RING OPERADS AND SYMMETRIC BIMONOIDAL CATEGORIES

KAILIN PAN

Abstract. We generalize the classical operad pair theory to a new model for E_∞ ring spaces, which we call ring operad theory, and relate it with the classical operad pair theory so that the classical multiplicative infinite loop machine can be applied to algebras over any E_∞ ring operad. As an application, we show that classifying spaces of all tight symmetric bimonoidal categories with strict zero and unit elements are directly homeomorphic to some E_∞ ring space in the ring operad sense. Therefore, beginning with the category of projective modules over some commutative ring R, we get an alternative construction of the algebraic K-theory ring spectrum $K(R)$.

Contents

1. Introduction

In [7], Peter May established the operad theory to detect zero spaces of spectra. Roughly speaking, any E_∞ space (algebra over some E_∞ operad) admits a group completion that has an n-fold delooping for any natural number n, and thus homotopy equivalent to the zero space of some connective spectrum. Therefore, any E_∞ space determines a connected spectrum and thus a generalized cohomology theory. As an example, the classifying space of some symmetric monoidal category admits an E_∞ space structure as shown in [1]. In particular, beginning with the category of projective modules over some commutative ring R with direct sum, we get the additive algebraic K-theory spectrum $K(R)$.

In practice, instead of additive generalized cohomology theory, we are more interested in generalized cohomology theory with a good cup-product, which is represented by an E_∞ ring spectrum. This motivates a multiplicative infinite loop space theory to detect zero spaces of E_∞ ring spectra. Several different models for multiplicative infinite loop space theory have been established. We list some of them.

(1) The operad pair theory [9, 11].
(2) The infinite categorical multiplicative infinite loop space machine [2].
(3) The moperad and bioperad theory [14].

Each of these three theories defines a notion E_∞ ring space and constructs for any E_∞ ring space a ring completion that is homotopic to the zero space of some E_∞ ring spectrum. However, the comparison between these three definitions of E_∞ ring space is still unknown.

As mentioned above, the additive algebraic K-theory spectrum $K(R)$ comes from the symmetric monoidal category of projective modules over some commutative ring R with direct sum. It turns out that $K(R)$ admits an E_∞ ring structure that is closely related to the symmetric bimonoidal structure of the category of projective R-modules with direct sum and tensor product. Hence, we expect the classifying space of any symmetric bimonoidal category admits an E_∞ ring space structure.

This holds for the infinite categorical machine. In the bioperad theory, there exists a bioperad $\mathscr{P}^{bi}$ in the category of small categories whose algebras are precisely bipermutative categories, so classifying spaces of bipermutative categories are E_∞ ring spaces in the bioperad sense. As shown in [9, Section VI.3], any symmetric bimonoidal category is functorially equivalent to some bipermutative category, so the bioperad theory also provides a construction of algebraic K-theory ring spectra from symmetric bimonoidal categories.

However, a similar construction does not work directly in the classical operad pair theory. There is no operad pair in the category of small categories whose algebra category is either symmetric bimonoidal categories or bipermutative categories. A construction from bipermutative categories to E_∞ ring spaces in the operad pair sense is shown in [11, 13], in which Peter May proves that the classifying of a bipermutative category is homotopy equivalent to some E_∞ ring spaces in the operad pair sense.

In this paper, we generalize the classical operad pair theory to a new model for E_∞ ring spaces, which we call ring operads. This new theory provides an alternative construction from bipermutative categories to E_∞ ring spectra.

Informally, we develop a new definition of "E_∞ ring spaces" using this new ring operad theory. It turns out that this new notion of "E_∞ ring spaces" coincides with the classical one in the operad pair sense up to homotopy, so we can apply the classical delooping machine to E_∞ ring spaces in the new sense. As an application, we show that classifying spaces of all tight symmetric bimonoidal categories with strict zero and unit elements are directly homeomorphic to some E_∞ ring space in the new sense. Therefore, we get an alternative construction from bipermutative categories to E_∞ ring spectra.

To compare this new construction with the classical one, recall that in [11] the classical construction is based on a passage from bipermutative categories to $\mathscr{F}\wr\mathscr{F}$-categories. Here $\mathscr{F}\wr\mathscr{F}$ is a specific small category defined in Definition 3.2.1. This passage is done by first construct a lax functor from $\mathscr{F}\wr\mathscr{F}$ to the category of small categories and then strictify it to get a genuine functor, and we will show that in this new construction, the ring operad theory gives a new interpretation of this lax functor. See Remark 4.2.7 for detail.

1.1. **Statement of results.** In analogy with classical operad theory, starting with any symmetric monoidal category, we define a ring operad $\mathscr{C}$ to be a collection of objects $\{\mathscr{C}(f)\}$ indexed not by natural numbers but by some collection of polynomials, together with structure maps satisfying several commutative diagrams. See Definition 2.1.7. We also define a notion of algebras over ring operads in Definition 2.1.9 compared with algebras over classical operads.

Similarly, in the category of spaces, we call a ring operad to be E_∞ if its all components are contractible and the structure maps satisfy several freeness and cofibration conditions. Less formally, an algebra over such an E_∞ ring operad is precisely a topological space with two binary operations $+, \times$ such that all unit, associativity, commutativity, and distributivity laws hold up to all higher homotopies.

The following comparison theorem plays an essential role in the ring operad theory.

Theorem 1.1.1 (2.2.4)**.** *The categories of algebras over any two E_∞ ring operads $\mathscr{C}, \mathscr{C}'$ have equivalent homotopy categories. Moreover, any $\mathscr{C}$ algebra is homotopy equivalent to some $\mathscr{C}'$ algebra.*

To compare the ring operad theory with classical operad pair theory, we prove the following theorem.

Theorem 1.1.2 (3.1.6, 3.1.7)**.** *For any operad pair $(\mathscr{C}, \mathscr{G})$, there is a ring operad $\mathscr{R}_{\mathscr{C},\mathscr{G}}$ which is E_∞ when $(\mathscr{C}, \mathscr{G})$ is E_∞ such that their categories of algebras are isomorphic.*

As a corollary, combining the above two theorems and the classical multiplicative infinite loop machine, we get a multiplicative infinite loop machine for all E_∞ ring operad.

Theorem 1.1.3 (3.1.8)**.** *For any E_∞ ring operad $\mathscr{C}$, there exists a functor $\mathbb{E}$ from the category of $\mathscr{C}$-algebras to the category of E_∞ ring spectra such that*

$$X \to \Omega^\infty \mathbb{E}(X)$$

is a ring completion.

As applications, we construct two ring operads in the category of small categories whose algebras are precisely tight symmetric bimonoidal categories with strict zero and unit elements and bipermutative categories.

Theorem 1.1.4 (4.1.4, 4.1.5, 4.1.7, 4.2.2)**.** *There exists two ring operads in the category of small categories $\mathscr{S}, \mathscr{P}$ such that $\mathscr{S}$-algebras are precisely tight symmetric bimonoidal categories with strict zero and unit elements and $\mathscr{P}$-algebras are precisely bipermutative categories. Moreover, after applying the classifying space functor $B = |N(-)|$, both $B\mathscr{S}$ and $B\mathscr{P}$ are E_∞. Therefore, the classifying space* *of a tight symmetric bimonoidal category with strict zero and unit elements admits a ring completion to the zero space of some E_∞ ring spectrum.*

Here tight means the distributivity laws are given by natural isomorphisms. Moreover, this new construction coincides with the classical one.

Theorem 1.1.5 (4.2.7)**.** *The two constructions from bipermutative categories to E_∞ ring spectra coincide up to homotopy.*

With this operadic description of tight symmetric bimonoidal categories with strict zero and unit elements and bipermutative categories, we reprove the strictification theorem in [9, Section VI.3] from symmetric bimonoidal categories to bipermutative categories.

Theorem 1.1.6 (4.2.6)**.** *There is a functor Φ from the category of tight symmetric bimonoidal categories (with strict unit and zero objects) to the category of bipermutative categories and a natural equivalence $\eta : \Phi C \to C$ of symmetric bimonoidal categories.*

1.2. **Organization of the paper.** We now summarize the contents of the paper.

In Section 2, we build up the foundations of ring operad theory.

The precise definition of ring operad and algebras over ring operads are given in Section 2.1. Recall that an operad can be regarded as a functor on the category Σ of finite sets and symmetric groups together with some structure maps. However, in the ring operad theory, the category Σ is generalized to a rather complicated category $\widehat{\mathcal{R}}$.

In section 2.2, we define the notion of E_∞ ring operad and state the most fundamental and useful theorem in the theory of ring operads: the Comparison Theorem 1.1.1. We defer its proof to Appendix A.

More precisely, we analyze the combinatorial properties of $\widehat{\mathcal{R}}$ in Section A.1. In particular, we construct a filtration of a sub-category $\widehat{\mathcal{R}}_{n.d.}$ of $\widehat{\mathcal{R}}$ in Proposition A.1.4. Using this filtration, we construct a filtration of the monad $\mathbb{C}X$ associated to an E_∞ ring operad $\mathscr{C}$ in Section A.2. This plays an essential role in the proof of the Comparison Theorem 1.1.1.

In Section 3, we compare ring operad theory with two classical theories: operad pair theory and category of ring operator theory.

In Section 3.1, we construct for any (E_∞) operad pair $(\mathscr{C}, \mathscr{G})$, a (an E_∞) ring operad $\mathscr{R}_{\mathscr{C},\mathscr{G}}$ such that their categories of algebras are isomorphic. Therefore, applying the Comparison Theorem 1.1.1, we get the multiplicative infinite loop machine Theorem 3.1.8 for any E_∞ ring operad.

In Section 3.2, we compare ring operad theory with category of ring operator theory that was used in the construction from bipermutative categories to E_∞ ring spaces in the operad pair sense in [13]. More precisely, we construct for any E_∞ ring operad $\mathscr{C}$ a category of ring operators $\tilde{\mathscr{C}}$ such that their categories of algebras have the same homotopy categories. We summarize this by the following diagram in which all functors induce equivalences on homotopy categories.

$$\begin{array}{ccc}
\mathscr{C}\text{-spaces} & \xrightarrow{\nu} & \text{special } \tilde{\mathscr{C}}\text{-spaces} \\
\downarrow & & \downarrow \\
(\mathscr{C} \times \mathscr{R}_{\mathscr{K},\mathscr{L}})\text{-spaces} & \xrightarrow{\nu} & \text{special } (\widetilde{\mathscr{C} \times \mathscr{R}_{\mathscr{K},\mathscr{L}}}) = (\tilde{\mathscr{C}} \times_{\mathscr{F}\wr\mathscr{F}} \tilde{\mathscr{R}}_{\mathscr{K},\mathscr{L}})\text{-spaces} \\
\uparrow & & \uparrow \\
\mathscr{R}_{\mathscr{K},\mathscr{L}}\text{-spaces} & \xrightarrow{\nu} & \text{special } \tilde{\mathscr{R}}_{\mathscr{K},\mathscr{L}} = (\hat{\mathscr{L}} \wr \hat{\mathscr{K}})\text{-spaces} \\
 & \nwarrow \quad (\mathscr{K},\mathscr{L})\text{-spaces} \quad \nearrow &
\end{array}$$

In Section 4, we apply the ring operad theory to classical category theory.

In Section 4.1, we construct a ring operad $\mathscr{S}$ in the category of small categories whose algebras are precisely tight symmetric bimonoidal categories (defined in [4, Volume I, Definition 2.1.2]) with strict zero and unit elements such that the classifying space $B\mathscr{S}$ is E_∞. Here tight means the distributivity maps are isomorphisms. Therefore, we get a multiplicative infinite loop machine for tight symmetric bimonoidal categories with strict zero and unit elements.

In Section 4.2, we mimic the above construction to get a ring operad $\mathscr{P}$ whose algebras are bipermutative categories. Hence, we get a multiplicative infinite loop machine for bipermutative categories. We state a comparison between this new construction and the classical one in Remark 4.2.7. Lastly, we use this description to reprove the strictification theorem from symmetric bimonoidal categories to bipermutative categories.

2. Foundations of ring operads

2.1. **Definition of ring operads.** Recall that in classical operad theory, to describe the higher homotopy, we put all operators that we want to identify up to higher homotopy into a contractible space. This strategy also works in the multiplicative context. Let X be a space with two binary operators $+, \times$. In order to describe a higher homotopical distributivity law

$$a \times b + a \times c \simeq a \times (b + c),$$

we construct a contractible space of trinary operators over X which contains the above two trinary operators

$$(a, b, c) \to a \times b + a \times c,$$
$$(a, b, c) \to a \times (b + c).$$

Therefore, to describe all higher homotopical associativity, commutativity and distributivity laws, we need a collection of contractible spaces of multivariable operators over X.

Different from classical operad theory, this collection of contractible spaces is not indexed by natural numbers but by polynomials.

Warning 2.1.1. Less formally, each polynomial f determines a type of multivariable operators over X and then we can assign to each f a space that parameterizes all multivariable operator operators of that type. However, for some polynomials, the associated space is not necessarily contractible. For example, consider a path in the space associated to $x+y$ connecting

$$(x,y)\mapsto x+y \quad \text{and} \quad (x,y)\mapsto y+x.$$

Then after evaluating both x and y to be x, the above path gives a loop in the space associated to $2x$ starting at

$$x\mapsto x+x.$$

This loop is not necessarily contractible in general.

Therefore, to define a ring operad as a collection of contractible spaces indexed by polynomials, we expect the index set to only consist of those good polynomials whose associated spaces are indeed contractible. It turns out the following collection of good polynomials is the largest one that is closed under composition.

Notation 2.1.2. Fix a set of variables $\{a_{i,j}: 1\leqslant i\leqslant j\}$ and let $\mathbb{Z}[a_{1,n},a_{2,n},\cdots,a_{n,n}]$ for $n\geqslant 0$ be the polynomial ring on n variables. Consider the subset $\mathcal{R}(n)\subset \mathbb{Z}[a_{1,n},a_{2,n},\cdots,a_{n,n}]$, which consists of all polynomials f such that f is a finite sum of different monic monomials with positive degree and each monomial in this sum is a product of different variables; that is,

$$\mathcal{R}(n)=\left\{\sum_{I=(i_1,i_2,\cdots,i_n)\in\{0,1\}^n\setminus\{0\}^n}\varepsilon_I a_{1,n}^{i_1}\cdots a_{n,n}^{i_n}\in\mathbb{Z}[a_{n,1},\cdots,a_{n,n}]:\varepsilon_I=0\text{ or }1\right\}.$$

To avoid confusion, let $0_n\in\mathcal{R}(n)$ denote the zero element in $\mathcal{R}(n)$. We write $|f|=n$ for $f\in\mathcal{R}(n)$.

Note that the collection of all monomials with positive degree that is a product of different variables is one-to-one corresponding to the set of non-empty subset of n variables $\{a_{1,n},a_{2,n},\cdots,a_{n,n}\}$, so $\mathcal{R}(n)$ is one-to-one corresponding to the power set $P(P\{a_{1,n},a_{2,n},\cdots,a_{n,n}\}\setminus\{\emptyset\})$, and thus $\mathcal{R}(n)$ is finite with cardinality 2^{2^n-1}.

There is also a non-symmetric operad structure on $\mathcal{R}(n)$ defined by the composition of polynomials, which generalize the sum of natural numbers in the classical theory.

Notation 2.1.3. The composition of polynomials is given by

$$\circ:\mathcal{R}(k)\times\mathcal{R}(j_1)\times\cdots\times\mathcal{R}(j_k)\to\mathcal{R}(j_+)$$
$$(g,f_1,\cdots,f_k)\mapsto g(f_1(a_{j_+,1},\cdots,a_{j_+,j_1}),\cdots,f_k(a_{j_+,j_1+\cdots+j_{k-1}+1},\cdots,a_{j_+,j_+}))$$

where $j_+:=j_1+j_2+\cdots+j_k$.

Lemma 2.1.4. *$\coprod\mathcal{R}(n)$ is closed under composition.*

Proof. By definition, a polynomial $f\in\mathbb{Z}[a_{1,n},a_{2,n},\cdots,a_{n,n}]$ is contained in $\mathcal{R}(n)$ if and only if the following conditions hold:

(1) $\frac{\partial^2}{\partial a_{k,n}^2}f=0$ for each $k=1,2,\cdots,n$,

(2)
$$\frac{\partial^{|I|}}{\partial a_{I,n}}f(0,\cdots,0)=0\text{ or }1$$
for each sequence of finite length $I=(i_1,\cdots,i_l)$, $1\leqslant i_1<\cdots<i_l\leqslant n$, $l\geqslant 1$,

(3) $f(0,0,\cdots,0)=0$.

Using this criterion, we can check that $g(f_1,\cdots,f_k)$ is contained in $\mathcal{R}(j_+)$ and the associativity diagram commutes.

Indeed, when $l=j_1+\cdots+j_{s-1}+r$, $1\leqslant r\leqslant j_s$,

$$\begin{aligned}&\frac{\partial^2}{\partial a_{l,j_+}^2}g(f_1,\cdots,f_k)\\=&\frac{\partial}{\partial a_{l,j_+}}\left(\frac{\partial}{\partial a_{s,k}}g(f_1,\cdots,f_k)\cdot\frac{\partial}{\partial a_{r,j_s}}f\right)\\=&\frac{\partial^2}{\partial a_{s,k}^2}g(f_1,\cdots,f_k)\cdot\left(\frac{\partial}{\partial a_{r,j_s}}f\right)^2+\frac{\partial}{\partial a_{s,k}}g(f_1,\cdots,f_k)\cdot\frac{\partial^2}{\partial a_{r,j_s}^2}f\\=&0.\end{aligned}$$

Also, when $I=(i_{1,1},\cdots,i_{r_1,1},\cdots,i_{1,k},\cdots,i_{r_k,k})$ with

$$j_1+\cdots+j_{s-1}<i_{1,s}<\cdots<i_{r_s,s}\leqslant j_1+\cdots+j_s$$

for any $1\leqslant s\leqslant k$, we write $i'_{t,s}:=i_{t,s}-(j_1+\cdots+j_{s-1})$ and

$$L:=(l_1,\cdots,l_m)=\{s:j_s\geqslant 1\}.$$

Then

$$\begin{aligned}&\frac{\partial^{|I|}}{\partial a_{I,j_+}}g(f_1,\cdots,f_k)(0,\cdots,0)\\=&\left(\frac{\partial^m}{\partial a_{L,k}}g(f_1,\cdots,f_k)\cdot\prod_{j_s\geqslant 1}\frac{\partial^{r_s}}{\partial a_{i'_{1,s},j_s}\cdots\partial a_{i'_{r_s,s},j_s}}f_s\right)(0,\cdots,0)\\=&0\text{ or }1.\end{aligned}$$

Here we need to use the fact that $f_s(0,\cdots,0)=0$.

Lastly, we automatically have

$$g(f_1,\cdots,f_k)(0,\cdots,0)=0.$$

Thus γ gives a non-symmetric operad structure. □

The Σ_j actions in the classical operad theory seem to be more difficult in the multiplicative context because there are more relations between these polynomials. To summarize these relations, we define the following small category.

Notations 2.1.5. Let $\widehat{\mathcal{R}}$ be a small category with object set $\coprod_{n\geqslant 0}\mathcal{R}(n)$. Let $\mathbf{Set}_e$ be the category of set with two based points $\{0,e\}$ (that is the under category of $\{0,e\}$) and

$$\mathbf{n}_e:=\{0,e,1,2\cdots,n\}$$

be and object in $\mathbf{Set}_e$ for $n\geqslant 0$. Note that for each map $\phi\in\mathbf{Set}_e(\mathbf{m}_e,\mathbf{n}_e)$ the induced map

$$\begin{aligned}\phi_*:\mathbb{Z}[a_{1,m},a_{2,m},\cdots,a_{m,m}]&\to\mathbb{Z}[a_{1,n},a_{2,n},\cdots,a_{n,n}]\\f(a_{1,m},a_{2,m},\cdots,a_{m,m})&\mapsto f(a_{\phi(1),n},a_{\phi(2),n},\cdots,a_{\phi(m),n})\end{aligned}$$

is functorial; that is $\phi_*\circ\psi_*=(\phi\circ\psi)_*$. Here $a_{0,n}=0$ and $a_{e,n}=1$.

For each $f_m\in\mathcal{R}(m)$, $f_n\in\mathcal{R}(n)$, we define the hom-set to be

$$\widehat{\mathcal{R}}(f_m,f_n):=\{(f_m,\phi,f_n):\phi\in\mathbf{Set}_e(\mathbf{m}_e,\mathbf{n}_e),\phi_*(f_m)=f_n\}.$$

The unit and composition are induced by those in $\mathbf{Set}_e$.

We say a map $\phi \in \mathbf{Set}_e(\mathbf{m}_e, \mathbf{n}_e)$ is effective if $\phi^{-1}(0) = \{0\}, \phi^{-1}(e) = \{e\}$ and denote the sub-category of effective morphisms by $\widehat{\mathcal{R}}_{eff}$. Moreover, we say a map $\phi \in \mathbf{Set}_e(\mathbf{m}_e, \mathbf{n}_e)$ singular if it is surjective and if $1 \leqslant i < j \leqslant m$ with $\phi(i), \phi(j) \geqslant 1$, then $\phi(i) < \phi(j)$. Then any $\phi \in \mathbf{Set}_e(\mathbf{m}_e, \mathbf{n}_e)$ has a unique decomposition $\phi = p \circ \sigma$ such that σ is singular and p is effective. We say $\phi = p \circ \sigma$ is the canonical decomposition of ϕ.

Lemma 2.1.6. *(1) For $f \in \mathbb{Z}[a_{1,m}, a_{2,m}, \cdots, a_{m,m}]$, if $\phi_* f \in \mathcal{R}(n)$ for some injection $\phi : \mathbf{m}_e \to \mathbf{n}_e$, then $f \in \mathcal{R}(m)$.*

(2) For $(f_m, \phi, f_n) \in \widehat{\mathcal{R}}(f_m, f_n)$ with $\phi = p \circ \sigma$ the canonical decomposition of ϕ, $\sigma_* f_m \in \coprod \mathcal{R}(n)$.

Proof. When ϕ is injective, ϕ_* is also injective on monomials. Therefore, to show (1), it suffices to check this lemma for monomials f, which holds immediately by definition.

To see (2), we write f_m as a sum of different monic monomials m_k. Then it suffices to show that nonzero elements in $\sigma_* m_k$ are distinct monic monomials in $\coprod \mathcal{R}(n)$.

Indeed, since σ is singular, $\sigma_* m_k$ is either 1 or a monic monomial in $\coprod \mathcal{R}(n)$ (probably zero). If $\sigma_* m_k = 1$, then

$$p_* \sigma_* f_m(0, \cdots, 0) = f_n(0, \cdots, 0) \neq 0.$$

If $\sigma_* m_k = \sigma_* m_l \notin \{0, 1\}$, then the coefficient of monomial $p_* \sigma_* m_k$ in the standard decomposition of f_n is at least 2. Both contradict with $f_n \in \coprod \mathcal{R}(n)$. □

We will further focus on the properties of $\widehat{\mathcal{R}}_{eff}$ in section A.1.

In the rest of this section, we modify the definitions in classical operad theory to multiplicative context. Compare with [7, Chapter 1,2].

Let $(\mathscr{V}, \otimes, *)$ be a symmetric monoidal category such that the unit object $*$ is also terminal.

Definition 2.1.7. A ring operad $\mathscr{C}$ in $\mathscr{V}$ consists of a functor $\mathscr{C} : \widehat{\mathcal{R}} \to \mathscr{V}$, a unit map $\eta : * \to \mathscr{C}(a_{1,1})$, together with compositions

$$\gamma : \mathscr{C}(g) \otimes \mathscr{C}(f_1) \otimes \cdots \otimes \mathscr{C}(f_k) \to \mathscr{C}(g(f_1, f_2, \cdots, f_k))$$

for all $f_1, f_2, \cdots, f_k \in \mathrm{Obj}(\widehat{\mathcal{R}})$, $g \in \mathcal{R}(k) \subset \mathrm{Obj}(\widehat{\mathcal{R}})$, $k = 1, 2, \cdots$, such that $\mathscr{C}(0_n) \cong *$ for all $n \geqslant 0$ and the following diagrams commute:

(1) Associativity diagram:

For $g \in \mathcal{R}(k)$, $f_s \in \mathcal{R}(j_s)$, $s = 1, 2, \cdots, k$, $h_t \in \mathrm{Obj}(\widehat{\mathcal{R}})$, $t = 1, 2, \cdots, \Sigma j_s$, let $J_s := j_1 + \cdots + j_s$, $F_s := f_s(h_{J_{s-1}+1}, \cdots, h_{J_s})$, $G := g(f_1, f_2, \cdots, f_k)$. Then we have $G(h_1, \cdots, h_J) = g(F_1, \cdots, F_k) =: \tilde{G}$,

$$\begin{array}{ccc}
\mathscr{C}(g) \otimes (\bigotimes_{s=1}^{k} \mathscr{C}(f_s)) \otimes (\bigotimes_{t=1}^{j} \mathscr{C}(h_t)) & \xrightarrow{\gamma\otimes \mathrm{id}} & \mathscr{C}(G) \otimes (\bigotimes_{t=1}^{j} \mathscr{C}(h_t)) \\
\downarrow \text{shuffle} & & \\
\mathscr{C}(g) \otimes \bigotimes_{s=1}^{k} (\mathscr{C}(f_s) \otimes (\bigotimes_{t=1}^{j_s} \mathscr{C}(h_{J_{s-1}+t}))) & & \downarrow \gamma \\
\downarrow \mathrm{id}\otimes\gamma^k & & \\
\mathscr{C}(g) \otimes \bigotimes_{s=1}^{k} (\mathscr{C}(F_s)) & \xrightarrow{\gamma} & \mathscr{C}(\tilde{G})
\end{array}$$

(2) Unit diagrams:
For $g \in \mathcal{R}(k)$,

$$\begin{array}{ccc}
\mathscr{C}(g) \otimes (*)^{\otimes k} & \xrightarrow{\cong} & \mathscr{C}(g) \\
\downarrow \mathrm{id}\otimes\eta^k & \nearrow \gamma & \\
\mathscr{C}(g) \otimes (\mathscr{C}(a_{1,1}))^{\otimes k} & &
\end{array}$$

and

$$\begin{array}{ccc}
* \otimes \mathscr{C}(g) & \xrightarrow{\cong} & \mathscr{C}(g) \\
\downarrow \eta\otimes\mathrm{id} & \nearrow \gamma & \\
\mathscr{C}(a_{1,1}) \otimes \mathscr{C}(g) & &
\end{array}$$

(3) Equivariance diagrams:
Let $g_m \in \mathcal{R}(m)$, $g_n \in \mathcal{R}(n)$, $f_1 \in \mathcal{R}(j_1)$, $h_1 \in \mathcal{R}(r_1)$, $\cdots$, $f_n \in \mathcal{R}(j_n)$, $h_n \in \mathcal{R}(r_n)$ and $(g_m, \psi, g_n) \in \mathrm{Mor}(\widehat{\mathcal{R}})$. When $\phi^{-1}(e) = e$, let $f_0 = 0_0$. Then,

$$\begin{array}{ccc}
\mathscr{C}(g_m) \otimes \mathscr{C}(f_1) \otimes \cdots \otimes \mathscr{C}(f_n) & \xrightarrow{\mathscr{C}(\psi)\otimes\mathrm{id}} & \mathscr{C}(g_n) \otimes \mathscr{C}(f_1) \otimes \cdots \otimes \mathscr{C}(f_n) \\
\downarrow \mathrm{id}\otimes\Delta & & \\
\mathscr{C}(g_m) \otimes \mathscr{C}(f_1)^{\otimes|\psi^{-1}(1)|} \otimes \cdots \otimes \mathscr{C}(f_n)^{\otimes|\psi^{-1}(n)|} & & \\
\downarrow \text{shuffle} & & \downarrow \gamma \\
\mathscr{C}(g_m) \otimes \mathscr{C}(f_{\psi(1)}) \otimes \cdots \otimes \mathscr{C}(f_{\psi(m)}) & & \\
\downarrow \gamma & & \\
\mathscr{C}(g_m(f_{\psi(1)}, \cdots, f_{\psi(m)})) & \xrightarrow{\mathrm{id}} & \mathscr{C}(g_n(f_1, f_2, \cdots, f_n))
\end{array}$$

When ϕ is singular with $\phi^{-1}(0) = 0$, let $f_e = a_{1,1}$. Then

$$\begin{array}{ccc}
\mathscr{C}(g_m)\otimes\mathscr{C}(f_1)\otimes\cdots\otimes\mathscr{C}(f_n) & \xrightarrow{\mathscr{C}(\psi)\otimes\mathrm{id}} & \mathscr{C}(g_n)\otimes\mathscr{C}(f_1)\otimes\cdots\otimes\mathscr{C}(f_n) \\
\downarrow{\scriptstyle \mathrm{id}\otimes\eta} & & \\
\mathscr{C}(g_m)\otimes\mathscr{C}(f_{\psi(1)})\otimes\cdots\otimes\mathscr{C}(f_{\psi(m)}) & & \downarrow{\scriptstyle \gamma} \\
\downarrow{\scriptstyle \gamma} & & \\
\mathscr{C}(g_m(f_{\psi(1)},\cdots,f_{\psi(m)})) & \xrightarrow{\mathscr{C}(\tilde{\psi})} & \mathscr{C}(g_n(f_1,f_2,\cdots,f_n))
\end{array}$$

Here we let $\phi^{-1}(e)=\{e\}\cup\{i_1<\cdots<i_s\}$ and let $j'_t:=j_{\psi(t)}$ for $t=1,2,\cdots,m$ with $j_e=1$. Then we define $\tilde{\psi}:\mathbf{j'_1}+\cdots+\mathbf{j'_m}\to\mathbf{j_1}+\cdots+\mathbf{j_n}$ by sending

$$\sum_{l=1}^{i_k} j'_l\mapsto e$$

for $k=1,\cdots,s$ and sending other elements bijectively order-preservingly to $\{1,\cdots,n\}\subset\mathbf{n}_e$.

For $(f_1,\phi_1,h_1),\cdots,(f_n,\phi_n,h_n)\in\mathrm{Mor}(\widehat{\mathcal{R}})$,

$$\begin{array}{ccc}
\mathscr{C}(g_n)\otimes\mathscr{C}(f_1)\otimes\cdots\otimes\mathscr{C}(f_k) & \xrightarrow{\gamma} & \mathscr{C}(g_n(f_1,\cdots,f_k)) \\
\downarrow{\scriptstyle \mathrm{id}\otimes\mathscr{C}(\phi_1)\otimes\cdots\otimes\mathscr{C}(\phi_k)} & & \downarrow{\scriptstyle \mathscr{C}(\phi_1\oplus\cdots\oplus\phi_k)} \\
\mathscr{C}(g_n)\otimes\mathscr{C}(h_1)\otimes\cdots\otimes\mathscr{C}(h_k) & \xrightarrow{\gamma} & \mathscr{C}(g_n(h_1,\cdots,h_k))
\end{array}$$

Here the diagonal morphisms $\mathscr{C}(f_s)\to\mathscr{C}(f_s)^{\otimes|\psi^{-1}(s)|}$ is defined to be the counit $\mathscr{C}(f_s)\to *$ (recall that $*$ is terminal) if $\psi^{-1}(s)$ is empty, and

$$\phi_1\oplus\cdots\oplus\phi_k:\mathbf{j_1}+\cdots+\mathbf{j_n}\to\mathbf{r_1}+\cdots+\mathbf{r_n}$$
$$x+\sum_{t=1}^{k}j_t\mapsto\begin{cases}\phi_{k+1}(x)+\sum_{t=1}^{k}r_t & \text{if } 1\leqslant x\leqslant j_{k+1} \text{ and } \phi_{k+1}(x)\notin\{0,e\},\\ \phi_{k+1}(x) & \text{if } \phi_{k+1}(x)\in\{0,e\}\end{cases}$$

defines morphism

$$(g_n(f_1,\cdots,f_k),\phi_1\oplus\cdots\oplus\phi_n,g_n(h_1,\cdots,h_k))\in\mathrm{Mor}(\widehat{\mathcal{R}}).$$

Moreover, a morphism between two ring operads is a natural transformation between functors which preserves all other structure maps.

From now on, we assume our ground category $\mathscr{V}$ to be cocomplete and also assume that the symmetric monoidal structure $\bullet\otimes\bullet$ is cocontinuous on each variable. Let κ_0,κ_e be two copies of $*$ and let $S^0:=\kappa_0\coprod\kappa_e$. We define $\mathscr{V}_e$ to be the under category of $\mathscr{V}$ with respect to S^0; that is the category of objects X in $\mathscr{V}$ with two specific morphisms

$$\epsilon:\kappa_e\to X$$
$$\iota:\kappa_0\to X$$

and morphisms that preserve ϵ and ι.

Remark 2.1.8. In the ring operad context, slightly different from the classical operad theory, we regard a ring operad $\mathscr{C}$ as a covariant functor from $\widehat{\mathcal{R}}$ and $X^{\otimes n}$

for some (X, ϵ, ι) in $\mathscr{V}$ a contravariant functor by shuffling the components; that is, for $(f_m, \phi, f_n) \in \widehat{\mathcal{R}}(f_m, f_n)$, $f_m \in \mathcal{R}(m)$, $f_n \in \mathcal{R}(n)$,

$$\phi^* : X^{\otimes n} = X_1 \otimes X_2 \otimes \cdots \otimes X_n \to X_{\phi(1)} \otimes X_{\phi(2)} \otimes \cdots \otimes X_{\phi(m)} \to X^{\otimes m},$$

Here $X_0 = \kappa_0$, $X_e = \kappa_e$ with

$$\epsilon : X_e \to X$$
$$\iota : X_0 \to X$$

the structure maps of X.

Definition 2.1.9. Let $\mathscr{C}$ be a ring operad. A $\mathscr{C}$-algebra is an object (X, ϵ, ι) in $\mathscr{V}_e$ together with morphisms in $\mathscr{V}$

$$\theta : \mathscr{C}(f) \otimes X^{\otimes j} \to X$$

for $f \in \mathcal{R}(j)$ such that the following diagrams commute in $\mathscr{V}$:
(1) Associativity diagram:
For $g \in \mathcal{R}(k)$, $f_s \in \mathcal{R}(j_s)$, $s = 1, 2, \cdots, k$, let $J := j_1 + \cdots + j_k$,

$$\begin{array}{ccc}
\mathscr{C}(g) \otimes (\bigotimes_{s=1}^{k} \mathscr{C}(f_s)) \otimes X^{\otimes J} & \xrightarrow{\gamma \otimes \mathrm{id}} & \mathscr{C}(g(f_1, f_2, \cdots, f_k)) \otimes X^{\otimes J} \\
\downarrow \text{shuffle} & & \\
\mathscr{C}(g) \otimes \bigotimes_{s=1}^{k} (\mathscr{C}(f_s) \otimes X^{\otimes j_s}) & & \downarrow \theta \\
\downarrow \mathrm{id} \otimes \theta^k & & \\
\mathscr{C}(g) \otimes X^{\otimes k} & \xrightarrow{\theta} & X
\end{array}$$

(2) Unit diagram:
For $g \in \mathcal{R}(k)$,

$$\begin{array}{ccc}
* \otimes X & \xrightarrow{\cong} & X \\
\downarrow \eta \otimes \mathrm{id} & \nearrow \theta & \\
\mathscr{C}(a_{1,1}) \otimes X & &
\end{array}$$

(3) Equivariance diagram:
For $g_m \in \mathcal{R}(m)$, $g_n \in \mathcal{R}(n)$, $(g_m, \psi, g_n) \in \mathrm{Mor}(\widehat{\mathcal{R}})$,

$$\begin{array}{ccc}
\mathscr{C}(g_m) \otimes X^{\otimes n} & \xrightarrow{\mathscr{C}(\psi) \otimes \mathrm{id}} & \mathscr{C}(g_n) \otimes X^{\otimes n} \\
\downarrow \psi^* & & \downarrow \theta \\
\mathscr{C}(g_m) \otimes X^{\otimes m} & \xrightarrow{\theta} & X
\end{array}$$

The category of all $\mathscr{C}$-algebras will be denoted by $\mathscr{C}[\mathscr{V}_e]$.

Then we can define a monad $\mathbb{C}$ associated to any ring operad $\mathscr{C}$ in $\mathscr{V}_e$.

Definition 2.1.10. Let $\mathscr{C}$ be a ring operad in $\mathscr{V}$. We define the monad $\mathbb{C} : \mathscr{V}_e \to \mathscr{V}_e$ associated to $\mathscr{C}$ as the following coend. For any $(X, \epsilon, \iota) \in \mathscr{V}_e$,

$$\mathbb{C}X := \mathscr{C}(\bullet) \otimes_{\widehat{\mathcal{R}}^{op}} X^{\bullet}.$$

The unit and composition are induced by

$$\begin{aligned}
\eta :& X = * \otimes X \to \mathscr{C}(a_{1,1}) \otimes X \to \mathbb{C}X,\\
\mu :& \mathscr{C}(f) \otimes \bigotimes_i (\mathscr{C}(g_i) \otimes X^{|g_i|}) \to \mathscr{C}(f(g_1, \cdots, g_{|f|})) \otimes X^{|g_1|+\cdots+|g_{|f|}|}.
\end{aligned}$$

The structure map $S^0 \to \mathbb{C}X$ is the unique one such that $\eta : X \to \mathbb{C}X$ is a morphism in $\mathscr{V}_e$.

We denote the category of $\mathbb{C}$-algebras in $\mathscr{V}$ by $\mathbb{C}[\mathscr{V}_e]$.

Lemma 2.1.11. *$\mathbb{C} : \mathscr{V}_e \to \mathscr{V}_e$ is a well-defined monad.*

Proof. This holds immediately from Definition 2.1.7. More precisely, the multiplication μ is well-defined by the equivariance diagrams, the associativity and unit diagrams

$$\begin{array}{ccc}
\mathbb{C}\mathbb{C}\mathbb{C}X & \xrightarrow{\mu} & \mathbb{C}\mathbb{C}X \\
\downarrow{\scriptstyle \mathbb{C}\mu} & & \downarrow{\scriptstyle \mu} \\
\mathbb{C}\mathbb{C}X & \xrightarrow{\mu} & \mathbb{C}X
\end{array}
\qquad
\begin{array}{ccccc}
\mathbb{C}X & \xrightarrow{\eta} & \mathbb{C}\mathbb{C}X & \xleftarrow{\mathbb{C}\eta} & \mathbb{C}X \\
& {\scriptstyle id}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle id} & \\
& & \mathbb{C}X & &
\end{array}$$

commute from the associativity and unit diagrams in Definition 2.1.7. ☐

Proposition 2.1.12. *Let $\mathscr{C}$ be a ring operad in $\mathscr{V}$ with associated monad $\mathbb{C}$. Then their categories of algebras are isomorphic; that is,*

$$\mathbb{C}[\mathscr{V}_e] \cong \mathscr{C}[\mathscr{V}_e].$$

Proof. The proof of the corresponding proposition in the classical operad theory also works here, see [7, Proposition 2.8].

In short, a morphism $\theta : \mathbb{C}X \to X$ is precisely a collection of morphisms $\theta_f : \mathscr{C}(f) \otimes X^{|f|} \to X$ such that the equivariance diagram in Definition 2.1.9 commutes. Moreover, such a morphism θ defines a $\mathbb{C}$-algebra structure on X if and only if the associativity and unit diagrams in Definition 2.1.9 commute. ☐

Example 2.1.13. *Strict ring operads.* Let $\mathscr{R}_{st}$ be a ring operad with $\mathscr{R}_{st}(g) = *$ for each $g \in \mathrm{Obj}(\widehat{\mathcal{R}})$. The image of morphisms in $\widehat{\mathcal{R}}$ is determined uniquely since $*$ is terminal. This ring operad is called the strict ring operad based in $\mathscr{V}$ because algebras over $\mathscr{R}_{st}$ are precisely strict rig objects in $\mathscr{V}$. Also, note that each component $\mathscr{R}_{st}(g)$ is terminal, so $\mathscr{R}_{st}$ itself is terminal in the category of ring operads based in $\mathscr{V}$.

2.2. **The Comparison Theorem.** In this section, we define a notion of E_∞ ring operad whose algebras, as desired, are spaces with addition and multiplication satisfying higher homotopical associativity, commutativity and distributivity laws. Then we state the Comparison Theorem 2.2.4 which says the categories of algebras over any two E_∞ ring operad have equivalent homotopy categories. The Comparison Theorem is the most fundamental and useful theorem in the theory of ring operads. However, the proof of this theorem is not easy to understand at first reading due to the extensive combinatorial details it contains, and since it will not be used elsewhere, we will defer it to Appendix A.

We first introduce some notations.

Definition 2.2.1. (1) A polynomial $f \in \mathcal{R}(n)$ is called non-degenerate if $\frac{\partial}{\partial a_{n,s}} f \neq 0$ for each $s = 1, 2, \cdots, n$ and it is called degenerate otherwise.

(2) We say a non-degenerate $f \in \mathcal{R}(n)$ is special if it is of the form

$$f = a_{1,n} \cdots a_{k_1,n} + a_{k_1+1,n} \cdots a_{k_1+k_2,n} + \cdots + a_{k_1+\cdots+k_{l-1}+1,n} \cdots a_{k_1+\cdots+k_l,n}$$

for some $0 < k_1 \leqslant k_2 \leqslant \cdots \leqslant k_l$ with $k_1 + k_2 + \cdots + k_l = n$.

Now, we fix our ground category to be the category of unbased topological spaces $(\mathscr{U}, \times, *)$.

Definition 2.2.2. When the ground category $(\mathscr{V}, \otimes, *)$ is the category of unbased spaces $(\mathscr{U}, \times, *)$, an E_∞ ring operad $\mathscr{C}$ is a ring operad such that:

(1) All $\mathscr{C}(f)$ are contractible.

(2) Let $(f_m, \phi, f_n) \in \widehat{\mathcal{R}}(f_m, f_n)$ be a morphism such that $\phi : \mathbf{m} \to \mathbf{n}$ is an injection as a map between sets. Then $\phi_* : \mathscr{C}(f_m) \to \mathscr{C}(f_n)$ is a homeomorphism.

(3) Given non-degenerate objects f_1, f_2 and $\alpha_1 \in \mathscr{C}(f_1), \alpha_2 \in \mathscr{C}(f_2)$, if there exists some non-degenerate g with effective morphisms $(f_1, \phi_1, g), (f_2, \phi_2, g)$ in $\widehat{\mathcal{R}}_{eff}$ such that

$$\phi_{1*}\alpha_1 = \phi_{2*}\alpha_2 \in \mathscr{C}(g),$$

then there must exist a non-degenerate h with effective morphisms

$$(h, \psi_1, f_1), (h, \psi_2, f_2)$$

in $\widehat{\mathcal{R}}_{n.d.}$ and a $\beta \in \mathscr{C}(h)$ such that

$$\psi_{1*}\beta = \alpha_1,$$
$$\psi_{2*}\beta = \alpha_2.$$

(4) Given any two effective morphisms

$$(f, \phi_1, g), (f, \phi_2, g)$$

with f non-degenerate, if there exists $\alpha \in \mathscr{C}(f)$ such that $\phi_{1*}\alpha = \phi_{2*}\alpha$, then $\phi_1 = \phi_2$.

(5) For any morphism (f, ϕ, g) in $\widehat{\mathcal{R}}_{n.d.}$, ϕ_* is a cofibration.

In this definition, condition (1) arises from our motivation to require elements in $\mathscr{C}(f)$ represent higher homotopically equivalent multivariable operators. Condition (2) arises from the interpretation that only those spaces indexed by non-degenerate objects provide essential information, as we have shown in previous sections. Condition (5) is a technical requirement which is necessary in the proof of the Comparison Theorem.

Condition (3) and (4) are generalizations of the Σ_j free condition in the classical operad theory. Moreover, these conditions imply the following result.

Lemma 2.2.3. *Let $\mathscr{C}$ be an E_∞ ring operad. Then Σ_n acts (on the left) freely on*

$$\coprod_{f\in\mathcal{R}_{n.d.}(n)}\mathscr{C}(f).$$

Proof. By Lemma A.1.2, for any $f\in\mathcal{R}_{n.d.}(n)$ and $\phi\in\Sigma_n$, $\phi_*f\in\mathcal{R}_{n.d.}(n)$, so the Σ_n action is well defined. The freeness follows directly from condition (4). □

Now we state the Comparison Theorem.

Theorem 2.2.4 (Comparison Theorem)**.** *Let $\mathscr{C},\mathscr{C}'$ be any two E_∞ ring operads. Then the homotopy categories of $\mathscr{C}'[\mathscr{U}_e]$ and $\mathscr{C}[\mathscr{U}_e]$ are equivalent.*

Moreover, any $\mathscr{C}$-algebra X is equivalent to some $\mathscr{C}'$-algebra Y.

3. Comparison with classical theories

3.1. **Ring operads and operad pairs.** In this section, we construct a ring operad $\mathscr{R}_{\mathscr{C},\mathscr{G}}$ from any classical operad pair $(\mathscr{C},\mathscr{G})$ such that their categories of algebras coincide. See [13] for the definition of an operad pair. Then we show that the ring operad $\mathscr{R}_{\mathscr{C},\mathscr{G}}$ is E_∞ when $(\mathscr{C},\mathscr{G})$ is an E_∞ operad pair. Therefore, the Comparison Theorem 2.2.4 generalizes the classical multiplicative infinite loop machine to be applied on algebras over any E_∞ ring operad.

Before the definition, we need some notations first.

Notations 3.1.1. (1) For any

$$I=(i_1,\cdots,i_n)\in\{0,1\}^n,$$

let Γ_I be the totally ordered set $\{j\in\{1,2,\cdots,n\}:i_j=1\}$ with order induced by that of integers. Then

$$\begin{aligned}\{0,1\}^n&\longleftrightarrow P(\{1,2,\cdots,n\})\\ I&\longleftrightarrow \Gamma_I\end{aligned}$$

is a bijection.

(2) For any

$$f=\sum_{I=(i_1,i_2,\cdots,i_n)\in\{0,1\}^n}\varepsilon_I a_{1,n}^{i_1}\cdots a_{n,n}^{i_n}\in\mathcal{R}(n),$$

let Λ_f be the totally ordered set $\{I=(i_1,\cdots,i_n)\in\{0,1\}^n:\varepsilon_I=1\}$ with lexicographical order. Then

$$\begin{aligned}\mathcal{R}(n)&\longleftrightarrow P(\{0,1\}^n)\\ f&\longleftrightarrow\Lambda_f\end{aligned}$$

is a bijection.

Lemma 3.1.2. *Let $(f_m,\phi,f_n)\in\widehat{\mathcal{R}}(f_m,f_n)$ be a morphism, say*

$$f_m=\sum_{I=(i_1,i_2,\cdots,i_m)\in\{0,1\}^m}\varepsilon_I a_{1,m}^{i_1}\cdots a_{m,m}^{i_m}\in\mathcal{R}(m).$$

If ϕ is effective, then there is a bijection $\tilde{\phi} : \Lambda_{f_m} \to \Lambda_{f_n}$ such that for each $I \in \Lambda_{f_m}$,

$$\begin{aligned}\phi|_{\Gamma_I} : \Gamma_I &\to \Gamma_{\tilde{\phi}(I)}\\ j &\mapsto \phi(j)\end{aligned}$$

is a bijection.

In general, if ϕ is not necessarily effective, then ϕ induces an injection $\phi' : \Lambda_{f_n} \to \Lambda_{f_m}$ sending each monomial summand $m \in \Lambda_{f_n}$ to the unique monomial summand $m' \in \Lambda_{f_m}$ such that $\phi_* m' = m$.

Moreover, for each $J \in \Lambda_{f_n}$, ϕ induces an injection $\phi_J : \Gamma_J \to \Gamma_{\phi' J}$ defined by the restriction of ϕ^{-1}.

Proof. By definition,

$$\begin{aligned}f_n &= \sum_{I=(i_1,i_2,\cdots,i_m)\in\{0,1\}^m} \varepsilon_I a^{i_1}_{\phi(1),n} \cdots a^{i_n}_{\phi(m),n}\\ &= \sum_{J=(j_1,j_2,\cdots,j_n)\in\{0,1\}^n} \varepsilon_J a^{j_1}_{1,n} \cdots a^{j_n}_{n,n} \in \mathcal{R}(n).\end{aligned}$$

The second line of the above expression is the decomposition of f_n under the monomial basis, in which all coefficients are zero or one. The first line is also a sum of monomials, so the fact that these two expressions of f_n coincide implies the above lemma. □

Here is a useful property of $\widehat{\mathcal{R}}_{eff}$.

Lemma 3.1.3. *Let f be a special object. Then for any two effective morphisms (f, ϕ_1, h), (f, ϕ_2, h), there exists an automorphism (f, σ, f) such that $\phi_1\sigma = \phi_2$.*

Proof. Assume

$$f = a_{1,n} \cdots a_{k_1,n} + a_{k_1+1,n} \cdots a_{k_1+k_2,n} + \cdots + a_{k_1+\cdots+k_{l-1}+1,n} \cdots a_{k_1+\cdots+k_l,n}.$$

Note that if $k_i = k_j$, then the following permutation

$$(k_1+\cdots+k_{i-1}+1, k_1+\cdots+k_{j-1}+1)\circ\cdots\circ(k_1+\cdots+k_{i-1}+k_i, k_1+\cdots+k_{j-1}+k_j)$$

induces an automorphism of f.

Therefore, without lost of generality, we assume the induced maps $\phi_{1*} = \phi_{2*} : \Lambda_f \to \Lambda_h$ coincide.

Also, note that any permutation of $\{k_1 + \cdots + k_{i-1} + 1, \cdots, k_1 + \cdots + k_{i-1} + k_i\}$ induces an automorphism of f for all i, so there exists an automorphism (f, σ, f) induced by a composition of permutation such that $\phi_1\sigma = \phi_2$. □

Now, we give the abstract Definition 3.1.4 and then show some Examples 3.1.1 to explain it.

Definition 3.1.4. Let $(\mathscr{C}, \mathscr{G})$ be an operad pair. Let $\tilde{\gamma}$ be the structure maps of both operads and λ be the action of $\mathscr{G}$ on $\mathscr{C}$. The associated ring operad $\mathscr{R}_{\mathscr{C},\mathscr{G}}$ is defined as follows.

(1) For any $f = \sum_{I=(i_1,i_2,\cdots,i_n)\in\{0,1\}^n} \varepsilon_I a^{i_1}_{1,n} \cdots a^{i_n}_{n,n} \in \mathcal{R}(n)$, we define

$$\mathscr{R}_{\mathscr{C},\mathscr{G}}(f) := \mathscr{C}(|\Lambda_f|) \otimes \bigotimes_{I\in\Lambda_f} \mathscr{G}(|\Gamma_I|).$$

(2) For any $(f_m, \phi, f_n) \in \widehat{\mathcal{R}}(f_m, f_n)$, say

$$f_m = \sum_{I=(i_1,i_2,\cdots,i_m)\in\{0,1\}^m\setminus\{0\}^m} \varepsilon_I a_{1,m}^{i_1}\cdots a_{m,m}^{i_m} \in \mathcal{R}(m).$$

By Lemma 3.1.2, ϕ induces an injection $\phi' : \Lambda_{f_n} \to \Lambda_{f_m}$ sending each monomial summand $m \in \Lambda_{f_n}$ to the unique monomial summand $m' \in \Lambda_{f_m}$ such that $\phi_* m' = m$.

Moreover, for each $J \in \Lambda_{f_n}$, ϕ induces an injection $\phi_J : \Gamma_J \to \Gamma_{\phi' J}$ defined by the restriction of ϕ^{-1}.

Therefore, we define

$$\phi_* : \mathscr{C}(|\Lambda_{f_m}|) \otimes \bigotimes_{I\in\Lambda_{f_m}} \mathscr{G}(|\Gamma_I|) \xrightarrow{\phi'^* \otimes \bigotimes_{J\in\Lambda_{f_n}} \phi_J^*} \mathscr{C}(|\Lambda_{f_n}|) \otimes \bigotimes_{J\in\Lambda_{f_n}} \mathscr{G}(|\Gamma_J|)$$

(3) The unit $\eta : * \to \mathscr{R}_{\mathscr{C},\mathscr{G}}(a_{1,1}) = \mathscr{C}(1) \times \mathscr{G}(1)$ is defined to be the product of two unit maps of $\mathscr{C}$ and $\mathscr{G}$.

(4) The composition map is defined as

$$\mathscr{C}(|\Lambda_f|) \otimes \bigotimes_{I\in\Lambda_f} \mathscr{G}(|\Gamma_I|) \otimes \bigotimes_{i=1}^{k} (\mathscr{C}(|\Lambda_{g_i}|) \otimes \bigotimes_{J\in\Lambda_{g_i}} \mathscr{G}(|\Gamma_J|))$$

$$\downarrow \Delta$$

$$\mathscr{C}(|\Lambda_f|) \otimes \bigotimes_{I\in\Lambda_f} \left(\left(\mathscr{G}(|\Gamma_I|) \otimes \bigotimes_{i\in\Gamma_I} \mathscr{C}(|\Lambda_{g_i}|) \right) \otimes \mathscr{G}(|\Gamma_I|) \otimes \bigotimes_{i\in\Gamma_I} \bigotimes_{J\in\Lambda_{g_i}} \mathscr{G}(|\Gamma_J|) \right)$$

$$\downarrow \lambda$$

$$\mathscr{C}(|\Lambda_f|) \otimes \bigotimes_{I\in\Lambda_f} \mathscr{C}(\prod_{i\in\Gamma_I} |\Lambda_{g_i}|) \otimes \bigotimes_{I\in\Lambda_f} \bigotimes_{(J_1,\cdots,J_{|\Gamma_I|})\in\prod_{i\in\Gamma_I}\Lambda_{g_i}} \left(\mathscr{G}(|\Gamma_I|) \otimes \bigotimes_{k=1}^{|\Gamma_I|} \mathscr{G}(|\Gamma_{J_k}|) \right)$$

$$\downarrow \tilde{\gamma}$$

$$\mathscr{C}(\sum_{I\in\Lambda_f} \prod_{i\in\Gamma_I} |\Lambda_{g_i}|) \otimes \bigotimes_{I\in\Lambda_f} \bigotimes_{(J_1,\cdots,J_{|\Gamma_I|})\in\prod_{i\in\Gamma_I}\Lambda_{g_i}} \mathscr{G}(\sum_{k=1}^{|\Gamma_I|} |\Gamma_{J_k}|)$$

$$\downarrow \text{shuffle}$$

$$\mathscr{C}(|\Lambda_{f(g_1,\cdots,g_k)}|) \otimes \bigotimes_{I\in\Lambda_{f(g_1,\cdots,g_k)}} \mathscr{G}(|\Gamma_I|)$$

Proposition 3.1.5. *$\mathscr{R}_{\mathscr{C},\mathscr{G}}$ is a well-defined ring operad.*

Proof. This is just a reformulation of the definition of an operad pair.

$$\mathscr{R}_{\mathscr{C},\mathscr{G}}(0_n) = \mathscr{C}(0) = *$$

by definition. The unit and equivariance diagrams commute from that of both $\mathscr{C}$ and $\mathscr{G}$, and the associativity diagram commutes from the relation between λ with the internal structure $\tilde{\gamma}$ of both $\mathscr{C}$ and $\mathscr{G}$. ∎

Examples 3.1.1. (1) Consider a morphism

$$(f = a_{1,5}a_{2,5}a_{3,5} + a_{1,5}a_{4,5} + a_{5,5}, \phi, g = a_{1,2}a_{2,2} + a_{1,2})$$

where

$$\begin{aligned} \phi : \{0, e, 1, \cdots, 5\} &\to \{0, e, 1, 2\}, \\ 1 &\mapsto e, \\ 2, 4 &\mapsto 1, \\ 3 &\mapsto 2, \\ 5 &\mapsto 0. \end{aligned}$$

Then we have

$$\begin{aligned} \mathscr{R}_{\mathscr{C},\mathscr{G}}(f) &= \mathscr{C}(3) \otimes \mathscr{G}(3) \otimes \mathscr{G}(2) \otimes \mathscr{G}(1), \\ \mathscr{R}_{\mathscr{C},\mathscr{G}}(g) &= \mathscr{C}(2) \otimes \mathscr{G}(2) \otimes \mathscr{G}(1), \end{aligned}$$

together with

$$\begin{gathered} \phi' : \Lambda_g = \{\{1\} < \{1, 2\}\} \to \Lambda_f = \{\{5\} < \{1, 4\} < \{1, 2, 3\}\} \\ \{1\} \mapsto \{1, 4\} \\ \{1, 2\} \mapsto \{1, 2, 3\} \end{gathered}$$

and

$$\begin{aligned} \phi_{\{1\}} : \{1\} &\to \{1, 4\} \\ 1 &\mapsto 4 \\ \phi_{\{1,2\}} : \{1, 2\} &\to \{1, 2, 3\} \\ 1 &\mapsto 2 \\ 2 &\mapsto 3. \end{aligned}$$

Therefore, $\phi_* : \mathscr{R}_{\mathscr{C},\mathscr{G}}(f) \to \mathscr{R}_{\mathscr{C},\mathscr{G}}(g)$ is the tensor of

$$\begin{aligned} {\phi'}^* : \mathscr{C}(3) &\to \mathscr{C}(2), \\ \phi^*_{\{1\}} : \mathscr{G}(2) &\to \mathscr{G}(1), \\ \phi^*_{\{1,2\}} : \mathscr{G}(3) &\to \mathscr{G}(2), \\ \mathscr{G}(1) &\to *. \end{aligned}$$

(2) Consider polynomials

$$\begin{aligned} f &= a_{1,2} + a_{1,2}a_{2,2}, \\ g_1 &= a_{1,2} + a_{1,2}a_{2,2}, \\ g_2 &= a_{1,2}a_{2,2}. \end{aligned}$$

Then we have $f(g_1, g_2) = a_{1,4} + a_{1,4}a_{3,4}a_{4,4} + a_{1,4}a_{2,4} + a_{1,4}a_{2,4}a_{3,4}a_{4,4}$ and

$$\begin{aligned} \mathscr{R}_{\mathscr{C},\mathscr{G}}(f) &= \mathscr{C}(2) \otimes \mathscr{G}(1) \otimes \mathscr{G}(2), \\ \mathscr{R}_{\mathscr{C},\mathscr{G}}(g_1) &= \mathscr{C}(2)' \otimes \mathscr{G}(1)' \otimes \mathscr{G}(2)', \\ \mathscr{R}_{\mathscr{C},\mathscr{G}}(g_2) &= \mathscr{C}(1)'' \otimes \mathscr{G}(2)'', \\ \mathscr{R}_{\mathscr{C},\mathscr{G}}(f(g_1, g_2)) &= \mathscr{C}(4) \otimes \mathscr{G}(1) \otimes \mathscr{G}(3) \otimes \mathscr{G}(2) \otimes \mathscr{G}(4). \end{aligned}$$

In this case, $\gamma : \mathscr{R}_{\mathscr{C},\mathscr{G}}(f) \otimes \mathscr{R}_{\mathscr{C},\mathscr{G}}(g_1) \otimes \mathscr{R}_{\mathscr{C},\mathscr{G}}(g_2) \to \mathscr{R}_{\mathscr{C},\mathscr{G}}(f(g_1, g_2))$ is gven by the tensor product of

$$\begin{aligned} \tilde{\gamma} &: \mathscr{G}(1) \otimes \mathscr{G}(1)' \to \mathscr{G}(1), \\ \tilde{\gamma} &: \mathscr{G}(2) \otimes \mathscr{G}(1)' \otimes \mathscr{G}(2)'' \to \mathscr{G}(3), \\ \tilde{\gamma} &: \mathscr{G}(1) \otimes \mathscr{G}(2)' \to \mathscr{G}(2), \\ \tilde{\gamma} &: \mathscr{G}(2) \otimes \mathscr{G}(2)' \otimes \mathscr{G}(2)'' \to \mathscr{G}(4) \end{aligned}$$

and

$$\begin{array}{c} \mathscr{C}(2) \otimes (\mathscr{G}(1) \otimes \mathscr{C}(2)') \otimes (\mathscr{G}(2) \otimes \mathscr{C}(2)' \otimes \mathscr{C}(1)'') \\ \downarrow {\scriptstyle id\otimes\lambda\otimes\lambda} \\ \mathscr{C}(2) \otimes (\mathscr{C}(2) \otimes \mathscr{C}(2)) \\ \downarrow {\scriptstyle \tilde{\gamma}} \\ \mathscr{C}(4) \end{array}$$

Proposition 3.1.6. *Let $(\mathscr{C}, \mathscr{G})$ be an operad pair with associated ring operad $\mathscr{R}_{\mathscr{C},\mathscr{G}}$. Then the category of $(\mathscr{C}, \mathscr{G})$-algebras is isomorphic to the category of $\mathscr{R}_{\mathscr{C},\mathscr{G}}$-algebras.*

Proof. Let (X, θ) be an $\mathscr{R}_{\mathscr{C},\mathscr{G}}$-algebra. We define a $(\mathscr{C}, \mathscr{G})$-algebra structure as follows.

$$\theta_+ : \mathscr{C}(j) \otimes X^{\otimes j} \xrightarrow{\mathrm{id}\times\eta^j\times\mathrm{id}} \mathscr{C}(j) \otimes \mathscr{G}(1)^j \otimes X^{\otimes j} = \mathscr{R}_{\mathscr{C},\mathscr{G}}(a_{1,j} + \cdots + a_{j,j}) \otimes X^{\otimes j} \xrightarrow{\theta} X,$$

$$\theta_\times : \mathscr{G}(j) \otimes X^{\otimes j} \xrightarrow{\eta\times\mathrm{id}\times\mathrm{id}} \mathscr{C}(1) \otimes \mathscr{G}(j) \otimes X^{\otimes j} = \mathscr{R}_{\mathscr{C},\mathscr{G}}(a_{1,j} \cdots a_{j,j}) \otimes X^{\otimes j} \xrightarrow{\theta} X.$$

Conversely, let $(Y, \theta_+, \theta_\times)$ be a $(\mathscr{C}, \mathscr{G})$-algebra. We define a $\mathscr{R}_{\mathscr{C},\mathscr{G}}$-algebra structure as follows.

For any

$$f = \sum_{I=(i_1,i_2,\cdots,i_n)\in\{0,1\}^n} \varepsilon_I a_{1,n}^{i_1} \cdots a_{n,n}^{i_n} \in \mathcal{R}(n),$$

assume $\Lambda_f = \{I_1 < \cdots < I_{|\Lambda_f|}\}$ under the lexicographical order, and assume $I_j = \{i_{1,j} < \cdots < i_{|\Gamma_{I_j}|,j}\}$. Then we define

$$\theta : \mathscr{R}_{\mathscr{C},\mathscr{G}}(f) \otimes X^{\otimes|f|} = \mathscr{C}(|\Lambda_f|) \otimes \bigotimes_{I\in\Lambda_f} \mathscr{G}(|\Gamma_I|) \otimes X^{\otimes|f|} \to X$$

to be the composition

$$\begin{array}{c}
\mathscr{R}_{\mathscr{C},\mathscr{G}}(f) \otimes X^{\otimes|f|} = \mathscr{C}(|\Lambda_f|) \otimes \bigotimes_{I\in\Lambda_f} \mathscr{G}(|\Gamma_I|) \otimes X_1 \otimes \cdots \otimes X_{|f|} \\
\Big\downarrow \text{shuffle} \\
\mathscr{C}(|\Lambda_f|) \otimes \bigotimes_{j=1}^{|\Lambda_f|} \left(\mathscr{G}(|\Gamma_{I_j}|) \otimes X_{1,j} \otimes \cdots \otimes X_{i_{|\Gamma_{I_j}|},j} \right) \\
\Big\downarrow \theta_\times \\
\mathscr{C}(|\Lambda_f|) \otimes X^{\otimes|\Lambda_f|} \\
\Big\downarrow \theta_+ \\
X
\end{array}$$

The above correspondence gives an isomorphism between the category of $(\mathscr{C},\mathscr{G})$-algebras and the category of $\mathscr{R}_{\mathscr{C},\mathscr{G}}$-algebras. ☐

Now we assume $(\mathscr{C},\mathscr{G})$ is an E_∞ operad pair in $\mathscr{U}$.

Proposition 3.1.7. *Let $(\mathscr{C},\mathscr{G})$ be an E_∞ operad pair. Then $\mathscr{R}_{\mathscr{C},\mathscr{G}}$ is an E_∞ ring operad.*

Proof. First, $\mathscr{R}_{\mathscr{C},\mathscr{G}}(f)$ is contractible since it is a finite product of contractible spaces. Conditions (2) and (5) in Definition 2.2.2 hold since $\phi_* : \mathscr{R}_{\mathscr{C},\mathscr{G}}(f) \to \mathscr{R}_{\mathscr{C},\mathscr{G}}(g)$ is a homeomorphism for all effective morphism (f,ϕ,g) in $\widehat{\mathcal{R}}_{eff}$. Condition (4) follows from the freeness of both Σ actions on $\mathscr{C}$ and $\mathscr{G}$.

For condition (3), given non-degenerate objects f_1, f_2 and $\alpha_1 \in \mathscr{C}(f_1), \alpha_2 \in \mathscr{C}(f_2)$, if there exists some g with effective morphisms $(f_1,\phi_1,g), (f_2,\phi_2,g)$ in $\widehat{\mathcal{R}}_{eff}$ such that

$$\phi_{1*}\alpha_1 = \phi_{2*}\alpha_2 \in \mathscr{C}(g),$$

then f_1, f_2, g are connected.

Let h be the special object of the same type as g. Then there must exist morphisms $(h,\psi_1,f_1), (h,\psi_2,f_2)$ in $\widehat{\mathcal{R}}_{n.d.}$ such that $\phi_1\psi_1 = \phi_2\psi_2$ by Lemma 3.1.3. Since ψ_{1*} is a homeomorphism, it follows that there exists $\beta \in \mathscr{C}(h)$ such that $\psi_{1*}\beta = \alpha_1$. Now

$$\phi_{2*}\psi_{2*}\beta = \phi_{1*}\psi_{1*}\beta = \phi_{1*}\alpha_1 = \phi_{2*}\alpha_2$$

implies that $\psi_{2*}\beta = \alpha_2$ since ϕ_{2*} is a homeomorphism, hence condition (3) follows. ☐

Therefore, applying the Comparison Theorem 2.2.4 and the classical multiplicative infinite loop machine as shown in [11], we get a multiplicative infinite loop machine defined on algebras over any E_∞ ring operad.

Theorem 3.1.8. *Let $(\mathscr{K},\mathscr{L})$ be the canonical operad pair with associated monad pair $(\mathbb{K},\mathbb{L})$ and associated ring operad $\mathscr{R}$. Let $\mathscr{C}$ be an arbitrary E_∞ ring operad. We denote the monad associated to $\mathscr{C}\times\mathscr{R}$ and $\mathscr{R}$ by $\mathbb{D}$ and $\mathbb{R}$, respectively.*

Then for any X in $\mathscr{C}[\mathscr{U}_e]$, the following composition is a group completion.

$$X \simeq B(\mathbb{D},\mathbb{D},X) \to B(\mathbb{R},\mathbb{D},X) \simeq B(\mathbb{K},\mathbb{K},B(\mathbb{R},\mathbb{D},X)) \to \Omega^\infty B(\Sigma^\infty,\mathbb{K},B(\mathbb{R},\mathbb{D},X))$$

Moreover, $B(\Sigma^\infty,\mathbb{K},B(\mathbb{R},\mathbb{D},X))$ is an E_∞ ring spectrum.

3.2. **Ring operads and categories of ring operators.** Historically, the construction of E_∞ ring spaces from bipermutative categories is given in [11] and [13] in which an intermediate theory is used, namely, the theory of categories of ring operators. We will give an alternative construction in Section 4.1 using the theory of ring operads. Before that, we briefly describe a comparison between ring operads and categories of ring operators. We first recall some notations defined in [13].

Let $\mathscr{F}$ be the category of finite based sets $\mathbf{n} = \{0, 1, 2, \cdots, n\}$, with 0 as basepoint, and based functions. Let $\Pi \subset \mathscr{F}$ be the subcategory whose morphisms are the based functions $\phi : \mathbf{m} \to \mathbf{n}$ such that $|\phi^{-1}(j)| \leqslant 1$ for $1 \leqslant j \leqslant n$, where $|S|$ denotes the cardinality of a finite set S.

Definition 3.2.1. Let $\varepsilon : \mathscr{K} \to \mathscr{F}$ and $\mathscr{D} \to \mathscr{F}$ be two topological categories over $\mathscr{F}$ which have the same objects as $\mathscr{F}$. Then the objects of $\mathscr{K} \wr \mathscr{D}$ are n-tuples of nonnegative integers for all $n \geqslant 0$. We write such an object as $(n, S) = (n, s_1, \cdots, s_n)$. Moreover, morphisms are defined as

$$\mathscr{K} \wr \mathscr{D}((m,R),(n,S)) := \coprod_{\phi \in \mathscr{F}(\mathbf{m}_*, \mathbf{n}_*)} \varepsilon^{-1}(\phi) \times \prod_{1 \leqslant j \leqslant n} \mathscr{D}\big(\bigwedge_{\phi(i)=j} \mathbf{r}_{i*}, \mathbf{s}_{j*}\big)$$

where the empty smash product is $\mathbf{1}_*$.

Definition 3.2.2. A category of ring operators is a topological category $\mathscr{J}$ with objects those of $\Pi \wr \Pi$ such that the inclusion $\Pi \wr \Pi \subset \mathscr{F} \wr \mathscr{F}$ factors as the composite of an inclusion $\Pi \wr \Pi \subset \mathscr{J}$ and a surjection $\mathscr{J} \to \mathscr{F} \wr \mathscr{F}$, both of which are the identity on objects. We require the maps $\mathscr{J}((l, Q), (m, R)) \to \mathscr{J}((l, Q), (n, S))$ induced by an injection $(\phi, \chi) : (m, R) \to (n, S)$ to be $\Sigma(\phi, \chi)$-cofibrations.

Here $\Sigma(\phi, \chi)$ is the group of automorphisms $(\sigma, \tau) : (n, S) \to (n, S)$ such that $(\sigma, \tau)\mathrm{Im}(\phi, \chi) \subset \mathrm{Im}(\phi, \chi)$ where $\mathrm{Im}(\phi, \chi) = \sqcup_i \mathrm{Im}\chi_i \subset \sqcup_i \mathbf{s_i}$.

We denote the category of $\mathscr{J}$-spaces (functor category from $\mathscr{J}$ to $\mathscr{U}$) by $\mathscr{J}[\mathscr{U}]$ and the category of special $\mathscr{J}$-spaces (see [13, Definition 5.5]) by $\mathscr{J}^s[\mathscr{U}]$

Now we assign to each E_∞ ring operad a category of ring operators $\tilde{\mathscr{C}}$. We give the abstract Definition 3.2.3 and then show an Example 3.2.4 to explain it.

Definition 3.2.3. Let $\mathscr{C}$ be an E_∞ ring operad. We define a category of ring operators $\varepsilon : \tilde{\mathscr{C}} \to \mathscr{F} \wr \mathscr{F}$ as follows.

Let $(\phi, d) = (\phi, d_1, \cdots, d_n) \in \mathscr{F}((m, R), (n, S))$ be any morphism in $\mathscr{F}$. To each pair (h, j) with $1 \leqslant j \leqslant n$, $1 \leqslant h \leqslant s_j$, we assign a polynomial $f_{\phi,d,h,j}$ in

$$\mathcal{R}(|R|) \cup \{1\} \subset \mathbb{Z}[a_{1,|R|}, \cdots, a_{|R|,|R|}]$$

as follow. Here $|R| := \sum_i r_i$.

(1) If $\phi^{-1}(j) \neq \emptyset$, then we assume $\phi^{-1}(j) = \{i_1 < \cdots < i_l\}$ and define

$$f_{\phi,d,h,j} = \sum_{(k_1,\cdots,k_l) \in d_j^{-1}(h)} \prod_{t=1}^{l} a_{\sum_{s=1}^{i_t - 1} r_s + k_1, |R|}.$$

(2) If $\phi^{-1}(j) = \emptyset$, then

$$f_{\phi,d,h,j} := \begin{cases} 0_{|R|} & \text{if } d_j^{-1}(h) = 0 \in \mathbf{1}_*, \\ 1_{|R|} & \text{if } d_j^{-1}(h) = 1 \in \mathbf{1}_*. \end{cases}$$

Then we define $\varepsilon^{-1}(\phi, d) := \prod_{(h,j)} \mathscr{C}(f_{\phi,d,h,j})$. Here $\mathscr{C}(1)$ consists of a single point.

Note that for

$$(\phi, d) = (\phi, d_1, \cdots, d_n) \in \mathscr{F} \wr \mathscr{F}((m, R), (n, S)),$$
$$(\phi', d') = (\phi', d'_1, \cdots, d'_n) \in \mathscr{F} \wr \mathscr{F}((l, Q), (m, R)),$$

the polynomial associated to their composition is

$$\begin{aligned} &f_{(\phi,d)(\phi',d'),h,j}(a_{1,|Q|}, \cdots, a_{|Q|,|Q|}) \\ =&f_{\phi,d,h,j}(f_{\phi',d',1,1}, \cdots, f_{\phi',d',q_1,1}, \cdots, f_{\phi',d',1,l}, \cdots, f_{\phi',d',q_l,l})(a_{1,|Q|}, \cdots, a_{|Q|,|Q|}) \end{aligned}$$

The composition of morphisms in $\tilde{\mathscr{C}}$ is defined to be the induced maps of $f_{\phi,d,h,j}$'s by composition of polynomials and evaluation of 1's.

When $(\phi, d) = (\phi, d_1, \cdots, d_n) \in \wr((m, R), (n, S))$, all polynomials $f_{\phi,d,h,j}$'s are of the form $0_{|R|}$, $1_{|R|}$, $a_{k,|R|}$, so $\varepsilon^{-1}(\phi, d)$ is a product of $\mathscr{C}(0_{|R|}) \cong *$, $\mathscr{C}(1_{|R|}) \cong *$, $\mathscr{C}(a_{k,|R|})$. Therefore, there is a well-defined functor $\Pi \wr \Pi \to \tilde{\mathscr{C}}$ induced by the unit $\eta : * \to \mathscr{C}(a_{1,1}) \to \mathscr{C}(a_{k,|R|})$.

Moreover, when $(\phi, \chi) : (m, R) \to (n, S)$ is an injection and $\mathscr{C}$ is E_∞, for any $(\psi, d) : (l, Q) \to (m, R)$ the induced map $\varepsilon^{-1}(\psi, d) \to \varepsilon^{-1}((\phi, \chi) \circ (\psi, d))$ is a permutation on components together with a product with $* \to \mathscr{C}(0)$ and $* \to \mathscr{C}(1)$, so the induced map $\tilde{\mathscr{C}}((l, Q), (m, R)) \to \tilde{\mathscr{C}}((l, Q), (n, S))$ is a $\Sigma(\phi, \chi)$-cofibration.

Example 3.2.4. For example, consider

$$(\phi, d) = (\phi, d_1) \in \mathscr{F} \wr \mathscr{F}((2, (2, 1)), (1, 1)),$$
$$(\phi', d') = (\phi', d'_1, d'_2) \in \mathscr{F} \wr \mathscr{F}((2, (2, 2)), (2, (2, 1))),$$

where

$$\begin{aligned} \phi(1) = \phi(2) &= 1 \\ d_1(1,1) = d_1(2,1) &= 1 \\ \phi'(1) = \phi'(2) &= 1 \\ d'_1(1,1) = d'_1(2,2) &= 1 \\ d'_1(1,2) &= 2 \\ d'_1(2,1) &= 0 \\ d'_2(1) &= 1 \end{aligned}$$

Then

$$\begin{aligned} f_{\phi,d,1,1} &= a_{1,3}a_{3,3} + a_{2,3}a_{3,3} \\ f_{\phi',d',1,1} &= a_{1,4}a_{3,4} + a_{2,4}a_{4,4} \\ f_{\phi',d',2,1} &= a_{1,4}a_{4,4} \\ f_{\phi',d',1,2} &= 1_4 \end{aligned}$$

and

$$f_{(\phi,d)(\phi',d'),1,1} = a_{1,4}a_{3,4} + a_{2,4}a_{4,4} + a_{1,4}a_{4,4}$$

while

$$\begin{aligned} &f_{\phi,d,1,1}(f_{\phi',d',1,1}, f_{\phi',d',2,1}, a_{1,1}) \\ =&(a_{1,13}a_{3,13} + a_{2,13}a_{4,13})a_{13,13} + (a_{5,13}a_{8,13})a_{13,13} \\ =&(a_{1,13}a_{3,13} + a_{2,13}a_{4,13} + a_{5,13}a_{8,13})a_{13,13} \end{aligned}$$

Therefore, the composition of morphisms in $\tilde{\mathscr{C}}$ is given by

$$\begin{array}{c}\mathscr{C}(f_{\phi,d,1,1}) \times \mathscr{C}(f_{\phi',d',1,1}) \times \mathscr{C}(f_{\phi',d',2,1}) \times * \\ \downarrow \eta \\ \mathscr{C}(f_{\phi,d,1,1}) \times \mathscr{C}(f_{\phi',d',1,1}) \times \mathscr{C}(f_{\phi',d',2,1}) \times \mathscr{C}(a_{1,1}) \\ \downarrow \gamma \\ \mathscr{C}((a_{1,13}a_{3,13} + a_{2,13}a_{4,13} + a_{5,13}a_{8,13})a_{13,13}) \\ \downarrow \psi_* \\ \mathscr{C}(a_{1,4}a_{3,4} + a_{2,4}a_{4,4} + a_{1,4}a_{4,4})\end{array}$$

where

$$\begin{aligned}\psi : \{0, e, 1, 2, \cdots, 12, 13\} &\to \{0, e, 1, 2, \cdots, 4\} \\ n &\mapsto [n \pmod 4] \text{ for } n = 1, 2, \cdots, 12 \\ 13 &\mapsto e.\end{aligned}$$

Here we require $[n \pmod 4] \in \{1, 2, 3, 4\}$.

With this definition, we can construct a special $\tilde{\mathscr{C}}$ space from any $\mathscr{C}$ algebra in $\mathscr{U}_e$.

Definition 3.2.5. Let $\mathscr{C}$ be an E_∞ ring operad with the associated category of ring operad $\tilde{\mathscr{C}}$. Then there is a canonical functor $\nu : \mathscr{C}[\mathscr{U}_e] \to \tilde{\mathscr{C}}^s[\mathscr{U}]$ defined as follows.

Let (X, θ) be an object in $\mathscr{C}[\mathscr{U}_e]$. Then we define

$$\begin{aligned}\nu X : \tilde{\mathscr{C}} &\to \mathscr{U} \\ (n, S) &\mapsto X^{s_1} \times \cdots \times X^{s_n}.\end{aligned}$$

Moreover, for any $(\phi, d) = (\phi, d_1, \cdots, d_n) \in \mathscr{F}((m, R), (n, S))$ and $(\alpha_{h,j}) \in \varepsilon^{-1}(\phi, d) := \prod_{(h,j)} \mathscr{C}(f_{\phi,d,h,j})$, we define

$$\begin{aligned}\nu X(\alpha_{h,j}) : X^{r_1} \times \cdots \times X^{r_m} &\to X^{s_1} \times \cdots \times X^{s_n} \\ (x_{1,1}, \cdots, x_{r_1,1}, \cdots, x_{1,m}, \cdots, x_{r_m,m}) &\mapsto (y_{1,1}, \cdots, y_{s_1,1}, \cdots, y_{1,n}, \cdots, y_{s_n,n})\end{aligned}$$

where

$$y_{h,j} = \theta(\alpha_{h,j}, x_{1,1}, \cdots, x_{r_1,1}, \cdots, x_{1,m}, \cdots, x_{r_m,m})$$

for $f_{\phi,d,h,j} \neq 1$ and $y_{h,j} = e$ otherwise.

Theorem 3.2.6. *Let $\mathscr{C}$ be an E_∞ ring operad with the associated category of ring operators $\tilde{\mathscr{C}}$. Then $\nu : \mathscr{C}[\mathscr{U}_e] \to \tilde{\mathscr{C}}^s[\mathscr{U}]$ induces an equivalence on homotopy categories.*

Proof. Let $(\mathscr{K}, \mathscr{L})$ be the canonical operad pair with the associated ring operad $\mathscr{R}_{\mathscr{K},\mathscr{L}}$. Let $\mathscr{D} := \mathscr{C} \times \mathscr{R}_{\mathscr{K},\mathscr{L}}$.

Consider the following diagram

$$\begin{array}{ccc}
\mathscr{C}[\mathscr{U}_e] & \xrightarrow{\quad\nu\quad} & \tilde{\mathscr{C}}^s[\mathscr{U}] \\
\downarrow & & \downarrow \\
\mathscr{D}[\mathscr{U}_e] & \xrightarrow{\quad\nu\quad} & \tilde{\mathscr{D}}^s[\mathscr{U}] \\
\uparrow & & \uparrow \\
\mathscr{R}_{\mathscr{K},\mathscr{L}}[\mathscr{U}_e] = (\mathscr{K},\mathscr{L})[\mathscr{U}_e] & \xrightarrow{\nu} & \tilde{\mathscr{R}}^s_{\mathscr{K},\mathscr{L}}[\mathscr{U}] = (\hat{\mathscr{L}} \wr \hat{\mathscr{K}})^s[\mathscr{U}],
\end{array}$$

Here the above diagram commutes by Definition 3.2.5. Comparing with Definition 3.1.4, Definition 3.2.3 and [13, Definition 1.2, Definition 5.1], we get

$$\tilde{\mathscr{R}}_{\mathscr{K},\mathscr{L}} = (\hat{\mathscr{L}} \wr \hat{\mathscr{K}}).$$

Moreover, comparing Definition 3.2.5 and [13, Definition 6.1], the bottom horizontal functor ν coincides with $R = R''R'$ defined in [13].

All vertical functors in the above diagram induce equivalences on homotopy categories by Theorem 2.2.4 and [13, Theorem 5.11]. The bottom horizontal functor induces an equivalence on homotopy categories by [13, Theorem 8.6, Theorem 10.6].

Therefore, all functors in the above diagram induce equivalences on homotopy categories. □

Remark 3.2.7. In the proof of the above theorem, we reduced it to the special case when $\mathscr{C}$ is the ring operad associated to some E_∞ operad pair. The special case is also not easy to prove but it has been proved in [13]. However, the proof of the special case cannot (at least not in an obvious way) be generalized to prove the theorem because the monad in $\Pi \wr \Pi$-spaces associated to $\tilde{\mathscr{C}}$ for a general E_∞ ring operad $\mathscr{C}$ is hard to describe. Only the monad associated to $\hat{\mathscr{L}} \wr \hat{\mathscr{K}}$ for some E_∞ operad pair $(\mathscr{K},\mathscr{L})$ has been described in [13].

4. Applications in category theory

4.1. **Ring operad for symmetric bimonoidal categories.** As an application, we show that the classifying space of a symmetric bimonoidal category is equivalent to some $(\mathscr{K},\mathscr{L})$-algebra, where $(\mathscr{K},\mathscr{L})$ is the canonical operad pair. This is originally proved in [13].

In [1], Elmendorf constructs an operad in the category of small categories whose algebras are precisely symmetric monoidal categories. We modify this construction to get a ring operad $\mathscr{S}$ in $(\mathbf{Cat}, \times, *)$ such that $\mathscr{S}$-algebras are precisely tight symmetric bimonoidal categories (with strict zero object and unit object). This ring operad cannot be induced by any operad pair, and it shows the difference between operad pairs and ring operads.

Originally, the coherence theorem for symmetric bimonoidal categories was first proved by Laplaza in [5], in which some details are omitted. A complete proof is given in [4], and our notations in this section are also due to [4].

Notation 4.1.1. Let $E : (\mathbf{Set}, \times, *) \to (\mathbf{Cat}, \times, *)$ be the functor sending a set X to its indiscrete category; that is, objects in EX are elements in X and each hom-set consists of exactly one element.

Here are some propositions about this functor E. All of them can be easily checked by definition.

Proposition 4.1.2. *(1) E is fully faithful.*

(2) E is symmetric monoidal.

(3) The nerve of EX is precisely the free simplicial set generated by vertexes X. Here the free simplicial set functor is the left adjoint of $X_* \mapsto X_0$.

(4) The classifying space BEX of EX is always contractible.

(5) If $X \to X'$ is an injection between sets, then the induced map $BEX \to BEX'$ is a cofibration.

Therefore, if $\mathscr{C}$ is a ring operad in $(\mathbf{Set}, \times, *)$, then $E\mathscr{C}$ is a ring operad in $(\mathbf{Cat}, \times, *)$. We first construct a ring operad $\mathscr{S}_{set}$ in $(\mathbf{Set}, \times, *)$ as follows.

As defined in [4], a $\{+, \times\}$-algebra is a set with two specific elements $0, 1$ and two binary operations $+, \times$. Let A_n be the free $\{+, \times\}$-algebra generated by $\{0_n, 1_n, a_{1,n}, \cdots, a_{n,n}\}$ quotient out relations

$$\begin{aligned} 0_n + x &= x + 0_n = x, \\ 0_n \times x &= x \times 0_n = 0_n, \\ 1_n \times x &= x \times 1_n = x, \end{aligned}$$

Therefore, there is a canonical $\{+, \times\}$-algebra morphism

$$\begin{aligned} p_n : A_n &\to \mathbb{Z}_{\geqslant 0}[a_{1,n}, \cdots, a_{n,n}] \\ a_{k,n} &\mapsto a_{k,n}, \\ 1_n &\mapsto 1_n, \\ 0_n &\mapsto 0_n. \end{aligned}$$

Moreover, for any $\phi : \mathbf{m}_e \to \mathbf{n}_e$, there is a induced $\{+, \times\}$-algebra morphism

$$\begin{aligned} \phi_* : A_m &\to A_n \\ a_{k,m} &\mapsto a_{\phi(k),n} \end{aligned}$$

with $a_{0,n} = 0$ and $a_{e,n} = 1$ such that the following diagram commutes:

$$\begin{array}{ccc} A_m & \xrightarrow{\phi_*} & A_n \\ \downarrow{\scriptstyle p_m} & & \downarrow{\scriptstyle p_n} \\ \mathbb{Z}_{\geqslant 0}[a_{1,m}, \cdots, a_{m,m}] & \xrightarrow{\phi_*} & \mathbb{Z}_{\geqslant 0}[a_{1,n}, \cdots, a_{n,n}] \end{array}$$

Less formally, the ring operad $\mathscr{S}_{set}$ is defined as the preimage of $\coprod \mathcal{R}(n)$ under $\coprod p_n$.

Definition 4.1.3. We define $\mathscr{S}_{set}$ as follows:

(1) For any $f \in \mathcal{R}(n)$,

$$\mathscr{S}_{set}(f) := p_n^{-1}(f);$$

(2) For any $(f_m, \phi, f_n) \in \widehat{\mathcal{R}}(f_m, f_n)$, $\phi_* : \mathscr{S}_{set}(f_m) \to \mathscr{S}_{set}(f_n)$ is the restriction of ϕ_* on $\mathscr{S}_{set}(f_m)$;

(3) The unit element is $a_{1,1} \in \mathscr{S}_{set}(a_{1,1}) = p_1^{-1}(a_{1,1})$;

(4) The composition map

$$\gamma : \mathscr{S}_{set}(f) \times \mathscr{S}_{set}(g_1) \times \cdots \times \mathscr{S}_{set}(g_k) \to \mathscr{S}_{set}(f(g_1, \cdots, g_k))$$

is induced by the composition of elements in $\coprod A_n$.

It's easy to check $\mathscr{S}_{set}$ above is a well-defined ring operad, and we let $\mathscr{S}$ to be the ring operad in $(\mathbf{Cat}, \times, *)$ defined by $E\mathscr{S}_{set}$.

Theorem 4.1.4. *The algebras over $\mathscr{S}$ are precisely tight symmetric bimonoidal categories (defined in* [4, Volume I, Definition 2.1.2]*) with strict zero and unit objects.*

Proof. Note that the polynomial rig over non-negative integers $\mathbb{Z}_{\geqslant 0}[a_{1,n}, \cdots, a_{n,n}]$ is by definition the quotient of A_n by relation given by associativity, commutativity, distributivity, and unit laws. So morphisms in $\mathscr{S}$ are generated by morphisms of one of the following forms and their inverses

$$\begin{aligned}
\alpha_{A,B,C} &: A \times (B \times C) \to (A \times B) \times C,\\
\alpha'_{A,B,C} &: A + (B + C) \to (A + B) + C,\\
\gamma_{A,B} &: A \times B \to B \times A,\\
\gamma'_{A,B} &: A + B \to B + A,\\
\lambda_{A,n} &: 1_n \times A \to A,\\
\rho_{A,n} &: A \times 1_n \to A,\\
\delta_{A,B,C} &: A \times (B + C) \to A \times B + A \times C,\\
\delta^{\#}_{A,B,C} &: (A + B) \times C \to A \times C + B \times C.
\end{aligned}$$

Therefore, if C is a $\mathscr{S}$-algebra with structure map $\lambda : \mathscr{S}(f) \to \mathrm{Func}(C^n, C)$, then

$$(C, \gamma(a_{1,2} + a_{2,2}), \gamma(a_{1,2} + a_{2,2}))$$

gives a tight symmetric bimonoidal category (with strict zero object) structure on C. Here all the coherence diagrams commute since there is precisely one morphism in each hom-set in $\mathscr{S}$.

Conversely, let $\tilde{\mathscr{S}}(f)$ be the free category generated by morphisms of one of the forms in the above list and their inverse except for the inverses of $\delta_{A,B,C}$ and $\delta^{\#}_{A,B,C}$. If $(C, \oplus, \otimes, 0, 1)$ is a tight symmetric bimonoidal category with strict zero object, then we can define a functor $\coprod_{|f|=n} \tilde{\mathscr{S}}(f) \to \mathrm{Func}(C^n, C)$ sending

$$\begin{aligned}
a_{i,n} &\mapsto \{(x_1, \cdots, x_n) \mapsto x_i\}\\
\alpha + \beta &\mapsto \{(x_1, \cdots, x_n) \mapsto \alpha(x_1, \cdots, x_n) \oplus \beta(x_1, \cdots, x_n)\}\\
\alpha \times \beta &\mapsto \{(x_1, \cdots, x_n) \mapsto \alpha(x_1, \cdots, x_n) \otimes \beta(x_1, \cdots, x_n)\}\\
0_n &\mapsto \{(x_1, \cdots, x_n) \mapsto 0\}\\
1_n &\mapsto \{(x_1, \cdots, x_n) \mapsto 1\}
\end{aligned}$$

and sending the morphisms listed above to the structure maps of $\mathrm{Func}(C^n, C)$. Here the tight symmetric bimonoidal of $\mathrm{Func}(C^n, C)$ is induced by that of C.

Comparing our Notation 2.1.2 and [4, Volume I, Definition 3.1.25], we get an element $x \in A_n$ is regular in the sense of [4] if and only if $p_n(x) \in \mathcal{R}(n)$.

Then by the coherence theorem [4, Volume I, Theorem 3.9.1], the above functor $\tilde{\mathscr{S}}(f) \to \mathrm{Func}(C^n, C)$ factors through the image of $\tilde{\mathscr{S}}(f)$ in $\mathscr{S}(f)$, denoted by $\mathscr{S}'(f)$.

To define an $\mathscr{S}$ action on C, it suffices to extend the above functor defined on $\mathscr{S}'(f)$ to $\mathscr{S}(f)$. Note that any morphism in $\mathscr{S}'(f)$ is sent to an isomorphism in $\mathrm{Func}(C^n, C)$, so we only need to show that $\mathscr{S}(f)$ is the free groupoid generated by $\mathscr{S}'(f)$.

Note that each hom-set in $\mathscr{S}'(f)$ contains exactly zero or one element, so $\mathscr{S}'(f)$ is equivalent to some poset. Also, by [3, Proposition 1], the free groupoid of some poset P is equivalent to the fundamental group of the classifying space BP. Therefore, it remains to show that $B\mathscr{S}'(f)$ is contractible.

Indeed, $B\mathscr{S}'(f)$ is contractible because $\mathscr{S}'(f)$ has a terminal object. For

$$f = \sum_{k=1}^{l} a_{i_{1,k},n} a_{i_{2,k},n} \cdots a_{i_{j_k,k},n},$$

the following element

$$\Bigg(\Bigg(((a_{i_{1,1},n} a_{i_{2,1},n}) \cdots a_{i_{j_1,1},n}) + ((a_{i_{1,2},n} a_{i_{2,2},n}) \cdots a_{i_{j_2,2},n})\Bigg) + \cdots + ((a_{i_{1,l},n} a_{i_{2,l},n}) \cdots a_{i_{j_l,l},n})\Bigg)$$

is terminal in $\mathscr{S}'(f)$, so the theorem holds. □

Note that the classifying space functor B is product preserving, so the classifying space of $\mathscr{S}$ gives a ring operad in $\mathscr{U}$.

Proposition 4.1.5. *$B\mathscr{S}$ is an E_∞ ring operad.*

Proof. First, all $B\mathscr{S}(f)$ are contractible by Proposition 4.1.2.

To check conditions (2), (3) and (4), note that if they hold for $\mathscr{S}_{set}$, they also hold after taking the free simplicial sets generated by $\mathscr{S}_{set}$, and therefore hold after taking geometric realization. So it suffices to check them on $\mathscr{S}_{set}$.

By Proposition 4.1.2, to check condition (5), it suffices to check all ϕ_* are injective on $\mathscr{S}_{set}$.

Note that an element in A_n is one-to-one corresponding to a sequence with length l of variables in $\{a_{1,n}, \cdots, a_{n,n}, 1_n\}$, a sequence with length $l-1$ of operators $\{+, \times\}$, together with a parenthesization such that this expression is not of the form

$$\cdots (1_n \times (\alpha)) \cdots$$

or

$$\cdots ((\alpha) \times 1_n) \cdots .$$

Here 0_n does not appear because it has been cancelled by strict nullity.

Moreover, if $\phi_* a = b$ for two elements a, b in $\coprod A_n$ with ϕ effective, then a and b share the same sequence of operators and the same parenthesization. Only the sequences of variables are changed by ϕ, but the length of the sequences of variables and the 1's in these sequences are also the same.

With this prospective, we introduce the following notation.

Notation 4.1.6. For any element $\alpha \in A_n$, let

$$(a_{i_1,n}, a_{i_2,n}, \cdots, a_{i_l,n})$$

be the sequence of variables associated to α. Here $i_j \in \{e, 1, 2, \cdots, n\}$ for $j = 1, 2, \cdots, l$ and $a_{e,n} = 1_n$. We denote

$$\chi(j) = \begin{cases} e & \text{if } i_j = e, \\ j & \text{if } i_j \neq e, \end{cases}$$

for $j = 1, 2, \cdots, l$.

Let $\tilde{\alpha}$ be the element in A_l with the same sequence of operators and parenthesization as α and with the corresponding sequence of variables

$$(a_{\chi(1),l}, a_{\chi(2),l}, \cdots, a_{\chi(l),l}).$$

We say $1 \leqslant k_1 < k_2 < \cdots < k_r \leqslant l$ is a fundamental sequence of α if and only if:
(1) $i_{k_j} \neq e$, i.e. $\chi(k_j) = k_j$, for $j = 1, 2, \cdots, r$;
(2) in the monomial decomposition of $p_l(\tilde{\alpha}) \in \mathbb{Z}_{\geqslant 0}[a_{1,l}, \cdots, a_{l,l}]$, the coefficient of the monomial $a_{k_1,l} a_{k_2,l} \cdots a_{k_r,l}$ is positive.

Now we return to the proof of Proposition 4.1.5. Note that A_n is only a $\{+, \times\}$-algebra and there is no minus operations. So if $a_{m,n}$ does not appear in some $f \in \mathcal{R}(n)$, $a_{m,n}$ cannot appear in the expression of any element in $\mathscr{S}_{set}(f)$. Therefore, condition (2) holds.

For condition (3), let f_1, f_2 be non-degenerate objects and

$$\alpha_1 \in \mathscr{S}_{set}(f_1), \alpha_2 \in \mathscr{S}_{set}(f_2).$$

Suppose there exists some non-degenerate g with effective morphisms $(f_1, \phi_1, g), (f_2, \phi_2, g)$ in $\widehat{\mathcal{R}}_{eff}$ such that

$$\phi_{1*}\alpha_1 = \phi_{2*}\alpha_2 \in \mathscr{S}_{set}(g).$$

With the argument above, we regard α_1, α_2 as sequences $(a_{i_1,|f_1|}, \cdots, a_{i_l,|f_1|})$, $(a_{i'_1,|f_2|}, \cdots, a_{i'_l,|f_l|})$, respectively.

Therefore, $\phi_{1*}\alpha_1 = \phi_{2*}\alpha_2$ implies $\phi_1(i_k) = \phi_2(i'_k)$ for $k = 1, 2, \cdots, l$.

Consider the following pull-back diagram in the category of based finite sets with based point e.

$$\begin{array}{ccc}
\{e, 1, \cdots, m\} & \xrightarrow{\psi_2} & \{e, 1, \cdots, |f_2|\} \\
\Big\downarrow{\scriptstyle \psi_1} & & \Big\downarrow{\scriptstyle \phi_2} \\
\{e, 1, \cdots, |f_1|\} & \xrightarrow{\phi_1} & \{e, 1, \cdots, |g|\}
\end{array}$$

Let β the element in A_m corresponding to the sequence

$$(a_{j_1,m}, \cdots, a_{j_l,m})$$

where j_k is the unique element in $\{e, 1, \cdots, m\}$ such that $(\psi_1(j_k), \psi_2(j_k)) = (i_k, i'_k)$. Then we get

$$\psi_{1*}\beta = \alpha_1,$$
$$\psi_{2*}\beta = \alpha_2.$$

Here the projection $p_m(\beta) \in \mathbb{Z}_{\geqslant 0}[a_{1,m}, \cdots, a_{m,m}]$ must be contained in $\mathcal{R}(m)$ since otherwise $f_1 = \psi_{1*}p_m(\beta) = p_m\psi_{1*}(\beta)$ is not contained in $\mathcal{R}(|f_1|)$. So condition (3) holds.

For condition (4), let

$$(f, \phi_1, g), (f, \phi_2, g)$$

be effective morphisms with f non-degenerate, and $\alpha \in \mathscr{C}(f)$ such that $\phi_{1*}\alpha = \phi_{2*}\alpha$. We also regard α as a sequence of variables

$$(a_{i_1,|f|}, \cdots, a_{i_l,|f|}).$$

Now $\phi_{1*}\alpha = \phi_{2*}\alpha$ implies $\phi_1(i_k) = \phi_2(i_k)$, and f non-degenerate implies

$$\{e\} \cup \{i_1, \cdots, i_l\} = \{e, 1, 2, \cdots, |f|\},$$

so condition (4) holds.

For condition (5), let (f, ϕ, g) be a morphism in $\widehat{\mathcal{R}}_{n.d.}$. If there exists α_1, α_2 in $\mathscr{P}_s(f)$ such that $\phi_*\alpha_1 = \phi_*\alpha_2$, then we denote the corresponding sequences of operators with α_1, α_2 by $(a_{i_1,|f|}, \cdots, a_{i_l,|f|})$, $(a_{i'_1,|f|}, \cdots, a_{i'_l,|f|})$, respectively. $\phi_*\alpha_1 = \phi_*\alpha_2$ implies $\phi_*(i_k) = \phi_*(i'_k)$.

We also have the projections $p_{|f|}(\alpha_1) = p_{|f|}(\alpha_2) = f$. Therefore, for a monomial $a_{s_1,|f|} \cdots a_{s_r,|f|}$ which has coefficient 1 in the monomial decomposition of f, there must exist a fundamental sequence $\{k_1 < \cdots < k_r\}$ of α_1 such that $(i_{k_1}, \cdots, i_{k_r})$ is a permutation of $(s_1, \cdots, s_r)$. Moreover, $\phi_* f = g \in \mathcal{R}(|g|)$ implies $\phi(s_1), \cdots, \phi(s_r)$ are distinct.

Similarly, there is a fundamental sequence $\{k'_1 < \cdots < k'_r\}$ of α_2 such that $(i'_{k'_1}, \cdots, i'_{k'_r})$ is a permutation of $(s_1, \cdots, s_r)$. If $\{k'_1 < \cdots < k'_r\} \neq \{k_1 < \cdots < k_r\}$, then applying $p_{|g|}\phi_*\alpha_1 = p_{|g|}\phi_*\alpha_2$, we get two same monomials in the summation of g, and thus the coefficient of monomial $a_{\phi(s_1),|g|}a_{\phi(s_2),|g|} \cdots a_{\phi(s_r),|g|}$ is at least 2, which gives a contradiction.

Therefore, $\{k'_1 < \cdots < k'_r\} = \{k_1 < \cdots < k_r\}$. Also, since $\phi(s_1), \cdots, \phi(s_r)$ are distinct, $\phi(i_{k_t}) = \phi(i'_{k_t}) \in \{\phi(s_1), \cdots, \phi(s_r)\}$ together with $i_{k_t}, i'_{k_t} \in \{s_1, \cdots, s_r\}$ implies $i_{k_t} = i'_{k_t}$ for $t = 1, 2, \cdots, r$.

Moreover, each $k = 1, 2, \cdots, l$ is either contained in some fundamental sequence of α_1 or $i_k = i'_k = e$, so $i_k = i'_k$ holds for all k. Therefore, $\alpha_1 = \alpha_2$, so condition (5) holds. □

Therefore, applying the group completion theorem 3.1.8, we get an alternative proof of the following result.

Theorem 4.1.7. *Let C be a tight symmetric bimonoidal category (with strict zero and unit object) and let 0 be the based-point of the classifying space BC. Then there is a group completion*

$$BC \to \Omega^\infty \mathbb{E}(BC)$$

where $\mathbb{E}(BC)$ is an E_∞ ring spectrum depending functorially on C.

In particular, if we begin with a skeleton of the category of free R-modules with operators $\{\oplus, \otimes\}$ over some commutative ring R, then we get an alternative approach to construct the algebraic K-theory ring spectrum.

4.2. From symmetric bimonoidal categories to bipermutative categories.

The classical construction of algebraic K-theory ring spectrum [13] only concerns bipermutative categories, while we focus on general symmetric bimonoidal categories since the category of projective R-modules is only symmetric bimonoidal. The classical construction works due to a strictification functor sending each symmetric bimonoidal category to some bipermutative category with equivalent classifying space, see [9, Section VI.3].

In this section, we modify the construction in Section 4.1 to get a ring operad for bipermutative categories and define an operadic strictification functor. Then we use this to description a comparison between the the two construction from bipermutative categories to E_∞ ring spectra.

Recall that a bipermutative category is a tight symmetric bimonoidal category with strict unit and associativity for both addition and multiplication and also a strict right distributivity. Hence, we define A'_n to be the $\{+, \times\}$-algebra generated by $\{a_{1,n}, \cdots, a_{n,n}, 1_n, 0_n\}$ with relations

$$\begin{aligned}
0_n + x = x + 0_n &= x, \\
0_n \times x = x \times 0_n &= 0_n, \\
1_n \times x = x \times 1_n &= x, \\
(x+y)+z &= x+(y+z), \\
(x \times y) \times z &= x \times (y \times z), \\
(x+y) \times z &= x \times z + y \times z.
\end{aligned}$$

Therefore, the projection map $p_n : A_n \to \mathbb{Z}[a_{1,n}, \cdots, a_{n,n}]$ factor through A'_n uniquely

$$\begin{array}{ccc}
A_n & \xrightarrow{p_n} & \mathbb{Z}[a_{1,n}, \cdots, a_{n,n}] \\
\downarrow{\scriptstyle \nu} & \nearrow_{p'_n} & \\
A'_n & &
\end{array}$$

In analogy with Section 4.1, we define the ring operad $\mathscr{P}_{set}$ as the preimage of $\coprod \mathcal{R}(n)$ under $\coprod p'_n$.

Definition 4.2.1. We define $\mathscr{P}_{set}$ as follows:

(1) For any $f \in \mathcal{R}(n)$,

$$\mathscr{P}_{set}(f) := {p'}_n^{-1}(f);$$

(2) For any $(f_m, \phi, f_n) \in \widehat{\mathcal{R}}(f_m, f_n)$, $\phi_* : \mathscr{P}_{set}(f_m) \to \mathscr{P}_{set}(f_n)$ is the restriction of ϕ_* on $\mathscr{P}_{set}(f_m)$;

(3) The unit element is $a_{1,1} \in \mathscr{P}_{set}(a_{1,1}) = {p'}_1^{-1}(a_{1,1})$;

(4) The composition map

$$\gamma : \mathscr{P}_{set}(f) \times \mathscr{P}_{set}(g_1) \times \cdots \times \mathscr{P}_{set}(g_k) \to \mathscr{P}_{set}(f(g_1, \cdots, g_k))$$

is induced by the composition of elements in $\coprod A'_n$.

Moreover, we let $\mathscr{P}$ be the ring operad in $(\mathbf{Cat}, \times, *)$ defined by $E\mathscr{P}_{set}$.

With the same proof as Theorem 4.1.4 and Proposition 4.1.5, we get the following result.

Theorem 4.2.2. *The algebras over $\mathscr{P}$ are precisely bipermutative categories. Moreover, after applying the classifying space functor, $B\mathscr{P}$ is an E_∞ ring operad.*

Proof. Note that A'_n is a quotient of A_n as a set. We define a section map $s : A'_n \to A_n$ as follows.

For each $a \in A_n$, we express a as a sequence with length l of variables in $\{a_{1,n}, \cdots, a_{n,n}, 1_n\}$, a sequence with length $l-1$ of operators $\{+, \times\}$, together with a parenthesization. For each operator $\bullet \in \{+, \times\}$ in this sequence, a must be locally of the form

$$\cdots((\alpha) \bullet (\beta))\cdots$$

for some $\alpha, \beta \in A_n$. In this case, we say this operator $\bullet$ acts on (α, β) in a. Moreover, we say $a \in A_n$ is reduced if

(1) for any operator $\times$ acting on α, β in a,

$$\alpha \in \{a_{1,n}, \cdots, a_{1,n}\},$$
$$\beta \notin \{1_n, 0_n\},$$

(2) for any operator $+$ acting on α', β' in a,

$$\alpha' \neq 0_n,$$
$$\beta' \neq 0_n.$$

By applying the associativity and distributivity law, for each $b \in A'_n$, there exists a unique reduced $a \in A'_n$ such that $\nu(a) = b$. We define $s : A'_n \to A_n$ such that $s(x)$ is the unique reduced element in $\nu^{-1}(x)$.

Regarding $\mathscr{P}(f)$ as the full sub-category of reduced objects in $\mathscr{S}(f)$, we can modify the proof of Theorem 4.1.4 to show that algebras over $\mathscr{P}$ are precisely bipermutative categories.

Moreover, since the $\widehat{\mathcal{R}}$ action on $\mathscr{P}$ only changes the sequence of variables in the reduced expression of objects in $\mathscr{P}$, by the same proof of Proposition 4.1.5, it follows that $B\mathscr{P}$ is an E_∞ ring operad. □

Corollary 4.2.3. *Let C be a bipermutative category and let 0 be the based-point of the classifying space BC. Then there is a group completion*

$$BC \to \Omega^\infty \mathbb{E}(BC)$$

where $\mathbb{E}(BC)$ is an E_∞ ring spectrum depending functorially on C.

Remark 4.2.4. So far we have two different constructions from bipermutative category to E_∞ ring spectra: one shown in [13] and the other is given by ring operad theory. Now we will describe a comparison between them.

To begin with, note that Definition 3.2.3 assigns to each E_∞ ring operad a category of ring operators, but this construction also work on the category level. Starting with the ring operad $\mathscr{P}$, applying the construction in Definition 3.2.3, we get a categorically enriched category of ring operators $\tilde{\mathscr{P}}$, that is, a category enriched in **Cat**, i.e. a strict 2-category, which is over $\mathscr{F} \wr \mathscr{F}$ and under $\Pi \wr \Pi$ satisfying the same conditions in Definition 3.2.2. Moreover, we regard a $\tilde{\mathscr{P}}$-category as a 2-functor from $\tilde{\mathscr{P}}$ to the 2-category of small categories and we define the notion of a special $\tilde{\mathscr{P}}$-category similarly. Under the same argument as Definition 3.2.5, we also get a canonical functor $\nu : \mathscr{P}[\mathbf{Cat}_e] \to \tilde{\mathscr{P}}^s[\mathbf{Cat}]$ from the category of $\mathscr{P}$-algebras in **Cat** to the category special $\tilde{\mathscr{P}}$-categories.

Now we summarize the comparison between the two constructions by the following figure.

$$
\begin{array}{ccccc}
h(\text{bipermutative cats}) & \xrightarrow{\cong} & h(\mathscr{P}[\mathbf{Cat}_e]) & = & h(\mathscr{P}[\mathbf{Cat}_e]) \\
\downarrow & & \downarrow \nu & & \\
h(\text{special } \mathscr{F}\wr\mathscr{F}\text{-cats}) & \longrightarrow & h(\text{special } \tilde{\mathscr{P}}\text{-cats}) & & \downarrow B \\
\downarrow B & & \downarrow B & & \\
h(\text{special } \mathscr{F}\wr\mathscr{F}\text{-spaces}) & \xrightarrow{\simeq} & h(\text{special } \widetilde{B\mathscr{P}}\text{-spaces}) & \xleftarrow{\simeq} & h(B\tilde{\mathscr{P}}\text{-spaces}) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \\
h(\text{special } \hat{\mathscr{L}}\wr\hat{\mathscr{K}}\text{-spaces}) & \xrightarrow{\simeq} & h(\text{special } \hat{\mathscr{L}}\wr\hat{\mathscr{K}} \times_{\mathscr{F}\wr\mathscr{F}} \widetilde{B\mathscr{P}}\text{-spaces}) & \xleftarrow{\simeq} & h(\mathscr{R}_{\mathscr{K},\mathscr{L}}\text{-spaces}) \\
& \nwarrow \simeq & & \simeq \nearrow & \\
& & h((\mathscr{K},\mathscr{L})\text{-spaces}) & & \\
& & \downarrow & & \\
& & h(E_\infty \text{ ring spectra}) & &
\end{array}
$$

Here hC means the homotopy category of C. In particular, the category of special $\tilde{\mathscr{P}}$-categories is by definition a full sub-2-category of the 2-category of 2-functors between the 2-category $\tilde{\mathscr{P}}$ and the 2-category of categories, but here we only consider its homotopy category.

The whole diagram commutes automatically by definition except for the top left square. Moreover, each homotopy equivalence arrow in the above diagram induces an homotopy equivalence on the underlying space. Here the underlying space of a special $\mathscr{J}$-space for some category of ring operators $\mathscr{J}$ is its value on the object $(1,1) \in \mathscr{J}$.

Note that the composite of all these left arrows is precisely the construction in [13] and the composite of all these right arrows is precisely the construction given by ring operad theory. Therefore, to compare these two constructions, we only need to consider the top left square.

Proposition 4.2.5. *The following square commutes up to natural equivalence. As a corollary, the two constructions from bipermutative categories to E_∞ ring spectra coincide up to homotopy.*

$$
\begin{array}{ccc}
h(\textit{bipermutative cats}) & \xrightarrow{\cong} & h(\mathscr{P}[\mathbf{Cat}_e]) \\
\downarrow & & \downarrow \nu \\
h(\textit{special } \mathscr{F}\wr\mathscr{F}\textit{-cats}) & \longrightarrow & h(\textit{special } \tilde{\mathscr{P}}\textit{-cats})
\end{array}
$$

Proof. Recall that in [11], the passage from bipermutative categories to $\mathscr{F}\wr\mathscr{F}$-categories is constructed as follows. We first construct a lax functor from $\mathscr{F}\wr\mathscr{F}$ to **Cat**, and then apply [10, Theorem 3.4] to strictify it to a genuine functor.

Actually this proposition holds directly from [10, Theorem 3.4]. To see this, note that each 2-functor F from $\tilde{\mathscr{P}}$ to the 2-category of categories determines a lax functor $\bar{F}$ from $\mathscr{F} \wr \mathscr{F}$ to **Cat** as follows:

(1) $\bar{F}(n, S) = F(n, S)$;

(2) For any $f \in \coprod \mathcal{R}(n)$, say

$$f = \sum_{I=(i_1,i_2,\cdots,i_m)\in\{0,1\}^m\setminus\{0\}^m} \varepsilon_I a_{1,m}^{i_1} \cdots a_{m,m}^{i_m} \in \coprod \mathcal{R}(n).$$

Let $t(f) := \sum_{I=(i_1,i_2,\cdots,i_m)\in\{0,1\}^m\setminus\{0\}^m} \varepsilon_I a_{1,m}^{i_1} \cdots a_{m,m}^{i_m} \in \mathrm{Obj}(\mathscr{P}(f))$ under the order of both Λ_f and Γ_I. By definition, for each morphism $(\phi, d) : (m, R) \to (n, S)$ in $\mathscr{F} \wr \mathscr{F}$, the preimage of (ϕ, d) under the canonical map $\tilde{\mathscr{P}} \to \mathscr{F} \wr \mathscr{F}$ is

$$\prod_{h,j} \mathscr{P}(f_{\phi,d,h,j}).$$

We define $\bar{F}(\phi, d) := F((tf_{\phi,d,h,j})_{h,j})$.

(3) $\sigma((\phi, d), (\psi, d')) : \bar{F}(\phi, d) \circ \bar{F}(\phi', d') \to \bar{F}((\phi, d) \circ (\phi', d'))$ is defined to be the canonical equivalence.

Let $(A, \oplus, \otimes, 0, 1)$ be an arbitrary bipermutative category. Then the composite of the top and the right arrow in this diagram sends A to a $\tilde{\mathscr{P}}$-category A', which determines a lax functor $\bar{A}'$. By carefully checking definitions, we have this lax functor $\bar{A}'$ coincides with that appears in the passage from bipermutative categories to $\mathscr{F} \wr \mathscr{F}$-categories in [11]. Therefore, this proposition holds directly from the strictification theorem [10, Theorem 3.4] from lax functor to genuine functor. □

We end this section with a proof of the strictification theorem from symmetric bimonoidal categories to bipermutative categories. Recall that when $\nu' : \mathscr{C} \to \mathscr{C}'$ is a morphism between E_∞ ring operads, any $\mathscr{C}$-algebra is homotopy equivalent to some $\mathscr{C}'$-algebra. A similar result holds for the functor $\nu : \mathscr{S} \to \mathscr{P}$. Of course, we can prove this by modifying the proof in Section A.2. However, in the special case $\nu : \mathscr{S} \to \mathscr{P}$, a simpler proof is possible.

Theorem 4.2.6. *There is a functor Φ from the category of tight symmetric bimonoidal categories (with strict unit and zero objects) to the category of bipermutative categories and a natural equivalence $\eta : \Phi C \to C$ of symmetric bimonoidal categories.*

Proof. Let $\mathbb{S}$, $\mathbb{P}$ be the monads in **Cat** associated to $\mathscr{S}$, $\mathscr{P}$, respectively.

By definition of the section map $s : \mathscr{P} \to \mathscr{S}$ sending each object x to the unique reduced object in $\nu^{-1}(x)$, we have $s : \mathscr{P} \to \mathscr{S}$ is a natural transformation between functors over $\widehat{\mathcal{R}}$, although it is not a morphism between ring operads. Therefore, there is a well-defined functor $s : \mathbb{P}X \to \mathbb{S}X$.

When X is symmetric bimonoidal, we define

$$\tilde{\eta} : \mathbb{P}X \xrightarrow{s} \mathbb{S}X \xrightarrow{\theta} X.$$

Moreover, we define a relation on each hom-set $\mathbb{P}X(A, B)$ such that two morphisms $f, g \in \mathbb{P}X(A, B)$ are equivalent if and only if $\tilde{\eta}(f) = \tilde{\eta}(f)$. Then we denote the quotient category by ΦX.

Because $\tilde{\eta} : \mathbb{P}X \to X$ is symmetric bimonoidal, we get the following commutative diagram in the category of symmetric bimonoidal categories:

$$\begin{array}{ccc} \mathbb{P}X & \xrightarrow{\tilde{\eta}} & X \\ \Big\downarrow{\scriptstyle \text{quotient}} & \nearrow_{\eta} & \\ \Phi X & & \end{array}$$

Here η is faithful by construction and it is full and essentially surjective since

$$X \xrightarrow{\text{unit}} \mathscr{P}(a_{1,1}) \times X \longrightarrow \mathbb{P}X \xrightarrow{\text{quotient}} \Phi X \xrightarrow{\eta} X$$

is the identity. Therefore, the theorem holds. □

Remark 4.2.7. Our construction is different from [9, Section VI.3], but there is a canonical comparison equivalence

$$\Phi X \to \Phi' X$$

where $\Phi' X$ is the bipermutative category constructed in [9].

Appendix A. Proof of the comparison theorem

A.1. **Properties of the category $\widehat{\mathcal{R}}_{eff}$.** In this section, we describe the structure of $\widehat{\mathcal{R}}_{eff}$ more precisely. These combinatorial descriptions will be used in the next section to prove the Comparison Theorem 2.2.4.

Recall that $(f_m, \phi, f_n) \in \widehat{\mathcal{R}}_{eff}(f_m, f_n)$ if and only if

$$f_n(a_{1,n}, \cdots, a_{n,n}) = f_m(a_{\phi(1),n}, \cdots, a_{\phi(m),n}).$$

In general, the hom-set $\widehat{\mathcal{R}}_{eff}(f_m, f_n)$ between two arbitrary objects f_m and f_n might be empty. We regard a morphism from f_m to f_n as a relation between them and two related objects should have something in common.

More precisely, we say two objects in a small category are connected if they are path-connected in the classifying space; that is, there is a zig-zag diagram of morphisms connecting these two objects. Now, we describe the connected components of $\widehat{\mathcal{R}}_{eff}$ precisely.

To begin with, by Lemma 3.1.2, if two objects f_m, f_n are contained in the same connected component, we must have:

(1) $|\Lambda_{f_m}| = |\Lambda_{f_n}|$;

(2) the ordered $|\Lambda_{f_m}|$-tuple $(|\Gamma_I|)_{I\in\Lambda_{f_m}}$ is a permutation of $(|\Gamma_J|)_{J\in\Lambda_{f_n}}$.

For a non-negative integer k and a non-decreasing sequence of l positive integers $0 < k_1 \leqslant i_2 \leqslant \cdots \leqslant k_l$, we say an object f in $\widehat{\mathcal{R}}_{eff}$ is of type $(l; k_1, \cdots, k_l)$ if and only if $|\Lambda_f| = l$ and $(|\Gamma_I|)_{I\in\Lambda_{f_m}}$ is a permutation of $(k_1, \cdots, k_l)$. Therefore, two connected objects are of the same type.

In fact, the converse statement is also true. To see this, note that by definition 2.2.1, the special object

$$f = a_{1,n} \cdots a_{k_1,n} + a_{k_1+1,n} \cdots a_{k_1+k_2,n} + \cdots + a_{k_1+\cdots+k_{l-1}+1,n} \cdots a_{k_1+\cdots+k_l,n}$$

is of type $(l; k_1, \cdots, k_l)$ and two different special objects are of different types. More precisely, we have the following result.

Proposition A.1.1. *For each $f \in \coprod \mathcal{R}(n)$, there exists a unique special $g \in \coprod \mathcal{R}(n)$ such that $\widehat{\mathcal{R}}_{eff}(g, f)$ is non-empty. Moreover, two different special objects are in different connected components. Therefore, the collection of special objects gives a collection of representatives for connected components in $\widehat{\mathcal{R}}_{eff}$.*

Proof. Different special objects are in different connected components because they are of different types, so it suffices to construct for any object f a morphism from the special object of the same type as f to f itself.

Indeed, suppose

$$f = \sum_{I=(i_1,i_2,\cdots,i_n)\in\{0,1\}^n} \varepsilon_I a_{1,n}^{i_1} \cdots a_{n,n}^{i_n}$$

is of type $(l, k_1, \cdots, k_l)$. Then we want to find a morphism from the special object

$$g = a_{1,n} \cdots a_{k_1,n} + a_{k_1+1,n} \cdots a_{k_1+k_2,n} + \cdots + a_{k_1+\cdots+k_{l-1}+1,n} \cdots a_{k_1+\cdots+k_l,n}$$

to f.

Let

$$\Lambda_f =: \{I_1, \cdots, I_l\}$$

be such that $|\Gamma_{I_j}| = k_j$ and let

$$I_j = \{t_{1,j} < t_{2,j} < \cdots < t_{k_j,j}\}.$$

Then

$$\begin{aligned} \phi : \{1, 2, \cdots, k_1 + \cdots + k_l\} &\to \{1, 2, \cdots, n\} \\ \sum_{h=1}^{j-1} k_h + s &\mapsto t_{s,j} \text{ if } 1 \leqslant s \leqslant k_j \end{aligned}$$

gives a well-defined morphism $(g, \phi, f) \in \widehat{\mathcal{R}}(g, f)$ by definition. □

Now we focus on these non-degenerate objects. Actually, those degenerate objects appear in the definition of ring operad only because the non-degenerate ones are not closed under composition. In practice, for a ring operad $\mathscr{C}$, the collection of $\mathscr{C}(f)$ indexed by non-degenerate objects f already provides enough information. For example, $\mathscr{C}(a_{1,3}a_{2,3} + a_{1,3}a_{3,3})$ is regarded as a collection of operators of the form

$$(a, b, c) \to (ab + ac),$$

and $\mathscr{C}(a_{1,4}a_{2,4} + a_{1,4}a_{3,4})$ is regarded as a collection of operators of the form

$$(a, b, c, d) \to (ab + ac).$$

Therefore, we do not expect $\mathscr{C}(a_{1,4}a_{2,4} + a_{1,4}a_{3,4})$ to contain more information than $\mathscr{C}(a_{1,3}a_{2,3} + a_{1,3}a_{3,3})$. This interpretation leads to one of the conditions in Definition 2.2.2 of E_∞ ring operads.

Now we describe the structure of the full sub-category $\widehat{\mathcal{R}}_{n.d.}$ of $\widehat{\mathcal{R}}_{eff}$ generated by non-degenerate objects.

Consider morphisms in $\widehat{\mathcal{R}}_{n.d.}$ first. We have the following lemma.

Lemma A.1.2. *Let (f, ϕ, g) be an effective morphism in $\widehat{\mathcal{R}}_{eff}$ with f non-degenerate. Then g is non-degenerate if and only if $\phi : \{0, e, 1, 2, \cdots, |f|\} \to \{0, e, 1, 2, \cdots, |g|\}$ is surjective. In this case, $|f| \geqslant |g|$.*

In particular, for any map $\psi : \{0, e, 1, 2, \cdots, |f|\} \to \{0, e, 1, 2, \cdots, |f|\}$, $(f, \psi, \psi_* f)$ defines a morphism in $\widehat{\mathcal{R}}_{n.d.}$ if and only if ψ is a bijection.

Proof. Suppose ϕ is not surjective. Pick a $k \in \{1, 2, \cdots, |g|\} \setminus \mathrm{Im}(\phi)$. Then since

$$g(a_{1,|g|}, \cdots, a_{|g|,|g|}) = f(a_{\phi(1),|g|}, \cdots, a_{\phi(|g|),|g|}),$$

$\frac{\partial}{\partial a_{k,|g|}} g = 0$, hence g is degenerate.

Conversely, if ϕ is surjective and $\psi_* f = g \in \mathcal{R}(|g|)$, then $\frac{\partial}{\partial a_{k,|f|}} \psi_* f \neq 0$ follows from $\frac{\partial}{\partial a_{\phi^{-1}(k),|f|}} f \neq 0$, so $\psi_* f$ is non-degenerate.

In particular, note that a bijection ψ on $\{1, 2, \cdots, |f|\}$ induces a bijection on monomials in $\mathcal{R}(|f|)$, so we get $\psi_* f \in \mathcal{R}(|f|)$. Therefore, the lemma holds. □

We end this section with a description of the connected components of $\widehat{\mathcal{R}}_{n.d.}$. As shown in Proposition A.1.1, each non-degenerate object f admits a morphism from the special object g of the same type as f to f itself. Such a morphism exists implies $|g| \geqslant |f|$. Therefore, the following results hold.

Notation A.1.3. Let f be a special object. Let $\widehat{\mathcal{R}}_{n.d.}(f)$ be the connected component containing f with objects $\mathcal{R}_{n.d.}(f)$ and $\widehat{\mathcal{R}}_{n.d.}(\geqslant n)$ ($\widehat{\mathcal{R}}_{n.d.}(n)$, resp.) be the sub-category generated by non-degenerate g with $|g| \geqslant n$ ($|f| = n$, resp.) with objects $\mathcal{R}_{n.d.}(\geqslant n)$ ($\mathcal{R}_{n.d.}(n)$, resp.). Denote

$$\widehat{\mathcal{R}}_{n.d.}(f, \geqslant n) := \widehat{\mathcal{R}}_{n.d.}(f) \cap \widehat{\mathcal{R}}_{n.d.}(\geqslant n),$$
$$\widehat{\mathcal{R}}_{n.d.}(f, n) := \widehat{\mathcal{R}}_{n.d.}(f) \cap \widehat{\mathcal{R}}_{n.d.}(n).$$

with objects

$$\mathcal{R}_{n.d.}(f, \geqslant n) := \mathcal{R}_{n.d.}(f) \cap \mathcal{R}_{n.d.}(\geqslant n),$$
$$\mathcal{R}_{n.d.}(f, n) := \mathcal{R}_{n.d.}(f) \cap \mathcal{R}_{n.d.}(n).$$

Proposition A.1.4. *The collection of special objects gives a collection of representatives for connected components in* $\widehat{\mathcal{R}}_{n.d.}$.

Moreover, there is a finite filtration on each connected component.

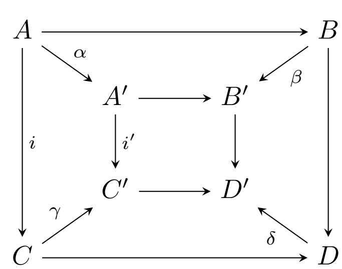
Then we get

$$\emptyset = \widehat{\mathcal{R}}_{n.d.}(f, \geqslant |f| + 1) \subset \widehat{\mathcal{R}}_{n.d.}(f, \geqslant |f|) \subset \cdots \subset \widehat{\mathcal{R}}_{n.d.}(f, \geqslant 0) = \widehat{\mathcal{R}}_{n.d.}(f).$$

This filtration plays an essential role in section A.2.

A.2. **Algebras over different E_∞ ring operads.** We finally prove the Comparison Theorem 2.2.4 in this section. To show the proof, we need to introduce two lemmas first.

Lemma A.2.1. *Given the following commutative diagram of spaces*



in which i and i' are cofibrations, α, β, γ are equivalences, both the outer and inner squares are push-out squares, then δ is also an equivalence.

Proof. We functorially factorize all horizontal morphisms in the above diagram as a composition of an acyclic fibration and a cofibration. Therefore it suffices to prove this lemma when all horizontal morphisms are also cofibrations. This follows from [12, Pages 80-81]. □

The next lemma is also a fundamental result in the theory of model category, see [6].

Lemma A.2.2. *Given the following pull-back diagram of spaces*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & X \end{array}$$

in which morphisms $B \to X$, $C \to X$, and the composition $A \to X$ are cofibrations, then so is the universal morphism

$$D := B \coprod_A C \to X.$$

Applying these lemmas, we get the following property about E_∞ ring operads.

Proposition A.2.3. *Let $\mathscr{C}$ be an E_∞ ring operad, f be a special object, and $g \in \mathcal{R}(f)$. Let $\{\phi_i(\mathscr{C}(h_i))\}_{i=1}^N$ be a finite collection of distinct sub-spaces in $\mathscr{C}(g)$ where (h_i, ϕ_i, g) is a morphism in $\widehat{\mathcal{R}}_{n.d.}$. Then both $\bigcap_{i=1}^N \phi_i(\mathscr{C}(h_i))$ and $\bigcup_{i=1}^N \phi_i(\mathscr{C}(h_i))$ are contractible and their inclusions into $\mathscr{C}(g)$ are cofibrations.*

Proof. We prove this by induction first on $|g|$ and then on $\min\{|h_i|\}$.

The induction begins with $\min\{|h_i|\} = |f|$. Note that for $h_i \in \mathcal{R}(f)_{n.d.}$ with $|h_i| = |f|$, there must be some $\sigma_i \in \Sigma_{|f|}$ such that $h_i = \sigma_{i*} f$. $\sigma_{i*} : \mathscr{C}(f) \to \mathscr{C}(h_i)$ are homeomorphisms, so without lost of generality, we assume all $h_i = f$.

By Lemma 3.1.3, two morphisms (f, ϕ_i, g) and (f, ϕ_j, g) only differ by an automorphism of f, which induces a homeomorphism on $\mathscr{C}(f)$. So $\phi_i(\mathscr{C}(h_i))$ are all the same space in this case. Therefore, the proposition holds by condition (5) in Definition 2.2.2 when $\min\{|h_i|\} = |f|$, and hence when $|g| = |f|$.

Now we assume the proposition holds when $|g'| > |g|$ and when $|g'| = |g|$ with $\min\{|h_i'|\} > \min\{|h_i'|\}$. There is nothing to prove when $N = 1$ since it follows from condition (5).

When $N \geqslant 2$, for any $\alpha \in \phi_i(\mathscr{C}(h_i)) \cap \phi_j(\mathscr{C}(h_j))$ for $i \neq j$, say $\alpha = \phi_{i*}\alpha_1 = \phi_{j*}\alpha_j$. By condition (3) in Definition 2.2.2, there exists h_{ij}' with morphisms (h_{ij}', ψ_i, h_i) (h_{ij}', ψ_j, h_j) and $\beta \in \mathscr{C}(h_{ij}')$ such that $\psi_{i*}\beta = \alpha_i$, $\psi_{j*}\beta = \alpha_j$. By condition (4), $\phi_i\psi_i = \phi_j\psi_j =: \phi_{ij}$. By Lemma A.1.2, $|h_{ij}'| \geqslant \max\{|h_i|, |h_j|\}$. If $|h_{ij}'| = |h_i| = |h_j|$, then both ψ_i, ψ_j are invertible, so

$$\phi_i(\mathscr{C}(h_i)) = \phi_{ij}(\mathscr{C}(h_{ij}')) = \phi_j(\mathscr{C}(h_j)),$$

which contradicts with $i \neq j$.

Therefore, $|h_{ij}'| > \min\{|h_i|, |h_j|\}$, so

$$\bigcap_{i=1}^N \phi_i(\mathscr{C}(h_i)) = \bigcap_{i,j} \phi_{ij}(\mathscr{C}(h_{ij}'))$$

with $\min\{|h_{ij}'|\} > \min\{|h_i|\}$. Hence, $\bigcap_{i=1}^N \phi_i(\mathscr{C}(h_i))$ is contractible and its inclusion into $\mathscr{C}(g)$ is a cofibration.

To prove the union part, we further assume that $|g| > \min\{|h_i|\}$. Otherwise, if $|h_i| = |g|$ for some i then ϕ_i is invertible, so $\phi_{i*} : \mathscr{C}(h_i) \to \mathscr{C}(g)$ is a homeomorphism. Thus the union $\bigcup_{i=1}^N \phi_i(\mathscr{C}(h_i))$ is the whole space $\mathscr{C}(g)$ and the proposition holds.

Consider the following commutative diagram

$$\begin{array}{ccc}
\phi_k(\mathscr{C}(h_k)) \cap \left(\bigcup_{i=1}^{k-1} \phi_i(\mathscr{C}(h_i))\right) & \longrightarrow & \bigcup_{i=1}^{k-1} \phi_i(\mathscr{C}(h_i)) \\
\downarrow & & \downarrow \\
\phi_k(\mathscr{C}(h_k)) & \longrightarrow & \bigcup_{i=1}^{k} \phi_i(\mathscr{C}(h_i))
\end{array}$$

This diagram is both a pull-back and a push-out diagram. Here

$$\begin{aligned}
&\phi_k(\mathscr{C}(h_k)) \cap \left(\bigcup_{i=1}^{k-1} \phi_i(\mathscr{C}(h_i))\right) \\
=&\bigcup_{i=1}^{k-1} (\phi_k(\mathscr{C}(h_k)) \cap \phi_i(\mathscr{C}(h_i)))
\end{aligned}$$

and $\phi_k(\mathscr{C}(h_k)) \cap \phi_i(\mathscr{C}(h_i))$ is a union of some $\bigcap_{i,j} \phi_{ij}(\mathscr{C}(h'_{ij}))$ with $\min\{|h'_{ij}|\} > |g|$. Then the proposition holds for $\phi_k(\mathscr{C}(h_k)) \cap \left(\bigcup_{i=1}^{k-1} \phi_i(\mathscr{C}(h_i))\right)$.

The left arrow in the above diagram is a cofibration by applying the inductive hypothesis to $g = h_k$ (note that $|h_k| > |g|$). Therefore, the proposition follows by applying Lemma A.2.1 and A.2.2 inductively on the above diagram. ☐

In particular, we have the following corollary.

Corollary A.2.4. *Let $\mathscr{C}$ be an E_∞ ring operad. For a special object f and $n \geqslant 0$, let $L(f,n)$ be the subspace*

$$L(f,n) \subset \coprod_{g \in \mathcal{R}_{n.d.}(f,n)} \mathscr{C}(g),$$

such that $\alpha \in \mathscr{C}(g)$ belongs to $L(f,n)$ if and only if there exists

$$h \in \mathcal{R}_{n.d.}(f, \geqslant n+1)$$

with morphism (h, ϕ, g) in $\widehat{\mathcal{R}}_{n.d.}$ and $\beta \in \mathscr{C}(h)$ such that $\phi_* \beta = \alpha$. Then each $L(f,n) \cap \mathscr{C}(g)$ is contractible for any $g \in \mathcal{R}_{n.d.}(f)$ and the inclusion $L(f,n) \subset \coprod_{g \in \mathcal{R}_{n.d.}(f,n)} \mathscr{C}(g)$ is a cofibration.

Now let $\mathscr{C}$ be an E_∞ ring operad and we focus on the associated monad $\mathbb{C}$. Our argument here is closely related to [8, Appendix], but is much more complicated. The following lemma gives a strict description of the interpretation that only those spaces indexed by non-degenerate objects provide essential information.

Lemma A.2.5. *If $\mathscr{C}$ is an E_∞ ring operad, then the canonical map*

$$i : \mathscr{C}(\bullet) \times_{\widehat{\mathcal{R}}^{op}_{n.d.}} X^\bullet \to \mathscr{C}(\bullet) \times_{\widehat{\mathcal{R}}^{op}_{eff}} X^\bullet =: \mathbb{C}_{\mathscr{U}} X$$

is a homeomorphism.

Proof. By definition of coend,

$$\mathscr{C}(\bullet) \times_{\widehat{\mathcal{R}}^{op}_{eff}} X^\bullet = \coprod_{f\in\mathcal{R}(n),\ n\geqslant 0} \mathscr{C}(f) \times X^n/(\sim)$$

where $(\sim)$ is generated by $(\alpha, \phi^*\mathbf{x}) \sim (\phi_*\alpha, \mathbf{x})$ for all effective morphisms (f, ϕ, g) with $\alpha \in \mathscr{C}(f)$ and $\mathbf{x} \in X^{|g|}$.

We define an inverse j of i as follows. For any $(\alpha, \mathbf{x}) \in \mathscr{C}(f) \times X^{|f|}$ if f is non-degenerate, we define $j(\alpha, \mathbf{x})$ to be the quotient of $(\alpha, \mathbf{x})$ in $\mathscr{C}(\bullet) \times_{\widehat{\mathcal{R}}^{op}_{eff}} X^\bullet$.

When f is degenerate, then we pick an effective morphism (g, ϕ, f) such that g is special by Proposition A.1.1. We further decomposite $\phi = \psi \circ p$ such that ψ is injective and p is surjective. Then by Lemma 2.1.6, $\psi_* p_* g \in \mathrm{Obj}(\widehat{\mathcal{R}})$ and ψ injective implies $p_*g \in \mathrm{Obj}(\widehat{\mathcal{R}})$. Furthermore, p surjective and g non-degenerate implies p_*g non-degenerate. Therefore, for any degenerate f, there exists a non-degenerate object p_*g and a morphism (p_*g, ψ, f) in which ψ is injective.

Note that the pair (p_*g, ψ) is not necessarily unique, so we just choose such a pair for each degenerate f and denote it by (h_f, ψ_f). By assumption, $\mathscr{C}$ is E_∞, so $\psi_{f*} : \mathscr{C}(h_f) \to \mathscr{C}(f)$ is a homeomorphism. Then we define $j(\alpha, \mathbf{x}) := ((\psi_{f*})^{-1}\alpha, \psi_f^*\mathbf{x})$.

To check $j : \mathscr{C}(\bullet) \times_{\widehat{\mathcal{R}}^{op}_{eff}} X^\bullet \to \mathscr{C}(\bullet) \times_{\widehat{\mathcal{R}}^{op}_{n.d.}} X^\bullet$ is well-defined, it suffices to check $j(\alpha, \phi^*\mathbf{x}) = j(\phi_*\alpha, \mathbf{x})$ for all effective morphisms (f, ϕ, g) with $\alpha \in \mathscr{C}(f)$ and $\mathbf{x} \in X^{|g|}$.

This holds by definition when both f and g are non-degenerate. If f is non-degenerate but g is degenerate, we get $\phi_*\alpha = \psi_{g*}(\psi_{g*})^{-1}\phi_*\alpha$.

By definition, $\mathrm{Im}(\phi)$ is contained in $\mathrm{Im}(\phi_g)$, Moreover, (f, ϕ, g) factors through $(f, \phi, g) = (h_g, \phi_g, g) \circ (f, \tilde{\phi}, h_g)$.

Therefore,

$$\begin{aligned} j(\alpha, \phi^*\mathbf{x}) =& j(\alpha, (\phi_g\tilde{\phi})^*\mathbf{x}) \\ =& j(\alpha, \tilde{\phi}^*\phi_g^*\mathbf{x}) \\ =& j(\tilde{\phi}_*\alpha, \phi_g^*\mathbf{x}) \\ =& j((\phi_g^{-1})_*\phi_{g*}\tilde{\phi}_*\alpha, \phi_g^*\mathbf{x}) \\ =& j((\psi_{g*})^{-1}\phi_*\alpha, \psi_g^*\mathbf{x}) \\ =& j(\phi_*\alpha, \mathbf{x}). \end{aligned}$$

The case when f is degenerate can be reduced to the above cases by composing with ϕ_f.

Therefore, j is well-defined. By definition of j, it follows that both $i \circ j$ and $j \circ i$ are identities, so i is a homeomorphism. □

In the rest of this section, we define $\mathbb{C}_{\mathscr{U}} X$ to be $\mathscr{C}(\bullet) \times_{\widehat{\mathcal{R}}^{op}_{n.d.}} X^\bullet$.

We apply the filtration defined in Proposition A.1.4 to get a filtration on $\mathbb{C}_{\mathscr{U}} X$.

Lemma A.2.6. *Let $\mathscr{C}$ be an E_∞ ring operad with associated monad $\mathbb{C}_{\mathscr{U}}$. Then*

$$\mathbb{C}_{\mathscr{U}} X = \coprod_{f\ special} \mathscr{C}(\bullet) \times_{\widehat{\mathcal{R}}_{n.d.}(f)^{op}} X^\bullet$$

Moreover, for each f special and $n = 0, 1, \cdots, |f| - 1$ there is a push-out diagram

$$\begin{array}{ccc}
L(f,n)\times_{\Sigma_n^{op}} X^n & \xrightarrow{\quad e \quad} & \mathscr{C}(\bullet)\times_{\widehat{\mathcal{R}}_{n.d.}(f,\geqslant n+1)^{op}} X^{\bullet} \\
\downarrow & & \downarrow \\
\left(\coprod_{g\in\mathcal{R}_{n.d.}(f,n)} \mathscr{C}(g)\right)\times_{\Sigma_n^{op}} X^n & \longrightarrow & \mathscr{C}(\bullet)\times_{\widehat{\mathcal{R}}_{n.d.}(f,\geqslant n)^{op}} X^{\bullet}
\end{array}$$

Proof. The attaching map e is defined as follows.

$$e: L(f,n)\times_{\Sigma_n^{op}} X^n \to \mathscr{C}(\bullet)\times_{\widehat{\mathcal{R}}_{n.d.}(f,\geqslant n+1)^{op}} X^{\bullet}$$
$$(\phi_*\beta,\mathbf{X})\mapsto(\beta,\phi^*\mathbf{X})$$

for some morphism (h,ϕ,g) in $\widehat{\mathcal{R}}_{n.d.}$ and $\beta\in\mathscr{C}(h)$ with $h\in\mathcal{R}_{n.d.}(f,\geqslant n+1)$, $g\in\mathcal{R}_{n.d.}(f,n)$.

This attaching map e is well-defined. Indeed, for any $\sigma\in\Sigma_n$, we have

$$e(\sigma_*\phi_*\beta,\mathbf{X})=(\beta,\phi^*\sigma^*\mathbf{X})=e(\phi_*\beta,\sigma^*\mathbf{X}).$$

Also, if there exist another (h',ϕ',g) in $\widehat{\mathcal{R}}_{n.d.}$ and $\beta'\in\mathscr{C}(h')$ with $h\in\mathcal{R}_{n.d.}(f,\geqslant n+1)$ such that $\phi'_*\beta'=\phi_*\beta$, then there exists some non-degenerate h'' and $\beta''\in\mathscr{C}(h'')$ together with morphisms (h'',ψ,h), (h'',ψ',h') such that $\psi_*\beta''=\beta$, $\psi'_*\beta''=\beta'$. By Lemma A.1.2, $|h''|\geqslant|h|$, so $h''\in\mathcal{R}_{n.d.}(f,\geqslant n+1)$. By condition (4) in Definition 2.2.2, $\phi_*\psi_*\beta''=\phi'_*\psi'_*\beta''$ implies $\phi\psi=\phi'\psi'$, so the following equation holds in $\mathscr{C}(\bullet)\times_{\widehat{\mathcal{R}}_{n.d.}(f,\geqslant n+1)^{op}} X^{\bullet}$.

$$\begin{aligned}
(\beta,\phi^*\mathbf{X}) &=(\psi_*\beta'',\phi^*\mathbf{X})\\
&=(\beta'',\psi^*\phi^*\mathbf{X})\\
&=(\beta'',\psi'^*\phi'^*\mathbf{X})\\
&=(\psi'_*\beta'',\phi'^*\mathbf{X})\\
&=(\beta',\phi'^*\mathbf{X}).
\end{aligned}$$

Note that

$$\left(\coprod_{g\in\mathcal{R}_{n.d.}(f,n)}\mathscr{C}(g)\right)\times_{\Sigma_n^{op}} X^n=\mathscr{C}(\bullet)\times_{\widehat{\mathcal{R}}_{n.d.}(f,n)^{op}} X^{\bullet}.$$

So

$$\begin{aligned}
&\mathscr{C}(\bullet)\times_{\widehat{\mathcal{R}}_{n.d.}(f,\geqslant n)^{op}} X^{\bullet}\\
=&\left(\coprod_{g\in\mathcal{R}_{n.d.}(f,n)}\mathscr{C}(g)\right)\times_{\Sigma_n^{op}} X^n\coprod\mathscr{C}(\bullet)\times_{\widehat{\mathcal{R}}_{n.d.}(f,\geqslant n+1)^{op}} X^{\bullet}/(\sim)
\end{aligned}$$

where $(\sim)$ is generated by $(\alpha,\phi^*\mathbf{x})\sim(\phi_*\alpha,\mathbf{x})$ for all morphisms (f,ϕ,g) with $f\in\mathcal{R}_{n.d.}(f,\geqslant n+1)$, $g\in\mathcal{R}_{n.d.}(f,n)$.

By definition of the attaching map e, the above diagram is a push-out diagram. $\square$

Applying Lemma A.2.1 inductively on the construction of $\mathbb{C}_{\mathscr{U}}X$ described in Lemma A.2.6, we get the following result.

Proposition A.2.7. *Let $\nu : \mathscr{C} \to \mathscr{C}'$ be a morphism between E_∞ ring operads with corresponding morphism between monads $\nu : \mathbb{C}_{\mathscr{U}} \to \mathbb{C}'_{\mathscr{U}}$. Then*

$$\nu : \mathbb{C}_{\mathscr{U}} X \to \mathbb{C}'_{\mathscr{U}} X$$

is a homotopy equivalence for all space X.

Proof. By Definition 2.2.2, both the following maps

$$\nu : \Big(\coprod_{g \in \mathcal{R}_{n.d.}(f,n)} \mathscr{C}(g) \Big) \to \Big(\coprod_{g \in \mathcal{R}_{n.d.}(f,n)} \mathscr{C}'(g) \Big)$$
$$\nu : L(f,n)_{\mathscr{C}} \to L(f,n)_{\mathscr{C}'}$$

are equivalences. So applying Lemma A.2.1 and Lemma A.2.6, by induction on n, this proposition holds. □

Proposition A.2.7 is also true for the monad $\mathbb{C}$ defined in Definition 2.1.10. To see this, note that if some non-degenerate f is of type $(l; k_1, \cdots, k_l)$ with $(f, \sigma, \sigma_* f)$ a singular morphism and $\sigma_* f$ is of type $(l'; k'_1, \cdots, k'_{l'})$, then $k_1 + \cdots + k_l > k'_1 + \cdots + k'_{l'}$ because we have evaluated some variable in f to be zero. This gives a filtration on $\mathbb{C}X$ so that we can again apply Lemma A.2.1 inductively.

We denote the full sub-category of $\widehat{\mathcal{R}}_{eff}$ generated by objects of the same type as some special f to be $\widehat{\mathcal{R}}_{eff}(f)$ with objects $\mathcal{R}(f)$. Moreover, we denote

$$\widehat{\mathcal{R}}_{n.d.}^{=n(\leqslant n, <n,\text{resp.})} := \coprod_{f \text{ special}, \ |f| = n(\leqslant n, <n, \text{resp.})} \widehat{\mathcal{R}}_{n.d.}(f)$$
$$\widehat{\mathcal{R}}_{eff}^{=n(\leqslant n, <n,\text{resp.})} := \coprod_{f \text{ special}, \ |f| = n(\leqslant n, <n, \text{resp.})} \widehat{\mathcal{R}}_{eff}(f)$$
$$\mathcal{R}_{n.d.}^{=n(\leqslant n, <n,\text{resp.})} := \coprod_{f \text{ special}, \ |f| = n(\leqslant n, <n, \text{resp.})} \mathcal{R}_{n.d.}(f)$$
$$\mathcal{R}_{eff}^{=n(\leqslant n, <n,\text{resp.})} := \coprod_{f \text{ special}, \ |f| = n(\leqslant n, <n, \text{resp.})} \mathcal{R}_{eff}(f)$$

Proposition A.2.8. *Let $\nu : \mathscr{C} \to \mathscr{C}'$ be a morphism between E_∞ ring operads with corresponding morphism between monads*

$$\nu : \mathbb{C} \to \mathbb{C}'.$$

Then

$$\nu : \mathbb{C}X \to \mathbb{C}'X$$

is a homotopy equivalence for all space X.

Proof. Let

$$F_n \mathbb{C}X := \operatorname{Im}(\mathscr{C}(\bullet) \times_{\widehat{\mathcal{R}}_{eff}^{\leqslant n, op}} X^\bullet \to \mathbb{C}X).$$

Then there is a push-out diagram

$$\begin{array}{ccc}
\mathscr{C}(\bullet) \times_{\widehat{\mathcal{R}}^{=n,op}_{n.d.}} sX^\bullet & \xrightarrow{\quad e \quad} & F_{n-1}\mathbb{C}X \\
\downarrow & & \downarrow \\
\mathscr{C}(\bullet) \times_{\widehat{\mathcal{R}}^{=n,op}} X^\bullet = \mathscr{C}(\bullet) \times_{\widehat{\mathcal{R}}^{=n,op}_{n.d.}} X^\bullet & \longrightarrow & F_n\mathbb{C}X
\end{array}$$

Here sX^m consists of $(x_1, \cdots, x_m)$ with some x_i lies in the image of $S^0 \to X$, $e(\alpha, \sigma^*\mathbf{x}) = (\sigma_*\alpha, \mathbf{x})$, and all $\sim$ are generated by $(\sigma_*\alpha, \mathbf{x}) \sim (\alpha, \sigma^*\mathbf{x})$.

Beginning with $F_0\mathbb{C}X = F_0\mathbb{C}'X = *$, applying Lemma A.2.1 inductively, it suffices to show both

$$\begin{aligned}
\nu : &\mathscr{C}(\bullet) \times_{\widehat{\mathcal{R}}^{=n,op}_{n.d.}} sX^\bullet \to \mathscr{C}'(\bullet) \times_{\widehat{\mathcal{R}}^{=n,op}_{n.d.}} sX^\bullet, \\
\nu : &\mathscr{C}(\bullet) \times_{\widehat{\mathcal{R}}^{=n,op}_{n.d.}} X^\bullet \to \mathscr{C}'(\bullet) \times_{\widehat{\mathcal{R}}^{=n,op}_{n.d.}} X^\bullet.
\end{aligned}$$

are equivalences. The latter has been proved in Lemma A.2.6 while the former follows with the same proof. □

Using the same filtration and applying Lemma A.2.1 inductively, we can also prove the follow result.

Lemma A.2.9. *Let $\mathscr{C}$ be a ring operad in $\mathscr{U}$ with associated monad $\mathbb{C}$ and let $X \to X'$ be an equivalence in $\mathscr{U}_e$. Then so is $\mathbb{C}X \to \mathbb{C}X'$.*

Now applying [7, Proposition 9.8, Corollary 11.10] we get

$$\eta : X \to B(\mathbb{C}, \mathbb{C}, X)$$

is a homotopy equivalence for all $\mathbb{C}$-algebra X in $\mathscr{U}_e$ with homotopy inverse ε.

Proposition A.2.10. *Let $\nu : \mathscr{C} \to \mathscr{C}'$ be a morphism between E_∞ ring operads. Then the pull-back of action functor $\nu^* : \mathscr{C}'[\mathscr{U}_e] \to \mathscr{C}[\mathscr{U}_e]$ induces an equivalence between homotopy categories.*

In particular, any $\mathscr{C}$-algebra X is equivalent to some $\mathscr{C}'$-algebra Y since ν^* is the identity map on underlying pointed spaces.

Proof. We define

$$\begin{aligned}
\nu_* : \mathscr{C}[\mathscr{U}_e] &\to \mathscr{C}'[\mathscr{U}_e] \\
X &\mapsto B(\mathbb{C}', \mathbb{C}, X).
\end{aligned}$$

Then we get the following equivalences

$$X \xrightarrow{\eta} B(\mathbb{C}, \mathbb{C}, X) \xrightarrow{B(\nu,\mathrm{id},\mathrm{id})} \nu^* B(\mathbb{C}', \mathbb{C}, X) = \nu^*\nu_* X$$

and

$$\nu_*\nu^* Y = B(\mathbb{C}', \mathbb{C}, \nu^* Y) \xrightarrow{B(\mathrm{id},\nu,\mathrm{id})} B(\mathbb{C}', \mathbb{C}', Y) \xrightarrow{\varepsilon} Y.$$

Here $B(\nu$,id,id$)$ and $B($id,ν,id$)$ are equivalences by Lemma A.2.9 together with [7, Theorem A.4]. □

Therefore, we get the Comparison Theorem.

Theorem A.2.11 (Comparison Theorem 2.2.4)**.** *Let $\mathscr{C}, \mathscr{C}'$ be any two E_∞ ring operads. Then the homotopy categories of $\mathscr{C}'[\mathscr{U}_e]$ and $\mathscr{C}[\mathscr{U}_e]$ are equivalent.*

Moreover, any $\mathscr{C}$-algebra X is equivalent to some $\mathscr{C}'$-algebra Y.

Proof. Note that the projections $\mathscr{C} \times \mathscr{C}' \to \mathscr{C}$ and $\mathscr{C} \times \mathscr{C}' \to \mathscr{C}$ are morphisms between E_∞ ring operads, so the Comparison Theorem follows from Proposition A.2.10. ☐

Acknowledgement

It is a pleasure to thank my mentor, Peter May, for his guidance and encouragement. His insightful advice greatly contributed to this paper, particularly in the comparison between two constructions of bipermutative categories to E_∞ ring spectra and in the details in the proof of the Comparison Theorem.

I would also like to thank Weinan Lin for his helpful comments and suggestions. The original idea of defining ring operads emerged from our early discussions, and the naming of ring operads is also credited to him.

References

[1] A.D. Elmendorf. Operads for Symmetric Monoidal Categories. *arXiv e-prints*, January 2023.

[2] David Gepner, Moritz Groth, and Thomas Nikolaus. Universality of multiplicative infinite loop space machines. *Algebraic & Geometric Topology*, 15(6):3107–3153, December 2015.

[3] J. F. Jardine. Persistent homotopy theory, 2020.

[4] Niles Johnson and Donald Yau. Bimonoidal categories, E_n-monoidal categories, and algebraic k-theory. *arXiv preprint arXiv:2107.10526*, 2021.

[5] Miguel L Laplaza. Coherence for distributivity. In *Coherence in categories*, pages 29–65. Springer, 2006.

[6] Joachim Lillig. A union theorem for cofibrations. *Archiv der Mathematik*, 24:410–415, 1973.

[7] J.P. May. *The Geometry of Iterated Loop Spaces.* Lecture notes in mathematics, 271. Springer-Verlag, 1972.

[8] J.P. May. *E_∞ spaces, group completions, and permutative categories*, pages 61–94. London Mathematical Society Lecture Note Series. Cambridge University Press, 1974.

[9] J.P. May. E_∞ ring spaces and E_∞ ring spectra. *Lecture Notes in Mathematics*, 1977.

[10] J.P. May. Pairings of categories and spectra. *Journal of Pure and Applied Algebra*, 19:299–346, 1980.

[11] J.P. May. Multiplicative infinite loop space theory. *Journal of pure and applied algebra*, 26(1):1–69, 1982.

[12] J.P. May. *A concise course in algebraic topology.* University of Chicago press, 1999.

[13] J.P. May. The construction of E_∞ ring spaces from bipermutative categories. *Geometry & Topology Monographs*, 16:283–330, 2009.

[14] J.P. May. Operads, moperads, and bioperads. 2024.