Providing a bound to pertubations of reduced density matrices in disordered quantum systems using percolation theory

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August 2024

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1 Percolation Theory

1.1 Elementary Percolation Theory

Percolation is a stochastic model in graphs that describes spatial disorder, most often used in a lattice system. The most intuitive way to think of it is through coffee brewing. In the beginning, water cannot permeate the ground coffee beans. However, the cohesion is inversely correlated to the temperature. As the temperature increases, the bonds will randomly break, allowing water to start seeping through. Then, there exists a temperature at which the cohesion breaks down enough such that water can fully permeate the coffee, allowing the brewing process to begin.

We will now go through a rigorous definition of percolation. As in [6], we will consider percolation on the hypercube lattice vertex-edge set: $G = (V, E) \subset \mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$. Similarly, we will define our percolation process to be an edge process. As such, each edge $e \in E$ is designated 1 (open) with probability p, or 0 (closed) with probability $1 - p$. Our state space is $\Omega = \{0, 1\}^E$, where elements ω of this space are vectors known as "configurations." That is, they denote the 0/1 arrangement on E of open or closed edges. The law of this Bernoulli (discrete) probability process is $\mathbb{P}_p$:

$$\mathbb{P}_p(\omega) = \frac{1}{Z_p} \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \tag{1.1}$$

where Z_p is the normalization constant, and $\omega(e)$ is a function that returns 1 if the edge e is open and 0 if e is closed. If an edge is open, this means that two endpoints are connected. We denote a cluster K_x as the set of all points reachable from a vertex x. That is, $K_x = \{y \in \mathbb{Z}^d : x \leftrightarrow y\}$. Let K_0 be the cluster starting from the origin. Here is the primary concern of percolation: does there exist some infinite cluster containing the origin to infinity? Let $\theta(p) = \mathbb{P}_p(|K_0| = \infty)$, where this is the probability of the existence of an infinite cluster starting from the origin.

Definition 1.1. *The critical probability p_c is the probability that satisfies:*

$$p_c = \sup\{p : \theta(p) = 0\}. \tag{1.2}$$

That is, there exists some supercritical regime $p_c < p$, in which the probability that there exists an infinite open cluster starting at the origin is nonzero. Similarly, there exists some subcritical regime $p < p_c$ in which such a cluster almost surely does not exist. There are two very important facts to remember about the infinite open cluster: it is unique and translationally invariant [6]. By ergodicity, this means that the probability that there exists an infinite open cluster is either 0 in the subcritical case or 1 in the supercritical case because the probability of an infinite cluster from any $x \in V$ is the same.

Let $\Lambda_m = [-m, m]^d \subset \mathbb{L}^d$. In the subcritical regime, an important idea that we will use is that of exponential decay. That is, for some $\gamma > 0, m \geq 1$,

$$\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_m) \leq e^{-\gamma m}. \tag{1.3}$$

Essentially, because there is no infinite open cluster, there is some end, and the longer the cluster goes on, the more there is a ”cost” to keep going.

2 Random Cluster Model

First introduced by Fortuin and Kastelyn [4], the random cluster model is a generalization of the classical percolation process which attempts to unify many aspects of random graph theory, including electrical networks and percolation. Take $\varphi_{p,q}$ to be the law of this process defined on a finite set where:

$$\phi_{p,q}(\omega) = \frac{q^{|K|}}{Z_p} \prod_{e \in E} p^{\omega(e)}(1-p)^{1-\omega(e)} \tag{2.1}$$

where K is the set of all clusters K_x, q is a certain weight attached to cluster and $q^{|K|}$ reflects the fact that each connected component contributes a factor of q to the probability measure. When $q = 1$, we find ourselves back in classic percolation (1.1). For $q > 1$ for $q \in \mathbb{R}$, we find that opening an edge increases the likelihood that additional edges are open in the same component. This introduces a positive “correlation” between edges, which is a fundamental notion that we will expand upon.

2.1 Random Cluster Measures

We write $\phi_{\Lambda,p,q}$ when we are concerned with the measure defined on a finite set $\Lambda = (V, E) \subset \mathbb{L}^d$ where $\mathcal{F}$ is the σ-field = Power set of $\{0,1\}^E$. When dealing with the laws induced by various systems, how do we compare the various measures? In this section, we define and prove various concepts that will be instrumental in helping us work with a changing system, only assuming that $\phi_{\Lambda,p,q}$ is strictly positive. That is, $\phi_{\Lambda,p,q}(\omega) > 0$ for any $\omega \in \Omega$.

Let $\partial_e W$ be the set of all edges with one endpoint in W and one outside W. That is, $\partial_e W = \{e =< x, y >\in E | e : x \leftrightarrow y, x \in W, y \notin W\}$. On a finite set, we call a fixed configuration on $\partial_e W$ a “boundary condition.” For some boundary condition ξ,

$$\Omega^{\xi}_{\Lambda} = \{\omega \in \Omega : \omega(e) = \xi(e) \text{ for } e \in \partial_e \Lambda\} \tag{2.2}$$

we condition on the configuration space Ω to only contain configurations ω that satisfy a certain requirement on the boundary of Λ. That is, we take the conditional measure of an event given some restriction on the edges on $\partial_e W$. We define the random cluster measure conditioned on Ω^{ξ}_{Λ} as $\phi^{\xi}_{\Lambda,p,q}$. Here, we define two boundary conditions: $\xi = 0, 1$. We emphasize that this does not restrict whether the connections are all closed or open, respectively. Rather, $\xi = 0$ corresponds to the “free” boundary condition, which means there is no particular boundary condition imposed, and 1 is the “wired” boundary condition, where all edges are required to be the same on $\partial\Lambda$. This means $\omega(e) = 0$ OR $\omega(e) = 1$ for $e \in \partial_e\Lambda$. These are two specific types of boundary conditions, and we will define our own whenever necessary.

Definition 2.1. *We define $\omega \leq \omega'$ if $\omega(e) \leq \omega'(e)$ for all $e \in E$. A function f increases if $f(\omega) \leq f(\omega')$ for $\omega \leq \omega'$. Similarly, an event $A \subset \Omega$ increases if $\omega \in A$ and $\omega \leq \omega'$ implies $\omega' \in A$.*

By this definition, we note that for two boundary conditions ξ, ψ, $\xi \leq \psi$ implies more open edges imposed by the boundary conditions. We also define a separate boundary condition, $\phi^p_{\Lambda,p,q}$. In this case, the p in the exponent corresponds to a "periodic boundary condition. This means that points are identified on opposite boundaries of the box $\Lambda_m = \{x \in \mathbb{Z}^d | x \in [-m, m]\} \subseteq \mathbb{Z}^d$, creating a toroidal structure. Specifically, for any vertex $\omega \in \Lambda_m$, connections are determined modulo $2m+1$. That is, two vertices x_1 and x_2 are considered neighbors if:

$$|x_i - x_j| \equiv 1 \pmod{2m+1} \quad \text{for some } i, \text{ and } x_j = x_j \quad \forall j \neq i.$$

This essentially "wraps" the box Λ_m, creating a periodic structure. One can visualize this by imagining Λ_m as a torus.

Definition 2.2. *Let $\phi_{\Lambda,p,q}$ be a measure on Λ. We say that $\phi_{\Lambda,p,q}$ is strongly positively associated if for all $F \subseteq E$, with the boundary condition ξ where $\phi_{\Lambda,p,q}(\omega) > 0$ for all $\omega \in \Omega^\xi_F$, then for all increasing events $A, B \subset \Omega^\xi_F$, $\phi^\xi_{\Lambda,p,q}(A \cap B) \geq \phi^\xi_{\Lambda,p,q}(A)\phi^\xi_{\Lambda,p,q}(B)$.*

Otherwise stated, for any boundary condition or regime, $\phi_{\Lambda,p,q}$ is such that the measure of two events becomes more likely when they are taken together. We clarify that this is different than regular positive associativity in that we are directly involving the probabilities of events as opposed to the expectations of functions.

Definition 2.3. *Let $F \subseteq E$ where $\Lambda = (V, E) \subset (\mathbb{Z}^d, \mathbb{E}^d)$. For $\xi, \psi \in \Omega$ where $\xi \leq \psi$, we say $\phi_{\Lambda,p,q}$ is monotonic if $\phi^\xi_{\Lambda,p,q}(A) \leq \phi^\psi_{\Lambda,p,q}(A)$ for all increasing events $A \in \Omega_F$.*

That is, one measure will tend to give greater values than another. Another term for this (that we will use from now on) is one "stochastically dominates" the other. That is, for $\xi \leq \psi$ and $F \subseteq E$

$$\phi^\xi_{\Lambda,p,q} \leq_{st} \phi^\psi_{\Lambda,p,q} \tag{2.3}$$

where this is equivalent to

$$\phi^\xi_{\Lambda,p,q}(X_1 < t) \leq \phi^\psi_{\Lambda,p,q}(X_2 < t)$$

for some $t \in \mathbb{R}$. We similarly define 1-monotonicity, which is equivalent to (2.3) except that $F = \{e\}$. Finally, we will introduce an inequality about stochastic domination

Theorem 2.1. *(Holley's inequality) Let ϕ_1, ϕ_2 be strictly positive probability measures on Ω. If for $\omega_1, \omega_2 \in \Omega$, ϕ_1, ϕ_2 satisfy*

$$\phi_1(\max(\omega_1, \omega_2))\phi_2(\min(\omega_1, \omega_2)) \geq \phi_1(\omega_1)\phi_2(\omega_2) \tag{2.4}$$

Then $\phi_2 \leq_{st} \phi_1$.

To clarify, max and min in this case refer to the maximum and minimum connectivity.

Theorem 2.2. *Let ϕ be a strictly positive measure where p, q are omitted. The following are equivalent:*

1. *ϕ is strongly positively associated.*
2. *ϕ satisfies Holley's inequality.*
3. *ϕ is monotonic.*
4. *ϕ is 1-monotonic.*

Proof.

- ($1 \implies 2$) Let us take ω_1, ω_2 on Λ that only disagree on two edges, as any further amount will hold by induction by Theorem 2.2 from [7]. First, let A (B respectively) be the event that the left (right respectively) vertex is open. By strong positive association,

$$\phi(A \cap B) \geq \phi(A)\phi(B) \tag{2.5}$$

where this is equivalent to

$$\phi(11) \geq [\phi(10) + \phi(11)][\phi(01) + \phi(11)]. \tag{2.6}$$

Next, we multiply the LHS by 1 (the summation of all possible events

$$\phi(11)[\phi(10)+\phi(11)+\phi(01)+\phi(00)] \geq [\phi(10)+\phi(11)][\phi(01)+\phi(11)] \tag{2.7}$$

This simplifies to

$$\phi(11)\phi(00) \geq \phi(10)\phi(01)$$

as required.

- ($2 \implies 1$) Next, we prove (2.1) gives us strict positive association via a proof in [6]. Here, we define ϕ and ϕ' to be measures with arbitrary boundary conditions (BC) ξ. Take f, g to be increasing random variables and let ω, ω' be arbitrary configurations. Let

$$\phi_F^{\xi'} = \frac{g(\omega)\phi_F^{\xi}(\omega)}{\sum_{\omega'} g(\omega')\phi_F^{\xi}(\omega')}.$$

Because g is an increasing function, we have that by (2.1),

$$\phi_F^{\xi} \leq_{st} \phi_F^{\xi'}.$$

This extends to taking the expectation value of a random variable as well. Therefore, let us take the expectations of f with both measures. This gives us

$$\sum_{\omega} f(\omega)\phi_F^{\xi}(\omega) \leq \frac{\sum_{\omega} f(\omega)g(\omega)\phi_F^{\xi}(\omega)}{\sum_{\omega'} g(\omega')\phi_F^{\xi}(\omega')}.$$

Strong positive association then follows from our assumption that f, g are increasing functions.

- (2 $\implies$ 3) Comes from Theorem 4.4 in [6].
- (3 $\implies$ 4) Monotonicity trivially proves 1-monotonicity.
- (4 $\implies$ 2) This follows from Theorem 2.3 in [7].

□

2.2 Coupling

What if we wanted to directly relate two measures? In this case, we could create a joint probability measure in which we condition the two individual state spaces.

Definition 2.4. *Let μ, ν be measures on the space $(\Omega, \mathcal{F})$. A coupling $\kappa(\mu, \nu)$ is a joint probability measure in $(\Omega \times \Omega), (\mathcal{F} \times \mathcal{F})$ where the marginals of κ are μ and ν.*

Essentially, a coupling allows us to relate two random variables in a certain way. This paper will use various examples of couplings.

Theorem 2.3. *Let $\omega, \omega' \in \Omega$ and κ be a joint probability measure. If there exists some coupling $\kappa(\omega \geq \omega') = 1$, then it follows that if μ (respectively, μ') is the measure in ω (respectively, ω') then $\mu' \leq \mu$.*

Proof. $\kappa(\omega \geq \omega') = 1$ implies a general global monotonicity of the configurations and of the coupling. If the joint distribution is monotonic, this implies that the marginals μ, μ' are also monotonic. This inherited monotonicity implies $\mu' \leq \mu$ as required. □

This is a very general proof, and if we wanted to prove this for an explicit coupling, we would have to define boundary conditions to more directly compare ω and ω'. Intuitively, if you condition a measure to have greater connectivity, at the very least the probability of an event happening will not decrease. This is a direct result of the positive association. Generally, if you condition your measure on the event that there is more connectivity, that induces a greater chance of events happening.

3 Ratio Weak-Mixing

An important consequence of the exponential decay of connectivity functions is that the correlation between subsystems also decreases exponentially. Furthermore, the correlation between two events also decreases exponentially as the distance increases. Given this correlation decay, we can state that the system has the "weak mixing" property. Let $\delta(P, Q) = \sup_A |P(A) - Q(A)|$ be the total variation distance, where P, Q are arbitrary measures, and $A \in \mathbb{E}^d$. We use the definition of the weak mixing property as seen in [1].

Definition 3.1. *Let φ be a measure of Ω. We say φ has the weak mixing property for every $F \subseteq E$ and for any arbitrary boundary conditions ξ, ψ:*

$$\delta(\varphi_F^{\xi}, \varphi_F^{\psi}) \leq C \sum_{x \in \Delta, y \in \Lambda^c} \exp(-\lambda |x-y|). \tag{3.1}$$

This means that the effect of two different boundary conditions ξ, ψ decays exponentially over a long distance. This implies a far more interesting result, which is that as the distance between events grows, they become "essentially" independent. Otherwise, $|\varphi(A \cap B) - \varphi(A)\varphi(B)|$ is less than or equal to some exponential decay as in (1.3). (3.1) is an expression of weak-mixing we call "ratio weak-mixing." The rest of this section is dedicated to proving the "ratio weak-mixing" result for our continuum lattice. We only assume that we are in the subcritical regime ($p < p_c$), such that the weak mixing property holds. For a proof that the weak mixing property holds, refer to [1].

Let $\Lambda_m = \{x \in \mathbb{Z}^d | x \in [-m, m]\} \subseteq \mathbb{Z}^d$. We define a path as connected components on our lattice. We now define two disjoint subsets of Λ_m, $\Gamma$and Δ. We also define D to be a region that "separates" Γ and Δ in the sense that any path $\Gamma \leftrightarrow \Delta$ must also pass through D. D is minimal, such that no strict subset also separates Γ and Δ.

Theorem 3.1. *(Ratio weak-mixing) Let $\Gamma \subseteq \Lambda$ be measurable where Λ is finite, let $\Delta \subseteq \Lambda$ be finite such that $\Delta \cap \Gamma = \varnothing$, and let D be a linear subset of Λ that separates Δ and Γ. Let $\lambda, C \in (0, \infty)$. Let $t(\lambda, C, \Lambda, \Gamma) = Ce^{-\lambda|x-y|}$ for $x \in \Delta$, $y \in \Gamma$. If Σ_Γ (Σ_Δ) is the state space of Γ (Δ respectively), then for $\tau, \tau' \in \Sigma_\Gamma$ and $\alpha_\Delta \in \Sigma_\Delta$,*

$$\left| \frac{\phi_\Lambda^{\tau}(\sigma_\Delta = \alpha)}{\phi_\Lambda^{\tau'}(\sigma_\Delta = \alpha)} - 1 \right| \leq t(\lambda, C, \Lambda, \Gamma) \tag{3.2}$$

whenever the right side is less than or equal to 1.

This is the form we will use for the proof, but a more enlightening version is obtained as we average out over τ', leaving us with just

$$\frac{\phi_\Lambda^{\tau}(\sigma_\Delta = \alpha)}{\phi_\Lambda(\sigma_\Delta = \alpha)}.$$

We then use the following conditional relation:

$$\phi_\Lambda^{\beta,n}(\sigma_\Delta = \alpha) = \frac{\phi_\Lambda^n(\sigma_\Delta = \alpha, \sigma_\Gamma = \beta)}{\phi_\Lambda^n(\sigma_\Gamma = \beta)}$$

and define $\mathcal{F}_\Delta$ ($\mathcal{F}_\Gamma$) to be the sigma-fields of Δ (respectively, Γ). Now, we generalize and state that for any $A \in \mathcal{F}_\Delta, B \in \mathcal{F}_\Gamma$,

$$\left| \frac{\phi_\Lambda(A, B)}{\phi_\Lambda(A)\phi_\Lambda(B)} - 1 \right| \leq t(\lambda, C, \Lambda, \Gamma). \tag{3.3}$$

For the proof that follows, the reader can find alternate formulations in [1] and [9].

Proof. Let I (respectively, E) be the region of Λ that is reachable from Δ (respectively, Γ) along the paths of Λ that do not intersect D. From Theorem (2.3), we know that $\varphi^b \leq_{st} \varphi^\xi$ if $b \leq \xi$. The important consequence of this is that, for $\omega, \omega', \omega'' \in \Omega_\Lambda$, where $\omega', \omega'' \leq \omega$, we can create a coupling such that $\kappa(\omega \geq \omega', \omega'') = 1$. Let φ be the law of ω, and let φ^α_Λ (respectively, φ^η_Λ) be the law of ω' (respectively, ω''). Furthermore, we construct ω and ω' in such a way that if $\bar{\omega} \in E_2 = \{D \leftrightarrow \Gamma\}$, then $\bar{\omega}, \omega$, and ω' are identical on $D \cup I$. For the clusters of $\omega, \omega', \omega''$ we assign spins $\sigma, \sigma', \sigma''$, respectively, such that: on event E_2, the functions $\sigma, \sigma', \sigma''$ are equal on $D \cup I$. Let H be an event that satisfies

$$H \subset \{\sigma_\Delta = \sigma'_\Delta\}. \tag{3.4}$$

For the moment, let us assume that the following equations hold:

$$\kappa(\sigma_\Delta = \alpha_\Delta), \kappa(\sigma'_\Delta = \alpha_\Delta) > 0 \tag{3.5}$$

$$\kappa(H^C | \sigma_\Delta = \alpha_\Delta), \kappa(H^C | \sigma'_\Delta = \alpha_\Delta) \leq \frac{t}{2} \tag{3.6}$$

where κ is the appropriate measure. Using the fraction from (3.3), we state:

$$\frac{\kappa(\sigma_\Delta = \alpha_\Delta)}{\kappa(\sigma'_\Delta = \alpha_\Delta)} = \frac{\kappa(H \cap \{\sigma_\Delta = \alpha_\Delta\})}{\kappa(H \cap \{\sigma'_\Delta = \alpha_\Delta\})} \cdot \frac{\kappa(H \cap \{\sigma' = \alpha_\Delta\})}{\kappa(H \mid \sigma_\Delta = \alpha_\Delta)} \tag{3.7}$$

$$= \frac{\kappa(H \mid \sigma'_\Delta = \alpha_\Delta)}{\kappa(H \mid \sigma_\Delta = \alpha_\Delta)}. \tag{3.8}$$

where in (3.7) we simply turned each individual measure into a fraction using the definition of conditional probability and the fact that, by definition,

$$H \cap \{\sigma_\Delta = \alpha_\Delta\} = H \cap \{\sigma'_\Delta = \alpha_\Delta\}.$$

If (3.5),(3.6) hold, then it follows that (3.8) is bounded below by $1 - \frac{1}{t}$ and above by $1 + t$. As such, (3.1) follows.

The rest of the proof is dedicated to proving (3.6). If the reader is curious to understand why (3.5) is true, consult [1]. First, we note that (3.1) implies that the variance between the two configurations ω, ω' is less than or equal to some exponential decay. Define

$$t_1 = \varphi(\Delta \leftrightarrow D)$$

and

$$t_2 = \varphi(\Gamma \leftrightarrow D).$$

Therefore,

$$\kappa(\sigma_D \neq \sigma'_D) \leq 1 - t_1. \tag{3.9}$$

We note that due to our prior definition of $\kappa(\sigma, \sigma')$, we let $t_1 \leq t_2$. Similarly,

$$\kappa(\sigma_D = \sigma'_D) \geq 1 - t_2. \tag{3.10}$$

Next, let $\mathcal{F}_D(\mathcal{F}_D)$ be the sigma-field generated by σ_D, σ'_D. We define

$$q = \kappa(\sigma_D \neq \sigma'_D | \mathcal{F}_D), q' = \kappa(\sigma_D \neq \sigma'_D | \mathcal{F}'_D).$$

We let $a > 0$. We redefine $H = H_a$:

$$H_a = \{\sigma_D = \sigma'_D\} \cap \{q \leq a\} \cap \{q' \leq a\}.$$

We see that this redefinition of H is fine with our previous definition from (3.4). We will assign a value to a later on. Since we are in a regime with weak-mixing, for every $A \in \mathcal{F}_D$

$$\kappa(\sigma_D \in A | \sigma_\Delta = \alpha_\Delta) \leq \kappa(\sigma_D \in A) + t_1 \tag{3.11}$$

where t_1 comes from the fact that the asymptotic independance can be characterized by the exponential decay of the connectivity. By (3.9) and Markov's inequality:

$$\kappa(q > a) \leq \frac{1}{a}\mathbb{E}_\kappa(q) \leq \frac{1}{a}t_2 \tag{3.12}$$

where $\mathbb{E}_\kappa$ is the expectation with respect to κ. We then state that because the event $\{q > a\} \in \mathcal{F}_\mathcal{D}$, we can use (3.11) and (3.12) to get:

$$\begin{aligned} \kappa(q > a | \rho_\Delta = \alpha) &\leq \kappa(q > a) + t_1 \\ &\leq \frac{1}{a}t_2 + t_1 \end{aligned} \tag{3.13}$$

Because of our assumption of ω, ω' being identical on $D \cup I$, we determine that whatever we prove for σ_D, we prove for σ'_D. Next, we use the essential supremum and convenient upper bound we can derive by definition to state

$$\begin{aligned} \kappa(\sigma_D \neq \sigma'_D, q \leq a | \sigma_D) &\leq \text{ess sup}\{\kappa(\sigma_D \neq \sigma'_D | G_D) 1_{\{q \leq a\}}\} \\ &\leq a. \end{aligned} \tag{3.14}$$

Giving our conditioning on D, we know that

$$\{\sigma_\Delta = \alpha_\Delta\} \cap \{\sigma_D = \alpha'_D\} = \{\sigma'_\Delta = \alpha_\Delta] \cap [\sigma_D = \alpha'_D\}. \tag{3.15}$$

Next, let us analyze the probability of q and q' on other conditional events. By (3.15) and basic inequalities concerning intersections versus conditions, we have:

$$\kappa(q' > a, \sigma_D = \sigma'_D \mid \sigma_\Delta = \alpha_\Delta) \leq \kappa(q' > a \mid \sigma'_\Delta = \alpha_\Delta, \sigma_D = \sigma'_D). \tag{3.16}$$

Next, we use the law of total probability, set $a = t_2$, and finally use (3.14) and (3.16):

$$\begin{aligned} \kappa(q' > a \mid \sigma'_\Delta = \alpha_\Delta, \sigma_D = \sigma'_D) &\leq \frac{\kappa(q' > a \mid \sigma = \alpha_\Delta)}{\kappa(q' > a \mid \sigma_D = \sigma'_D)} \\ &\leq \frac{t_1 + t_2}{1 - t_1 - 2t_2}. \end{aligned} \tag{3.17}$$

Similarly, (3.17) holds with q' instead of q. Finally we combine ,(3.13),(3.14), (3.9), and (3.17) to get

$$\kappa(H^C|\sigma_\Delta = \alpha_\Delta) \leq t_1 + 3t_2 + \frac{t_1 + t_2}{1 - t_1 - 2t_2}. \tag{3.18}$$

Similarly, $\kappa(H^C|\sigma'_\Delta = \alpha_\Delta)$ from (3.6) also holds. Setting the RHS equal to t then concludes our proof. □

4 Quantum Ising Model

Define a graph $G = (V, E)$. For each vertex $v \in V$ we assign $\Sigma = \{-1, 1\}^V$ as before, but these are no longer the same quantities as before. In the quantum setting, we operate in a *Hilbert space*, a complete vector space equipped with an inner product where elements are vectors representing possible configurations of the system. To obtain the necessary configuration space Σ, the relevant Hilbert space will be $\mathbb{C}^2$, where elements are complex-valued two-dimensional vectors. As such, to properly define our configuration space with multiple vectors (corresponding to the vertices), our Hilbert configuration space: $\mathcal{H} = \otimes_{v\in V}\mathbb{C}^2$ and $\otimes$ is the tensor product, which ensures that every site can still be described individually.

Now, we define $SU(2)$ as the set of all 2×2 unitary matrices with determinant 1, which control rotations in $\mathbb{C}^2$. Its Lie algebra $\mathfrak{su}(2)$ is spanned by the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

governing spin transformations. By convention, we choose σ^z to be our "measurement basis" such that the bases of $\mathbb{C}^2$:

$$0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, 1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

correspond to the eigenvectors of σ^z, and the eigenvalues of σ^z, $\{-1, 1\}$, represent the physical measurement outcomes. Then, this structure allows us to define Σ on $\mathcal{H}$.

We call elements of $\mathcal{H}$ "wavefunctions", which we denote via the "ket": $|\cdot\rangle$, and we denote elements of the dual space $\mathcal{H}'$ via the "bra": $\langle\cdot|$. Let $|\psi\rangle \in \mathcal{H}$ be the linear combination:

$$|\psi\rangle = \sum_{\sigma\in\{-1,1\}^V} c_\sigma \, |\sigma\rangle \tag{4.1}$$

where $c_\sigma \in \mathbb{C}$ represents an assignment of $\{-1, 1\}$ to each vector. To recover the classical notion of probability, we use the Born rule, which states that in quantum mechanics, the probability p_i is given by:

$$p_i = |c_i|^2 \tag{4.2}$$

We define the density matrix ρ as a positive semi-definite, trace-1 operator which is the outer product between $|\psi\rangle$ with its dual conjugate, which we denote as:

$$\rho = |\psi\rangle \langle\psi| \tag{4.3}$$

We define each element in the density matrix as

$$\rho_{i,j} = \langle i|\rho|j\rangle$$

where we can similarly define this as

$$\rho_{i,j} = e_i^* \rho e_j,$$

where e_i represents the i_{th} basis vector, and e_j^* the j_{th} conjugate basis vector. By (4.2), it follows that the diagonals correspond to the probability of state $|i\rangle$. Given this, the proper way to normalize a density matrix must be to take its trace

$$\text{trace}(\rho) = \sum_{i=1}^{n} \langle i|\rho|i\rangle$$

assuming $|\psi\rangle$ is a n-dimensional vector. Define

$$\beta = \frac{1}{k_B T} \tag{4.4}$$

as the inverse temperature and H as the Hamiltonian operator, a matrix describing the energy of a specific configuration on Σ:

$$H(\sigma) = -\frac{1}{2}\lambda \sum_{e=\langle i,j\rangle \in E} \sigma_i^z \sigma_j^z - \delta \sum_{j\in V} \sigma_j^x. \tag{4.5}$$

Given β and H, the density matrix can be similarly defined as:

$$\rho_{\beta,\lambda,h}(\sigma) = \frac{1}{Z} e^{-\beta H(\sigma)} \tag{4.6}$$

where $\rho_{\beta,\lambda,h}(\cdot)$ is the probability mass function on the graph. The $\frac{1}{2}$ comes from the spin value of the electrons. σ^z, σ^x correspond to the spin algebras, in this case the z and x bases of the Pauli matrices. We interpret the left summation in (4.5) to be the edge operator, acting on the interaction of spins between two sites. However, by the properties of the spin operators:

$$[\sigma_i^z, \sigma_i^x] = 2i\sigma^y.$$

Because they do not commute, the z and x bases can not be simultaneously measured, implying that they do not have the same eigenbases. Since simultaneous eigenstates do not exist, applying the x-basis at a vertex induces a transition to the other basis state. As such, we can interpret the x basis as an operator denoting "quantum fluctuations."

We interpret λ (respectively δ) to be the strength of spin-coupling between two sites (respectively, the intensity of quantum fluctuations). However, it is important to mention that anticommutativity only holds if applied at the same site. If $i \neq j$, then

$$\sigma_i^z \sigma_j^x = \sigma_i^x \sigma_j^z. \tag{4.7}$$

This commutation implies that operators are independent of each other if applied to different sites or edges. Finally, Z is the partition (normalization) function of the density matrix:

$$Z = \text{trace}(e^{-\beta H}). \tag{4.8}$$

5 Unifying the Ising Model and Percolation

5.1 Edwards-Sokal Representation

It is possible to unite the Ising and random cluster models through a joint probability distribution known as the Edwards-Sokal representation. Define the coupling μ as taking from the probability distribution $\Omega \times \Sigma$ where Ω is our edge configuration state space $\Omega = \{0,1\}^{E^d}$ as before, and Σ is a vertex configuration space $\Sigma = \{1, 2, ..., q\}^V$ where q could denote some characteristic like spin, color, etc. We define the law of the coupling μ as:

$$\mu(\sigma, \omega) = \frac{1}{Z_{p,q}} \rho(\sigma) \phi_{p,q}(\omega) 1_F. \tag{5.1}$$

where F is the event where if for $\omega(e) = 1$, then $\sigma_x = \sigma_y$. That is, they are part of the same cluster. Now, we show that the marginal distributions in σ, ω give us the Ising model and the percolation, respectively.

The marginal over Ω is only constrained by the requirement that σ is constant on clusters. Therefore, this is a summation of all the clusters, which is equal to some weight of possible values raised to the amount of clusters, this being $q^{|K|}$ where K is the set of clusters. Therefore, this gives us back the random cluster model. To compute the marginal distribution Σ, we will require the use of

$$\delta_e(\sigma) = \delta_{\sigma_x, \sigma_y}$$

where $\delta_{u,v}$ is a delta function. This leaves us with

$$\prod_{e \in E} \frac{1}{Z} (1-p)^{1-\delta_e(\sigma)}.$$

If we take the density p to be $1 - e^{-\beta\lambda}$, then our marginal distribution becomes proportional to

$$\mu(\sigma, \cdot) = \frac{1}{Z} \exp\{\beta\lambda \sum_{e \in E} \delta_e(\sigma)\}.$$

This gives us a generalized version of the Ising model known as the Potts model, where there are an arbitrary amount of values (which we denoted as q) the spin can take. If we are to take $q = 2$ and let $\delta_e(\sigma) = \frac{1}{2}(1 + \sigma_i\sigma_j)$, then the measure reduces to

$$\mu(\sigma, \cdot) = \frac{1}{Z} e^{\beta\lambda \sum_{<i,j>\in E} \sigma_i\sigma_j} \tag{5.2}$$

which is the Ising model as required. Via the construction proposed in [2], it is possible to construct an Edwards-Sokal representation that includes a generalized Potts model, with the necessary site operator. Therefore, we have recovered the Ising model via the marginal distribution on Σ. The importance of this result is that the long-range order of correlations in the quantum Ising model is equivalent to a percolation process.

Now, if we want to properly use percolation theory, we need to adapt certain other ideas like the connectivity function. The following theorem explains how to translate the connectivity in the percolation theory to that of the Ising / Potts models.

Theorem 5.1. *For $q \in \{2, 3...\}$ and $\beta \geq 0$, the connectivity function in $\tau_{\beta,q}$ is given by*

$$\tau_{\beta,q}(x, y) = (1 - \frac{1}{q})\varphi_{p,q}(x \leftrightarrow y).$$

Proof. Let $\pi_{\beta,\lambda,\delta}$ be the law of the generalized Potts model. Here, we take the mean value of some observable $\tau : \Omega \to \mathbb{R}$

$$\pi_{\beta,\lambda,\delta}(\tau) = \sum_{\sigma} \tau(\sigma)\pi_{\beta,\lambda,\delta}(\sigma).$$

$\pi_{\beta,\lambda,\delta}(\sigma)$ represents the distribution on Σ. However, we can represent $\pi_{\beta,\lambda,\delta}(\sigma)$ as a conditional measure of the coupling $\mu(\sigma, \omega)$. Given ω, the conditional measure in Σ is performed by uniformly placing spins on clusters $k(\omega)$. Therefore, the conditional measure gives us an expression of the connectivity of specific clusters. Therefore,

$$\pi_{\beta,\lambda,\delta}(\tau) = \sum_{\sigma,\omega} \tau(\sigma)\mu(\sigma|\omega)\varphi(\omega). \tag{5.3}$$

Let us define τ as the two-point connectivity operator:

$$\tau_{\beta,q}(x, y) = \delta_{x,y}(\sigma) - \frac{1}{q}, \tag{5.4}$$

where q^{-1} represents the probability that two independent sites have the same spin. Remembering that $\mu(\sigma|\omega)$ gives us an expression of the connectivity of two points, we use (5.4) and (5.3) to get:

$$\pi_{\beta,\lambda,\delta}(\tau) = \sum_{\omega} \mu(\tau|\omega)\varphi(\omega) = \sum_{\sigma,\omega} \tau(\sigma)\mu(\sigma|\omega)\varphi(\omega) = (1 - \frac{1}{q})\varphi_{p,q}(x \leftrightarrow y) \tag{5.5}$$

as required. □

5.2 Critical Probability of the Quantum Ising Model

Something we have not yet addressed is that although percolation can be connected to the Quantum Ising model via the Edward-Sokal representation, we do not actually have the necessary conditions for it. If we have a finite lattice, even in the subcritical phase, there is a non-zero probability that all edges are open. Therefore, percolation is only well defined in the infinite case. This is tricky for a random cluster measure, as the existence of boundary conditions requires extra care in this "infinite-volume limit". In fact, it is possible to find that the infinite limit of the measure depends on which boundary condition you define on your measure. Refer to [6] for further information on the limit and when it is unique.

If percolation is achieved in the limit of the random cluster measure, then we similarly have long-range order in the Quantum Ising model. This implies that we can use the same theorems and concepts we have developed up until now in the Ising regime. This includes the idea of the critical probability. From [3], we can show that, using the notation from (4.5), for any $\delta < \delta_c$, there exists a $\lambda_c < \lambda$ such that the Quantum Ising model expresses global correlation between spins, which we call "long-range order". This result implies that in the Quantum Ising Model, it is neither λ nor δ that matters, but rather the ratio

$$\theta = \lambda/\delta \tag{5.6}$$

that determines long-range order. Henceforth, we shall only concern ourselves with θ_c. The rest of this section will be dedicated to an analysis of θ_c and the subcritical regime $\theta < \theta_c$.

Theorem 5.2. *The critical ratio of the Quantum Ising model is the self-dual point* $\theta = 2$

We will need to use the concept of "planar duality" to prove this. The dual graph of any lattice Λ, Λ_d is defined by placing a vertex in every face of Λ. For each edge $e \in E$ and $e_d \in E_d$, e_d is open if and only if the corresponding e in the primal (original) graph is also open. This implies that if long-range order is not possible, then every finite open cluster is enveloped by a closed cycle in the dual graph. There also exists the notion of a self-dual point, p_{sd}. That is, the probability at which there is a natural isomorphism from the vertex-edge set to its dual set.

Proof. 1. Assume $\theta_{sd} \leq \theta_c$

Let $\theta = \theta_{sd}$. Define a sublattice $\Lambda_m = [-m, m]^2$ where on Λ we define mutually exclusive events A, corresponding to a horizontal open crossing in the primal graph, and B, a vertical close crossing in the dual graph. By the isomorphism from Λ to Λ_{d}, there is a "scale invariance" in the equality of probability for any square lattice Λ_m. That is, self-duality ensures symmetry between the primal and dual crossings:

$$\varphi(A) = \varphi(B). \tag{5.7}$$

By (1.3), there exists some $\gamma > 0$ such that there is an exponential decay in the primal graph such that

$$\varphi_{p,q}(0 \leftrightarrow \partial(\Lambda_m)) \leq Ce^{-m\gamma}. \tag{5.8}$$

However, the isomorphism implies that there would also be an exponential decay in the dual graph as we go to large m. However, by scale invariance the distance does not matter. Therefore $\theta_c \leq \theta_{sd}$.

2. Assume $\theta_c \leq \theta_{sd}$

Let $\theta = \theta_{sd}$. Define a sublattice $\Lambda_m = [-m, m]^2$. Let there be an event that horizontally running through this sublattice is an infinite open cluster that stretches to infinity in both directions. By the isomorphism from Λ to Λ_d, there similarly exists the event that vertically running through the sublattice is an infinite closed cluster in the dual graph. Within Λ_m a horizontal open crossing is mutually exclusive with a vertical closed crossing. In the event that there is a horizontal crossing, this prevents a vertical closed path. This forces the infinite closed cluster to be divided into 2. By the uniqueness of the infinite open cluster (proven in [8]), this is impossible. Therefore, $\theta_{sd} \leq \theta_c$.

Therefore, $\theta_{sd} = \theta_c$. □

5.3 Continuous Time percolation

We take the graph $G = (V, E)$ as before. The quantum Ising model naturally evolves in time, necessitating the development of a continuous-time percolation process to describe its dynamics. Unlike classical percolation, where connections form independently based on a fixed probability, the quantum setting introduces a temporal component where transitions occur stochastically according to Poisson processes.

For any observable A, the projective expectation value $\langle \eta' | A | \eta \rangle$ is 1 if and only if η_i'=η_i. This can be interpreted as the probability that a given quantum state remains unchanged during the evolution. For the density matrix ρ, recalling the Hamiltonian form in (4.5), we express the σ^z (respectively σ^x) operators as a bridge (respectively cut) operator in a Poisson process with intensities λ (respectively δ). The bridge operator B_e, for $e = \langle i, j \rangle \in E$ creates a connection between two sites that share the same eigenvalue in the computational basis, reinforcing coherence between states. In contrast, the cut operator D_x, for $x \in V$ represents a quantum fluctuation that flips the eigenvalue, effectively breaking correlations between connected clusters.

Therefore, $\langle \eta' | \rho | \eta \rangle$ quantifies the probability that, given the stochastic effects of cuts and bridges, each element in the quantum state remains unchanged. This formulation provides an intuitive path integral representation, where the evolution from an initial quantum state η to a final state η is governed by a sum over possible percolation histories in time. This perspective links the probabilistic structure of percolation with the quantum mechanical evolution of the Ising model.

5.4 Reduced Density Matrix

As quantum systems grow, their Hilbert spaces expand exponentially, making direct analysis intractable. To manage this complexity, we employ a technique inspired by Markov chains: rather than analyzing the full state space, we focus on a reduced section. While we lose information, it sometimes gives us more than we originally thought: how much what is left of the system depended on what we conditioned over. Let $W \subset G = (V, E)$. We define the reduced density matrix:

$$\rho_G^W = \text{trace}_{G \setminus W}(\rho). \tag{5.9}$$

Similarly, we adapt the continuous-time percolation to the reduced case to find:

$$\langle \eta' | \rho_G^W | \eta \rangle = \frac{\varphi_{G,\beta}(\sigma_{V \setminus W,0} = \sigma_{V \setminus W,\beta} | F)}{\varphi_{G,\beta}(\sigma_0 = \sigma_\beta | F)}. \tag{5.10}$$

where $F = \{\sigma_{V \setminus W,0} = \sigma_{V \setminus W,\beta}\}$ To see how this works, imagine a partially periodic boundary condition. That is, impose periodicity on the original set G, now remove W from it. This will produce a hole in the cylinder that represents $G \setminus W$. This will become a very important notion shortly.

We also introduce another way of stating (5.10). Recalling (4.1), for each quantum state $|\eta\rangle$, we can decompose it as $|\eta\rangle = \sum_{\psi \in \Sigma_W} c(\psi) |\psi\rangle$ for $c : \{-1, +1\}^W \to \mathbb{C}$.

$$\langle \eta' | \rho_G^W | \eta \rangle = \frac{\varphi_m(c(\sigma_{W,0})\overline{c(\sigma_{W,\beta})})}{a_{m,\beta}} \tag{5.11}$$

where $a_{m,\beta} = \varphi_m(\sigma_{W,0} = \sigma_{W,\beta})$. We will use this version of (5.10) to prove our main result with perturbations of reduced density matrices.

6 Disordered Quantum System

As of now, we have assumed λ, δ are constant throughout our system. We allow the bond and site intensities λ, δ to be random variables, leading to what is known as a "disordered system". Instead of assuming constant intensities throughout the system, we now index them as λ_e and δ_x, summing over edges and sites. In the previous sections, we have worked with the ratio $\theta = \lambda/\delta$, and if the event $\{\lambda_e/\delta_x \leq \theta\}$ holds with probability 1, then there is no need to make adjustments.

However, if we assume that $\lambda = \lambda_{i,j}, \delta = \delta_x$ are independent, identically distributed random variables, then this ratio is no longer globally well-defined. Instead, interactions depend only on their immediate neighbors. Another immediate consequence is that we need to modify θ: we define

$$\theta(x) = \frac{1}{\delta(x)} \max_{||x-y||_2=1} \{\lambda(x, y)\} \tag{6.1}$$

$$\overline{\theta}(x) = \frac{1}{\delta(x)} \min_{||x-y||_2=1} \{\lambda(x, y)\}. \tag{6.2}$$

We are strictly in the subcritical regime, that is, connectivity decays exponentially, when $\theta(x) < \theta_c$ and in long-range order, that is, connectivity persists over large distances, when $\overline{\theta}(x) > \theta_c$ [10]. Although we will not explore them in this paper, when $\overline{\theta}(x) < \theta_c < \theta(x)$, connectivity fluctuates throughout the system despite global disorder. We call these "Griffiths singularities", as seen in [5].

6.1 Multiscale analysis

To analyze subgraphs $\Lambda \subseteq \mathbb{Z} \times \mathbb{R}$, we will need to use the following theorem:

Theorem 6.1. *Consider continuous percolation on $\mathbb{Z} \times \mathbb{R}$ in a random environment satisfying*

$$\Gamma = \max\left\{P\left([\ln(1+\Lambda)]^{\beta}\right), P\left([\ln(1+\Delta^{-1})]^{\beta}\right)\right\} < \infty$$

for some

$$\beta > 5 + \frac{7}{2}\sqrt{2}. \tag{6.3}$$

There exists $Q = Q(\beta) > 1$ such that the following holds. For $q \in [1, Q)$ and $\gamma > 0$, there exists $\epsilon = \epsilon(\beta, \Gamma, \gamma, q) > 0$ and $\eta = \eta(\beta, q) > 1$ such that: if

$$P\left([\ln(1 + (\lambda/\delta))]^{\beta}\right) < \epsilon, \tag{6.4}$$

there exist independant, identically distributed, positive random variables $D_x \in L^{\eta}(P), x \in \mathbb{Z}$, such that

$$\mathbb{P}_{\lambda,\delta}((x,s) \leftrightarrow (y,t)) \leq \exp[-\gamma d_q(x,s;y,t)] \quad \text{if } d_q(x,s;y,t) \geq D_x \tag{6.5}$$

for $(x,s), (y,t) \in \mathbb{Z} \times \mathbb{R}$.

The lower bound (6.3) for β is enough to imply that $P(D_x^{\eta}) < \infty$ for some $\eta > 1$. The larger β, the larger η may be taken.

This is the version of the theorem from [9] and in [10] for d=1 where we are not as concerned with generality. Here we present the idea for the general proof as seen in [10].

The proof employs a "multiscale analysis" approach, where we establish that exponential decay over different ranges holds even when the decay parameter is random, provided a suitable bound. The strategy involves an inductive process:

1. *Base Case*: Define an initial lattice Λ_0 and ensure subcritical exponential decay for connections from any $x \in \Lambda_0$ to $\partial\Lambda_n$. WLOG, we focus on the origin.

2. *Inductive Hypothesis*: Assume for scale n, there exists a coefficient m_n and a length scale L_n (in terms of the dimensions of the lattice Λ_n) such that exponential decay holds.

3. *Inductive Step*: We now define new parameters $M < m_n$ and $L > L_n$ such that $M = m_n - \frac{1}{L_n^\alpha}$ for some $\alpha > r$, where r represents the spatial dimension of Λ_n. Thus, as we increase the length scale, the decay coefficient decreases accordingly to keep the decay approximately constant. We then employ the limit $n \to \infty$ to show that $m_\infty = \lim_{n\to\infty} m_n$ remains positive such that for all $n \in \mathbb{N}$, $m_\infty \leq m_n$. If this holds, it guarantees that exponential decay persists at arbitrarily large scales.

4. The final step is to show that the number of points x that violate the decay condition is summable. Since the probability of an exceptional point at scale n is at most $O(L_n^d e^{-cL_n^\alpha})$, if $\alpha > d$, the sum $\sum_n L_n^d e^{-cL_n^\alpha}$ converges. By the Borel-Cantelli lemma, only finitely many exceptional points exist almost surely, ensuring that at sufficiently large scales, exponential decay holds universally.

6.2 Disordered ratio weak-mixing

We define a slit S_L on a lattice Λ_m to be a rectangle $[0, L] \times [0, k]$, where we are specifically interested in the bottom and top horizontal strips $[0, L], [k, k+L]$ respectively. Using (5.10), our reduced density matrix will impose partial periodic boundary conditions on the space $\Lambda_m \setminus S_L$, analyzing how our wavefunction on $[0, L]$ evolves to $[k, k + L]$. The result we proved about ratio-weak mixing holds in any environment due to its generality, regardless of how we define λ and δ. Therefore, it will hold for the slit. However, this is not true for the following lemmas in [9]. As such, we need to adjust them before we can properly work with the slits in the disordered quantum system.

Lemma 6.1. *Let $\lambda, \delta \in (0, \infty)$. If $\theta < 2$ and the event A_L holds, there exists $\alpha, C, M, m, n \in (0, \infty)$, depending on θ only, such that the following holds. There exists $\gamma(\theta) > 0$ such that, for all $L \geq 1$ and $M \leq m \leq n < \infty$,*

$$\sup_{||c||=1} |\frac{\varphi_m(c(\sigma_L^+)c(\sigma_L^-))}{a_m} - \frac{\varphi_n(c(\sigma_L^+)c(\sigma_L^-))}{a_n}| \leq CL^\alpha e^{-\gamma m} \tag{6.6}$$

where the supremum is over all functions $c : \Sigma_L \to \mathbb{R}$ with $||c|| = 1$. The function γ may be chosen to satisfy $\gamma(\theta) \to \infty$ as $\theta \downarrow 0$.

Proof. First, define $S_{0,\beta}$ as the slit box where S_0 (S_β) is region Δ (Γ respectively) and Ω_L as the separating region. Let D_x be the death random variable (in the sense of quantum fluctuations destroying a connection as per (4.5)) and D its distribution as defined in [9]. For the lemma to work, we need there to be no death on Δ or Γ:

$$A_L = \bigcap_{x=K}^{L-K} \{D_x \leq \min\{x, L - x\}\}. \tag{6.7}$$

We know that due to the symmetry of the slit box:

$$\varphi(A_L) = 1 - 2\varphi(\overline{A_L}) \geq 1 - 2\sum_{x=0}^{K} \varphi(D \geq x).$$

Because of the bound on β and therefore η, we have that $\varphi(D^{\eta}) < \infty$. Since D_x is independant from each other, by the 2nd Borel-Cantelli lemma:

$$\lim_{L\to\infty}(A_L) = 1. \tag{6.8}$$

Take t_1 as in 3.1. By (1.3) and the symmetry of the slit box we have that as long as A_L holds (S is the distance between the slits):

$$t_1 \leq\leq 2\sum^{S/2} Ce^{-\gamma i} \leq C' e^{-\gamma \frac{S}{2}}$$

where the final expression on the right is an expression of the dominance of the final term, which just has to be shifted by a constant to account for the approximation. By the symmetry of the slit box, $t_1 = t_2$ The claim follows from (3.1). □

Lemma 6.2. *Let $\lambda, \delta \in (0, \infty)$, event B_L hold, and $\theta < 2$. Then there exists constants $\rho, C_1, C_2 \in (0, \infty)$ such that, for all $L \geq 0, m \geq 1, \beta > 2m + L$, and all $\epsilon^+, \epsilon^- \in \Sigma_L$,*

$$C_1 L^{-\rho} \leq \frac{\varphi_{m,\beta}(\sigma_L^+ = \epsilon^+, \sigma_L^- = \epsilon^-)}{\varphi_{m,\beta}(\sigma_L^+ = \epsilon^+)\varphi_{m,\beta}(\sigma_L^- = \epsilon^-)} \leq C_2 L^{\rho}. \tag{6.9}$$

We will not prove the full lemma here. Instead, we will summarize the proof as seen in [9], and then adapt it to the disordered setting.

Proof. We want to whittle down our slit S_L to a suitable region. In this case, the probability of the event that there is no path from a vertex x to another part of S_L is at least as large as the probability that the first event is a death. We can express this as

$$d \geq x_l = \{\frac{\delta_x}{\delta_x + \lambda_{x,x+1} + \lambda_{x,x-1}}\}.$$

We iterate over all the vertices to get

$$X_L = \prod_{x\in S_L}\{\frac{\delta_x}{\delta_x + \lambda_{x,x+1} + \lambda_{x,x-1}}\}. \tag{6.10}$$

To aid in the analysis of X_L, we take the natural logarithm. We rearrange the expression using the properties of logarithms to find

$$\ln(X_L) = \sum^{S_L} \ln(1 + \frac{\lambda_{x,x+1} + \lambda_{x,x-1}}{\delta_x}). \tag{6.11}$$

It is important to note that by (4.7), this is a sequence of 1-dependent random variables. That is, they only rely on their immediate neighbors. Next we state via the logarithmic triangle inequality Theorem:

$$Z_x = \ln(1 + \frac{\lambda_{x,x+1} + \lambda_{x,x-1}}{\delta_x}) \leq \ln(1 + \frac{\lambda_{x,x+1}}{\delta_x}) + \ln(1 + \frac{\lambda_{x,x-1}}{\delta_x}). \tag{6.12}$$

Next, given that we are in a Hilbert space, we want to show that the second moment, that is, $\sqrt{\mathbb{E}(Z_x^2)}$, converges. To do this, we use the Minkowski inequality and (6.12) to get

$$\sqrt{\mathbb{E}(Z_x^2)} \leq \sqrt{\mathbb{E}(\ln(1+\frac{\lambda_{x,x-1}}{\delta_x})^2)} + \sqrt{\mathbb{E}(\ln(1+\frac{\lambda_{x,x+1}}{\delta_x})^2)}. \tag{6.13}$$

Finally, we can apply (6.4) to show

$$\sqrt{\mathbb{E}(Z_x^2)} \leq 2\sqrt{\mathbb{E}(\ln(1+(\lambda/\delta))^2)} < \infty. \tag{6.14}$$

By symmetry, the reduction process proceeds uniformly across the domain. Since the domain is split into two disjoint regions, we iterate over four steps, each corresponding to an expectation computed from the origin. We define the event:

$$B_L = \{X_L \geq L^{-\rho}\}. \tag{6.15}$$

where

$$\rho > 4\mathbb{E}(Z_0). \tag{6.16}$$

Since (6.14) ensures the finiteness of $E(Z_x^2)$, we apply the the Central-Limit Theorem of 1-dependent sequences. Under event (6.15), $L^{-\rho}$ provides a both a lower and upper bound for the measure on the slit, giving us 6.2. □

The following lemma is fairly unchanged from the disordered setting.

Lemma 6.3. *Let $\lambda, \delta \in (0,\infty)$ and let the event $C_L \cap D_L$ hold. There exists constants $C, \gamma \in (0,\infty)$ that satisfies $0 < \gamma < 1$ when $\theta < 2$ such that: for all $L \geq 0$, $m \geq 1$, $\beta \geq 5e^{m+\frac{1}{2}L}$, all events $A \subseteq \Sigma_L \times \Sigma_L$, and all admissable random-cluster boundary conditions τ and spin boundary conditions η of $\Lambda_{m,\beta}$*

$$\left| \frac{\phi^{\alpha}_{m,\beta}\left((\sigma_L^+, \sigma_L^-) \in A\right)}{\phi_{m,\beta}\left((\sigma_L^+, \sigma_L^-) \in A\right)} - 1 \right| \leq Ce^{-\frac{2}{7}\gamma m}, \text{for } \alpha = \tau, \eta$$

whenever the right side of the inequality is less than or equal to 1. The function γ depends only on θ and $\infty > \gamma(\theta) > 0$.

The only thing that is different in this case is the form of the circuit used to finish the proof. Please refer to [9] for further information.

7 bound on operators in disordered setting

We return to the reduced density matrix. How do we analyze small perturbations in a reduced density matrix? If we recall the definition of the density matrix in (4.6), we can use the properties of exponential matrices to see

$$e^A = I + A + \frac{A^2}{2} + ... \tag{7.1}$$

where I is the identity matrix. If we subtract one density matrix from another, the 0th degree will disappear, and anything above the 1st degree will be very small. We define the operator norm:

$$||A|| = sup_{||\Psi||=1} | \langle \Psi | A | \Psi \rangle |$$

We let ρ_m^L, ρ_n^L be two reduced density matrices. Using the definition of the exponential matrix from (7.1) and the reduced density matrix from (5.10) we have that

$$||\rho_m^L - \rho_n^L|| = \beta ||(H_m - H_n) - o(H_m^2)||. \tag{7.2}$$

The supremum is taken over all vectors $\langle \Psi | \in \mathcal{H}_M$, where $\mathcal{H}$ is the L^2-norm space belonging to a Hilbert space defined on a spin space M.

Theorem 7.1. *Let $\lambda, \delta \in (0, \infty)$. If $\theta < 2$ and the event $E = \{A_L \cap B_L \cap C_L \cap D_L\}$ holds, then there exists constants $\alpha, C \in (0, \infty)$ depending on θ only, and a constant $\infty > \gamma(\theta) > 0$ such that, for all $L \geq 1$,*

$$||\rho_m^L - \rho_n^L|| \leq min\{2, CL^{\rho} e^{-\gamma m}\}, \; 2 \leq m \leq n \tag{7.3}$$

This theorem originates from [9] but here, we adapt it to a disordered regime.

Proof. The upper bound of 2 comes from the inherent properties of the operator norm and the fact that we are taking the supremum over the set of Ψ such that $||\Psi|| = 1$. Let $\theta < \theta_c$, and let γ be the same as in lemma 6.1. Let $m \leq n < \infty$. Take the box Λ_m with configuration ω', and Λ_n with configuration ω, where $\Lambda_m \subseteq \Lambda_n$. Define $\varphi_{m,\beta}$ ($\varphi_{n,\beta}$) to be the measure defined on Λ_m (respectively Λ_n). Finally, we also define a box $B \subseteq \Lambda_m$, for which we will also define a measure φ_B. We will work within B, using it to compare $\varphi_{m,\beta}, \varphi_{n,\beta}$. By positive association, it follows that $\varphi_{m,\beta} \leq_{st} \varphi_{n,\beta}$. By 2.3, it follows that we can find some coupling κ of pairs (ω', ω) such that $(\omega' \leq \omega) = 1$. In this coupling, we may demand the configuration include identical configurations in Λ_m such that there exists a set W of all the different configurations $(\omega', \omega) \in \Omega_n \times \Omega_n$ where there exists no path from ∂B to $\partial h \Lambda_m$, where ∂h is the horizontal boundary. We make no condition on the configuration of B.

Now, define a slit $S_L \subseteq \Lambda_m$, where we understand one end as being the initial state at time $t = 0$, and the other as the final state at time $t = \beta$. Denote σ_0 (respectively σ_β) as the spin configuration in the clusters of ω' (respectively, ω), whereas we denote $\sigma_{L,0}$ (respectively, $\sigma_{L,\beta}$) for the spin configurations on the reduced density matrices of ω (respectively, ω') in the slit S_L.

By (5.11), we can define our reduced density matrix on S_L as:

$$\langle \psi | \rho_m^L - \rho_n^L | \psi \rangle = \frac{\varphi_m(c(\sigma_{L,0})\overline{c(\sigma_{L,\beta})})}{a_{m,\beta}} - \frac{\varphi_n(c(\sigma_{L,0})\overline{c(\sigma_{L,\beta})})}{a_{n,\beta}} \tag{7.4}$$

where $c : \Sigma_L \to [0, \infty)$ with $||c|| = 1$. Similarly, we define the random variable

$$S_c = \frac{c(\sigma_{L,0})\overline{c(\sigma_{L,\beta})}}{a_{m,\beta}} - \frac{c(\sigma_{L,0})\overline{c(\sigma_{L,\beta})}}{a_{n,\beta}}.$$

We separate into two cases, one where the spin configurations are in D (coupled such that they are the same) and one where they are not. Defining $\kappa(f;F) = \kappa(f1_F)$:

$$Sc = \kappa(S_c; D) + \kappa(S_c; \overline{D}). \tag{7.5}$$

$$\begin{aligned}|\kappa(S_c; D)| &\leq \left| \frac{\varphi_m\big(c(\sigma_{L,0})\overline{c(\sigma_{L,\beta})}\big)}{a_{m,\beta}} - \frac{\varphi_n\big(c(\sigma_{L,0})\overline{c(\sigma_{L,\beta})}\big)}{a_{n,\beta}} \right| \\ &= \left|1 - \frac{a_{m,\beta}}{a_{n,\beta}}\right| \varphi_m\big(c(\sigma_{L,0})\overline{c(\sigma_{L,\beta})}\big).\end{aligned}$$

Let us first analyze the case of the event D. On the event D: $\sigma_{W,0} = \sigma_{W,\beta}$.

$$\begin{aligned}|\kappa(S_c; D)| &\leq \left| \frac{\varphi_m\big(c(\sigma_{L,0})\overline{c(\sigma_{L,\beta})}\big)}{a_{m,\beta}} - \frac{\varphi_n\big(c(\sigma_{L,0})\overline{c(\sigma_{L,\beta})}\big)}{a_{n,\beta}} \right| \\ &= \left|1 - \frac{a_{m,\beta}}{a_{n,\beta}}\right| \varphi_m\big(c(\sigma_{L,0})\overline{c(\sigma_{L,\beta})}\big).\end{aligned} \tag{7.6}$$

Recalling the definition of $a_{m,\beta}, a_{n,\beta}$ from (5.11), we get in the absolute values:

$$|1 - \frac{\varphi_m(\sigma_0 = \sigma_\beta)}{\varphi_n(\sigma_0 = \sigma_\beta)}|.$$

By lemma 6.3, we then get that

$$|1 - \frac{a_{m,\beta}}{a_{n,\beta}}| \leq C_1 e^{-\frac{1}{2}\gamma m}. \tag{7.7}$$

Next, we use the definition of the measures to get

$$\varphi_{m,\beta}(c(\sigma_{L,0})\overline{c(\sigma_{L,\beta})}) = \sum_{\epsilon^{0,\beta} \in \Sigma_L} (c(\epsilon^0)c(\epsilon^\beta)\varphi_{m,\beta}(\sigma_{L,0} = \epsilon^0, \sigma_{L,\beta} = \epsilon^\beta)) \tag{7.8}$$

Using Lemma 6.2 we get

$$\varphi_{m,\beta}(c(\sigma_{L,0})\overline{c(\sigma_{L,\beta})}) \leq C_2 L^\rho \phi_{m,\beta} c(\sigma_{L,0})\overline{c(\sigma_{L,\beta})}).$$

In the next step, we first use (7.8), then we apply the symmetry of the system, and finally use Cauchy Schwartz:

$$= C_2 L^\rho \left(\sum_{\epsilon \in \Sigma_L} c(\epsilon)\phi_{m,\beta}\left(\sigma_{L,0} = \epsilon\right) \right)^2 \leq C_2 L^\rho \sum_{\epsilon \in \Sigma_L} \phi_{m,\beta}\left(\sigma_{L,\beta} = \epsilon\right)^2, \tag{7.9}$$

using our assumption that $||c|| = 1$. Revisiting the definition of $a_{m,\beta}$ in (5.11) and using lemma 6.2, we have that

$$\varphi_{m,\beta}(\sigma^+ = \sigma^- = \epsilon) \geq C_3 L^\rho \sum \varphi(\sigma^+ = \epsilon)^2. \tag{7.10}$$

Combining (7.9), (7.7), and (7.10), we get that

$$\kappa(Sc;D) \le C_4 L^{2\rho} e^{-\frac{1}{2}\gamma m}. \tag{7.11}$$

Now, we focus on $\kappa(Sc;\overline{D})$ as in (7.5). The configurations are no longer conditioned on being equal to each other. As such we can decompose the coupling into the two possible configurations:

$$\kappa(Sc;\overline{D}) \le A_m + B_n \tag{7.12}$$

where

$$A_m = \frac{\kappa\left(c(\sigma^1_{L,0})\overline{c(\sigma^1_{L,\beta})};\overline{D}\right)}{a_{m,\beta}}, \quad B_n = \frac{\kappa\left(c(\sigma^2_{L,0})\overline{c(\sigma^2_{L,\beta})};\overline{D}\right)}{a_{n,\beta}}.$$

We use a another way to state the coupling in B_n via our definition in (7.5):

$$\begin{aligned}
&\frac{\kappa(\overline{D})}{a_{n,\beta}}\kappa\left(c(\sigma^1_{L,0})\overline{c(\sigma^1_{L,\beta})} \mid \overline{D}\right) \\
&= \frac{\kappa(\overline{D})}{a_{n,\beta}}\phi_{n,\beta}\left(\phi^T_B\left(c(\sigma^1_{L,0})\overline{c(\sigma^1_{L,\beta})}\right) \mid \overline{D}\right) \\
&\le \frac{\kappa(\overline{D})}{a_B} C_5 \phi_B\left(c(\sigma^1_{L,0})\overline{c(\sigma^1_{L,\beta})}\right).
\end{aligned} \tag{7.13}$$

We used conditional expectation given the configuration τ on $\Lambda_{m,\beta}\setminus B$ in the intermediate step. By (7.10) and (7.9) applied to the measure ϕ_B, there exists $C_6 = C_6(\lambda,\delta)$ such that

$$\frac{1}{a_B}\phi_B\left(c(\sigma^1_{L,0})\overline{c(\sigma^1_{L,\beta})}\right) \le C_6 L^{2\rho}. \tag{7.14}$$

We can undergo a similar procedure with A_m. Finally, we use (7.14) and (7.13) to get a bound of

$$|\kappa(S_c;\overline{D})| \le \kappa(\overline{D}) C_6 L^{2\rho}. \tag{7.15}$$

All that is left is to obtain some bound for $\kappa(\overline{D})$. By (1.3) we know that we will get exponential decay:

$$\kappa(\overline{D}) \le C_7 L e^{-\frac{1}{2}\gamma m}. \tag{7.16}$$

We combine (7.16) and (7.13) to obtain a final bound for the $\overline{D}$ case of:

$$|\kappa(S_c;\overline{D})| \le C_8 L^{2\rho} L e^{-\frac{1}{2}\gamma m}. \tag{7.17}$$

We combine (7.11) and (7.17) as in (7.5). The constants C,γ may be amended to obtain the required inequality in Theorem 7.1. □

8 Conclusion

There are two immediate consequences of Theorem 7.1. The first is a bound on perturbations on the entanglement entropy.

Definition 8.1. *The entanglement of the spins of W relative to its complement $V \setminus W$ is defined as the entropy*

$$S_V^W = -tr(\rho_V^W log_2 \rho_V^W). \tag{8.1}$$

This definition of entropy is the Von Neumann entropy, which is the quantum analog to the classical Shannon entropy in information theory. To analyze the entanglement entropy, we make use of the following theorem:

Theorem 8.1. *(Schmidt Decomposition) Let $\langle C|$ be a pure ground state of the composite system $A \otimes B$. There exist orthonormal bases $\{|u_j\rangle, |v_j\rangle\}$ for the states of A,B respectively, such that*

$$\langle C| = \sum_{j=1}^{s} \sqrt{\lambda_j^{\downarrow}(\rho)}\, |u_j\rangle\, |v_j\rangle \tag{8.2}$$

where s, the Schmidt rank, is given by $s = min\{2^m, 2^n\}$

Essentially, if you are dealing with some lattice C which is a product of A and B, we denote this as $C = A \otimes B$. Summing over elements in C therefore gives us a tensor which, using Singular-Value Decomposition, allows us to find a single coefficient for the product of each element. The important result is that we use the bound on the trace distance to serve as a bound on the summation on the spectra resulting in the entanglement entropy.

A second interesting consequence of Theorem 7.1 is using the Fannes-Audenärt inequality.

Theorem 8.2. *(Fannes inequality) For any two density matrices ρ, ρ' of dimension d:*

$$|S(\rho) - S(\rho')| \leq 2Tlog_2(d) - 2Tlog_2(T)$$

Where $T = \frac{1}{2}||\rho - \rho'||$.

The original Fannes inequality provides a bound for small pertubations in the entropy of density matrices. The important idea is that this holds for any density matrix, including a reduced density matrix. Please refer to [11] for a quick proof.

Audenärt amended the proof, finding a completely optimal bound using completely orthogonal states.

Corollary 8.1. *(Fannes-Audenärt inequality) For any two density matrices ρ, ρ' of dimension d:*

$$|S(\rho) - S(\rho')| \leq Tlog_2(d-1) + H[\{T, 1-T\}] \tag{8.3}$$

Where $T = \frac{1}{2}||\rho - \rho'||$ and $H[\{p_i\}] = -\sum p_i log_2(p_i)$ is the classical Shannon entropy.

In either case, the major application is providing a bound on the trace distance.

However, there still lies the issue of the methodology used to bound the trace distance. This is not an optimal bound, and merely uses convenient starting points of both measure theory and disordered systems. The next logical step would be to sharpen or solve for a completely optimal bound for the trace distance. In this case, this would help make the applicative bounds more impactful as well.

9 Acknowledgements

I would like to thank Peter May for organizing this wonderful REU opportunity and for providing detailed corrections to this paper. Thank you to my REU mentor Jinwoo Sung, who very dutifully guided me through the complex topics of percolation theory, was available even on weekends and in South Korea, and for all of his comments on my paper. Thank you as well to my parents for all their support. Finally, I would like to thank Brittany Wisniewski and the UChicago Career Advancement office for funding my REU experience.

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