_{S^*\mathbf{X}_\Gamma} a\, dW_{ir_j} = \langle \mathrm{Op}(a)\phi_{ir_j}, \phi_{ir_j}\rangle_{L^2(\mathbf{X}_\Gamma)}$, where $\mathrm{Op}(a)$ denotes the hyperbolic quantization, arise in quantum chaos and are invariant under the wave group (quantum evolution). The following is the relationship between the Patterson-Sullivan distributions and Wigner distributions, where L_r is an operator that transforms normalized Patterson-Sullivan distributions $\widehat{PS}_{ir_j}$ into W_{ir_j}. In the semi-classical limit, namely, as $\lambda_j \to \infty$, the distributions $\widehat{PS}_{ir_j}$ and W_{ir_j} are asymptotically equivalent.

Theorem 2.1 ([1]). *For any $a \in C^\infty(\Gamma\backslash G)$, we have the exact formula*

$$\langle Op(a)\phi_{ir_j}, \phi_{ir_j}\rangle_{S\mathbf{X}_\Gamma} = 2^{(1+2ir_j)} \int_{S\mathbf{D}} (L_{r_j}\chi a)(g) PS_{ir_j}(dg)$$

and the asymptotic formula

$$\int_{S\boldsymbol{X}_\Gamma} a(g) W_{ir_j}(dg) = \int_{S\boldsymbol{X}_\Gamma} a(g) \widehat{PS}_{ir_j}(dg) + O(r_j^{-1}).$$

Through similar methods, we can prove the asymptotic equivalence of the off-diagonal Γ-invariant Patterson-Sullivan distributions and Wigner distributions in the semi-classical limit.

Theorem 2.2 ([2])**.** *Let $a \in C^\infty(\Gamma\backslash G)$. Given a sequence of pairs (v_{j_n}, v_{k_n}) of spectral parameters with $-iv_{j_n} \to +\infty$ and $|v_{j_n} - v_{k_n}| \le \tau_0$ for some $\tau_0 \ge 0$, we have the asymptotic formula*

$$\int_{S\mathbf{X}_\Gamma} a(g) W^\Gamma_{j_n,k_n}(dg) = 2^{1+v_{j_n}-\overline{v_{k_n}}} \left(\frac{\pi}{r_{k_n}}\right)^{1/2} e^{\frac{-i\pi}{4}} \int_{S\mathbf{X}_\Gamma} a(g) PS^\Gamma_{v_{j_n},-v_{k_n}}(dg) + O(v_{k_n}^{-1}).$$

In the preceding results, we showed the asymptotic equivalence of two phase distributions associated with the classical and quantum dynamics. Below, we show the correspondence between the classical and quantum dynamics holds at a deeper level. In particular, we explicitly construct an operator $\mathcal{L}$ that intertwines the geodesic flow G^t and Schrödinger group $e^{it\frac{\Delta}{2}}$ at the level of symbols. The quantum evolution is given by $\alpha^t(\mathrm{Op}(a)) = e^{-it\frac{\Delta}{2}}\mathrm{Op}(a)e^{it\frac{\Delta}{2}}$ and this operator on symbols V^t is defined formally by $\alpha^t(\mathrm{Op}(a)) = \mathrm{Op}(V^t(a))$. The following theorem describes the intertwining relation on the universal cover.

Theorem 2.3 ([2])**.** *For $a \in \mathcal{S}^0_0$, $\mathcal{L}a$ is a continuous function, and we have the pointwise equality*

$$\mathcal{L} \circ V^t a = G^t \circ \mathcal{L}a.$$

Furthermore, the intertwining operator $\mathcal{L}$ extends to an isometry from $L^2_W(G \times \mathbf{R}, dg \times dp(r))$ to the space $\mathcal{H}_{PS}(\mathbf{D})$ of functions such that

$$\frac{1}{4}\int |PSa(ir, b, ir', b')|^2 \, db\, db' p(dr) p(dr') < +\infty,$$

and we have

$$\mathcal{L} \circ V^t = G^t \circ \mathcal{L}$$

where both sides are bounded operators from $L^2_W(G \times \mathbf{R}, dg \times dp(r))$ to $\mathcal{H}_{PS}(\mathbf{D})$.

On the quotient $\mathbf{X}_\Gamma$, we define Hilbert spaces $\mathcal{H}_W = \mathcal{H}_W(\mathbf{X}_\Gamma)$ and $\mathcal{H}_{PS} = \mathcal{H}_{PS}(\mathbf{X}_\Gamma)$ of Γ-invariant symbols and their dual spaces $\mathcal{H}^*_W$ and $\mathcal{H}^*_{PS}$. The Wigner distributions $W^\Gamma_{j,k}$ form an orthonormal basis of $\mathcal{H}^*_W$ while the Patterson-Sullivan distributions form an orthonormal basis of $\mathcal{H}^*_{PS}$. We define an operator between these spaces that satisfies the intertwining relation analogous to that on the universal cover.

Theorem 2.4 ([2])**.** *The operator $\mathcal{L}_{\Gamma\#} : \mathcal{H}^*_{PS} \to \mathcal{H}^*_W$ is an isometric isomorphism, and $\mathcal{L}_{\Gamma\#}$ sends $PS_{v_j,-v_k}$ to $W_{j,k}$. Dually, we have*

$$\mathcal{L}_\Gamma \circ V^t_\Gamma = G^t_\Gamma \circ \mathcal{L}_\Gamma,$$

as an equality between operators from $\mathcal{H}_W$ to $\mathcal{H}_{PS}$.

Although the Hilbert spaces $\mathcal{H}_{PS}$ and $\mathcal{H}_W$ are defined ad hoc so that the intertwining relation holds, this result is substantive in several ways. The Hilbert spaces of distributions $\mathcal{H}_W$ and $\mathcal{H}_{PS}$ contain large symbol classes. The intertwining operator clearly is inspired by the exact relation L_r between the Patterson-Sullivan distributions and Wigner distributions. Finally, it gives rise to a quantization procedure $a \mapsto \mathrm{Op}(\mathcal{L}_\Gamma^{-1} a)$ that satisfies the Egorov Theorem exactly (without a remainder term). This makes it natural in a sense because the only other quantization procedure known to have this property is the Weyl quantization on Euclidean space.

3. Asymptotic Equivalence of Classical and Quantum Dynamics in the Semi-Classical Limit

We devote this section to the proof of Theorem 2.1, which relates two types of phase space distributions associated to eigenfunctions of the Laplacian on a compact hyperbolic surface, namely, the Patterson-Sullivan and Wigner distributions. Patterson-Sullivan distributions arise as residues of dynamical zeta functions and are invariant under the geodesic flow (classical evolution). Wigner distributions arise in quantum chaos and are invariant under the wave group (quantum evolution).

Based on Fourier analysis on the hyperbolic disk introduced by Helgason in [9], a hyperbolic pseudo-differential calculus was introduced in [14]. We use this hyperbolic pseudo-differential calculus in the remainder of the paper. Let $\langle z, b\rangle$ denote the signed distance to 0 of the horocycle through the points $z \in \mathbf{D}, b \in B$. The hyperbolic plane waves $e_{\nu,b}(z) := e^{\left(\frac{1}{2}+\nu\right)\langle z,b\rangle}$ with $\nu \in \mathbf{C}$ and $b \in B$ are hyperbolic analogues of the Euclidean plane waves $x \mapsto e^{i\langle x,\xi\rangle}$ and are complex-valued eigenfunctions of the Laplacian $\Delta e_{\nu,b} = -\left(\frac{1}{4}-\nu^2\right) e_{\nu,b}$. The family of functions $\{e_{ir,b}(z)\}_{r>0,b\in B}$ forms a basis of generalized eigenfunctions of the Laplacian on $L^2(\mathbf{D})$ [9]. The Helgason-Fourier transform is defined as

$$\mathcal{F}f(b,r) = \int_{\mathbf{D}} e^{\left(\frac{1}{2}-ir\right)\langle z,b\rangle} f(z)\, d\mathrm{Vol}(z)$$

with $b \in B$ and $r \in \mathbf{R}$. The Fourier transform has the symmetry

$$\int_B \mathcal{F}f(b,r) e^{\left(\frac{1}{2}+ir\right)\langle z,b\rangle}\, db = \int_B \mathcal{F}f(b,-r) e^{\left(\frac{1}{2}-ir\right)\langle z,b\rangle}\, db. \tag{3.1}$$

The non-Euclidean Fourier inversion formula is given by

$$u(z) = \int_{\mathbf{R}_+} \int_B \mathcal{F}u(r,b) e^{\left(\frac{1}{2}+ir\right)\langle z,b\rangle}\, dp(r)\, |db|,$$

where the Plancherel measure on $\mathbf{R}$ is $dp(r) = \frac{1}{2\pi} r \tanh(\pi r)\, dr$. Note that the Plancherel formula $\|f\|_{L^2(\mathbf{D},\mathrm{Vol})} = \|\mathcal{F}f\|_{L^2(B\times\mathbf{R}_+, db\times dp(r))}$ holds for $f \in L^2(\mathbf{D})$.

Following [14], we define a hyperbolic quantization procedure on $\mathbf{D}$. For a symbol $a(z,b,r)$ that is polyhomogeneous in r, we define the action of pseudo-differential operators on $\mathbf{D}$ by

$$\mathrm{Op}(a) e^{\left(\frac{1}{2}+ir\right)\langle z,b\rangle} = a(z,b,r) e^{\left(\frac{1}{2}+ir\right)\langle z,b\rangle}.$$

With the non-Euclidean Fourier inversion formula, we extend the definition of $\mathrm{Op}(a)$ from the non-Euclidean plane waves $e_{\nu,b}(z)$ to $C_c^\infty(\mathbf{D})$

$$\mathrm{Op}(a)u(z) = \int_B \int_{\mathbf{R}_+} a(z,b,r) e^{\left(\frac{1}{2}+ir\right)\langle z,b\rangle} \mathcal{F}u(b,r)\, dp(r)\, db.$$

Helgason proved the following fundamental representation theorem for the eigenfunctions of the Laplacian on $\mathbf{D}$. It states that we can represent eigenfunctions via distributions that encode their boundary values. We will use these distributions to define the Patterson-Sullivan and Wigner distributions.

Theorem 3.2 (Helgason [9])**.** *Let ϕ be an eigenfunction for the eigenvalue $\lambda = -\left(\frac{1}{4} - \nu^2\right) \in \mathbf{C}$ with the property that ϕ has exponential growth, that is, there exists $C > 0$ such that $|\phi(z)| \leq Ce^{Cd_{\mathbf{D}}(0,z)}$. Then there exists a distribution $T_{\nu,\phi} \in \mathcal{D}'(B)$ such that*

$$\phi(z) = \int_B e^{\left(\frac{1}{2}+\nu\right)\langle z,b\rangle} T_{\nu,\phi}(db),$$

for all $z \in \mathbf{D}$. The distribution is unique if $\frac{1}{2} + \nu \neq 0, -1, -2, \ldots$.

The distribution $T_{\nu,\phi}$ is called the boundary values of the eigenfunction ϕ (for the spectral parameter ν). This terminology arises from the analogy to the theory of distributional boundary values of harmonic functions. Indeed, the kernel that arises in Helgason's Representation Theorem is the generalized Poisson kernel $P_{\mathbf{D}}^{\left(\frac{1}{2}+ir\right)}(z,b) = e^{\left(\frac{1}{2}+ir\right)\langle z,b\rangle}$ of the unit disk.

Definition 3.3. The Patterson-Sullivan distribution associated to a real eigenfunction ϕ_{ir_j} is the distribution on $B \times B \setminus \Delta$ defined by

$$ps_{ir_j}(db', db) := \frac{T_{ir_j}(db)T_{ir_j}(db')}{|b-b'|^{1+2ir_j}}.$$

If ϕ_{ir_j} is Γ-invariant, then ps_{ir_j} is Γ-invariant and time reversal invariant. We use the distributions ϕ_{ir_j} to construct geodesic flow-invariant distributions PS_{ir_j} on $S\mathbf{D}$. This requires the use of the Radon transform and smooth fundamental domain cutoffs, which we now define.

Definition 3.4. The Radon transform $\mathcal{R} : C_0(S\mathbf{D}) \to C_0(B \times B \setminus \Delta)$ is given by

$$\mathcal{R}f(b', b) = \int_{\gamma_{b',b}} f\, dt.$$

When dealing with integrals against irregular distributions, it is convenient to replace the characteristic function of a fundamental domain by a smooth (compactly supported) cutoff.

Definition 3.5. We say that $\chi \in C_0^\infty(\mathbf{D})$ is a smooth fundamental domain cutoff if

$$\sum_{\gamma \in \Gamma} \chi(\gamma z) = 1.$$

These cutoffs are defined to satisfy the property that $\int_{\mathcal{D}} f\, dVol(z) = \int_{\mathbf{D}} \chi f\, dVol(z)$ for any $f \in C(\Gamma \backslash \mathbf{D})$, where $\mathcal{D}$ is a fundamental domain for Γ in $\mathbf{D}$.

Definition 3.6. (1) On $S\mathbf{D}$ we define the Patterson-Sullivan distributions PS_{ir_j} by

$$PS_{ir_j}(db', db, dt) = ps_{ir_j}(db', db)|dt|$$

in the sense that

$$\langle a, PS_{ir_j} \rangle_{S\mathbf{D}} = \int_{B \times B \setminus \Delta} (\mathcal{R}a)(b', b) ps_{ir_j}(db', db).$$

(2) On the quotient $S\mathbf{X}_\Gamma = \Gamma\backslash\mathbf{D} = \Gamma \setminus PSU(1,1)$, we define the Patterson-Sullivan distributions $PS_{ir_j} \in \mathcal{D}'(S\mathbf{X}_\Gamma)$ by

$$\langle a, PS_{ir_j}\rangle_{S\mathbf{X}_\Gamma} = \langle \chi a, PS_{ir_j}\rangle_{S\mathbf{D}} = \int_{B\times B\setminus\Delta} \mathcal{R}(\chi a)(b',b)ps_{ir_j}(db',db),$$

where χ is a smooth fundamental domain cutoff.

(3) We define the normalized Patterson-Sullivan distributions by

$$\widehat{PS}_{ir_j} := \frac{1}{\langle \mathbf{1}, PS_{ir_j}\rangle_{S\mathbf{X}_\Gamma}} PS_{ir_j}.$$

It is apparent from this definition that the Patterson-Sullivan distributions are geodesic flow invariant on $S\mathbf{D}$.

Proposition 3.7. *Suppose that ϕ_{ir_j} is Γ-invariant, and let T_{ir_j} denote its radial boundary values. Then the distribution on $B \times B \setminus \Delta$ defined by*

$$ps_{ir_j}(db',db) := \frac{T_{ir_j}(db)T_{ir_j}(db')}{|b-b'|^{1+2ir_j}}$$

is Γ-invariant.

Proof. The Γ-invariance of ϕ_{ir_j} means that $\phi_{ir_j}(\gamma z) = \phi_{ir_j}(z)$ for $\gamma \in \Gamma$. By the uniqueness of the Helgason representation, $e^{\left(\frac{1}{2}+ir_j\right)\langle \gamma z, \gamma b\rangle} T_{ir_j}(d\gamma b) = e^{\left(\frac{1}{2}+ir_j\right)\langle z,b\rangle} T_{ir_j}(db)$. Using the identity $\langle \gamma\cdot z, \gamma\cdot b\rangle = \langle z,b\rangle + \langle \gamma\cdot 0, \gamma\cdot b\rangle$ from [9], the boundary values $T_{ir_j}(db)$ have the following invariance property: $T_{ir_j}(d\gamma b) = e^{-\left(\frac{1}{2}+ir_j\right)\langle\gamma\cdot 0,\gamma\cdot b\rangle} T_{ir_j}(db)$. Consequently,

$$T_{ir_j}(d\gamma b)T_{ir_j}(d\gamma b') = e^{-\left(\frac{1}{2}+ir_j\right)\langle\gamma\cdot 0,\gamma\cdot b\rangle} e^{-\left(\frac{1}{2}+ir_j\right)\langle\gamma\cdot 0,\gamma\cdot b'\rangle} T_{ir_j}(db)T_{ir_j}(db').$$

To continue the proof, we state a couple of identities from [11]:

$$|\gamma(x)-\gamma(y)| = |\gamma'(x)|^{1/2}|\gamma'(y)|^{1/2}|x-y|$$
$$1-|\gamma(x)|^2 = |\gamma'(x)|(1-|x|^2)$$

for every $x, y \in \mathbf{D}\cup B$, $\gamma\in\Gamma$. So, for $b \in B$ and $\gamma \in \Gamma$, we have

$$|\gamma(0)-\gamma(b)|^2 = |\gamma'(b)|(1-|\gamma(0)|^2).$$

Furthermore, using the formula for the Poisson kernel of the unit disk, we have

$$\begin{aligned} e^{-[\langle\gamma\cdot 0,\gamma\cdot b\rangle + \langle\gamma\cdot 0,\gamma\cdot b'\rangle]}|b-b'|^2 &= \frac{|\gamma(0)-\gamma(b)|^2}{1-|\gamma(0)|^2}\frac{|\gamma(0)-\gamma(b')|^2}{1-|\gamma(0)|^2}|b-b'|^2 \\ &= |\gamma'(b)||\gamma'(b')||b-b'|^2 \\ &= |\gamma b - \gamma b'|^2. \end{aligned}$$

Therefore,

$$|\gamma b-\gamma b'|^{1+2ir_j} = e^{-\left(\frac{1}{2}+ir_j\right)[\langle\gamma\cdot 0,\gamma\cdot b\rangle+\langle\gamma\cdot 0,\gamma\cdot b'\rangle]}|b-b'|^{1+2ir_j}.$$

By substitution,

$$T_{ir_j}(d\gamma b)T_{ir_j}(d\gamma b') = \frac{|\gamma b-\gamma b'|^{1+2ir_j}}{|b-b'|^{1+2ir_j}} T_{ir_j}(db)T_{ir_j}(db')$$

and hence

$$ps_{ir_j}(d\gamma b', d\gamma b) = ps_{ir_j}(db',db).$$

∎

Corollary 3.8. *PS_{ir_j} is a Γ-invariant distribution on $S\mathbf{D} = \mathbf{D} \times B$.*

The geodesic flow invariance of PS_{ir_j} on $S\mathbf{X}_\Gamma$ results from the following lemma.

Lemma 3.9. *Let $T \in \mathcal{D}'(S\mathbf{D})$ be a Γ-invariant distribution. Let a be a Γ-invariant smooth function on $S\mathbf{D}$. Then, for any $a_1, a_2 \in \mathcal{D}(S\mathbf{D})$ such that $\sum_{\gamma\in\Gamma} a_i(\gamma.(z,b)) = a(z,b)$ $(i = 1,2)$, we have*

$$\langle a_1, T\rangle_{S\mathbf{D}} = \langle a_2, T\rangle_{S\mathbf{D}}.$$

Proof. Let χ be a function on $C_0^\infty(\mathbf{D} \times B)$ such that $\sum_{\gamma\in\Gamma} \chi(\gamma.(z,b)) \equiv 1$. We choose χ to be independent of b. Then,

$$\begin{aligned}\langle a_i, T\rangle_{S\mathbf{D}} &= \int_{S\mathbf{D}} \left[\sum_{\gamma\in\Gamma} \chi(\gamma(z,b))\right] a_i(z,b)T(dz,db)\\ &= \int_{S\mathbf{D}} \sum_{\gamma\in\Gamma} \chi(z,b)a_i(\gamma(z,b))T(dz,db)\\ &= \int_{S\mathbf{D}} \chi(z,b)a(z,b)T(dz,db)\end{aligned}$$

□

Next, we define the Wigner distributions.

Definition 3.10. The Wigner measure of ϕ_{ir_j} is the distribution $W_{ir_j} \in \mathcal{D}'(S^*\mathbf{X}_\Gamma)$ defined by

$$\langle a, W_{ir_j}\rangle = \int_{S^*\mathbf{X}_\Gamma} a(g)W_{ir_j}(dg) := \langle \mathrm{Op}(a)\phi_{ir_j}, \phi_{ir_j}\rangle_{L^2(\mathbf{X}_\Gamma)}$$

for $a \in C^\infty(S^*\mathbf{X}_\Gamma)$.

Note that the Wigner distributions are normalized since $\langle \mathbf{1}, W_{ir_j}\rangle = 1$ and are invariant under quantum evolution: $\langle U_t^*\mathrm{Op}(a)U_t\phi_{ir_j}, \phi_{ir_j}\rangle_{L^2(\mathbf{X}_\Gamma)} = \langle \mathrm{Op}(a)\phi_{ir_j}, \phi_{ir_j}\rangle_{L^2(\mathbf{X}_\Gamma)}$ for $U_t = e^{it\sqrt{\Delta}}$. By Egorov's Theorem, W_{ir_j} is asymptotically invariant under the action of the geodesic flow g^t on $S^*\mathbf{X}_\Gamma$ in the large energy limit $r_j \to \infty$. This justifies having a result like Theorem 2.1, which asserts the asymptotic equivalence of W_{ir_j} and geodesic flow-invariant distributions in the semi-classical limit. At the heart of the proof of the theorem is the operator L_r, which transforms $\widehat{PS}_{ir_j}$ into W_{ir_j} and induces an asymptotic equality $W_{ir_j} \sim \widehat{PS}_{ir_j}$ between them. To make the notation more convenient, we will sometimes drop the j indices of r_j and index the eigenfunctions by r instead.

Definition 3.11. Define $L_r : C_0^\infty(G) \to C^\infty(G)$ by

$$L_r a(g) := \int_{\mathbf{R}} (1+u^2)^{-(\frac{1}{2}+ir)} a(gn_u)\, du.$$

The first part of the proof of Theorem 2.1 involves computing an explicit expression of W_{ir_j}. We will use the following identity.

Lemma 3.12 ([11])**.** *Let $z \in \mathbf{D}$, let $b_1, b_2 \in B$, and let $s_{b_1,b_2}(z)$ denote the hyperbolic distance from z to the geodesic γ_{b_1,b_2} defined by (b_1, b_2). Then*

$$\cosh s_{b_1,b_2}(z) = \frac{2|z-b_1||z-b_2|}{|b_1-b_2|(1-|z|^2)}.$$

Lemma 3.13. *We have*

$$\langle Op(a)\phi_{ir_j}, \phi_{ir_j}\rangle_{L^2(\mathbf{X}_\Gamma)} =$$
$$2^{(1+2ir_j)} \int_{B\times B} \left(\int_{\mathbf{D}} \chi a(z,b) [\cosh s_{b',b}(z)]^{-(1+2ir_j)}\, d\,Vol(z) \right) \frac{T_{ir_j}(db) T_{ir_j}(db')}{|b-b'|^{1+2ir_j}}.$$

The right side is independent of the choice of χ.

Proof. When a is a Γ-invariant function on $S\mathbf{D}$, then we have

$$\mathrm{Op}(a)\phi_{ir_j} := \int_B a(z,b) e^{\left(\frac{1}{2}+ir_j\right)\langle z,b\rangle} T_{ir_j}(db).$$

Using this formula and the generalized Poisson formula from Theorem 3.2, we obtain

$$\langle \mathrm{Op}(a)\phi_{ir_j}, \phi_{ir_j}\rangle = \int_{B\times B} \left(\int_{\mathbf{D}} \chi a(z,b) e^{\left(\frac{1}{2}+ir_j\right)\langle z,b\rangle} e^{\left(\frac{1}{2}+ir_j\right)\langle z,b'\rangle}\, d\mathrm{Vol}(z) \right) T_{ir_j}(db) T_{ir_j}(db').$$

Next, observe from Lemma 3.12 that

$$\begin{aligned} e^{\langle z,b\rangle} e^{\langle z,b'\rangle} &= \frac{(1-|z|^2)^2}{|z-b|^2|z-b'|^2} \\ &= \left(\frac{|z-b'||z-b|}{1-|z|^2} \right)^{-2} \\ &= 4[\cosh s_{b',b}(z)]^{-2} |b-b'|^{-2} \end{aligned}$$

so

$$e^{\left(\frac{1}{2}+ir_j\right)\langle z,b\rangle} e^{\left(\frac{1}{2}+ir_j\right)\langle z,b'\rangle} = \frac{2^{1+2ir_j} [\cosh s_{b',b}(z)]^{-(1+2ir_j)}}{|b-b'|^{1+2ir_j}}.$$

By substitution, the proof is complete. □

Next, we analyze the operator $\mathcal{L}_r : C_c(\mathbf{D}) \to C(B\times B)$, which we define as

$$\mathcal{L}_r(\chi a)(b',b) := \int_{\mathbf{D}} \chi a(z,b) [\cosh s_{b',b}(z)]^{-(1+2ir)}\, d\mathrm{Vol}(z).$$

To relate $\mathcal{L}_r$ to the intertwiner L_r, we perform a change of coordinates on $\mathbf{D}$ or $\mathbf{H}$ adapted to a particular geodesic $\gamma_{b',b}$. Given this geodesic, we write $z = (t,u)$, where t measures arclength on $\gamma_{b',b}$ and u measures arclength on horocycles centered at b. More precisely, we denote by $g(b',b)$ the vector on $\gamma_{b',b}$ which is closest to the origin, and the parametrization of $z = (t,u)$ is defined by $(z,b) = g(b',b)a_t n_u$. For any given (b',b), the volume element of z is $dVol = dtdu$. We check this for $b' = 0$ and $b = \infty$ using the upper half plane model of the hyperbolic plane. This is the geodesic represented by $x = 0$. The origin, which we denote by e, is the point $i \in \mathbf{C}$, which has an x-coordinate of 0 and a y-coordinate of 1. Therefore, $g(b',b) = e = (i,\infty)$. Horocycles centered at ∞ are represented by horizontal lines parallel to the real axis, so the horocycle through e is represented by the line $y = i$. So, the action of the geodesic and horocycle flows at the base point are $a_t n_u i = e^t(i+u)$. From this, we observe that $y = e^t$ and $x = ue^t$. Substituting this into the area form for the upper half plane model, which is $\frac{dxdy}{y^2}$, yields $dtdu$, as required. It follows that

$$\mathcal{L}_r(\chi a)(b',b) = \int \cosh s_{b',b}(t,u)^{-(1+2ir)} \chi a(g(b',b)a_t n_u)\, du\, dt.$$

The distance $s_{b',b}(z)$ from $z = (t, u)$ to the geodesic from b' to b depends only on u and satisfies $\cosh s_{b',b}(t, u) = \sqrt{1 + u^2}$. By substitution,

$$\mathcal{L}_r(\chi a)(b, b') = \int_{\mathbf{R}\times\mathbf{R}} (1 + u^2)^{-\left(\frac{1}{2}+ir\right)} \chi a(g(b', b)a_t n_u)\, du\, dt.$$

We also observe that

$$\mathcal{L}_r(\chi a)(b, b') = \int_{\mathbf{R}} L_r(\chi a)(g(b, b')a_t)\, dt.$$

Combining the preceding results, we can write the Wigner distributions in terms of the Patterson-Sullivan distributions.

Lemma 3.14.

$$\langle Op(a)\phi_{ir}, \phi_{ir}\rangle_{L^2(\boldsymbol{X}_\Gamma)} = 2^{1+2ir} \int_G L_r(\chi a)(g) PS_{ir}(dg)$$

Proof.

$$\begin{aligned}
&\langle \mathrm{Op}(a)\phi_{ir_j}, \phi_{ir_j}\rangle_{L^2(\mathbf{X}_\Gamma)} = \\
&2^{1+2ir} \int_{B\times B} \left[\int_{\mathbf{D}} \chi a(z, b)[\cosh s_{b',b}(z)]^{-(1+2ir)} |b - b'|^{-(1+2ir)}\, d\mathrm{Vol}(z)\right] T_{ir_j}(db) T_{ir_j}(db') = \\
&2^{1+2ir} \int_{B\times B} |b - b'|^{-(1+2ir)} \mathcal{L}_r(\chi a)(b', b) T_{ir_j}(db) T_{ir_j}(db') = \\
&2^{1+2ir} \int_{B\times B} |b - b'|^{-(1+2ir)} \mathcal{R}(L_r(\chi a))(b, b') T_{ir_j}(db) T_{ir_j}(db') = \\
&2^{1+2ir} \int_{B\times B} \mathcal{R}(L_r(\chi a))(b, b')\, dps_{ir_j} = \\
&2^{1+2ir} \langle L_r(\chi a), PS_{ir_j}\rangle_{S\mathbf{D}}
\end{aligned}$$

□

To complete the proof of Theorem 2.1, we fix an arbitrary geodesic $\gamma_{b',b}$ and perform the method of stationary phase for $L_r(\chi a)$. The critical set of $\mathcal{L}_r(\chi a)(b', b)$ is the geodesic $\gamma_{b',b}$, so the critical set of $L_r(\chi a)$ is the set $u = 0$ in the integral defining $L_r(\chi a)$. Since $(\log(1 + u^2)'')\Big|_{u=0} = 2$, the method of stationary phase provides the expansion

$$L_r(\chi a)(g) = (-ir/\pi)^{-1/2} \left(\sum_{n\geq 0} r^{-n} L_{2n}(\chi a)(g)\right), \tag{3.15}$$

where L_{2n} is a differential operator of order $2n$ on $S\mathbf{D}$. In particular, L_0 is the identity and the other L_{2n} are differential operators in the stable direction n_u. We integrate 3.15 with respect to PS_{ir_j} and substitute into Lemma 3.14 to obtain the asymptotic expansion

$$\langle \mathrm{Op}(a)\phi_{ir}, \phi_{ir}\rangle_{L^2(\mathbf{X}_\Gamma)} = 2^{(1+2ir)}(-ir/\pi)^{-1/2} \left(\sum_{n\geq 0} \int_{S\mathbf{D}} L_{2n}(\chi a)(g) PS_{ir}(dg)\right).$$

Note that the first term ($n = 0$) of the expansion is the Patterson-Sullivan distribution in the quotient $S\mathbf{X}_\Gamma$. On the left side of the equation above, we have the

Γ-invariant distribution

$$e^{(1/2+ir)\langle z,b\rangle}e^{(1/2+ir)\langle z,b'\rangle}\,d\mathrm{Vol}(z)T_{ir}(db)T_{ir}(db')$$

in the triple (b, b', z). This implies that the distributions on the right side of the preceding equation

$$f \mapsto \int_{S\mathbf{D}} L_{2n}(f)(g)PS_{ir}(dg)$$

are also Γ-invariant. By Lemma 3.9, the functional

$$a \mapsto \int_{G} L_{2n}(\chi a)(g)PS_{ir}(dg)$$

defines a distribution on $\Gamma\backslash G$ and the definition is independent of the choice of the cutoff χ. Now that we have confirmed well-definedness, we deduce from the stationary phase asymptotics in 3.15 that

$$\int_{S\mathbf{X}_\Gamma} a(g)W_{ir_j}(dg) = 2^{(1+2ir)}(-ir/\pi)^{-1/2}\sum_{n=0}^{N} r_j^{-n}\int_{S\mathbf{D}} L_{2n}(\chi a)(g)PS_{ir_j}(dg)+O(r_j^{-N-1+K}),$$

where $C, K > 0$ are numerical constants such that for $a \in C^2(S\mathbf{X}_\Gamma)$, $|\langle a, PS_{ir_j}\rangle_{S\mathbf{X}_\Gamma}| \leq C(1+|r_j|)^K\|a\|_{C^2}$ for all j. If we choose $N > K$ then the remainder term vanishes in the semi-classical limit. Since $L_0 = \mathrm{Id}$, the operator $L_r^{(N)} = \sum_{n=0}^N r^{-n}L_{2n}$ can be inverted up to $O(r^{-N-1})$, that is, one can find differential operators $M_r^{(N)} = \sum_{n=0}^N r^{-n}M_{2n}$ (with $M_0 = \mathrm{Id}$) and $R_r^{(N)}$ such that

$$L_r^{(N)}M_r^{(N)} = \mathrm{Id} + r^{-N-1}R_r^{(N)}.$$

We thus get

$$\begin{aligned}\int_{S\mathbf{X}_\Gamma} M_{r_j}^{(N)}a(g)W_{ir_j}(dg) &= \int_{S\mathbf{D}} L_{r_j}^{(N)}\chi M_{r_j}^{(N)}a(g)PS_{ir_j}(dg) + O(r_j^{-N-1+K})\\ &= \int_{S\mathbf{D}} L_{r_j}^{(N)}M_{r_j}^{(N)}\chi a(g)PS_{ir_j}(dg) + O(r_j^{-N-1+K})\\ &= \int_{S\mathbf{X}_\Gamma} a(g)PS_{ir_j}(dg) + O(r_j^{-N-1+K}),\end{aligned}$$

where the second step follows from Lemma 3.9. By standard estimates on pseudo-differential operators, the Wigner measures are uniformly bounded in $(C^k)^*$ for some k. Therefore,

$$\int_{S\mathbf{X}_\Gamma} M_{r_j}^{(N)}a(g)W_{ir_j}(dg) = \int_{S\mathbf{X}_\Gamma} a(g)W_{ir_j}(dg) + O(r_j^{-1}).$$

This shows that

$$2^{(1+2ir_j)}(-ir_j/\pi)^{-1/2}\int_{S\mathbf{X}_\Gamma} a(g)PS_{ir_j}(dg) = \int_{S\mathbf{X}_\Gamma} a(g)W_{ir_j}(dg) + O(r_j^{-1}).$$

The left side is asymptotically the same as $\langle a, \widehat{PS}_{ir_j}\rangle$ since the leading coefficients must match when $a = 1$. This completes the proof of Theorem 2.1.

The proof of Theorem 2.2 for the asymptotic equivalence in the semi-classical limit for the off-diagonal Patterson-Sullivan and Wigner distributions can be reduced to the above, so we do not present it here.

4. Explicit Intertwining of the Geodesic Flow and Schrodinger Flow

The objective of this section is to discuss the proofs of Theorems 2.3 and 2.4. These proofs involve microlocal analysis, so we start by describing several symbol classes that we will use for pseudo-differential calculus on the Poincaré disk.

Observe that the Schwarz kernel of $\mathrm{Op}(a)$ is given by

$$K_a(z,w) = \int_B \int_{\mathbf{R}^+} a(z,b,r) e^{\left(\frac{1}{2}+ir\right)\langle z,b\rangle} e^{\left(\frac{1}{2}-ir\right)\langle w,b\rangle}\, dp(r)\, db. \tag{4.1}$$

Assume that a has the following symmetry with respect to the transformation $(z,b,r) \mapsto (z,b,-r)$

$$\int a(z,b,r) e^{\left(\frac{1}{2}+ir\right)\langle z,b\rangle} e^{\left(\frac{1}{2}-ir\right)\langle w,b\rangle}\, db = \int a(z,b,-r) e^{\left(\frac{1}{2}-ir\right)\langle z,b\rangle} e^{\left(\frac{1}{2}+ir\right)\langle w,b\rangle}\, db \tag{4.2}$$

for all $z, w \in \mathbf{D}$ and $r \in \mathbf{R}$. Then, by the Plancherel formula for the non-Euclidean Fourier transform, we can recover the symbol from the kernel by

$$a(z,b,r) = e^{-\left(\frac{1}{2}+ir\right)\langle z,b\rangle} \int_{\mathbf{D}} K_a(z,w) e^{\left(\frac{1}{2}+ir\right)\langle w,b\rangle} \mathrm{Vol}(dw)$$

for all $r \in \mathbf{R}$. By $L^2_W(G\times\mathbf{R}, dg\times dp(r))$, we denote the space of functions in $L^2(G\times\mathbf{R}, dg\times dp(r))$ that have symmetry 4.2 with respect to the Weyl group. Furthermore, we will use symmetry 4.2 in the definition of the Wigner distributions on $\mathbf{D}$.

The pointwise intertwining in Theorem 2.3 occurs for functions on $S\mathbf{D}\times\mathbf{R}$ with appropriate growth and smoothness properties with respect to $(z,b) \in S\mathbf{D}$. The definitions of the Hilbert spaces that are intertwined in Theorem 2.4 are also based on this particular class of smoothing symbols. To define this symbol class, we first introduce Schwarz functions. Schwarz functions were first defined on G by Harish-Chandra [8] and were extended to the hyperbolic disk G/K by Eguchi [5, 6].

Definition 4.3 ([5, 6])**.** We say that f belongs to the Schwarz space $\mathcal{C}^p(G/K)$ for $0 < p \leq 2$ if and only if f is a smooth function on G which is right-K-invariant and

$$\sup_{g\in G} \varphi_0(gK)^{-2/p}(1+d(gK,o))^q |LRf(g)| < +\infty$$

for any $q > 0$ and for any differential operators L, R on G which are respectively left and right invariant. Here, φ_0 is the spherical function on G/K, given by $\varphi_0(z) = \int e^{\frac{1}{2}\langle z,b\rangle}\, db$.

Since functions on $\mathcal{C}^p(G/K)$ are L^p, they are known as Schwarz functions of L^p type. The following result is a Paley-Wiener theorem for Schwarz functions of L^p type.

Proposition 4.4 ([5, 6])**.** *Set $\epsilon = \epsilon(p) = \frac{2}{p} - 1$. The space $\mathcal{F}(\mathcal{C}^p(G/K))$ coincides with the space $\mathcal{C}(B\times\mathbf{R}^\epsilon)_W$ of functions u on $B\times\mathbf{R}$ such that*

- *if $\epsilon > 0$, then u extends holomorphically to the strip $\mathbf{R}^\epsilon = \{|Im(r)| < \frac{\epsilon}{2}\}$*
- *on $\mathbf{R}^\epsilon$, we have a bound*

$$\sup_{(b,r)} (1+|r|)^q \left|\frac{\partial^\alpha}{\partial r^\alpha} Du(b,r)\right| < +\infty$$

for all $q > 0$, every integer α, and every K-left-invariant differential operator D acting on B. Here we are using the identification $B \sim K$.

- *u must satisfy symmetry 3.1. This symmetry condition with respect to the Weyl group is indicated by the subscript W.*

We define the space $\mathcal{K}^p_q(G/K\times G/K)_W$ of kernels of operators sending $\mathcal{C}^p(G/K)$ continuously to $(\mathcal{C}^q(G/K))'$. On this space, formula 4.1 can be used to relate the kernel to the Fourier transform of the symbol. We denote the symbol class corresponding to the kernels $\mathcal{K}^p_q(G/K\times G/K)_W$ by $\mathcal{S}^p_q(G/K\times B\times \mathbf{R})$.

The space $\mathcal{K}^0_0(G/K\times G/K)$ of smoothing operators that send $(\cap \mathcal{C}^p(G/K))'$ to $\cap\mathcal{C}^p(G/K)$ is of particular interest to us. The corresponding space of smoothing symbols is defined as follows.

Definition 4.5. $\mathcal{C}(B\times\mathbf{R}^\epsilon;\mathcal{C}^p(G/K))_W$ is the space of functions $a(z,b,r)$ with the $\mathcal{C}(B\times\mathbf{R}^\epsilon)$-regularity in the (b,r) variables, taking values in $\mathcal{C}^p(G/K)$. We will denote the space of smoothing symbols

$$\mathcal{S}^0_0 := \mathcal{S}^0_0(G/K\times B\times\mathbf{R})_W := \bigcap_\epsilon\bigcap_p \mathcal{C}(B\times\mathbf{R}^\epsilon;\mathcal{C}^p(G/K))_W.$$

By this definition, we have for $a\in\mathcal{S}^0_0$ that

$$\mathrm{Op}(a)e^{\left(\frac12+\nu\right)\langle\bullet,b\rangle}(z) = a(z,b,-i\nu)e^{\left(\frac12+\nu\right)\langle z,b\rangle}$$

for any $\nu\in\mathbf{C}$.

4.1. **Pointwise Intertwining of Symbols on the Universal Cover.** In this part, we prove Theorem 2.3. Recall that we extend the Patterson-Sullivan distribution $PS_{(ir,b),(-ir',b')}$ for $r,r'\in\mathbf{R}$, which was originally defined on $S\mathbf{D}$, to $S\mathbf{D}\times\mathbf{R}$ by tensoring it by $\delta_{\frac{r+r'}{2}}$ on the $\mathbf{R}$ variable. We make precise the notions of geodesic and Schrödinger flow in this context. The geodesic flow on G extends to $G\times\mathbf{R}$ via the formula $G^t(g,r)=(ga_{rt},r)$. The geodesic flow can be considered an operator acting on functions by composition. For a function a on G, we have $g^ta := a\circ g^t$ and for a function a on $G\times\mathbf{R}$, we have $G^ta := a\circ G^t$. The Schrödinger group is the unitary 1-parameter group $(e^{it\frac{\Delta}{2}})$ induced by the Laplacian on the Hilbert space $L^2(\mathbf{D})$ and it preserves the Schwarz spaces $\mathcal{C}^p(G/K)$. We denote by V^t the operator on symbols, defined formally by $e^{-it\Delta/2}\mathrm{Op}(a)e^{it\Delta/2} = \mathrm{Op}(V^ta)$. Explicitly,

$$V^ta(z,b,r) = e^{-\left(\frac12+ir\right)\langle z,b\rangle}\int e^{\left(\frac12+ir\right)\langle w,b\rangle}e^{\left(\frac12-ir'\right)\langle w,b'\rangle}e^{\left(\frac12+ir'\right)\langle z,b'\rangle}a(w,b,r) \\ \times e^{\frac{it}{2}(r'^2-r^2)}\mathrm{Vol}(dw)\,db'\,dp(r').$$

The generator D^V of V^t is a skew-adjoint differential operator on $L^2(G\times\mathbf{R}^+,dg\times dp(r))$ that satisfies $\mathrm{Op}(D^Va) = \left[\mathrm{Op}(a),\frac{i\Delta}{2}\right]$.

Next, we define analogues of Wigner and Patterson-Sullivan distributions on the universal cover $\mathbf{D}$.

Definition 4.6. For $b,b'\in B$ and $\nu,\nu'\in i\mathbf{R}$, the Wigner distribution $W_{(\nu,b),(\nu',b')}\in\mathcal{D}'(S\mathbf{D}\times\mathbf{R})$ associated to the two eigenfunctions $e_{(\nu,b)}(z)=e^{\left(\frac12+\nu\right)\langle z,b\rangle}$ and $e_{(\nu',b')}(z) = e^{\left(\frac12+\nu'\right)\langle z,b'\rangle}$ is defined formally by

$$\int_{S\mathbf{D}\times\mathbf{R}} a(z,\tilde b,r)W_{(\nu,b),(\nu',b')}(dz,d\tilde b,dr) = \langle \mathrm{Op}(a)e_{\nu,b},e_{\nu',b'}\rangle$$

for a having the symmetry 4.2.

On the boundary B (endowed with the density db) we will denote $\delta_{b_0}(b)$ the distribution defined by the Dirac mass at a point b_0 and $\delta_{b_0}(db) = \delta_{b_0}(b)\, db$ the corresponding distribution density, defining the linear form $f \mapsto f(b_0)$ on $C^\infty(B)$. The Wigner and Patterson-Sullivan distributions are actually distribution densities. If $\nu = ir$, then

$$\langle \mathrm{Op}(a)e_{\nu,b}, e_{\nu',b'}\rangle = \int_{\mathbf{D}} a(z,b,r)e_{ir,b}(z)\overline{e_{\nu',b'}(z)}\mathrm{Vol}(dz).$$

Therefore,

$$W_{(\nu,b),(\nu',b')}(dz, d\tilde{b}, dr) = e_{\nu,b}(z)\overline{e_{\nu',b'}(z)}\delta_b(d\tilde{b})\delta_{-i\nu}(dr)\mathrm{Vol}(dz).$$

We can observe that supp $W_{(\nu,b),(\nu',b')} \subset S\mathbf{D} \times \{-i\nu\}$, so $W_{(\nu,b),(\nu',b')}$ can be extended to functions $a(z,b,r)$ that depend continuously on r with values in $C_c^\infty(S\mathbf{D})$, and in particular for functions a that do not depend on r and are continuous and compactly supported with respect to (z,b).

Definition 4.7. The Wigner transform of a function $a \in C_c^\infty(G \times \mathbf{R})$ that satisfies symmetry 4.2 is $\mathcal{W} : C_c^\infty(G \times \mathbf{R}) \to L^2(B \times i\mathbf{R} \times B \times i\mathbf{R}, db \otimes p(dr) \otimes db' \otimes p(dr'))$ and is given by

$$\mathcal{W}a(\nu, b, \nu', b') = W_{(\nu,b),(-\nu',b')}(a),$$

where p denotes the Plancherel measure.

The Wigner transformation is the non-Euclidean Fourier transform of $a(z,b,r)e^{\left(\frac{1}{2}+ir\right)\langle z,b\rangle}$ with respect to z, evaluated at $(b', -r')$. From the inversion formula for $\mathcal{F}$, we have

$$a(z,b,r) = \frac{1}{2}e^{-\left(\frac{1}{2}+ir\right)\langle z,b\rangle} \int_B \int_{\mathbf{R}} e^{\left(\frac{1}{2}-ir'\right)\langle z,b'\rangle}\mathcal{W}a(ir,b,ir',b')\, db'\, dp(r')$$

and by the Plancherel formula for $\mathcal{F}$, we have the isometry

$$\|a\|_{L^2_W(G\times\mathbf{R}, dg\times dp(r))} = \|\mathcal{W}a(ir,b,ir',b')\|_{L^2(B\times i\mathbf{R}_+\times B\times i\mathbf{R}_+, db\otimes p(dr)\otimes db'\otimes p(dr'))}. \tag{4.8}$$

The Patterson-Sullivan distributions and transform can be defined in a manner similar to the Wigner distributions and transform, respectively.

Definition 4.9. For $\nu, \nu' \in i\mathbf{R}$, the Patterson-Sullivan distribution $PS_{(\nu,b),(-\nu,b)} := PS_{e_{(\nu,b)},e_{(-\nu,b')}}$ associated to the two eigenfunctions $e_{(\nu,b)}(z) = e^{\left(\frac{1}{2}+\nu\right)\langle z,b\rangle}$ and $e_{(-\nu',b')}(z) = e^{\left(\frac{1}{2}-\nu'\right)\langle z,b'\rangle}$ is the distribution on $S\mathbf{D} = B^{(2)} \times \mathbf{R}$ defined by

$$PS_{e_{(\nu,b)},e_{(-\nu',b')}}(d\tilde{b}, d\tilde{b}', d\tau) = \frac{\delta_b(d\tilde{b})\delta_{b'}(d\tilde{b}')}{|\tilde{b}-\tilde{b}'|^{1+\nu-\overline{\nu'}}}e^{(\nu+\overline{\nu'})\tau}\, d\tau.$$

The Patterson-Sullivan distributions are eigendistributions of the geodesic flow in the sense that for $\nu, \nu' \in i\mathbf{R}$, we have

$$g^t_\# PS_{e_{(\nu,b)},e_{(-\nu',b')}} = e^{-t(\nu+\overline{\nu'})}PS_{e_{(\nu,b)},e_{(-\nu',b')}},$$

where $g^t_\#$ is the pushfoward by g^t. The Patterson-Sullivan transform of a test function consists of the pairing of a Patterson-Sullivan distribution with that test function.

Definition 4.10. The Patterson-Sullivan transform $PS : C_c^\infty(G\times\mathbf{R}) \to C^\infty(B^{(2)}\times i\mathbf{R}\times i\mathbf{R})$ on G is given by

$$\begin{aligned} PSa(\nu,b,\nu',b') &= PS_{(\nu,b),(-\nu',b')}\left(a_{\frac{\nu-\overline{\nu'}}{2i}}\right) \\ &= \frac{1}{|b-b'|^{1+\nu-\overline{\nu'}}}\int_{\mathbf{R}} a\left(g(b',b)a_\tau, \frac{\nu-\overline{\nu'}}{2i}\right)\times e^{(\nu+\overline{\nu'})\tau}\,d\tau. \end{aligned}$$

The Patterson-Sullivan transform can be related to the Fourier-Radon transform. The inversion formula for the Patterson-Sullivan transform, stated below, can be derived from the inversion formula for the Fourier-Radon transform

$$a(b',b,t,R) = \frac{e^{2iRt}|b-b'|^{1+2iR}}{\pi}\int_{\mathbf{R}} PSa(ir,b,i(2R-r),b')e^{-2irt}\,dr. \tag{4.11}$$

As in the diagonal case, we define an operator L_ν that sends the Patterson-Sullivan distributions to the Wigner distributions.

Definition 4.12. If a is a function on $S\mathbf{D}\simeq G$ and $\nu\in\mathbf{C}$, we define the function $L_\nu a$ on G by

$$L_\nu a(g) = \int_{\mathbf{R}} a(gn_u)(1+u^2)^{-\left(\frac{1}{2}+\nu\right)}\,du.$$

We then have the exact relation below. Its proof is an adaptation of the proof of the analogous result for the diagonal Patterson-Sullivan and Wigner distributions [1].

Proposition 4.13. *Let $a\in C_c^\infty(G)$, $\nu,\nu'\in i\mathbf{R}$ and $(b',b)\in B^{(2)}$. Then $L_{-\overline{\nu'}}(a)\in C^\infty(S\mathbf{D})$. Although $L_{-\overline{\nu'}}(a)$ is not compactly supported, the pairing $PS_{(\nu,b)(-\nu',b')}(L_{-\overline{\nu'}}(a))$ is well-defined, and we have*

$$PS_{(\nu,b)(-\nu',b')}(L_{-\overline{\nu'}}(a)) = 2^{-(1+\nu-\overline{\nu'})}W_{(\nu,b)(-\nu',b')}(a).$$

Proof. By definition,

$$\begin{aligned} W_{(\nu,b),(-\nu',b')}(a) &= \int_{\mathbf{D}} a(z,b)e^{\left(\frac{1}{2}+\nu\right)\langle z,b\rangle}e^{\left(\frac{1}{2}-\overline{\nu'}\right)\langle z,b'\rangle}\,Vol(dz) \\ &= \int_{\mathbf{D}} a(z,b)e^{\left(\frac{1}{2}-\overline{\nu'}\right)\langle z,b\rangle}e^{\left(\frac{1}{2}-\overline{\nu'}\right)\langle z,b'\rangle}e^{(\nu+\overline{\nu'})\langle z,b\rangle}\,Vol(dz) \\ &= \frac{2^{1-2\nu'}}{|b-b'|^{1-2\overline{\nu'}}}\int_{\mathbf{D}} a(z,b)[\cosh s_{b',b}(z)]^{-(1-2\overline{\nu'})}e^{(\nu+\overline{\nu'})\langle z,b\rangle}\,Vol(dz) \\ &= \frac{2^{1-2\overline{\nu'}}}{|b-b'|^{1+\nu-\overline{\nu'}}}\int_{\mathbf{D}} a(z,b)[\cosh s_{b',b}(z)]^{-(1-2\overline{\nu'})}\frac{e^{(\nu+\overline{\nu'})\langle z,b\rangle}}{|b-b'|^{-(\nu+\overline{\nu'})}}\,Vol(dz) \end{aligned}$$

Recall that s_{b_1,b_2} denotes the hyperbolic distance from z to the geodesic γ_{b_1,b_2} and satisfies the identity

$$e^{\langle z,b\rangle}e^{\langle z,b'\rangle} = 4[\cosh s_{b',b}(z)]^{-2}|b-b'|^{-2}.$$

As in [1], we use the adapted coordinates $(z,b)=g(b',b)a_\tau n_u$ for which $\mathrm{Vol}(dz)=d\tau du$ and $\cosh s_{b',b}(z)=\sqrt{1+u^2}$.

The expression for the hyperbolic distance in the upper half plane model is given by

$$\cosh d(z,w) = 1+\frac{|z-w|^2}{2\mathrm{Im}(z)\mathrm{Im}(w)} \tag{4.14}$$

and from this we obtain

$$e^{\langle g(b',b)a_\tau n_u, b\rangle}|b-b'| = 2e^{\tau}.$$

Therefore,

$$\begin{aligned} W_{(\nu,b),(-\nu',b')}(a) &= \frac{2^{(1+\nu-\overline{\nu'})}}{|b-b'|^{1+\nu-\overline{\nu'}}}\int_{\mathbf{R}\times\mathbf{R}} (1+u^2)^{-\left(\frac{1}{2}-\overline{\nu'}\right)} a(g(b,b')a_\tau n_u) e^{(\nu+\overline{\nu'})\tau}\, du\, d\tau \\ &= 2^{(1+\nu-\overline{\nu'})} PS_{(\nu,b)(-\nu',b')}(L_{-\overline{\nu'}}(a)). \end{aligned}$$

□

The intertwining operator $\mathcal{L} : C_c^\infty(G\times\mathbf{R}) \to C(G\times\mathbf{R})$ should satisfy $PS\mathcal{L}a(ir,b,ir',b') = \mathcal{W}a(ir,b,ir',b')$ for $a \in \mathcal{D}'(S\mathbf{D}\times\mathbf{R})$ and $r,r' \in \mathbf{R}$. It follows from Proposition 4.13 that this equality implies that $PS\mathcal{L}a(ir,b,ir',b') = 2^{1+ir+ir'}PS_{(ir,b),(-ir',b')}(L_{ir'}(a_r))$. By the inversion formula for the Patterson-Sullivan transform (equation 4.11), we have for all $(b',b) \in B^{(2)}$, $t \in \mathbf{R}$, $R \in \mathbf{R}$ that

$$\begin{aligned} \mathcal{L}a(b',b,t,R) &= \frac{2^{1+2iR}}{\pi} e^{2iRt}|b-b'|^{1+2iR}\int_{\mathbf{R}} PS(\mathcal{L}a)(ir,b,i(2R-r),b')e^{-2irt}\, dr \\ &= \frac{2^{1+2iR}}{\pi}\int_{\mathbf{R}} (1+u^2)^{-\left(\frac{1}{2}+iR\right)} a_r \circ h^u(b',b,\tau) e^{2i(R-r)\left(t-\tau-\frac{\log(1+u^2)}{2}\right)}\, dr\, du\, d\tau. \end{aligned}$$

Letting $g = (b',b,t)$, we have

$$\mathcal{L}a(g,R) = \frac{2^{1+2iR}}{\pi}\int (1+u^2)^{-\left(\frac{1}{2}+iR\right)} a\left(g a_{\tau-\frac{\log(1+u^2)}{2}} n_u, r\right) e^{2i(r-R)\tau}\, dr\, du\, d\tau.$$

Now, we prove Theorem 2.3. Let $a \in \mathcal{S}_0^0$. We first check that $\mathcal{L}a$ is a continuous function. For $g = (z,b)$, we see that

$$\begin{aligned} \mathcal{L}a(z,b,R) &= \frac{2^{1+2iR}}{\pi}\int (1+u^2)^{-\left(\frac{1}{2}+iR\right)} a\circ h^u \circ g^{\tau-\frac{\log(1+u^2)}{2}}(z,b,r) \times e^{2i(r-R)\tau}\, dr\, du\, d\tau \\ &= \frac{2^{1+2iR}}{\pi}\int (1+u^2)^{-\left(\frac{1}{2}+iR\right)} \hat{a}\circ h^u \circ g^{\tau-\frac{\log(1+u^2)}{2}}(z,b,2\tau) \times e^{-2iR\tau}\, du\, d\tau, \end{aligned}$$

where we define $\hat{a}(z,b,r) = \int a(z,b,r)e^{ir\tau}\, dr$. It follows from the definition of $\mathcal{S}_0^0$ that $|\hat{a}(z,b,\tau)| \le C_{N,M,x_0}e^{-N|\tau|}e^{-Md(z,x_0)}$ for any $N,M > 0$ and any given x_0.

Assume that z stays in a fixed compact set. Then for $(\tilde{z},b) = h^u \circ g^{\tau-\frac{\log(1+u^2)}{2}}(z,b)$, we have $\exp d(\tilde{z},x_0) \ge C(1+|u|)e^{|\tau|}$ with $C > 0$. We check this with a concrete example. For $(z,b) = e \in G$, we have

$$a_{\tau-\frac{\log(1+u^2)}{2}} n_u = \begin{pmatrix} \frac{e^{\tau/2}}{(1+u^2)^{1/4}} & \frac{ue^{\tau/2}}{(1+u^2)^{1/4}} \\ 0 & e^{-\tau/2}(1+u^2)^{1/4} \end{pmatrix}.$$

In the upper half plane model of $\mathbf{H}$, this element represents a unit tangent vector based at

$$\tilde{z} = \frac{e^\tau}{(1+u^2)^{1/2}} i + \frac{ue^\tau}{(1+u^2)^{1/2}}.$$

By formula 4.14, the hyperbolic distance of this point to the origin $x_0 = i$ is

$$\begin{aligned}\cosh d(\tilde{z}, i) &= 1 + \frac{1}{2}\left[\left(\frac{e^{\tau}}{(1+u^2)^{1/2}} - 1\right)^2 + \frac{u^2 e^{2\tau}}{1+u^2}\right](1+u^2)^{1/2} e^{-\tau} \\ &\geq \frac{1}{2}(1+u^2)^{1/2} e^{|\tau|}\end{aligned}$$

as required. By substitution,

$$|\hat{a}(\tilde{z}, b, 2\tau)| \leq C_{N,M}(1+u^2)^{-M/2} e^{-(N+M)|\tau|},$$

where $C_{N,M}$ is a uniform constant. Therefore, $\mathcal{L}a$ exists and is continuous in the variables (z, b, R). The geodesic flow is given by $g^t(b', b, t) = (b', b, t+\tau)$, so we deduce from the equality above that

$$|\hat{a} \circ g^t(\tilde{z}, b, 2\tau)| \leq C_{N,M}(1+u^2)^{-M/2} e^{-N|\tau| - M|t+\tau|}.$$

Using this estimate and our expression for $\mathcal{L}a$ above, we deduce that $|(\mathcal{L}a) \circ g^t(z, b, R)| \leq C_{z,b,R,M} e^{-M|t|}$ for fixed (z, b, R) and arbitrary $M > 0$. Hence, $PS\mathcal{L}a(ir, b, ir', b')$ is well-defined for any $r, r' \in \mathbf{C}$, $(b', b) \in B^{(2)}$. By definition of the geodesic flow G^t on $S\mathbf{D} \times \mathbf{R}$ and the fact that we defined $\mathcal{L}$ to satisfy $PS\mathcal{L}a(ir, b, ir', b') = \mathcal{W}a(ir, b, ir', b')$, it follows that

$$PS(G^t\mathcal{L}a)(ir, b, ir', b') = e^{-i\frac{(r^2 - r'^2)t}{2}} PS(\mathcal{L}a)(ir, b, ir', b') = e^{-i\frac{(r^2 - r'^2)t}{2}} \mathcal{W}a(ir, b, ir', b').$$

The definition of the quantum evolution implies that

$$\langle \mathrm{Op}(V^t a) e_{(ir,b)}, e_{(-ir',b')}\rangle = \langle \alpha^t(\mathrm{Op}(a)) e_{(ir,b)}, e_{(-ir',b')}\rangle = e^{\frac{it}{2}((-ir')^2 - (ir)^2)} \langle \mathrm{Op}(a) e_{(ir,b)}, e_{(-ir',b')}\rangle$$

and it follows from the definition of the Wigner distributions that

$$W_{(ir,b),(-ir',b')}(a) = e^{\frac{it}{2}(r^2 - r'^2)} W_{(ir,b),(-ir',b')}(V^t a).$$

By definition of the Wigner transform, we have

$$e^{-i\frac{(r^2 - r'^2)t}{2}} \mathcal{W}a(ir, b, ir', b') = \mathcal{W}(V^t a)(ir, b, ir', b').$$

By substitution and the fact that we defined $\mathcal{L}$ to satisfy $PS\mathcal{L}a(ir, b, ir', b') = \mathcal{W}a(ir, b, ir', b')$,

$$PS(G^t\mathcal{L}a)(ir, b, ir', b') = \mathcal{W}(V^t a)(ir, b, ir', b') = PS(\mathcal{L}V^t a)(ir, b, ir', b').$$

Inverting the Patterson-Sullivan transform (equation 4.11), we deduce the pointwise equality $G^t\mathcal{L}a = \mathcal{L}V^t a$.

Since the Wigner transform extends to $L^2_W(G \times \mathbf{R}, dg \times dp(r))$ as an isometry (equation 4.8) and $PS\mathcal{L}a(ir, b, ir', b') = \mathcal{W}a(ir, b, ir', b')$, we conclude that $\mathcal{L}$ is an isometry from $L^2_W(G \times \mathbf{R}, dg \times dp(r))$ onto its image $\mathcal{H}_{PS}(\mathbf{D}) = \mathcal{L}(L^2_W(G \times \mathbf{R}, dg \times dp(r)))$. Thus, $\mathcal{L} \circ V^t = G^t \circ \mathcal{L}$, where both sides are bounded operators from $L^2_W(G \times \mathbf{R}, dg \times dp(r))$ to $\mathcal{H}_{PS}(\mathbf{D})$.

4.2. **Intertwining of Γ-Invariant Symbols on the Quotient.** In this part, we investigate $\mathcal{L}$ on the quotient $\mathbf{X}_\Gamma$ by mimicking the constructions from the proof of Theorem 2.3. Based on the definition of the Hilbert space of Hilbert-Schmidt pseudo-differential operators, we provide an ad hoc definition of two Hilbert spaces $\mathcal{H}_W$ and $\mathcal{H}_{PS}$ that have the families $(W_{\nu_j,-\nu_k})$ and $(PS_{\nu_j,-\nu_k})$ as dual orthonormal bases and satisfy the following properties: V^t acts unitarily on $\mathcal{H}_W$, G^t acts unitarily on $\mathcal{H}_{PS}$, $\mathcal{L}$ sends $\mathcal{H}_W$ isometrically to $\mathcal{H}_{PS}$, and the intertwining relation $\mathcal{L} \circ V^t = G^t \circ \mathcal{L}$ holds on these spaces. We cannot directly define a Γ-invariant version of $\mathcal{L}$, which we denote by $\mathcal{L}_\Gamma$, on symbols because the kernel of $\mathcal{L}$ is not smooth and does not decay fast enough away from the diagonal. However, the adjoint intertwining operator is naturally defined on $PS_{\nu_j,-\nu_k}$, enabling us to define $\mathcal{L}_\Gamma$ by duality. This part culminates with the proof of Theorem 2.4.

To obtain intertwining on the quotient, we must adapt the symbols, quantization map, Patterson-Sullivan and Wigner distributions, and geodesic and Schrödinger flows. We proceed with each of these in turn.

Definition 4.15. Let χ on G be a smooth fundamental cutoff on G for the lift to G of the (irregular) fundamental domain of the action of Γ on G/K satisfying $\Pi\chi = 1$, where the periodization operator Π is defined by

$$\Pi\chi(g) = \sum_{\gamma\in\Gamma} \chi(\gamma g).$$

The definition of the periodization operator extends to functions of (z, b, r) as

$$\Pi\tilde{a}(z, b, r) = \sum_{\gamma\in\Gamma} \tilde{a}(\gamma \cdot z, \gamma \cdot b, r).$$

The image of $\mathcal{S}_0^0$ under Π is denoted by $\Pi\mathcal{S}_0^0$.

Definition 4.16. The space of symbols $\mathcal{S}_\alpha^0$ consists of the $a \in \mathcal{S}_0^0$ that satisfy a bound of the form

$$\sup_{(r,b)} e^{q|r|} \left| \frac{\partial^n}{\partial r^n} Da(\bullet, b, r) \right|_{\mathcal{C}^p(G/K)} < +\infty$$

in $\{|\mathrm{Im}(r)| < \epsilon/2\}$ for every $\epsilon > 0$, p, all $q > 0$, every nonnegative integer n, and every K-left-invariant differential operator D acting on B.

This definition strengthens the definition of $\mathcal{S}_0^0$ by requiring that symbols decay superexponentially fast in r instead of superpolynomially fast. For $a \in \mathcal{S}_\alpha^0$, and any fixed $g \in G$ and $R \in \mathbf{C}$ the map $t \mapsto \mathcal{L}a(ga_t, R)$, originally defined for $t \in \mathbf{R}$, has a holomorphic extension to $t \in \mathbf{C}$. This ensures that $(G^t\mathcal{L}a)(g, R)$ is well-defined for $R \in \mathbf{C}$, which is necessary for intertwining on the quotient.

Next, we describe the quantization map Op_Γ acting on Γ-invariant symbols. For a Γ-invariant symbol a, if $\mathrm{Op}(a)$ is a properly supported pseudo-differential operator, that is, $K_a(z, w)$ is supported in a fixed tube with $d(z, w) < R$ around the diagonal, where d denotes the hyperbolic distance between points, then $\mathrm{Op}(a)$ preserves the space of Γ-invariant functions and $\mathrm{Op}_\Gamma(a)$ is well-defined [14, 15]. We specify the action of such operators by examining their kernels. If $p \leq 1$, then the decay properties of the spherical function φ_o imply that $L^2(\mathbf{X}_\Gamma)$ can be continuously embedded in $(\mathcal{C}^p(G/K))'$. Therefore, a Γ-invariant kernel $K(z, w)$ with $\chi(z)K(z, w) \in \mathcal{K}_p^p(G/K \times G/K)$ naturally defines a bounded operator on the quotient $K_\Gamma : L^2(\mathbf{X}_\Gamma) \to L^2(\mathbf{X}_\Gamma)$. For $\phi \in L^2(\mathbf{X}_\Gamma)$, we define $K_\Gamma\phi$ by the identity

$$\langle K_\Gamma\phi, \psi\rangle := \langle \chi K\phi, \psi\rangle_{\mathbf{D}}$$

for all $\psi \in L^2(\mathbf{X}_\Gamma)$, where χ is a smooth fundamental cutoff for the lift of the fundamental domain. We rephrase this in terms of symbols. For a Γ-invariant symbol $a(z, b, r)$ with $\chi(z)a(z, b, r) \in \mathcal{S}_p^p$ for $p \leq 1$, we define the bounded operator $\mathrm{Op}_\Gamma(a) : L^2(\mathbf{X}_\Gamma) \to L^2(\mathbf{X}_\Gamma)$ by

$$\langle \mathrm{Op}_\Gamma(a)\phi, \psi\rangle_{\mathbf{X}_\Gamma} := \langle \chi\mathrm{Op}(a)\phi, \psi\rangle_{\mathbf{D}} = \langle \mathrm{Op}(\chi a)\phi, \psi\rangle_{\mathbf{D}}.$$

If $a(z,b,r)$ is Γ-invariant and $a(z,b,r) = \sum_{\gamma\in\Gamma} \tilde{a}(\gamma\cdot z, \gamma\cdot b, r)$ where $\tilde{a} \in \mathcal{S}^0_0$, we have

$$\langle \mathrm{Op}_\Gamma(a)\phi, \psi\rangle_{\mathbf{X}_\Gamma} = \langle \mathrm{Op}(\tilde{a})\phi, \psi\rangle_{\mathbf{D}}$$

for $\phi, \psi \in L^2(\mathbf{X}_\Gamma)$ and this is independent of the choice of $\tilde{a}$.

On the quotient, we will be undertaking similar analysis as that on the universal cover, replacing the distributions $PS_{e_{(\nu,b)},e_{(-\nu',b')}}$ by the family $PS_{\nu_j,-\nu_k}$.

Definition 4.17. $PS_{(j,\nu_j),(k,-\nu_k)}(db', dt, d\tau)$, abbreviated as $PS_{\nu_j,-\nu_k}$, is the right-Γ-invariant distribution on $B^{(2)} \times \mathbf{R} \sim G$ defined by

$$PS_{(j,\nu_j),(k,-\nu_k)}(db', db, d\tau) = \frac{T_{j,\nu_j}(db)\overline{T_{k,-\nu_k}(db')}}{|b-b'|^{1+\nu_j-\overline{\nu_k}}} e^{(\nu_j+\overline{\nu_k})\tau}\, d\tau$$

where $T_{j,\nu_j}(db)$ and $T_{k,-\nu_k}(db)$ are the Helgason boundary values for the pairs of eigenfunctions ϕ_j and ϕ_k, respectively.

The geodesic flow is given by $g^t(b', b, \tau) = (b', b, \tau+t)$, so the Patterson-Sullivan distributions are eigendistributions for the geodesic flow in the sense that

$$g^t_\# PS_{\nu_j,-\nu_k} = e^{-t(\nu_j+\overline{\nu_k})} PS_{\nu_j,-\nu_k}.$$

Therefore, $PS_{\nu_j,-\nu_k}$ induces an eigendistribution $PS^\Gamma_{\nu_j,-\nu_k}$ of the geodesic flow on $\Gamma\backslash G = S\mathbf{X}_\Gamma$ defined by

$$\int_{\Gamma\backslash G} a\, dPS^\Gamma_{\nu_j,-\nu_k} = \int_G (\chi a)\, dPS_{\nu_j,-\nu_k},$$

where χ is a smooth fundamental domain cutoff. We extend the PS-distributions to $S\mathbf{X}_\Gamma \times \mathbf{R}$ by considering $PS^\Gamma_{\nu_j,-\nu_k} \otimes \delta_{\frac{r_j+r_k}{2}}$ for real values of r_j, r_k. When r_j or r_k is imaginary, this formula will be generalized to $PS^\Gamma_{\nu_j,-\nu_k} \otimes \delta_{\frac{\nu_j-\overline{\nu_k}}{2i}}$, which technically is no longer a distribution. In order to exist, $PS^\Gamma_{\nu_j,-\nu_k} \otimes \delta_{\frac{\nu_j-\overline{\nu_k}}{2i}}$ must be paired with functions $a(z,b,r)$ that have a holomorphic extension to $r \in \mathbf{C}$. The Paley-Wiener Theorem indicates that this is the case if the Schwarz kernel $K_a(z,w)$ corresponding to the symbol a is smooth and decays rapidly away from the diagonal $\{z = w\}$.

As with the Patterson-Sullivan distributions, we replace the distributions $W_{(\nu,b),(\nu',b')}$ with the family $W_{j,k}$ on the quotient.

Definition 4.18. The Wigner distributions $W_{j,k}$ are defined on $S\mathbf{D} \times \mathbf{R} \simeq G \times \mathbf{R}$ by the formula

$$\int_{S\mathbf{D}\times\mathbf{R}} a\, dW_{j,k} := \langle \mathrm{Op}(a)\phi_j, \phi_k\rangle_{\mathbf{D}}$$

for $a \in \mathcal{S}^0_0$.

Since the distribution $W_{j,k}$ is invariant by the action of Γ on $S\mathbf{D}$, it can be used to define a distribution $W^\Gamma_{j,k}$ on the quotient $S\mathbf{X}_\Gamma \times \mathbf{R} \simeq \Gamma\backslash G \times \mathbf{R}$. If a is a smooth function on $\Gamma\backslash G \times \mathbf{R}$, we define

$$\int_{S\mathbf{X}_\Gamma\times\mathbf{R}} a\, dW^\Gamma_{j,k} = \int_{S\mathbf{D}\times\mathbf{R}} \chi a\, dW_{j,k},$$

where χ is a smooth fundamental cutoff for the action of Γ. In terms of the quantization on the quotient,

$$\int_{S\mathbf{X}_\Gamma\times\mathbf{R}} a\,dW^\Gamma_{j,k} = \langle \mathrm{Op}_\Gamma(a)\phi_j, \phi_k\rangle_{L^2(\mathbf{X}_\Gamma)}.$$

We describe the geodesic and Schrödinger flow on the quotient. We will denote by G^t_Γ the flow on $\Gamma\backslash G\times\mathbf{R}$ induced by G^t. On the quotient, V^t induces an operator V^t_Γ acting on $\Pi\mathcal{S}^0_0$, the space of Γ-invariant symbols obtained by periodizing elements of $\mathcal{S}^0_0$. More precisely, V^t_Γ is the unitary flow on $L^2(\Gamma\backslash G\times\mathbf{R}, dg\times dp(r))$ generated by D^V_Γ, which is a well-defined differential operator on $\Gamma\backslash G\times\mathbf{R}$ because it is induced by the left-invariant differential operator D^V on $G\times\mathbf{R}$. Its action can be described as follows: if $a=\Pi\tilde{a}$ with $\tilde{a}\in\mathcal{S}^0_0$, then we have $V^t_\Gamma a = \Pi V^t\tilde{a}$. This expression is independent of the choice of $\tilde{a}$.

Explicitly,

$$W^\Gamma_{j,k}(a) = W_{j,k}(\tilde{a}) = \int W_{e(\nu_j,b),e_{(-\nu_k,b')}}(\tilde{a})\,dT_{\nu_j}(db)\,\overline{dT_{-\nu_k}}(db')$$

and

$$PS^\Gamma_{\nu_j,-\nu_k}(a) = PS_{\nu_j,-\nu_k}(\tilde{a}) = \int PS_{e_{(\nu_j,b),e_{(-\nu_k,b')}}}(\tilde{a})\,dT_{\nu_j}(db)\,\overline{dT_{-\nu_k}}(db')$$

for any smooth and Γ-invariant a and any $\tilde{a}$ such that $a=\Pi\tilde{a}$ provided $\tilde{a}$ is smooth and decays fast enough so that all terms are well-defined.

We can finally motivate and define the Hilbert spaces that will be intertwined on the quotient. On the Hilbert space $\mathrm{HS}(\mathbf{X}_\Gamma)\simeq L^2(\mathbf{X}_\Gamma\times\mathbf{X}_\Gamma)$ of Hilbert-Schmidt operators on $\mathbf{X}_\Gamma$, the quantum evolution α^t has the orthonormal spectral expansion

$$\alpha^t = \sum_{j,k} e^{it\frac{(\nu_j-\overline{\nu_k}^2)}{2}}(\phi_j\otimes\phi_k^*)\otimes(\phi_j\otimes\phi_k^*)^*.$$

The Hilbert-Schmidt norm is defined by $\|A\|^2_{HS(\mathbf{X}_\Gamma)} = \mathrm{Tr}_{L^2(\mathbf{X}_\Gamma)}(AA^\dagger)$ and is associated with the scalar product $\langle A,B\rangle_{HS(\mathbf{X}_\Gamma)} = \mathrm{Tr}(AB^\dagger)$. For a Γ-invariant symbol a belonging to $\Pi\mathcal{S}^0_0$, we obtain the Hilbert-Schmidt operator $\mathrm{Op}_\Gamma(a)\in HS(\mathbf{X}_\Gamma)$ with norm

$$\|\mathrm{Op}_\Gamma(a)\|^2_{HS(\mathbf{X}_\Gamma)} = \mathrm{Tr}_{L^2(\mathbf{X}_\Gamma)}\mathrm{Op}_\Gamma(a)\mathrm{Op}_\Gamma(a)^\dagger = \sum_{j,k}|\mathrm{Tr}_{L^2(\mathbf{X}_\Gamma)}\mathrm{Op}_\Gamma(a)\phi_j\otimes\phi_k^*|^2 = \sum_{j,k}|W^\Gamma_{j,k}(a)|^2.$$

Intuitively, the Hilbert space $\mathcal{H}_W$ is the collection of all symbols a for which $\mathrm{Op}_\Gamma(a)$ is a Hilbert-Schmidt operator. More formally, we have the definition below.

Definition 4.19. The space $\mathcal{H}_W(\mathbf{X}_\Gamma)$ is the completion of the symbol space $\Pi\mathcal{S}^0_0$ with respect to the seminorm

$$\|a\|^2_W = \sum|W^\Gamma_{j,k}(a)|^2.$$

We define $\mathcal{H}_{PS}$ based on a seminorm that is analogous to the seminorm used in the definition of $\mathcal{H}_W$.

Definition 4.20. We define $\mathcal{H}_{PS}(\mathbf{X}_\Gamma)$ as the closure of $\Pi\mathcal{S}^0_0$ under the seminorm

$$\|f\|^2_{PS} = \sum_{j,k}|PS^\Gamma_{\nu_j,-\nu_k}(f)|^2.$$

Just as the definition of Patterson-Sullivan distributions depends on the choice of spectral parameters while the definition of the Wigner distributions does not, the definition of $\mathcal{H}_{PS}$ depends on the choice of spectral parameters ν_j for each j unlike $\mathcal{H}_W$. We make the standard choice: $\nu_j \in [0, 1/2] \cup i\mathbf{R}_+$.

Definition 4.21. The intertwining operator $\mathcal{L}$ induces an operator

$$\mathcal{L}_{\Gamma\#} : \mathcal{H}^*_{PS}(\mathbf{X}_\Gamma) \to \mathcal{H}^*_W(\mathbf{X}_\Gamma), \ \ \mathcal{L}_{\Gamma\#} PS_{\nu_j,-\nu_k} = W_{j,k}$$

on Γ-invariant distributions. Define the operator $\mathcal{L}_\Gamma : \mathcal{H}_W(\mathbf{X}_\Gamma) \to \mathcal{H}_{PS}(\mathbf{X}_\Gamma)$ to be the adjoint of $\mathcal{L}_{\Gamma\#}$.

Since $\mathcal{L}_{\Gamma\#}$ is an isometry from $\mathcal{H}^*_{PS}$ to $\mathcal{H}^*_W$, it follows that $\mathcal{L}_\Gamma$ is an isometry from $\mathcal{H}_W$ to $\mathcal{H}_{PS}$. The $W^\Gamma_{j,k}$ form an orthonormal basis of $\mathcal{H}^*_W$ and $\mathcal{L}_{\Gamma\#} PS^\Gamma_{\nu_j,-\nu_k} = W^\Gamma_{j,k}$, so the $PS^\Gamma_{\nu_j,-\nu_k}$ form an orthonormal basis of $\mathcal{H}^*_{PS}$.

We now prove Theorem 2.4. Let $a \in \Pi\mathcal{S}^0_\alpha$, that is, $a = \Pi\tilde{a}$ with $a \in \mathcal{S}^0_\alpha$. We choose this symbol class because it ensures that $G^t\mathcal{L}\tilde{a}(g, R)$ is well-defined for all $R \in \mathbf{C}$. It suffices to consider such symbols because it is proven in section 7 of [2] that $\Pi\mathcal{S}^0_\alpha$ is dense in $\Pi\mathcal{S}^0_0$ for the $\mathcal{H}_{PS}$-norm and is therefore dense in $\mathcal{H}_{PS}$. For all j and k, the definition of G^t implies that

$$PS_{\nu_j,-\nu_k}(G^t\mathcal{L}\tilde{a}) = e^{it\frac{\nu_j^2-\overline{\nu_k}^2}{2}} PS_{\nu_j,-\nu_k}(\mathcal{L}\tilde{a}).$$

Due to the definition of the intertwiner $\mathcal{L}$,

$$\begin{aligned} PS_{\nu_j,-\nu_k}(\mathcal{L}\tilde{a}) &= \int PS_{e_{(\nu_j,b)},e_{(-\nu_k,b')}}(\mathcal{L}\tilde{a})\, dT_{\nu_j}(db)\overline{dT_{-\nu_k}}(db') \\ &= \int W_{e_{(\nu_j,b)},e_{(-\nu_k,b')}}(\tilde{a})\, dT_{\nu_j}(db)\, \overline{dT_{-\nu_k}}(db') \\ &= W_{j,k}(\tilde{a}). \end{aligned}$$

By substitution,

$$PS_{\nu_j,-\nu_k}(G^t\mathcal{L}\tilde{a}) = e^{it\frac{\nu_j^2-\overline{\nu_k}^2}{2}} W_{j,k}(\tilde{a}) = W_{j,k}(V^t\tilde{a}) = PS_{\nu_j,-\nu_k}(\mathcal{L}\circ V^t\tilde{a}),$$

where the last two equalities use the definitions of V^t and $\mathcal{L}$ in the same manner as in the proof of Theorem 2.3. By the definition of $\mathcal{L}_\#$ and the fact that $PS^\Gamma_{\nu_j,-\nu_k}(\mathcal{L}_\Gamma a) = PS_{\nu_j,-\nu_k}(\mathcal{L}\tilde{a})$, we conclude that

$$\mathcal{L}_\Gamma \circ V^t_\Gamma = G^t_\Gamma \circ \mathcal{L}_\Gamma,$$

as an equality between operators from $\mathcal{H}_W(\mathbf{X}_\Gamma)$ to $\mathcal{H}_{PS}(\mathbf{X}_\Gamma)$.

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