SHARPENING THE CONVERGENCE TO EQUILIBRIUM ESTIMATE OF THE DISPERSION PROCESS

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Abstract. In this paper, we summarize the continuous time dispersion process introduced by Fei Cao and Jincheng Yang in [2] and apply the theory of Volterra integral equation to sharpen the convergence to equilibrium estimate.

Contents

1. Introduction

Let G_N be the complete graph on N vertices with vertex set labeling 1 to N. Let $M \in \mathbb{N}_+$ be the number of particles that inhabit the vertices of G_N. Denote by $X_i(t)$ the number of particles on vertex i at time $t \in \mathbb{R}^+$ and $\mu := M/N$ the average number of particles per vertex. At random times (generated by exponential law), each non-empty vertex i which is inhabited by at least two particles expels a particle at the rate X_i to another uniformly chosen vertex j.

We take the limit of the system as M and N go to infinity while keeping constant the ratio $\mu = M/N$. Consider the probability distribution function of particles:

$$\mathbf{p}(t) = (p_0(t), p_1(t), ..., p_n(t), ...)$$

with $p_n(t) = \{$"probability that a typical site has n particles at time t"$\}$. Observe that the evolution of $\mathbf{p}(t)$ is governed by the following system of nonlinear ordinary differential equations:

The probability distribution $\mathbf{p}(t)$ is governed by the deterministic system of nonlinear ordinary differential equations:

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$$\frac{d}{dt}\mathbf{p}_n(t) = \mathcal{L}[\mathbf{p}_n(t)] \tag{1.1}$$

with

$$\mathcal{L}[\mathbf{p}]_n = \begin{cases} -\left(\sum_{k\geq 2} kp_k\right)p_0 & n=0,\\ 2p_2 + \left(\sum_{k\geq 2} kp_k\right)p_0 - \left(\sum_{k\geq 2} kp_k\right)p_1, & n=1,\\ (n+1)p_{n+1} + \left(\sum_{k\geq 2} kp_k\right)p_{n-1} - \left(n + \sum_{k\geq 2} kp_k\right)p_n & n\geq 2. \end{cases}$$

For each case, the positive terms come from p_{n-1} gaining one particle and p_{n+1} losing one particle, and the negative term comes from p_n either gaining or losing a particle. The rigorous justification of the transition from stochastic process on finite graphs into the associated ODE system (1.1) requires the *propagation of chaos* property [3], which is beyond the scope of this paper.

This model can be utilized to analyze wealth distribution within an exchange economy. Let N represent the number of individuals in the economy and M the total capital. In this context, $X_i(t)$ denotes the wealth of individual i at time t, where a value of 1 corresponds to the subsistence-level income. The parameter μ represents the multiple of subsistence-level income per capita. To examine the behavior of an economy with large values of N and M, we consider the limit as N and M approach infinity, maintaining the ratio $\mu = M/N$ as constant. Variants of this model can be found in [6], [7], and [8].

2. Summary of the main results in [2]

We summarize (without proofs) the main results of [2] to set up for our improvement results in Section 3.

2.1. Properties of the ODE system.

In this subsection, we describe the conservation rules satisfied by (1.1) and derive an equivalent system of ODEs. We also discussed the equilibrium solution of (1.1) across different ranges of μ.

Lemma 2.1 (Conservation rules)**.** *If $\mathbf{p}(t)$ is a solution of the system (1.1), then*

$$\sum_{n=0}^{\infty} \mathcal{L}[\mathbf{p}]_n = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} n\mathcal{L}[\mathbf{p}]_n = 0.$$

In particular, the total probability mass and the average amount of particles per site are conserved. Thanks to these conservation relations, the solution $\mathbf{p}(t)$ lives in the space of probability distributions on $\mathbb{N}$, with prescribed mean value μ, defined by

$$\mathcal{S}_\mu := \left\{ \mathbf{p} \in [0,1]^{\mathbb{N}} \,\middle|\, \sum_{n=0}^{\infty} p_n = 1, \sum_{n=0}^{\infty} np_n = \mu \right\}.$$

More importantly, the system (1.1) will be equivalent to the following system of nonlinear ODEs:

$$\frac{d}{dt}\mathbf{p}(t) = \mathcal{L}[\mathbf{p}(t)] \tag{2.2}$$

in which

$$\mathcal{L}[\mathbf{p}]_n = \begin{cases} -(\mu - p_1)p_0 & n = 0, \\ 2p_2 + (\mu - p_1)p_0 - (\mu - p_1)p_1 & n = 1, \\ (n+1)p_{n+1} + (\mu - p_1)p_{n-1} - (n + \mu - p_1)p_n & n \geq 2. \end{cases} \tag{2.3}$$

Proposition 2.4 (Equilibrium distributions)**.** *The unique equilibrium solution of* (2.2) *and* (2.3) *in the space* $\mathcal{S}_\mu$*, for* $\mu \in (0, 1]$*, is the two-point Bernoulli distribution* $\mathbf{p}^* = (p_0^*, p_1^*, ..., p_n^*, ...)$ *where*

$$p_0^* = 1 - \mu, \quad p_1^* = \mu, \quad p_n^* = 0 \quad \textit{for } n \geq 2.$$

The unique equilibrium solution of (2.2) *and* (2.3) *in the space* $\mathcal{S}_\mu$*, when* $\mu \in (1, \infty)$*, is the zero-truncated Poisson distribution* $\overline{\mathbf{p}} = (\overline{p}_0, \overline{p}_1, ..., \overline{p}_n, ...)$ *where*

$$\overline{p}_0 = 0 \quad , \overline{p}_n = \frac{\nu^n}{n!} \cdot \frac{1}{e^\nu - 1} \quad \textit{for } n \geq 1$$

where $\nu = \mu + W_0(-\mu e^{-\mu})$ *and* $W_0(\cdot)$ *denotes the principal branch of the Lambert-W function.*

Proof. From the evolution equation defined by (2.2) and (2.3), it is straightforward to check that

$$np_n = (\mu - p_1)p_{n-1} \quad \text{for all } n \geq 2, \quad \text{and} \quad (\mu - p_1)p_0 = 0$$

must hold at equilibrium. This is done by setting all the equations of (2.3) to zero and recursively substitute $\mathcal{L}[\mathbf{p}]_{n-1}$ into $\mathcal{L}[\mathbf{p}]_n$.

On the one hand, if $\mu < 1$, then $p_0 \neq 0$ so $\mu - p_1$ must be zero. This implies that $p_n = 0$ for all $n \geq 2$ and we recover $\mathbf{p}^*$. If $\mu = 1$ and $p_0 \neq 0$, then we simply wait for p_0 to decrease. On the other hand, for $\mu > 1 \geq p_1$, we deduce that $p_0 = 0$, and the unique equilibrium distribution, is $\overline{\mathbf{p}}$ in the form of

$$\overline{p}_0 = 0, \quad \overline{p}_n = \frac{(\mu - \overline{p}_1)^{n-1}}{n!}\overline{p}_1 \quad \text{for } n \geq 2$$

where $\overline{p}_1 > 0$ is chosen such that $\overline{p} \in \mathcal{S}_\mu$. Since $\sum_{n\geq 0} \overline{p}_n = 1$, we deduce that $\overline{p}_1 e^{-\overline{p}_1} = \mu e^{-\mu}$, whence $\overline{p}_1 = -W_0(-\mu e^{-\mu})$. We finish the proof by introducing a new constant $\nu = \mu - \overline{p}_1$. □

2.2. **Main results of** [2]**.**

Theorem 2.5 (Convergence estimates)**.** *There exists a positive constant* C *depending only on* μ *and* $\mathbf{p}(0)$*, such that any solution* $\mathbf{p}(t)$ *to* (2.2) *and* (2.3) *with finite variance converges strongly (almost surely) to its equilibrium distribution as* $t \to \infty$*. To be precise, denoting* $\nu = \mu + W_0(-\mu e^{-\mu}) \in (\mu - 1, \mu)$ *for* $\mu > 1$ *and* $\langle t \rangle = \sqrt{1 + t^2}$ *for* $t \geq 0$*, we have:*

(1) If $0 < \mu < 1$*, then*

$$||\mathbf{p}(t) - \mathbf{p}^*||_{\ell^1} \leq Ce^{-2(1-\mu)t}.$$

(2) If $\mu = 1$*, then*

$$||\mathbf{p}(t) - \mathbf{p}^*||_{\ell^1} \leq Ct^{-1}.$$

(3) If $1 < \mu < 1 + \frac{1}{e-1}$ (equivalently, $0 < \nu < 1$), then

$$||\mathbf{p}(t) - \overline{\mathbf{p}}||_{\ell^1} \leq C\langle t\rangle^{\frac{1}{2}} e^{\nu t}.$$

(4) If $\mu \geq 1 + \frac{1}{e-1}$ (equivalently $\nu \geq 1$), then there exists $N > 0$ depending only on μ such that

$$||\mathbf{p}(t) - \overline{\mathbf{p}}||_{\ell^1} \leq C\langle t\rangle^{N+\frac{1}{2}} e^{-t}. \tag{2.6}$$

We intend to sharpen the estimate of the fourth case (2.6) to

$$||\mathbf{p}(t) - \overline{\mathbf{p}}||_{\ell^1} \leq C\langle t\rangle^{\frac{1}{2}} e^{-t}.$$

As such, we narrow our focus to results relating to the case $\nu \geq 1$.

2.3. **Probability generating function.** In this subsection, we repackage the ODE system (2.2)-(2.3) into a probability generating function and derive a PDE satisfied by said generating function.

Definition 2.7 (PGF of the dynamics)**.** Define the probability generating function $G : [0, +\infty) \times [-1, 1]$ of the solution $\mathbf{p}(t)$ to (2.2) and (2.3) by

$$G(t, z) = \sum_{n=0}^{\infty} p_n(t) z^n.$$

Since $p_n(t) \geq 0$ and $\sum_{n=0}^{\infty} p_n(t) = 1$, we know the above series is absolutely summable. Moreover, because $\sum_{n=0}^{\infty} np_n(t) = \mu$, we know that

$$\partial_z G(t, z) = \sum_{n=1}^{\infty} np_n(t) z^{n-1}$$

is absolutely summable. The ODE system (2.2) and (2.3) can thus be written as the following first-order PDE for G:

$$\partial_t G = (1 - z)[\partial_z G - (\mu - p_1(t))G - p_1(t)]. \tag{2.8}$$

Lemma 2.9 (Solution to the PDE for G)**.** *The probability generating function G can be expressed using the following explicit formula: for $z \in \mathbb{C}$ with $z \in [0, 2]$ and $t \in [0, \infty)$, we have*

$$G(t, 1 - z) = 1 + \left(G(0, 1 - ze^{-t}) - 1 - \mu z \int_0^t [v(s)]^{ze^{-t+s}} e^{-t+s} \mathrm{d}s\right) [v(t)]^{-z},$$

where the auxiliary function $v : [0, \infty) \to \mathbb{R}$ is defined by

$$v(t) := \exp\left(\int_0^t e^{-t+s}(\mu - p_1(s))\mathrm{d}s\right). \tag{2.10}$$

It turns out that this auxiliary function v holds the key to proving the convergence of (2.6).

2.4. **Convergence of the auxiliary function.** In this subsection, we show that the auxiliary function v satisfies an integral equation, and that it converges to a limit which depends on the value of μ. We also include some lemmas useful for our proof in Section 3

Lemma 2.11 (Integral equation of v)**.** *For $t \geq 0$, $v(t)$ satisfies*

$$v(t) = 1 - f_0(t) + f_0(0)e^{-\int_0^t \log v(s)\mathrm{d}s} + \mu \int_0^t [v(s)]^{e^{-t+s}} e^{-t+s}\mathrm{d}s, \tag{2.12}$$

where $f_0(t) = G(0, 1-e^{-t})$.

Lemma 2.13 (Fixed point of a integral expression)**.** *Let $\psi : (0,\infty) \to \mathbb{R}_+$ be a strictly increasing continuous function defined by*

$$\psi(x) := \mu \int_0^\infty x^{e^{-t}} e^{-t}\mathrm{d}t = \begin{cases} \mu \cdot \frac{x-1}{\log x}, & x > 0, x \neq 1 \\ \mu, & x = 1 \end{cases}.$$

For $\nu \geq 1$, we have that $\psi(e^\nu) = e^\nu$.

Lemma 2.14 (Limit of v)**.** *Let $\nu = \mu + W_0(-\mu e^{-\mu})$, then*

$$\lim_{t\to\infty} v(t) = e^\nu.$$

Lemma 2.15. *(Estimates for error function) Define*

$$e_2(t) := 1 - f_0(t) + f_0(0)e^{-\int_0^t \log v(s)\mathrm{d}s} s$$

to be the error function in the integral equation of v (2.12). We have that $e_2(t)$ is continuous and that $e_2(t) \leq Ce^{-t}$.

3. Sharpening convergence estimate when $\nu \geq 1$

In this subsection, we sharpen the estimate of (2.6) to

$$||\mathbf{p}(t) - \overline{\mathbf{p}}||_{\ell^1} \leq C\langle t\rangle^{\frac{1}{2}} e^{-t}. \tag{3.1}$$

Our main tool will be Theorem 11.7.2 [1]. This theorem deals with boundedness of solutions satisfying a large class of integral equations.

Theorem 3.2 (Theorem 11.7.2 [1])**.** *Assume that $\delta > 0$ and that the following hold:*

(i) For each $y \in \mathbb{R}^n$, the function $(t,s) \mapsto h(t,s,y)$ is Borel-measurable on $\mathbb{R}^+ \times \mathbb{R}^+$, and for each $(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+$ the function $y \mapsto h(t,s,y)$ is continuous on $\mathbb{R}^n$. The function b defined by

$$b(t,s) = \sup_{|y|\leq\delta} |h(t,s,y)|, \quad 0 \leq s \leq t < \infty,$$

satisfies $\sup_{t\in\mathbb{R}^+} \int_0^t |b(t,s)|\mathrm{d}s < \infty$ and for each $t > 0$,

$$\lim_{\tau\downarrow} \left(\int_t^{t+\tau} b(t,s)\mathrm{d}s + \int_0^t \sup_{|y|\leq\delta} |h(t+\tau,s,y) - h(t,s,y)|\mathrm{d}s \right) = 0.$$

(ii) $f \in \mathrm{BC}(\mathbb{R}^+;\mathbb{R}^n)$ *(f is bounded and continuous).*

(iii) $||f||_{\sup(\mathbb{R}^+)} + \sup_{t\geq 0}\int_0^t |b(t,s)|\mathrm{d}s \leq \delta$.

Then the equation

$$x(t) + \int_0^t h(t,s,x(s))\mathrm{d}s = f(t), \quad t \in \mathbb{R}^+,$$

has a continuous solution x satisfying $\sup_{t\in\mathbb{R}^+}|x(t)| \leq \delta$.

Proposition 3.3 (Sharpening convergence estimate of v)**.** *Recall that $\nu = \mu + W_0(-\mu e^{-\mu})$. For $\nu \geq 1$, we have the new estimate*

$$|v(t) - e^\nu| \leq Ce^{-t}.$$

Proof. By Lemma 2.14, $\lim_{t\to\infty} v(t) = e^\nu$. Consider $x(t) = (v(t) - e^\nu)e^t$. We aim to show that $x(t)$ is bounded using Theorem 3.2. We rewrite (2.12) as

$$x(t) = e_2(t)e^t - e^\nu e^t + \mu\int_0^t [x(s)e^{-s} + e^\nu]^{e^{-t+s}} e^s\mathrm{d}s \tag{3.4}$$

From Lemma 2.13, we have that $e^\nu = \mu\int_0^\infty (e^\nu)^{e^{-s}} e^{-s}\mathrm{d}s$. So (3.4) becomes

$$\begin{aligned} &e_2(t)e^t - \mu e^t\int_t^\infty e^{\nu e^{-s}}e^{-s}\mathrm{d}s - \mu\int_0^t e^{\nu e^{-t+s}}e^s\mathrm{d}s + \mu\int_0^t [x(s)e^{-s}+e^\nu]^{e^{-t+s}}e^s\mathrm{d}s \\ &= e_2(t)e^t + \frac{\mu}{\nu}\left(1 - e^{\nu e^{-t}}\right)e^t + \mu\int_0^t\left[[x(s)e^{-s}+e^\nu]^{e^{-t+s}} - e^{\nu e^{-t+s}}\right]e^s\mathrm{d}s.\end{aligned}$$

In the notation of Theorem 3.2, we let

$$f(t) = \left[e_2(t) + \frac{\mu}{\nu}\left(1 - e^{\nu e^{-t}}\right)\right]e^t \quad \text{and} \quad h(t,s,y) = -\mu\left[|ye^{-s}+e^\nu|^{e^{-t+s}} - e^{\nu e^{-t+s}}\right]e^s.$$

By (2.10), v is positive for all $t > 0$. As such, the solutions $x(t)$ to (3.4) must satisfy $x(s)e^{-s} + e^\nu > 0$. Therefore, we impose the absolute value on $h(t,s,y)$ so that y can take on negative value as in the condition of 3.2.

To apply Theorem 3.2, we show that for all sufficiently large $\delta > 0$, (i)-(ii)-(iii) are satisfied. For (iii) in particular, we show that the left-hand side, as a function of δ grows slower than δ. So let $\delta > 0$ be in the air for the moment. We have that

(i) For each $y \in \mathbb{R}$, $(t,s) \mapsto h(t,s,y)$ is continuous (as a composition of continuous functions) and therefore is Borel measurable on $\mathbb{R}^+ \times \mathbb{R}^+$. For each $(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+$, $y \mapsto h(t,s,y)$ is continuous on $\mathbb{R}$ since e^{-t+s} is non-negative. Define

$$b(t,s) = \sup_{|y|\leq\delta}|h(t,s,y)|, \quad 0 \leq s \leq t < \infty.$$

Observe that $h(t,s,y)$ is piecewise differentiable (in y) on $\mathbb{R}$, with a cusp at $y = -e^{\nu+s}$. Furthermore, h is monotonically increasing on the left of the cusp and monotonically decreasing on the right of the cusp. As such, the candidates for the maximizer of $|h|$ on $[-\delta,\delta]$ are $-e^{\nu+s}$ and $\pm\delta$. We can bound

$$b(t,s) \leq |h(t,s,-\delta)| + |h(t,s,\delta)| + |h(t,s,-e^{\nu+s})|.$$

Observe that $e^{\nu+s} \leq \delta$ only when $s \leq \log(\delta) - \nu$. As such, this bound is only significant for $t \leq \log(\delta) - \nu$. We compute separately

$$\begin{aligned} \int_0^{0\wedge(\log(\delta)-\nu)} |h(t,s,-e^{\nu+s})| \mathrm{d}s &= \mu \int_0^{0\wedge(\log(\delta)-\nu)} e^{\nu e^{-t+s}} e^s \mathrm{d}s \\ &\leq \mu \int_0^{0\wedge(\log(\delta)-\nu)} e^{\nu} e^s \mathrm{d}s \\ &\leq \mu e^{\nu} |\delta e^{-\nu} - 1|, \\ &= \mu|\delta - e^{\nu}| \end{aligned} \tag{3.5}$$

$$\begin{aligned} \int_0^t |h(t,s,\pm\delta)| \mathrm{d}s &= \mu \int_0^t \left| |\pm\delta e^{-s} + e^{\nu}|^{e^{-t+s}} - e^{\nu e^{-t+s}} \right| e^s \mathrm{d}s \\ &\leq \mu \int_0^t |\delta e^{-s}|^{e^{-t+s}} e^s \mathrm{d}s + 2\mu \int_0^t e^{\nu e^{-t+s}} e^s \mathrm{d}s \\ &\leq \mu\delta + 2\mu e^{\nu} \end{aligned} \tag{3.6}$$

Thus, $\sup_{t\in\mathbb{R}^+} \int_0^t b(t,s)\mathrm{d}s \leq \mu(\delta + 5e^{\nu}) < \infty$. Finally, observe that $h(t,s,\pm\delta)$ and $h(t,s,-e^{\nu+s})$ are continuous in s, and are bounded and continuous in t. As such

$$\lim_{\tau\to 0} \left(\int_t^{t+\tau} b(t,s)\mathrm{d}s + \int_0^t \sup_{|y|\leq\delta} |h(t+\tau,s,y) - h(t,s,y)| \mathrm{d}s \right) = 0.$$

(ii) From Lemma 2.15, we have that $e_2(t)$ is continuous and that $e_2(t) \leq \mu e^{-t}$. Next, we use the Taylor expansion to rewrite

$$\frac{\mu}{\nu}(1 - e^{\nu e^{-t}}) = -\frac{\mu}{\nu}\left(\nu e^{-t} + O(e^{-2t})\right) = -\mu e^{-t} + O(e^{-2t})$$

This suffices to show that $f \in BC(\mathbb{R}^+;\mathbb{R}^n)$

(iii) Using (3.5), (3.6) and the fact that f is bounded, we rewrite condition (iii) as

$$\begin{aligned} ||f||_{\sup(\mathbb{R}^+)} + \sup_{t\in\mathbb{R}^+} \int_0^t |b(t,s)|\mathrm{d}s &\leq ||f||_{\sup(\mathbb{R}^+)} + \sup_{t\in\mathbb{R}^+} \int_0^t |h(t,s,-e^{\nu+s})|\mathrm{d}s \\ &\quad + \sup_{t\in\mathbb{R}^+} \int_0^t |h(t,s,-\delta)|\mathrm{d}s + \sup_{t\in\mathbb{R}^+} \int_0^t |h(t,s,\delta)| ds \\ &\leq ||f||_{\sup(\mathbb{R}^+)} + 5\mu e^{\nu} + 2\mu \sup_{t\in\mathbb{R}^+} \int_0^t |\delta e^{-s}|^{e^{-t+s}} e^{-s} \mathrm{d}s \\ &= C + 2\mu \sup_{t\in\mathbb{R}^+} a(t,\delta) \leq \delta. \end{aligned} \tag{3.7}$$

To show that there exists a δ such that $\delta - 2\mu \sup_{t\in\mathbb{R}^+} a(t,\delta) \geq C$, it suffices to show that $\sup_{t\in\mathbb{R}^+} a(t,\delta)$ grows slower than $\frac{\delta}{2\mu}$. This is the content of Appendix A.

Having checked conditions (i)-(ii)- (iii), we obtain the estimate $\sup_{t\in\mathbb{R}^+} |x(t)| \leq \delta$ for some $\delta > 0$. Equivalently, we get that $|v(t) - e^\nu| \leq \delta e^{-t}$ for all $t \geq 0$, as desired. ☐

Proposition 3.8. *There exists constant $C > 0$ depending on μ such that*

$$y_2(t) + |v(t) - e^\nu| + |v'(t)| \leq Ce^{-t}.$$

Proof. First, we have that

$$y_2(t) = \int_0^t |v(s) - e^\nu| e^{-2t+2s} \mathrm{d}s \leq \int_0^t Ce^{-s} e^{-2t+2s} \mathrm{d}s \leq Ce^{-t}.$$

For the derivative estimate, we differentiate v using (2.12) and get

$$\begin{aligned} v'(t) &= e_2'(t) + \mu v(t) - \mu \int_0^t [v(s)]^{e^{-t+s}} e^{-t+s} \mathrm{d}s - \mu \int_0^t [v(s)]^{e^{-t+s}} \log(v(s)) e^{-2t+2s} \mathrm{d}s \\ &= e_2'(t) + (\mu - 1)v(t) + \mu e_2(t) - \mu \int_0^t [v(s)]^{e^{-t+s}} \log(v(s)) e^{-2t+2s} \mathrm{d}s, \end{aligned}$$

in which

$$e_2'(t) = -f_0'(t) - f_0(0) \exp\left(-\int_0^t \log v(s) \mathrm{d}s\right) \log v(t) = O\left(e^{-t}\right).$$

Note that $x \mapsto x^\alpha \log x$ is Lipschitz on $[1, e^\mu]$ uniformly for $\alpha \in [0,1]$, hence

$$\begin{aligned} &\left| \mu \int_0^t [v(s)]^{e^{-t+s}} \log(v(s)) e^{-2t+2s} \mathrm{d}s - \mu \int_0^t \nu e^{\nu e^{-t+s}} e^{-2t+2s} \mathrm{d}s \right|. \\ &\leq C\mu \int_0^t |v(s) - e^\nu| e^{-2t+2s} \mathrm{d}s = Cy_2(t). \end{aligned}$$

Therefore,

$$\begin{aligned} |v'(t)| \leq& |e_2'(t)| + (\mu - 1)|v(t) - e^\nu| + \mu e_2(t) + Cy_2(t) \\ &+ (\mu - 1)e^\nu - \mu\nu \int_0^t e^{\nu e^{-s}} e^{-2s} \mathrm{d}s \\ &\leq |e_2'(t)| + (\mu - 1)|v(t) - e^\nu| + \mu e_2(t) + Cy_2(t) + \mu\nu \int_t^\infty e^{\nu e^{-s}} e^{-2s} \mathrm{d}s \\ &\leq Ce^{-t}. \end{aligned}$$

This completes the proof. ☐

As a corollary, we deduce the following convergence rate of $p_1(t) \to \mu - \nu$ as $t \to \infty$.

Corollary 3.9. *For $t \geq 0$, we have*

$$|p_1(t) - \mu + \nu| \leq Ce^{-t}, \nu \geq 1.$$

Proof. It suffices to notice that $\mu - p_1(t)$ can be recovered from v via

$$\mu - p_1(t) = \frac{v'(t)}{v(t)} + \log v(t).$$

Since both $v'(t) \xrightarrow{t\to\infty} 0$ and $v(t) \xrightarrow{t\to\infty} e^{\nu}$ occur at this rate, the result follows. ☐

Theorem 3.10. *Let $\nu \geq 1$. Recall that*

$$||\mathbf{p}(t) - \overline{\mathbf{p}}||_{\ell^2}^2 = \sum_{n=0}^{\infty} (p_n(t) - \overline{p}_n).$$

There exists a constant C depending on μ such that for all $t \geq 0$, it holds that

$$||\mathbf{p}(t) - \overline{\mathbf{p}}||_{\ell^2}^2 \leq Ce^{-t}.$$

Proof. We first recall the classical Parseval's identity:

$$||\mathbf{p}(t) - \overline{\mathbf{p}}||_{\ell^2}^2 = \frac{1}{2\pi} \int_0^{2\pi} \left| G(t, e^{i\theta}) - G_{\overline{\mathbf{p}}}(e^{i\theta}) \right|^2 d\theta.$$

By Lemma 2.9, we have

$$(G(t, 1-z) - 1)[v(t)]^z = G(0, 1 - ze^{-t}) - 1 - \mu z \int_0^t [v(s)]^{ze^{-t+s}} e^{-t+s} \mathrm{d}s$$

for all $z \in \mathbb{C}$ with $|z - 1| \leq 1$. Notice that

$$|G(0, 1 - ze^{-t}) - 1| \leq \partial_z G(0, 1)|z|e^{-t} \leq Ce^{-t},$$

$$\left| \int_0^t [v(s)]^{ze^{-t+s}} e^{-t+s} \mathrm{d}s - \int_0^t e^{\nu z e^{-t+s}} e^{-t+s} \mathrm{d}s \right| \leq C|z| \int_0^t |v(s) - e^{\nu}| e^{-2t+2s} \mathrm{d}s \leq C y_2(t),$$

$$\left| \mu \int_0^t e^{\nu z e^{-t+s}} e^{-t+s} \mathrm{d}s - \psi\left(e^{\nu z}\right) \right| \leq e^{-t} \left| \psi\left(e^{\nu z e^{-t}}\right) \right| \leq Ce^{-t}.$$

On the other hand, we know for $z \in \mathbb{C}$ with $|z - 1| \leq 1$ that

$$|(G(t, 1-z) - 1)[v(t)]^z - (G(t, 1-z) - 1)e^{\nu z}| \leq C|v(t) - e^{\nu}|.$$

Assembling these estimates, we proved for $z \in \mathbb{C}$ with $|z - 1| \leq 1$ that

$$|(G(t, 1-z) - 1)e^{\nu z} + z\psi\left(e^{\nu z}\right)| \leq Ce^{-t}.$$

Since

$$G_{\overline{\mathbf{p}}}(1-z) = \frac{1}{e^{\nu} - 1} \sum_{n=1}^{\infty} \frac{\nu^n}{n!}(1-z)^n = \frac{e^{\nu(1-z)} - 1}{e^{\nu} - 1} = 1 - e^{\nu z} z\psi\left(e^{\nu z}\right),$$

the above implies uniform convergence of $G(t, 1-z)$ to $G_{\overline{\mathbf{p}}}(1-z)$ for all $z \in \mathbb{C}$ with $|z - 1| \leq 1$, which shows that $\mathbf{p}(t)$ converges to $\overline{\mathbf{p}}$ in ℓ^2 by Parseval's identity.

☐

Corollary 3.11. *Let $\nu \geq 1$. There exist a constant C depending on μ such that for all $t \geq 0$, it holds that*

$$||\mathbf{p}(t) - \overline{p}||_{\ell^1} \leq C\langle t\rangle^{\frac{1}{2}} e^{-t}.$$

Proof. For $x \in \mathbb{N}$ to be specified later, we have

$$||\mathbf{p}(t) - \overline{\mathbf{p}}||_{\ell^1} \leq ||\mathbf{p}(t) - \overline{\mathbf{p}}||_{\ell^1([0,x])} + ||\mathbf{p}(t) - \overline{\mathbf{p}}||_{\ell^1([x,\infty))}$$
$$\leq 2||\mathbf{p}(t) - \overline{\mathbf{p}}||_{\ell^1([0,x])} + 2||\overline{\mathbf{p}}||_{\ell^1([x,\infty))}$$

The first term is easily controlled by the ℓ^2 norm:

$$||\mathbf{p}(t) - \overline{\mathbf{p}}||_{\ell^1([0,x])} \leq \sqrt{x}||\mathbf{p}(t) - \overline{\mathbf{p}}||_{\ell^2}.$$

The second term is amenable to explicit computations, leading us to

$$||\mathbf{p}||_{\ell^1([x,\infty])} = \sum_{n=x}^{\infty} \overline{p}_n = \frac{1}{e^\nu - 1}\sum_{n=x}^{\infty}\frac{\nu^n}{n!}.$$

Thanks to the Chernoff bound for the Poisson distribution, for $x \geq \nu$ it holds that

$$\sum_{n=x}^{\infty}\frac{\nu^n e^{-\nu}}{n!} \leq \frac{(e\nu)^x e^{-\nu}}{x^x}.$$

We know for our zero-truncated Poisson distribution that

$$||\overline{p}||_{\ell^1([x,\infty])} \leq \frac{1}{e^\nu - 1}\left(\frac{e\nu}{x}\right)^x.$$

Finally, setting $x = [t \wedge \nu e^2]$ allows us to deduce that $||\overline{\mathbf{p}}||_{\ell^1([x,\infty])} \leq Ce^{-t}$, whence

$$||\mathbf{p}(t) - \overline{\mathbf{p}}||_{\ell^1} \leq C\langle t\rangle^{\frac{1}{2}} e^{-t}.$$

□

Appendix A. Growth of a in (3.7)

Lemma A.1. *Let $0 \leq a \leq b < \infty$. We have that*

$$\int_a^b \frac{e^w}{w^2}\mathrm{d}w = \frac{e^a}{a} - \frac{e^b}{b} + \mathrm{Ei}(b) - \mathrm{Ei}(a),$$

where Ei *is the exponential integral.*

Proof. By integration by parts, we can write

$$\int_a^b \frac{e^w}{w^2}\mathrm{d}w = \left[-\frac{e^w}{w}\right]_a^b + \int_a^b \frac{e^w}{w}\mathrm{d}w = \frac{e^a}{a} - \frac{e^b}{b} + \mathrm{Ei}(b) - \mathrm{Ei}(a).$$

□

Lemma A.2. *Suppose f_1, f_2, g_1 and g_2 are positive functions on $\mathbb{R}^+$ such that $f_1(x) \leq O(g_1(x))$ and $f_2(x) = O(g_2(x))$. Furthermore, suppose that f_1 is monotonically increasing, $g_1(cx) = O(g_1(x))$ for all $c > 0$ and g_2 is unbounded up at infinity. Then*

$$f_1 \circ f_2 = O(g_1 \circ g_2).$$

Proof. For sufficiently large x, we have that

$$f_1(f_2(x)) \leq f_1(Cg_2(x)) \leq C' g_1(Cg_2(x)) \leq C'' g_1(g_2(x)).$$

□

Proposition A.3. *Recall the function $a(t, \delta)$ from (3.7). We have that $\sup_{t \in \mathbb{R}^+} a(t, \delta)$ grows slower than $\frac{\delta}{2\mu}$.*

Proof. We can bound

$$\begin{aligned} a(t,\delta) &\leq \int_0^t (\delta)^{e^{-t+s}} e^{-s} \mathrm{d}s \\ &= e^{-t} \int_{e^{-t}}^1 \frac{\delta^u}{u^2} \mathrm{d}u =: b(t,\delta). \end{aligned} \tag{A.4}$$

For fixed δ, observe that $b(t, \delta)$ is differentiable for $t \in (0, \infty)$. As such, for each $\delta \in \mathbb{R}^+$, the supremum in t occurs either at 0, $+\infty$ or a stationary point. We show that at each of these points, $b(-, \delta)$ grows slower that than $\frac{\delta}{2\mu}$.

First, observe that $b(0, \delta) = 0$. Next, as t goes to infinity, we can split $b(t, \delta)$ as

$$\begin{aligned} e^{-t} \int_{e^{-t}}^c \frac{\delta^u}{u^2} \mathrm{d}u + e^{-t} \int_c^1 \frac{\delta^u}{u^2} \mathrm{d}u &\leq e^{-t} \int_{e^{-t}}^c \frac{\delta^c}{u^2} \mathrm{d}u + e^{-t} \int_c^1 \frac{\delta}{u^2} \mathrm{d}u \\ &= \left(1 - \frac{e^{-t}}{c}\right) \delta^c + \left(\frac{1}{c} - 1\right) \delta \\ &\leq C\delta^c + \left(\frac{1}{c} - 1\right) \delta \end{aligned}$$

where $c \in (0, 1)$ and C is some fixed constant. We can get $\frac{1}{c} - 1$ to be arbitrarily small by taking c close to 1. As such, $\lim_{t \to \infty} b(t, \delta)$ has arbitrarily small linear part. Using Lemma A.1, we can write explicitly

$$b(t,\delta) = e^{-t} \left[\delta^{e^{-t}} e^t - \delta + \log(\delta) \left[\mathrm{li}(\delta) - \mathrm{li}\left(\delta^{e^{-t}}\right)\right]\right],$$

where $\mathrm{li}(x) = \mathrm{Ei}(\log(x))$ is the logarithmic integral function. We can now find the stationary points of f by setting

$$\begin{aligned} 0 = \frac{\partial}{\partial t} f(t,y) = -e^{-t} \int_{e^{-t}}^1 \frac{\delta^u}{u^2} \mathrm{d}u + \delta^{e^{-t}} &= -b(t,\delta) + \delta^{e^{-t}} \\ &= \delta e^{-t} + e^{-t} \log(\delta) \left[\mathrm{li}\left(\delta^{e^{-t}}\right) - \mathrm{li}(\delta)\right] \end{aligned}$$

Rearranging gives us

$$\mathrm{li}(\delta) - \frac{\delta}{\log(\delta)} = \mathrm{li}\left(\delta^{e^{-t}}\right).$$

By [4], $\delta \mapsto \mathrm{li}(\delta) - \frac{\delta}{\log(\delta)}$ grows as $O\left(\frac{\delta}{\log(\delta)^2}\right)$. So for sufficiently large δ, the left-hand side is arbitrarily large. Since li is bijective for sufficiently large argument, we can take the inverse li^{-1} to get that the single stationary point (in t) of $b(t,\delta)$ is

$$t_* = -\log\left[\frac{\log\left(B(\delta)\right)}{\log(\delta)}\right],$$

where $B(\delta) = \mathrm{li}^{-1}\left(\mathrm{li}(\delta) - \frac{\delta}{\log(\delta)}\right)$. So for sufficiently large y, we have that

$$b(t_*, \delta) = B(\delta).$$

By [5], $\mathrm{li}^{-1}(\delta)$ grows as $O(\delta \log(\delta))$. Using Lemma A.2, we deduce that

$$B(\delta) = O\left(\frac{\delta}{\log(x)^2} \cdot \left[\log(\delta) - 2\log_2(\delta)\right]\right) = O\left(\frac{\delta}{\log(\delta)}\right),$$

where $\log_2 = \log \circ \log$. This concludes the proof. □

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