TOPOLOGICAL TOOLS FOR DISTINGUISHING KNOT TYPES

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Abstract. Knot Theory's central question is deceptively simple: when are two knots equivalent? One approach to this problem is to investigate the fundamental group of a knot's complement. In this expository paper, we introduce homotopies and fundamental groups, compute them for several knots, and finish with the construction of the Alexander polynomial, a powerful invariant arising from these concepts. We assume familiarity with point-set topology and basic group theory. In the final section we also use some elementary commutative algebra.

Contents

1. Introduction to Knot Theory

Detangling knots - be they in shoelaces, earbud wires, or necklace chains - is perhaps one of the most infuriating problems we encounter in our daily lives; and it is one which mathematicians have been wrestling with for centuries. For instance, consider the left knot in Figure 1 below:

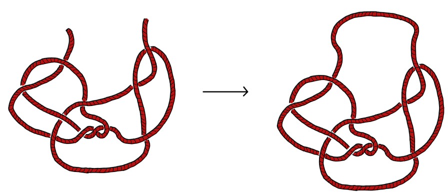

Figure 1.

With a small amount of effort, this rope could be untied. However, now consider what were to happen if we glue the two loose ends together, forming the loop to the right. Suddenly, our problem has become immensely more complex. Are we still able to untangle this knot into a simple circle, relying only on pulling, stretching, and twisting it? The answer, as it turns out, is no. Before we can prove results like this one, however, we must provide some important background.

Definition 1.1. A *knot k* is an embedding of the unit circle S^1 in $\mathbb{R}^3$.

Example 1.2. The most simple example of a knot is a simple unknotted loop. We call this the *unknot.* The next two most simple knots, the *trefoil* and the *figure eight* knots, are pictured, along with the unknot (far left) and the more complicated square knot (far right), in Figure 2.

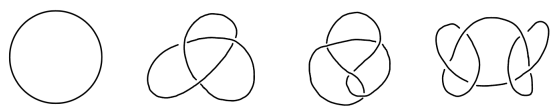

FIGURE 2.

Definition 1.3. Two knots k_1, k_2 are *equivalent* if there exists some orientation-preserving homeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(k_1) = k_2$.

Remark 1.4. For the application of some results, it is easier to think of a knot as S^1 embedded in the 3-sphere $S^3 \cong \mathbb{R}^3 \cup \{\infty\}$ i.e. the one-point compactification of $\mathbb{R}^3$. For this reason, the two may occasionally be interchanged.

Note that knot equivalence is still well-defined since there exists an orientation preserving homeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ sending k_1 to k_2 if and only if there exists such a homeomorphism $g : S^3 \to S^3$ sending k_1 to k_2 since the known homeomorphism can always be extended or restricted by one point.

The knot equivalence problem - determining if two knots are or are not equivalent - is a central question in Knot Theory. To this end, it is crucial to understand the intricacies of the above definition. In particular, we note that all knots are homeomorphic to S^1, and thus to each other. So, we are not concerned with the one-dimensional topology of a knot, which is trivial, but rather with *how it is embedded in space.* Our problem is one of three-dimensional topology - of determining how the knot changes the three-dimensional space around it. In a 1989 paper, C. McA. Gordon and J. Luecke formalized this point by proving the following statement:

Theorem 1.5. *Two knots k_1, k_2 are equivalent if and only if their complements, $S^3 \setminus k_1$ and $S^3 \setminus k_2$ are homeomorphic.*

The forward direction is simple. Given an orientation-preserving homeomorphism $f : S^3 \to S^3$ such that $f(k_1) = k_2$, restricting f to $S^3 \setminus k_1$ yields a homeomorphism from $S^3 \setminus k_1$ to $S^3 \setminus k_2$. The reverse direction, however, is much more complex. While we do not include the proof here, it may be found in Gordon and Luecke's original paper [1].

Theorem 1.5 is crucial for the rest of our project. It tells us that, to determine the equivalence of two knots, we need only consider their complements in S^3. This allows us to draw conclusions by analyzing topological invariants of this complement.

First, however, we must introduce a few more important ideas.

Definition 1.6. A knot is *polygonal* if it is the union of a finite number of line segments. We say a knot is *tame* if it is equivalent to a polygonal knot. If a knot is not tame, we say it is *wild*.

In this paper, we will only consider tame knots. One nice property of tame knots is that we can easily obtain a two-dimensional representation of them by projecting them onto a plane. We call this projection the *knot diagram* or the *knot projection*. For instance, the representations of the unknot, the trefoil knot, and the figure-eight knot in Fig. 2 are all knot diagrams of the knots in question.

It is important to note that two equivalent knots can have different knot diagrams. For instance, rotating, twisting, and stretching a knot will all change its knot diagram, but will not yield a different knot.

Furthermore, when analyzing knot diagrams, we would like to project the knot in a way such that every point where two strands overlap is an actual crossing. We formalize this idea in the definition below.

Definition 1.7. We say a polygonal knot k is in *regular position* if its knot diagram, P, satisfies the following conditions:

(i) At no point of P do three or more strands of k overlap.
(ii) There are only finitely many points of P where two strands of k overlap.
(iii) No points of overlap occur at a vertex of k.

Figure 3 shows the type of crossing we allow and the types we do not:

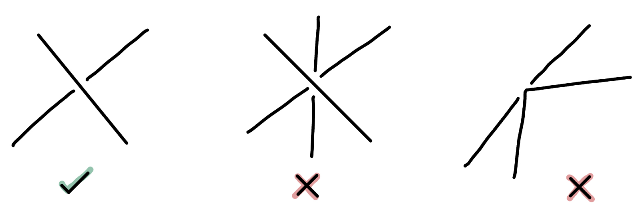

FIGURE 3.

Proposition 1.8. *Every polygonal knot is equivalent to a knot in regular position.*

The idea behind the proof is that, since a polygonal knot k is the union of finitely many line segments, if k is not already in regular position, we can rotate it in space by some arbitrarily small amount. This will eliminate the unwanted points of overlap. Furthermore, we can always make the rotation small enough so as to not create any new unwanted points of overlap.

To fully formalize this requires a fair bit of projective geometry. A full proof can be found in [2, p. 7].

Corollary 1.9. *Every tame knot is equivalent to a knot in regular position.*

We now have a sufficient introduction to Knot Theory to begin investigating the knot equivalence problem. There exist many approaches to differentiating between knots with combinatorial, algebraic, and topological invariants all being used. It is the latter two approaches that we will be interested in for the remainder of this paper. The next section develops the general theory we will use.

2. The Fundamental Group

We will now step away from knots for a moment, and consider an arbitrary topological space X. A *topological invariant* of X is a mathematical object or property associated with X that does not change under a homeomorphism. In this section, our goal is to find such an invariant that will allow us to distinguish the complement spaces of various knots. To this end, we will construct the fundamental group, one of the most powerful invariants used in algebraic topology.

Definition 2.1. A *path* in X is a continuous map $\alpha : [0,1] \to X$. We call $\alpha(0)$ and $\alpha(1)$ the *endpoints* of α, and say that α is a path from $\alpha(0)$ to $\alpha(1)$.

Additionally, we will define the path $\alpha^{-1} : [0,1] \to X$ by

$$\alpha^{-1}(t) = \alpha(1-t)$$

Definition 2.2. Let $\alpha, \beta : [0,1] \to X$ be two paths in X such that $\alpha(1) = \beta(0)$. We define the path $\alpha * \beta : [0,1] \to X$ as follows:

$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t) & 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} < t \le 1 \end{cases}$$

We call this operation *concatenation* of paths.

The concatenation of α with β traverses α and then β, each at twice the speed. Intuitively, it is the pasting of two paths together, one after another.

We now state a particularly important definition:

Definition 2.3. Let $f, g : X \to Y$ be continuous maps. A *homotopy* from f to g is a continuous map

$$F : [0,1] \times X \to Y$$

such that $F(0,-) = f$ and $F(1,-) = g$. If such a map exists, we say that f and g are *homotopic*.

For some $B \subseteq X$ such that for all $x \in B$, $f(x) = g(x)$, we say that f and g are homotopic *with respect to* B if there exists a homotopy F from f to g such that $F(s,x) = f(x) = g(x)$ for all $s \in [0,1]$ and $x \in B$.

For our purposes, we are specifically concerned with homotopies between paths which fix their endpoints. Such a homotopy is essentially a path *of paths* from a path α to a path β, since for any fixed value of $s \in [0,1]$, $h(s,-)$ is a path from $\alpha(0) = \beta(0)$ to $\alpha(1) = \beta(1)$. If there exists a homotopy from α to β then we say

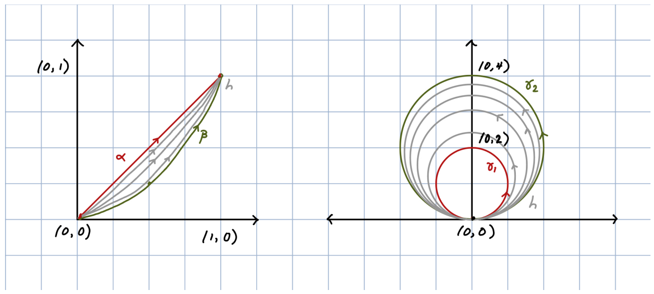

FIGURE 4.

that α and β are *homotopic*, denoted $\alpha \simeq \beta$. We now provide a short example illustrating this concept.

Example 2.4. Consider the paths $\alpha, \beta : [0,1] \to \mathbb{R}^2$ where $\alpha(t) = (t,t)$ and $\beta(t) = (t, t^2)$ (pictured on the left in Fig. 4). Then the map $h : [0,1] \times [0,1] \to \mathbb{R}^2$ where

$$h(s,t) = (t, t + s(t^2 - t))$$

defines a homotopy from α to β.

Definition 2.5. A path $\gamma : [0,1] \to X$ is a *loop* if $\gamma(0) = \gamma(1)$. We call the point $\gamma(0) = \gamma(1)$ the basepoint of the loop γ. The space of loops in X with basepoints at some fixed $x \in X$ is denoted $\Omega(X, x)$.

Homotopies between two loops with the same basepoint are defined in the same way as path homotopies. Consider the following example:

Example 2.6. Let $\gamma_1, \gamma_2 : [0,1] \to \mathbb{R}^2$ be loops in the real plane such that

$$\gamma_1(t) = (\cos(2\pi t) + 1, \sin(2\pi t) + 1) \quad \text{and} \quad \gamma_2(t) = (2\cos(2\pi t) + 2, 2\sin(2\pi t) + 2)$$

(pictured on the right in Fig. 4). In other words, γ_1 and γ_2 are the circlular loops with basepoints at $(0,0)$ and radii of 1 and 2 respectively. Then, the map $h : [0,1] \times [0,1] \to \mathbb{R}^2$ where

$$h(s,t) = ((2-s)\cos(2\pi t) + (2-s), (2-s)\sin(2\pi t) + (2-s))$$

defines a homotopy from γ_2 to γ_1. Here, h is shrinking γ_2 to γ_1, such that for each fixed $s \in [0,1]$, the map $h(s,t)$ defines the circle with basepoint $(0,0)$ and radius $2 - s$.

Now, consider what would happen in the above example if we were to remove the point $(0,3)$ from $\mathbb{R}^2$. We can then see γ_1 and γ_2 are no longer homotopic in this punctured plane, since any homotopy between them must pass through $(0,3)$. This suggests that "breaking" our space in some way alters which loops will be homotopic to one another.

Lemma 2.7. *Let X, Y be two topological spaces. Let $f : X \to Y$ be a continuous map between them such that for some $x \in X$ and $y \in Y$, $f(x) = y$. Now, consider two loops γ_1, γ_2 in X based at x. Then, their images under f, $f \circ \gamma_1$ and $f \circ \gamma_2$, are* *loops in Y based at y. Furthermore, if γ_1 and γ_2 are homotopic, then $f \circ \gamma_1$ and $f \circ \gamma_2$ are homotopic.*

Proof. The first part of the statement is easy to see: since f, γ_1, γ_2 are all continuous, we have that $f \circ \gamma_1$ and $f \circ \gamma_2$ are both continuous as well. Then, since γ_1 is a loop, we know that $\gamma_1(0) = x = \gamma_1(1)$ meaning that

$$(f \circ \gamma_1)(0) = f(\gamma_1(0)) = f(x) = f(\gamma_1(1)) = (f \circ \gamma_1)(1)$$

giving us that $f \circ \gamma_1$ is a loop in Y based at $f(x) = y$. By the same argument, $f \circ \gamma_2$ is also a loop in Y based at y.

Now, we will prove the second part of the statement. Let $h : [0,1] \times [0,1] \to X$ be a homotopy from γ_1 to γ_2. Then, consider the map $f \circ h : [0,1] \times [0,1] \to Y$. First, by continuity of both f and h we know that $f \circ h$ is continuous. Furthermore, by definition, $h(0,t) = \gamma_1(t)$ meaning that $(f \circ h)(0,t) = (f \circ \gamma_1)(t)$. Similarly, $(f \circ h)(1,t) = (f \circ \gamma_2)(t)$. So, we have that $f \circ h$ is a homotopy from $f \circ \gamma_1$ to $f \circ \gamma_2$. □

This behavior suggests that homotopy could be the basis for a topological invariant of the space X. This turns out to be precisely the case. Before we can construct this invariant, however, we must make the following key observation.

Proposition 2.8. *Let X be a topological space, and fix a basepoint $x \in X$. Then $\simeq$ defines an equivalence relation on $\Omega(X,x)$.*

Proof. We need to show that $\simeq$ satisfies reflexivity, symmetry, and transitivity:

Consider some $\gamma_1 \in \Omega(X,x)$. Then, the map $h : [0,1] \times [0,1] \to X$ where $h(s,t) = \gamma_1(t)$ defines a homotopy from γ_1 to itself. So, we have that $\gamma_1 \simeq \gamma_1$.

Now, take some other $\gamma_2 \in \Omega(X,x)$. Suppose that $\gamma_1 \simeq \gamma_2$ and let h be a homotopy from γ_1 to γ_2. Then, the map $h^{-1} : [0,1] \times [0,1] \to X$ defined by $h^{-1}(s,t) = h(1-s,t)$ is a homotopy from γ_2 to γ_1. Thus, $\gamma_2 \simeq \gamma_1$.

Finally, all that remains to check is transitivity. Consider a third $\gamma_3 \in \Omega(X,x)$ and suppose that $\gamma_1 \simeq \gamma_2$ and $\gamma_2 \simeq \gamma_3$ with h_1, h_2 being homotopies between them respectively. Then, we will define $h_3 : [0,1] \times [0,1] \to X$ as follows:

$$h_3(s,t) = \begin{cases} h_1(2s,t) & 0 \leq s \leq \frac{1}{2} \\ h_2(2s-1,t) & \frac{1}{2} < s \leq 1 \end{cases}$$

It is clear that h_3 defines a homotopy from γ_1 to γ_3 meaning that $\gamma_1 \simeq \gamma_3$. □

Now, consider the quotient space $\Omega(X,x)/\simeq$. We can then make the following observation: given loops f, f', g, g' in X based at the point x such that $f \simeq f'$ and $g \simeq g'$, we have that $f * g \simeq f' * g'$ (we leave checking this as an easy exercise for the reader). We can then adapt our definition of the operation $*$ as follows: given representatives f and g of two equivalence classes $[f], [g] \in \Omega(X,x)/\simeq$ we say that $[f] * [g] = [f * g]$.

If $f(t) = x$ for all $t \in [0,1]$ we call f the *constant loop* at x and we have the property that $[f] * [g] = [g] = [g] * [f]$ for all $[g]$. Furthermore, for any $[g]$ we have that $[g] * [g^{-1}] = [f] = [g^{-1}] * [g]$ where f is again the constant loop at x. Checking that these properties hold, as well as verifying the associativity of $*$ will be left as a short exercise for the reader. We now have all of the necessary information to define our invariant.

Definition 2.9. The *fundamental group* of X with basepoint x, denoted $\pi_1(X, x)$, is the group consisting of the set of equivalence classes of $\Omega(X, x)/\simeq$ with the operation $*$.

We will first prove two important theorems about the fundamental group and then finish the section by giving a few examples of $\pi_1(X, x)$ for different spaces.

Theorem 2.10. *Suppose X is path-connected. Then, for any two points $x_1, x_2 \in X$ we have that $\pi_1(X, x_1) \cong \pi_1(X, x_2)$.*

Proof. Let $\alpha : [0, 1] \to X$ be a path from x_1 to x_2. Then, for any loop γ based at x_1 we have that $\alpha^{-1} * \gamma * \alpha$ is a loop based at x_2. We can then define the map $\phi : \pi_1(X, x_1) \to \pi_1(X, x_2)$ where

$$\phi([\gamma]) = [\alpha^{-1} * \gamma * \alpha]$$

To verify that ϕ is a group homomorphism we will consider some $[\gamma_1], [\gamma_2] \in \pi_1(X, x_1)$. Then,

$$\phi([\gamma_1] * [\gamma_2]) = \phi([\gamma_1 * \gamma_2]) = [\alpha^{-1} * \gamma_1 * \gamma_2 * \alpha] = [\alpha^{-1} * \gamma_1 * \alpha * \alpha^{-1} * \gamma_2 * \alpha] = [\alpha^{-1} * \gamma_1 * \alpha] * [\alpha^{-1} * \gamma_2 * \alpha] = \phi([\gamma_1]) * \phi([\gamma_2])$$

meaning that ϕ is in fact a group homomorphism.

We will leave it to the reader to verify that ϕ is a bijection, and thus defines an isomorphism between $\pi_1(X, x_1)$ and $\pi_1(X, x_2)$. □

Result Theorem 2.10 tells us that, for path connected spaces, our choice of basepoint does not matter. In this case, we can simply denote the fundamental group of X by $\pi_1(X)$. However, note that when defining our isomorphism ϕ, no canonical choice of path α exists.

We have now established the fundamental group to be a well-defined object which we can associate with any path-connected space (which every space in this paper will be). Now, all that remains is to prove that the fundamental group of a space is invariant under homeomorphism.

Definition 2.11. Let X and Y be topological spaces with basepoints x and y, respectively. Let $f : X \to Y$ be a continuous map such that $f(x) = y$. Then, Lemma 2.7 tells us that the map $f_* : \pi_1(X, x) \to \pi_1(Y, y)$ where for $[\gamma] \in \pi_1(X, x)$ we have that $f_*([\gamma]) = [f \circ \gamma]$ is well-defined. We call f_* the *homomorphism induced by f*.

Observe that for $[\gamma_1], [\gamma_2] \in \pi_1(X, x)$ this means that $f_*([\gamma_1] * [\gamma_2]) = [f \circ (\gamma_1 * \gamma_2)] = [(f \circ \gamma_1) * (f \circ \gamma_2)]$, meaning that f_* is indeed a group homomorphism.

Remark 2.12. Suppose $f, g : X \to Y$ are two continuous, homotopic maps such that $f(x) = g(x) = y$. Let $F : [0, 1] \times X \to Y$ be a homotopy from f to g. Then, for any $[\gamma] \in \pi_1(X, x)$ we have that $F \circ \gamma$ defines a homotopy from $f \circ \gamma$ to $g \circ \gamma$, meaning that $f_*([\gamma]) = g_*([\gamma])$. So, f and g induce the same homomorphism.

Theorem 2.13. *Let X, Y be two topological spaces and let $f : X \to Y$ be a homeomorphism between them. Consider points $x \in X$ and $y \in Y$ such that $f(x) = y$. Then the induced homomorphism*

$$f_* : \pi_1(X, x) \to \pi_1(Y, y)$$

is an isomorphism.

Proof. All we must check is that f_* is a bijection. Note that for any $[\gamma'] \in \pi_1(Y, y)$ we have that $f_*([f^{-1} \circ \gamma']) = [\gamma']$ where $[f^{-1} \circ \gamma'] \in \pi_1(X, x)$. So, f_* is surjective. Additionally, if $f_*([\gamma_1]) = f_*([\gamma_2])$ for some $[\gamma_1], [\gamma_2] \in \pi_1(X, x)$, then $[f \circ \gamma_1] = [f \circ \gamma_2]$. By Lemma 2.7 we get that $[\gamma_1] = [\gamma_2]$. Therefore, f_* is also injective. This completes the proof. □

Theorem 2.13 allows us to conclude that $\pi_1(X)$ is a topological invariant. In other words, if two spaces are homeomorphic to one another, then they will have isomorphic fundamental groups. However, we caution that the converse is not true: two spaces with isomorphic fundamental groups need not be homeomorphic.

Now, before attempting to use fundamental groups to distinguish knots, we will first give examples of $\pi_1(X)$ for some simpler spaces.

Example 2.14. Let $X = \mathbb{R}^2 \cong \mathbb{C}$. Then, $\pi_1(X)$ is trivial since every loop in $\mathbb{R}^2$ is homotopic to the constant loop.

Note that $\mathbb{R}^2$ is a path-connected space with trivial fundamental group. We call such a space *simply connected*.

Example 2.15. Let $X = \mathbb{R}^2 \setminus \{(0, 0)\}$. In this case, it is easy to see that we no longer have a trivial $\pi_1(X)$, since any loop which winds around $(0, 0)$ will not be homotopic to the constant loop.

In fact, the homotopy class of a loop in X will directly correspond to the number of times that it winds around the point $(0, 0)$ in the counter-clockwise direction (winding around (0, 0) in the clockwise direction would correspond to "negative times" winding around). While this is somewhat intuitive, the proof requires some technical details, so we will omit it here. A good reference is [4, Theorem 3.2.7].

However, now we are virtually done, since this tells us that $\pi_1(X) \cong \mathbb{Z}$. This is because any loop winding around (0, 0) multiple times can be expressed as the concatenation of loops winding around (0,0) once and their inverses. So, $\pi_1(X)$ will be the infinite cyclic group generated by such a loop.

Example 2.16. Let X be $T^2 \cong S_1 \times S_1$ i.e. the one-holed torus, pictured in Fig. 5.

Let $[a]$ denote the homotopy class of loops circling the outside of the torus once (pictured in green) and $[b]$ denote the homotopy class of loops circling once through the inner hole of the torus (pictured in red). We then have that $\pi_1(T^2)$ is generated by $[a]$ and $[b]$. Observe that $[a]$ and $[b]$ each intuitively correspond to a loop around one copy of S^1.

Now, consider the loop $a * b$. This loop will be easiest to visualize on the square representation of the torus (pictured on the right). We let the basepoint be the

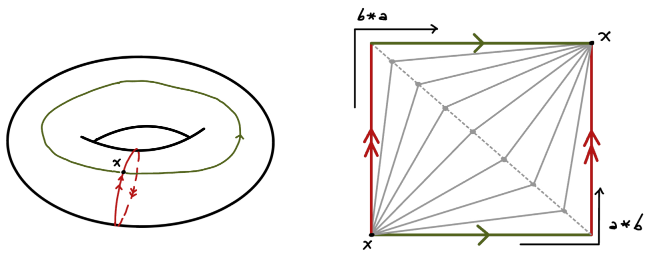

FIGURE 5.

bottom left corner of the square. Then, the path $a * b$ first traverses the bottom edge, then the right edge.

We are then able to "drag" this loop across the square by fixing its endpoints at the bottom left/top right corners and sliding the bottom right corner diagonally to the top left (drawn in grey). Note that this action is continuous and preserves the loop structure.

The resulting loop traverses the left edge and then the top edge. In other words, the resulting loop is $b * a$. So, we have that $[a * b] = [b * a]$. In other words, $\pi_1(T^2)$ is an abelian group. So, we can conclude by saying that $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$.

3. Seifert-Van Kampen Theorem

Before we can begin computing the fundamental groups of knot complements, we will need a very important result. While we will not be proving the original statement, we will be proving a refinement of it which will be essential for our project.

Theorem 3.1. *(Seifert - Van Kampen Theorem)* [5] *Let X be a topological space and let U_1, U_2 be open, path-connected subspaces of X such that $U_1 \cap U_2 \neq \emptyset$, $U_1 \cup U_2 = X$, and $U_1 \cap U_2$ is path-connected. Let $x \in U_1 \cap U_2$ be a basepoint of X. Then,*

$$\pi_1(X, x) \simeq \pi_1(U_1, x) *_{\pi_1(U_1 \cap U_2, x)} \pi_1(U_2, x).$$

First, we will deconstruct what the Seifert-Van Kampen Theorem says. Here, $\pi_1(U_1, x) *_{\pi_1(U_1 \cap U_2, x)} \pi_1(U_2, x)$ is the *amalgamated product*. What it means is that $\pi_1(X, x)$ is the group determined by the generators of $\pi_1(U_1, x)$ and $\pi_1(U_2, x)$ with the following set of relations: the relations of $\pi_1(U_1, x)$; the relations of $\pi_1(U_2, x)$; and the amalgamated relations.

Essentially, the amalgamated relations are included to resolve any disagreement between $\pi_1(U_1, x)$ and $\pi_1(U_2, x)$. For every generator γ of $\pi_1(U_1 \cap U_2, x)$, the corresponding amalgamated relation is $[\gamma]_1 = [\gamma]_2$ where $[\gamma]_1$ and $[\gamma]_2$ are the homotopy classes of γ in $\pi_1(U_1, x)$ and $\pi_1(U_2, x)$ respectively [5]. Together, the amalgamated relations determined by each of the generators of $\pi_1(U_1 \cap U_2, x)$ define the amalgamated product.

Now, in many cases, including that of knot complements, it is much easier to find suitable closed subspaces U_1, U_2 than open ones. To this end, we need to refine the Seifert-Van Kampen Theorem to allow for such a choice of U_1, U_2. This will be done as follows:

Definition 3.2. Let X be a topological space and $U \subset X$ be a subspace. Then a *retraction* $f : X \to U$ is a continuous map such that the restriction of f to U is the identity map. We call U a retract of X if such a map exists.

Definition 3.3. Let $f : X \to U$ be a retraction. We say that f is a *deformation retraction* if there exists a homotopy $F : [0,1] \times X \to X$ from f to the identity map. In other words, F is continuous and $F(0,x) = f(x)$ and $F(1,x) = x$ for all $x \in X$. We call U a *deformation retract* of X if such a homotopy F exists.

We say that f is a *strong deformation retraction* if f is homotopic to the identity map with respect to U.

Remark 3.4. Suppose $U \subseteq X$ is a deformation retract of X. Then, Remark 2.12 tells us that $\pi_1(X) \simeq \pi_1(U)$.

Definition 3.5. We say U is a *neighborhood deformation retract* of X if there exists some open neighborhood $V \subseteq X$ of U such that U is a strong deformation retract of V.

Using this new terminology, we will now prove an adapted version of the Seifert-Van Kampen Theorem:

Corollary 3.6. *Let X be a topological space and let U_1, U_2 be closed, path-connected subspaces of X such that $U_1 \cap U_2 \neq \emptyset$, $U_1 \cup U_2 = X$, $U_1 \cap U_2$ is path-connected, and $U_1 \cap U_2$ is a neighborhood deformation retract of both U_1 and U_2. Let $x \in U_1 \cap U_2$ be a basepoint of X. Then,*

$$\pi_1(X,x) \simeq \pi_1(U_1,x) *_{\pi_1(U_1 \cap U_2, x)} \pi_1(U_2,x).$$

[6]

Proof. Let $V_1 \subseteq U_1, V_2 \subseteq U_2$ be the neighborhoods of $U_1 \cap U_2$ such that $U_1 \cap U_2$ is a strong deformation retract of V_1 and V_2.

Now, let $U_1' = U_1 \cup V_2$ and $U_2' = U_2 \cup V_1$. Then, since $X \setminus U_1 \subseteq U_2$ and $X \setminus U_2 \subseteq U_1$ we have that $U_1' = (X \setminus U_2) \cup V_2$ and $U_2' = (X \setminus U_1) \cup V_1$. So, U_1' and U_2' are open subsets of X. Furthermore, $U_1' \cup U_2' = U_1 \cup U_2 = X$. Finally, we note that V_1 and V_2 must be path connected, since $U_1 \cap U_2$ is path connected, and path connectivity is preserved by homotopies.

So, by the Seifert-Van Kampen Theorem we have that

$$\pi_1(X,x) = \pi_1(U_1',x) *_{\pi_1(U_1' \cap U_2', x)} \pi_1(U_2',x).$$

But now, we know that U_1 and U_2 are retracts of U_1' and U_2' respectively since V_1, V_2 retract to $U_1 \cap U_2$ Thus, by Remark 3.4, we know that $\pi_1(U_1',x) = \pi_1(U_1,x)$ and $\pi_1(U_2',x) = \pi_1(U_2,x)$. Additionally, it is easy to see that $U_1' \cap U_2' = U_1 \cap U_2$. This gives us that

$$\pi_1(X,x) = \pi_1(U_1,x) *_{\pi_1(U_1 \cap U_2, x)} \pi_1(U_2,x).$$

□

4. Knot Groups

We will now apply fundamental groups to the problem of distinguishing knot types.

Definition 4.1. Let k be a knot in S^3. Then, we call $\pi_1(S^3 \setminus k)$ the *knot group* of k.

Recall that Theorem 1.5 says that two knots k_1 and k_2 are equivalent if and only if their complements are homeomorphic. Then, Theorem 2.11 tells us that if k_1 and k_2 are equivalent, their knot groups must be isomorphic. We will now explain how to compute knot groups, and then use these tools in some examples.

Let k be a knot in S^3. Without loss of generality, we can take k to be in the closed upper space of S^3, which we denote S^3_+. In other words, $k \subseteq \{(x, y, z) \in S^3 \mid z \geq 0\}$. Furthermore, by Corollary 1.9, there is no loss of generality in taking k to be in regular position with respect to the plane $z = 0$. We now modify our knot in such a way as to be able to apply Corollary 3.6.

To do this, we first divide our knot into overpasses and underpasses using its knot diagram as done in Fig. 6 for the trefoil, figure eight, and square knots. For each crossing on the diagram, we label the part crossing over an overpass (colored red in Fig. 6) and the part crossing under an underpass (colored green in Fig. 6). We then add in underpasses and overpasses as necessary so that each overpass is next to two underpasses and vice versa.

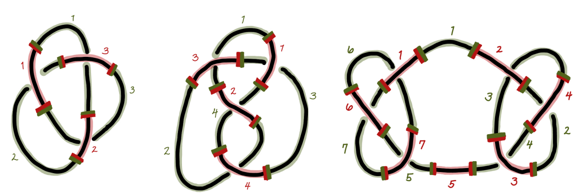

Figure 6.

Now, we can delete the underpasses, instead replacing them with their projections onto the plane $z = 0$. We then connect these curves to their neighboring overpasses by vertical line segments (see Fig. 7).

We call this resulting knot k'. Since we took k to be in regular position, k' is equivalent to k, as the former is simply a "stretching" of the latter. Thus, to compute the knot group of k, we just need to compute the knot group of k', which is much simpler to do.

For our final technical adjustment, we take an open tubular neighborhood of k' in $\mathbb{R}^3$ such that no self-intersections are created (the existence of such a neighborhood follows from the fact that k is tame). In other words, we are "thickening" k' by an infinitesimally small amount, and then taking the interior of this region, leaving us with a knotted open tube. We will call this new object k^*. Since we did not create any new self-intersections, it follows that the fundamental group of $S^3 \setminus k^*$ will be isomorphic to that of $S^3 \setminus k'$ and thus to that of $S^3 \setminus k$, see [7, p. 217].

Now, let U_0 be the closed set $S^3_+ \setminus k^*$ and fix some basepoint $x \in U_0$ (at this stage, since U_0 is path-connected, the choice of basepoint does not matter). Observe

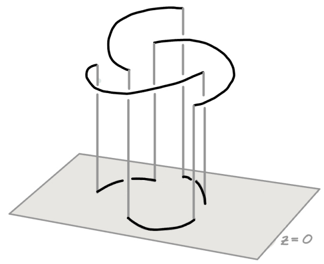

FIGURE 7.

that each overpass and its adjacent vertical segments defines an arch which is being removed from S^3_+. Let γ_i be a loop based at x winding around the ith overpass of k from the right-hand side (relative to the orientation of k) (see Fig. 8). Note that we can continuously drag γ_i along the arch defined by that overpass, meaning that γ_i will be homotopic to any other loop winding around the arch in this way. We will call this homotopy class $[\gamma_i]$. Then, a simple application of Corollary 3.6 gives us that $\pi_1(U_0, x)$ is the free group generated by $[\gamma_1], ..., [\gamma_n]$.

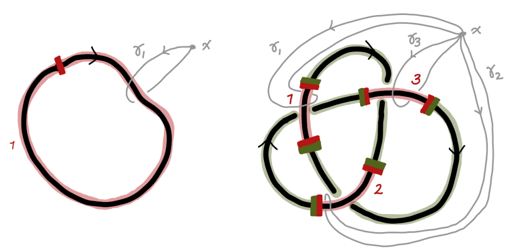

FIGURE 8.

Now, we need to add in the remainder of S^3 using Corollary 3.6. To make the process simpler, we add in each underpass one by one as follows: number the underpasses $1, ..., n$ so that the ith underpass lies between the ith and $i+1$th overpasses (see Fig. 6).

Next, consider the ith underpass and take a small closed path-connected neighborhood around it as pictured in Fig. 9. From this neighborhood, we then remove its intersection with k^* and call the resulting region U_i. We apply Corollary 3.6 to the union of U_0 and U_i. Note that, as in Fig. 9, $U_i \subseteq \mathbb{R}^3_- = \{(x, y, z) \in \mathbb{R}^3 \mid z \leq 0\}$.

By construction, we have that both U_0 and U_i are path-connected. Furthermore, if k^* has radius of ϵ, then we can let $\underline{\beta_i}$ and $\overline{\beta_i}$ be neighborhoods containing $B_i \cap U_0$ of U_i and U_0 respectively such that for all $(x, y, z) \in \underline{\beta_i} \cup \overline{\beta_i}$ we have that $|z| < \epsilon$ (see right image of Fig. 9). Then, the maps $\underline{f_i} : \underline{\beta_i} \to U_i \cap U_0$ and $\overline{f_i} : \overline{\beta_i} \to U_i \cap U_0$ defined by $\underline{f_i}(x, y, z) = (x, y, 0)$ and $\overline{f_i}(x, y, z) = (x, y, 0)$ are strong deformation

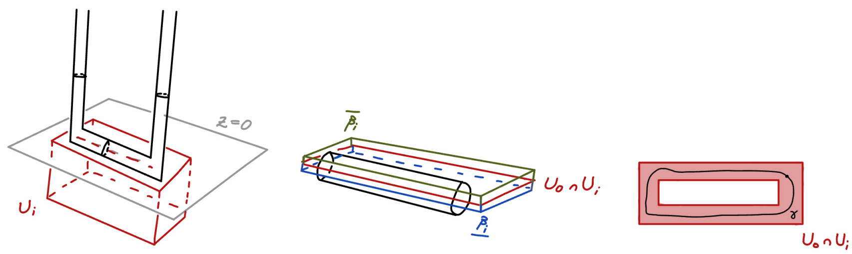

FIGURE 9.

retractions, of $\underline{\beta_i}$ and $\overline{\beta_i}$ resepectively, onto $U_0 \cap U_i$. In this way, we have that $U_i \cap U_0$ is a neighborhood deformation retract of both U_0 and U_i.

We are now able to apply Corollary 3.6 to find $\pi_1(U_i \cup U_0)$. First, we fix some basepoint $x_i \in U_i \cap U_0$. As we established, $\pi_1(U_0)$ is generated by $[\gamma_1], ..., [\gamma_n]$. Additionally, it is easy to see that $\pi_1(U_i)$ is trivial.

All that remains is to consider the relations determined by the intersection. To do so, we first notice that $\pi_1(U_i \cap U_0) \cong \mathbb{Z}$ (by the same reasoning as that used in Example 2.15 to compute $\pi_1(\mathbb{R}^2 \setminus \{(0,0)\})$). So, all we need is to consider the relation determined by $[\gamma]$ where γ is a loop in $U_i \cap U_0$ based at x_i which winds around the underpass once. Recall that this relation is of the form $[\gamma]_i = [\gamma]_0$ where $[\gamma]_i$ and $[\gamma]_0$ are the homotopy classes of γ in $\pi_1(U_i)$ and $\pi_1(U_0)$ respectively. Since $\pi_1(U_i)$ is trivial, we have that $[\gamma]_i = e$.

We now have two cases. The first, simpler case is where the ith underpass does not have an overpass crossing over it (top of Fig. 10). Instead, it merely exists to separate two adjacent overpasses. In this case, "pinching" the loop as depicted in Fig. 10 shows that γ is homotopic to $\gamma_i * \gamma_{i+1}^{-1}$. So, we get the following relation:

$$[\gamma_i][\gamma_{i+1}]^{-1} = e.$$

In the second case, the ith underpass has an overpass passing over it (bottom of Fig. 10). We say this overpass is the jth overpass. In this case, pinching γ on either side of the overpass as shown in Fig. 9 reveals that γ is homotopically equivalent to first traversing γ_i, then γ_j, then γ_{i+1} in the inverse direction, and, finally, γ_j in the inverse direction. This yields the following relation:

$$[\gamma_i][\gamma_j][\gamma_{i+1}]^{-1}[\gamma_j]^{-1} = e.$$

This procedure gives us that $\pi_1(U_i \cup U_0)$ is the group given by the generators $[\gamma_1], ..., [\gamma_n]$ and one of the two previous relations. We can then repeat these for each underpass, obtaining a new relation from each one. All one has to do is determine whether the underpass in question falls into the first case or the second, then note the resulting relation. This gives us a complete presentation for $\pi_1(S^3 \setminus k^*)$ since the remaining piece does not contain any part of k^*.

Below, we present a series of examples for some of the knots we have discussed thus far to enhance the reader's understanding of the knot group computation.

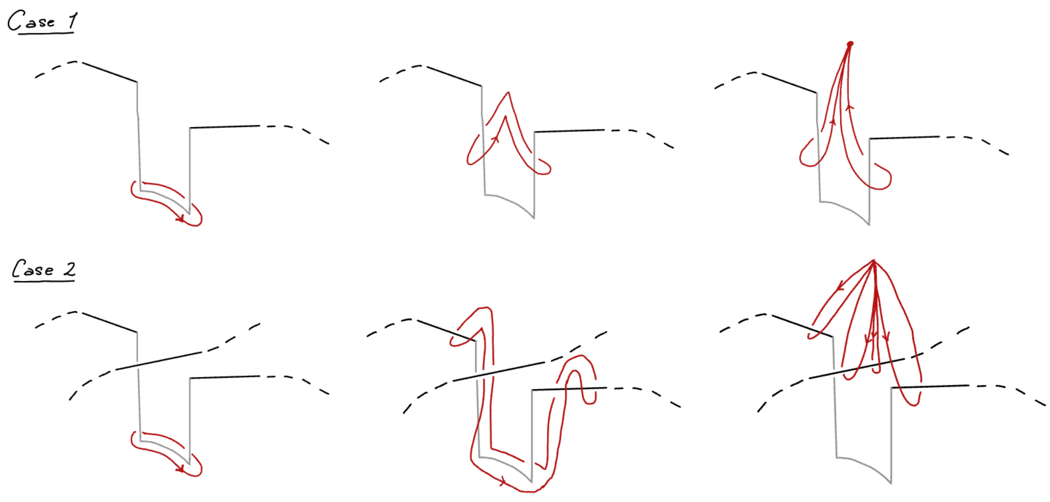

FIGURE 10.

Example 4.2. Let k be the unknot. Then, we can label k as being one overpass itself (see Fig. 11). So, we have that $\pi_1(S^3 \setminus k)$ is generated by $[\gamma_1]$ where $[\gamma_1]$ is a loop winding through k once.

Since k has no underpasses, there are no relations to add in, meaning that $\pi_1(S^3 \setminus k)$ is the infinite cyclic group generated by $[\gamma_1]$. In particular, $\pi_1(S^3 \setminus k) \cong \mathbb{Z}$.

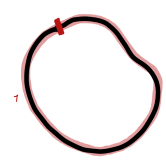

FIGURE 11.

Example 4.3. Now, let k be the trefoil knot (see Fig. 12). It has 3 overpasses, which give us three generators: $[\gamma_1]$, $[\gamma_2]$, and $[\gamma_3]$. Furthermore, it has three underpasses which give us the following relations:

$$[\gamma_1][\gamma_2] = [\gamma_2][\gamma_3]$$
$$[\gamma_3][\gamma_1] = [\gamma_1][\gamma_2]$$
$$[\gamma_2][\gamma_3] = [\gamma_3][\gamma_1].$$

Note that the third relation can be deduced from the first two. It turns out that this is always the case - if there are n underpasses, then taking the relations determined by $n-1$ of them is sufficient. The remaining relation will always follow from the others, see [7, p. 220].

We now have a complete presentation of the knot group of k. However, we are able to simplify it further. For instance, the first two relations tell us that $[\gamma_1][\gamma_2][\gamma_1]^{-1} = [\gamma_3] = [\gamma_2]^{-1}[\gamma_1][\gamma_2]$. This tells us that $[\gamma_1]$ and $[\gamma_2]$ are generators of $\pi_1(\mathbb{R}^3 \setminus k)$ and that $[\gamma_2][\gamma_1][\gamma_2] = [\gamma_1][\gamma_2][\gamma_1]$ (simply a rewriting of the first two relations).

So, we can say that

$$\pi_1(S^3 \setminus k) := \langle a, b \mid aba = bab \rangle.$$

Note that we have already made some progress in our problem of differentiating knots. We previously said that the knot group of the unknot was isomorphic to $\mathbb{Z}$. It is evident that the trefoil's knot group cannot be (for one, it is not abelian). This tells us that the two knots are not equivalent.

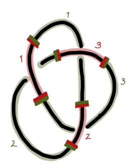

FIGURE 12.

While computing knot groups in this way can be useful for distinguishing the unknot from more complicated knots, when it comes to more complex problems this method proves to not be particularly effective. This is due to how difficult it is to check whether or not two groups are isomorphic. For instance, consider the following attempt to differentiate between the knot groups of the trefoil and the figure eight knot:

Example 4.4. Let k be the figure eight knot. We first divide k into underpasses and overpasses (see Fig. 13). This gives us a set of four generators: $[\gamma_1], [\gamma_2], [\gamma_3], [\gamma_4]$ and the following relations:

$$[\gamma_3][\gamma_1] = [\gamma_1][\gamma_4]$$
$$[\gamma_1][\gamma_2] = [\gamma_2][\gamma_4]$$
$$[\gamma_1][\gamma_3] = [\gamma_3][\gamma_2]$$
$$[\gamma_2][\gamma_4] = [\gamma_4][\gamma_3]$$

Recall that it is sufficient to consider only the first three relations. We can solve the second and third to get that $[\gamma_4] = [\gamma_2]^{-1}[\gamma_1][\gamma_2]$ and $[\gamma_3] = [\gamma_1][\gamma_4][\gamma_1]^{-1} = [\gamma_1][\gamma_2]^{-1}[\gamma_1][\gamma_2][\gamma_1]^{-1}$. In other words, we have that $[\gamma_1]$ and $[\gamma_2]$ are generators for $\pi_1(S^3 \setminus k)$. Finally, substituting in the previous expression for $[\gamma_3]$ into the third relation gives us the following presentation for the knot group of k:

$$\{a, b \mid ab^{-1}ab = b^{-1}aba^{-1}ba\}$$

While at first glance this presentation indeed does not look like that of the trefoil's knot group, proving they are not isomorphic is a difficult, and for more complex knots often impossible, task. So, while the knot group is a powerful invariant - in the sense that it can distinguish between many knots - in practice it is very difficult and inefficient to use. In the final section, we see a way of using basic

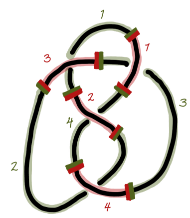

FIGURE 13.

commutative algebra to address this problem, allowing us to extract information from the knot group in a more easily comparable form.

5. The Alexander Polynomial

We now begin the construction of our invariant. First, we make a small remark: in this section we construct the Alexander Polynomial without a great deal of explanation as to why we decided to go down the path in question. In fact, much of the following stems from an analysis of an object called the "universal covering space" of $S^3 \setminus k$. We will not touch on this topic, but for those who wish to read about this background, a great reference is [7, p. 227].

We begin by letting k be a knot in S^3 as before. We let K denote the knot group of k and K' denote the commutator subgroup of K. Our first goal is to investigate the group structure of K'. Recall from Section 4 that K has generators $[\gamma_1], ..., [\gamma_n]$. An alternate set of generators is given by

$$\gamma = [\gamma_n],\ \ x_1 = [\gamma_1][\gamma_n]^{-1},\ \ x_2 = [\gamma_2][\gamma_n]^{-1}, \ldots,\ \ x_{n-1} = [\gamma_{n-1}][\gamma_n]^{-1}$$

Lemma 5.1. *The elements $x_1, ..., x_{n-1}$ and their conjugates by powers of γ generate K'.*

Proof. Let H be the subgroup of K generated by $x_1, ..., x_{n-1}$ and their conjugates by powers of γ. We wish to show that H is the commutator subgroup of K.

First, we show that H is normal. It is easy to see[1] that it is sufficient to show that H is invariant under conjugation by the generators of $K : [\gamma_1], ..., [\gamma_n]$. So, we consider any $[\gamma_i]$. We first show the inclusion $[\gamma_i]H[\gamma_i]^{-1} \subseteq H$: in the case where $i = n$, the inclusion is trivial. For $i \neq n$, consider some $h \in H$ and observe the following: $[\gamma_i]h[\gamma_i]^{-1} = x_i\gamma h\gamma^{-1}x_i^{-1}$ which means that $[\gamma_i]h[\gamma_i]^{-1} \in H$. This gives us that $[\gamma_i]H[\gamma_i]^{-1} \subseteq H$. To show the reverse inclusion, $H \subseteq [\gamma_i]H[\gamma_i]^{-1}$, we must show that for any $h \in H$, there exists some $g \in H$ such that $[\gamma_i]g[\gamma_i]^{-1} = h$. If $i = n$ then we just have $g = \gamma^{-1}h\gamma$. For $i \neq n$, letting $g = \gamma^{-1}x_i^{-1}hx_i\gamma = [\gamma_i]^{-1}h[\gamma_i]$ satisfies this condition. Therefore, we have that $H = [\gamma_i]H[\gamma_i]^{-1}$ for all γ_i, giving us that H is normal.

Next, we wish to show that K/H is abelian. It turns out that this is not at all difficult to see. For all $i, j = 1, \ldots, n$ we have that $\big[[\gamma_i], [\gamma_j]\big] = [\gamma_i][\gamma_j][\gamma_i]^{-1}[\gamma_j]^{-1}$. For $i, j \neq n$ this expression is equal to $x_i\gamma x_j\gamma\gamma^{-1}x_i^{-1}\gamma^{-1}x_j^{-1}$ which is evidently an

[1]If we know, for some $u, v \in K$, that $uHu^{-1} = H$ and $vHv^{-1} = H$, then it follows that $uvHv^{-1}u^{-1} = H$ and $u^{-1}Hu = H$.

element of H. If $i = n$ then we have that $\big[[\gamma_i],[\gamma_j]\big] = \gamma x_j \gamma\gamma^{-1}\gamma^{-1}x_j^{-1}$ which is also in H. This gives us that in K/H all the generators commute, hence K/H is abelian.

So, we have shown that H is a normal subgroup such that K/H is abelian, giving us the inclusion $K' \subseteq H$. We now wish to show that $H \subseteq K'$.

To do so, we recall that all[2] our relations were of the form $[\gamma_i][\gamma_j][\gamma_{i+1}]^{-1}[\gamma_j]^{-1} = e$. We thus have the following two equalities:

$$[\gamma_i] = [\gamma_j][\gamma_{i+1}][\gamma_j]^{-1} \quad \text{and} \quad [\gamma_{i+1}] = [\gamma_j]^{-1}[\gamma_i][\gamma_j].$$

First, consider x_1. We know that, for some fixed k, $[\gamma_n][\gamma_k][\gamma_1]^{-1}[\gamma_k]^{-1} = e$ meaning that

$$x_1 = [\gamma_1]\gamma^{-1} = [\gamma_1][\gamma_k][\gamma_1]^{-1}[\gamma_k]^{-1} = \big[[\gamma_1],[\gamma_k]\big].$$

So, $x_1 \in K'$. We will now show that $x_2 \in K'$. We know that there exists some j such that $[\gamma_2] = [\gamma_j]^{-1}[\gamma_1][\gamma_j]$. We then have the following:

$$x_2 = [\gamma_2]\gamma^{-1} = \big([\gamma_j]^{-1}[\gamma_1][\gamma_j]\big)\big([\gamma_k][\gamma_1]^{-1}[\gamma_k]^{-1}\big) =$$

$$\big([\gamma_j]^{-1}[\gamma_1][\gamma_j]\big)\big([\gamma_1]^{-1}[\gamma_1]\big)\big([\gamma_k][\gamma_1]^{-1}[\gamma_k]^{-1}\big) = \big[[\gamma_j]^{-1},[\gamma_1]\big]\big[[\gamma_k],[\gamma_1]^{-1}\big]$$

giving us that $x_2 \in K'$. The reader can check that $x_3, ..., x_{n-1} \in K'$ through an analogous procedure.

So, we have that $x_2, ..., x_{n-1} \in K'$, and it follows that their conjugates by powers of γ are also in K'. This gives us that $H \subseteq K'$ and allows us to conclude that $H = K'$. □

Now, let K'' be the commutator subgroup of K'. We are now going to look at the abelianization of K': K'/K''. First, consider the following map:

$$T : K'/K'' \to K'/K''$$
$$xK'' \mapsto \gamma x \gamma^{-1} K''$$

where xK'' is an element of K'/K''. Since K' is a normal subgroup of K, it is easy to check that T is well-defined and is, in fact, a bijection.

We now wish to consider the ring of Laurent Polynomials of t, denoted $\mathbb{Z}[t, t^{-1}]$. We define multiplication $\mathbb{Z}[t,t^{-1}] \times K'/K'' \to K'/K''$ as follows: for some $xK'' \in K'/K''$, we have that $P(t) \cdot xK'' = P(T)(xK'')$. Then, one can check that the group K'/K'' is a module over the ring $\mathbb{Z}[t,t^{-1}]$ with multiplication defined in this way. We call this module the *Alexander Invariant* of k and denote it M.

Before proceeding further, we briefly review some ideas about modules and presentation matrices. We consider a finitely generated module, N, over a ring, R, and let $y_1, ..., y_n$ be generators of N.

We can define a map $\varphi : R^n \to M$ as follows:

$$\varphi(a_1, ..., a_n) = \sum_{i=1}^{n} a_i y_i.$$

In this case, $\ker\varphi$ consists of the relations which define N. So (by the First Isomorphism Theorem), we have that $N \simeq R^n/\ker\varphi$. Moreover, we can take N to

[2]For simplicity, if we have a relation of the form $[\gamma_i] = [\gamma_{i+1}]$ we can re-index our generators so as to not include $[\gamma_i]$ twice.

be finitely presented, giving us that $\ker \varphi$ is finitely generated. This allows us to consider a set of generators $z_1, ..., z_m$ for $\ker \varphi$ and take the following matrix:

$$Z = \begin{bmatrix} | & | & & | \\ z_1 & z_2 & \cdots & z_m \\ | & | & & | \end{bmatrix}$$

We call Z a *presentation matrix* of N. Essentially, each column in the presentation matrix represents a relation among $y_1, ..., y_n$. For instance, consider some column of Z:

$$\begin{bmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix}$$

The relation represented by this column is $\sum_{i=1}^{n} a_i y_i = 0$. Together, the relations represented in the presentation matrix generate all the relations among $y_1, ..., y_n$.

Now, the matrix Z allows us to define another map $\psi : R^m \to R^n$ as follows:

$$v \overset{\psi}{\mapsto} Zv.$$

By construction, we have that $\text{im } \psi = \ker \phi$ giving us that $N \simeq R^n / \text{im } \psi$. We call ψ a *presentation* of N. Note that Z, and thus ψ, are not unique - a given module can have many different presentations.

We now return to our Alexander Module, M. Recall that K' has generators $x_1, ..., x_{n-1}$ and their conjugates by powers of γ. However, when we consider M, multiplying some $x_i K''$ by t gives us precisely the conjugate. So, we have that M is generated by $x_1 K'', \ldots, x_{n-1} K''$. Since we have $n-1$ generators and $n-1$ relations which define K, this will give us a $(n-1) \times (n-1)$ presentation matrix for M, which we call an *Alexander Matrix*.

At first glance, we face many of the same difficulties with the module M as we did with K. However, the structure of M as a module over a polynomial ring allows us to extract information about it in the form of an ideal. In particular, our goal is to obtain an invariant of M which is independent of our choice of presentation. We now introduce one method of doing this.

Definition 5.2. Let N be a finitely generated module over a ring R with a presentation $\phi : \mathbb{R}^m \to \mathbb{R}^n$. Let A_ϕ be the corresponding $n \times m$ presentation matrix. Then, for $-\infty < k \leq m$, $I_k(\phi)$ is the ideal generated by all k-minors of A_ϕ, with the convention that $I_k(\phi) = R$ if $k \leq 0$.

For every $0 \leq i < \infty$ we call $I_{n-i}(\phi)$ the ith *Fitting ideal* of N, denoted $\text{Fitt}_i(N)$, see [8].

In the definition above, we chose a presentation ϕ when defining A_ϕ and its ideals. However, it turns out that the Fitting ideals $\text{Fitt}_i(N)$ are independent of the choice of ϕ, see [8, Cor. 20.4].

Definition 5.3. Let k be a knot with Alexander Invariant M. Then, the *Alexander Polynomial* of k is a generator of $\text{Fitt}_0(M)$.

Note that $\mathrm{Fitt}_0(M)$ is the ideal generated by the determinant of the $n-1\times n-1$ Alexander Matrix. This makes it a *principal ideal*, meaning it is generated by a single element. Since $\mathbb{Z}[t,t^{-1}]$ is a domain and $\mathrm{Fitt}_0(M)$ is independent of presentation, this means that the generator of $\mathrm{Fitt}_0(M)$ is unique up to multiplication by an invertible element of $\mathbb{Z}[t,t^{-1}]$. Thus, the Alexander Polynomial is unique up to multiplication by $\pm t^k$ for $k\in\mathbb{Z}$.

We now briefly describe the procedure for computing the Alexander Polynomial, before walking the reader through a few examples. Our first step is to write down a presentation matrix for M. To do so, we must understand how the relations in the presentation of K translate to M. For instance, consider the following relation:

$$[\gamma_i][\gamma_j][\gamma_{i+1}]^{-1}[\gamma_j]^{-1}=e.$$

To understand it as a relation of M, we first wish to rewrite it in terms of γ and $x_1,...,x_{n-1}$:

$$x_i\gamma x_j\gamma\gamma^{-1}x_{i+1}^{-1}\gamma^{-1}x_j^{-1}=e.$$

Then, finally, we rewrite it again, this time in the language of the module M:

$$x_iK''+t\cdot x_jK''+t^{-1}\cdot x_{i+1}K''-x_jK''=x_iK''+(t-1)x_jK''+t^{-1}x_{i+1}K''.$$

So, the column of the presentation matrix corresponding to this relation would have 1 in the ith position, t^{-1} in the $(i+1)$th position, $t-1$ in the jth position, and zeros everywhere else. Repeating this procedure for all $n-1$ relations will give us our $(n-1)\times(n-1)$ Alexander Matrix. Taking the determinant of this matrix yields the Alexander Polynomial.

Example 5.4. Let k be the unknot. Recall from 4.2 that its knot group $K=\pi_1(\mathbb{R}^3\setminus k)\cong\mathbb{Z}$. In this case, M is the zero module and its presentation matrix is a 0×0 matrix. So, by definition, the Alexander Polynomial of k is 1.

Example 5.5. Let k be the trefoil knot. Recall from 4.3 that we had the following original presentation for its knot group, K:

$$\{[\gamma_1],[\gamma_2],[\gamma_3]\ \mid\ [\gamma_1][\gamma_3][\gamma_2]^{-1}[\gamma_3]^{-1}=e,\ \ [\gamma_2][\gamma_1][\gamma_3]^{-1}[\gamma_1]^{-1}=e\}$$

So, we let $\gamma=[\gamma_3],x_1=\gamma_1\gamma^{-1}$, and $x_2=\gamma_2\gamma^{-1}$. Now, we consider the first relation. Rewriting it in terms of γ,x_1, and x_2 gets us $x_1\gamma x_2^{-1}\gamma^{-1}$. In the language of the module M this is equal to $x_1K''-t\cdot x_2K''$. So, the first column of our presentation matrix is $\begin{bmatrix}1\\-t\end{bmatrix}$.

Performing the same operations on the second relation gets us that $[\gamma_2][\gamma_1][\gamma_3]^{-1}[\gamma_1]^{-1}$ becomes $x_2\gamma x_1\gamma^{-1}x_1^{-1}$ which in turn is equal to $x_2K''+(t-1)\cdot x_1K''$. So, we have the following Alexander Matrix:

$$\begin{bmatrix}1 & t-1\\-t & 1\end{bmatrix}.$$

Taking the determinant of this matrix gives us the Alexander Polynomial for the trefoil knot:

$$t^2-t+1.$$

Now, recall that our construction of the Alexander Polynomial stemmed from the difficulties of comparing presentations of knot groups; in particular, the presentations of the trefoil and figure-eight knots. In the following example, we see that the Alexander Polynomial is, in fact, capable of easily differentiating between the two.

Example 5.6. Let k be the figure-eight knot. Recall, we had the following presentation for K:

$$\left\{[\gamma_1],[\gamma_2],[\gamma_3],[\gamma_4] \mid [\gamma_3][\gamma_1][\gamma_4]^{-1}[\gamma_1]^{-1}=e,\ [\gamma_1][\gamma_3][\gamma_2]^{-1}[\gamma_3]^{-1}=e,\ [\gamma_3][\gamma_4][\gamma_2]^{-1}[\gamma_4]^{-1}=e\right\}$$

We then make the substitutions $\gamma=[\gamma_4], x_1=[\gamma_1]\gamma^{-1}, x_2=[\gamma_2]\gamma^{-1}$, and $x_3=[\gamma_3]\gamma^{-1}$ and get the following three relations:

$$\begin{gathered}
x_3\gamma x_1\gamma^{-1}x_1^{-1}=x_3K''+t\cdot x_1K''-x_1K''\\
x_1\gamma x_3\gamma\gamma^{-1}x_2^{-1}\gamma^{-1}x_3^{-1}=x_1K''+t\cdot x_3K''-t\cdot x_2K''-x_3K''\\
x_3\gamma x_2^{-1}\gamma^{-1}=x_3K''-t\cdot x_2K''
\end{gathered}$$

These relations give us the presentation matrix

$$\begin{bmatrix} t-1 & 1 & 0\\ 0 & -t & -t\\ 1 & t-1 & 1\end{bmatrix},$$

the determinant of which gives the Alexander Polynomial of the figure-eight knot:

$$t^3-3t^2+t.$$

Since the above polynomial for the figure-eight knot is not equivalent to the trefoil knot through any multiplication by $\pm t^k$, we have that the two are not equivalent. In this way, we see how the Alexander Polynomial greatly simplifies the problem of comparing knot groups.

That being said, it's differentiating ability is far from perfect, and even less so than that of the knot group. Each step - considering K', then abelianizing it, then taking the Fitting Ideal - results in loss of information, meaning it is possible to have knots with equivalent Alexander Polynomials whose knot groups are not isomorphic. For instance, there are infinitely many knots, which are not equivalent to the unknot, yet have Alexander Polynomial 1 [10, p. 167]. Nonetheless, for simple knots, this invariant is still a powerful one, capable of distinguishing between all prime knots with up to 8 crossings. [9]

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