GEOMETRIC MEASURE THEORY WITH METHODS FROM ALGORITHMIC COMPLEXITY

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Abstract. Since the development of Hausdorff dimension, there have been many attempts both to calculate the dimension of specific fractals and to find generalized descriptions of dimension by set transformations, classifications, etc. These calculation techniques and theorems have formed the basis for the field of geometric measure theory. More recently, the theory of computation has provided an alternative way to view this field: through the use of the point-to-set principle, mathematicians have been able to use the Kolmogorov complexity of binary strings to further generalize the fascinating theorems of the field.

Contents

1. Measure Theory

Before we begin applying effective methods to measure theory, we must first consider what the study of measure theory actually is.

1.1. **Measure Spaces.** Measure theory is an incredibly significant tool in modern probability and analysis. However, it is based on a relatively simple question known as *the problem of measure*: is there a suitable way one can assign "volume" or "size" to subsets of arbitrary spaces?

Definition 1.1. Let X be a set. A σ-algebra $\mathcal{M}$ on X is a collection of subsets of X with the following properties:

(1) $X \in \mathcal{M}$
(2) If $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$ (closure under complement)
(3) If $A_1, A_2, ...$ is a countable sequence of sets in $\mathcal{M}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$ (closure under countable union)

Example 1.2. Some examples of σ-algebras include

- The trivial σ-algebra on any set X, consisting of only the empty set $\emptyset$ and X itself.
- The power set $\mathcal{P}(X)$ of any set X, consisting of all possible subsets of X.
- The collection of all countable subsets, along with complements of such subsets, of any set X.

Definition 1.3. Let $\mathcal{M}$ be a σ-algebra on X. Then, the ordered pair $(X, \mathcal{M})$ is called a measurable space. The sets in $\mathcal{M}$ are called the measurable sets of X.

Notice that a measurable space has no notion of how to actually "measure" the sets it labels as "measurable." It merely formalizes intuitive notions of what sets can be measured: e.g., the whole space may be measured, the union of two disjoint sets whose measures are known may be measured, etc.

To actually measure sets, we must define a set function that assigns such a value to each set we are concerned with (i.e., those in the σ-algebra). The set function must satisfy certain desirable properties, however.

Definition 1.4. Let $(X, \mathcal{M})$ be a measurable space. A function $\mu : \mathcal{M} \to [0, \infty]$ on $(X, \mathcal{M})$ is called a (positive) measure if it satisfies the following properties:

(1) $\mu(\emptyset) = 0$
(2) If $A_1, A_2, ...$ is a countable sequence of pairwise disjoint sets in $\mathcal{M}$, then $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ (countable additivity)

Example 1.5. Some examples of measure include

- The counting measure on any set X, which is defined by $\mu(S)$ being the number of elements in any subset $S \subseteq X$.
- The Dirac measure on any set X, which is defined by fixing a point $a \in X$ and letting $\delta_a(S) = 1$ if $a \in S$ but $\delta_a(S) = 0$ if $a \notin S$ for any subset $S \subseteq X$.
- The Lebesgue measure on $\mathbb{R}^n$, which generalizes the notion of length of intervals or volume of products of intervals to a large class of sets.

Definition 1.6. Let μ be a measure on $(X, \mathcal{M})$. Then, the ordered triple $(X, \mathcal{M}, \mu)$ is called a measure space.

Proposition 1.7. *Let X be a set, and let $\mathcal{F}$ be a family of subsets in X. Then, there exists a smallest σ-algebra $\mathcal{M}$ on X such that $\mathcal{F} \subseteq \mathcal{M}$.*

Proof. Let $\mathcal{Y}$ be the collection of all σ-algebras that contain $\mathcal{F}$. $\mathcal{Y}$ is clearly non-empty as it contains the power set $\mathcal{P}(X)$. Then, let:

$$\mathcal{M} = \bigcap_{\Sigma \in \mathcal{Y}} \Sigma$$

I claim that $\mathcal{M}$ is the smallest σ-algebra that contains $\mathcal{F}$. First, we see that $\mathcal{M}$ satisfies all the conditions of a σ-algebra:

- Since $X \in \Sigma$ for all $\Sigma \in \mathcal{Y}$, $X \in \mathcal{M}$.
- Fixing $A \in \mathcal{M}$, $A \in \Sigma$ for all $\Sigma \in \mathcal{Y}$. Thus, by definition, $A^c \in \Sigma$ for all $\Sigma \in \mathcal{Y}$, and so $A^c \in \mathcal{M}$.
- Fixing a countable sequence $A_1, A_2, ...$ of sets in $\mathcal{M}$, $A_1, A_2, ...$ are in Σ for all $\Sigma \in \mathcal{Y}$. Thus, by definition, $\bigcup_{n=1}^{\infty} A_n \in \Sigma$ for all $\Sigma \in \mathcal{Y}$, and so $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$.

Then, we see that $\mathcal{F} \subseteq \Sigma$ for all $\Sigma \in \mathcal{Y}$, so naturally $\mathcal{F} \subseteq \mathcal{M}$. Moreover, any σ-algebra containing $\mathcal{F}$ is at least as large as $\mathcal{M}$ since $\mathcal{M}$ is contained in any such σ-algebra. □

Definition 1.8. We refer to the smallest σ-algebra containing all the open sets of some topological space X as the Borel σ-algebra, which we denote $\mathcal{B}$.

Most of the sets in $\mathbb{R}^n$ that we can readily think of as mathematicians are Borel: that is, such sets are the result of a countable sequence of basic set operations (union, intersection, and complement) on open and closed sets. This makes the Borel σ-algebra a very useful σ-algebra to make measurable and to equip with a measure, so much so that we give it its own name.

Definition 1.9. A Borel measure μ on a topological space X is one that is defined on all the open sets of X.

1.2. **Outer Measures.** To come up with a useful measure out of thin air is difficult. It must satisfy countable additivity, which is often tedious to verify due to the need for a strict equality, on a very specific collection of subsets of X.

A remedy to this is Carathéodory's process for the *construction of measures*. It makes use of a set function defined on all subsets of X, called an outer measure, which can become a measure when restricted to a particular σ-algebra. Before stating the theorem concerning this process, we lay out some necessary definitions.

Definition 1.10. Let X be a set. A function $\mu_* : \mathcal{P}(X) \to [0, \infty]$ is called an outer measure on X if it satisfies the following properties:

(1) $\mu_*(\emptyset) = 0$
(2) If $A \subseteq B$, then $\mu_*(A) \leq \mu_*(B)$ (monotonicity)
(3) If $A_1, A_2, ...$ is a sequence of subsets of X, then $\mu_*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu_*(A_n)$ (countable subadditivity)

These should intuitively feel like easier conditions to satisfy for arbitrary subsets of X, compared to the condition of strict additivity for a particular collection of subsets of X (as with a measure).

Definition 1.11. Let μ_* be an outer measure on X. A set $E \subseteq X$ is called (Carathéodory) measurable if we have

$$\mu_*(A) = \mu_*(A \cap E) + \mu_*(A \cap E^c)$$

for all $A \subseteq X$.

Clearly, to show that E is measurable with respect to μ_*, it suffices to shows that $\mu_*(A) \geq \mu_*(A \cap E) + \mu_*(A \cap E^c)$ for an arbitrary subset A as the other direction of inequality follows from subadditivity.

With this, we state Carathéodory's main theorem.

Theorem 1.12. *Let μ_* be an outer measure on X, and let $\mathcal{M}$ be the set of all measurable subsets of X. Then, μ_* restricted to $\mathcal{M}$, denoted by μ, is a measure on X.*

The proof of this theorem is involved but can be found in many introductory texts on measure theory, including [7].

It now helps to have a way to ensure our measures are useful in some way. The concept of a Borel measure would satisfy this need, and so the next step is to develop a method to ensure that some outer measure has the property that it can be restricted to such a measure. It turns out that there is a simple criterion that allows outer measures on metric spaces to satisfy this property.

Definition 1.13. Let A, B be subsets of a metric space (X, d). The separation between subsets A and B is defined as

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$$

Disjoint subsets A and B are said to be positively separated if $d(A, B) > 0$.

Definition 1.14. A metric outer measure μ_* on a metric space X is an outer measure such that

$$\mu_*(A \cup B) = \mu_*(A) + \mu_*(B)$$

for all positively separated subsets A and B.

Theorem 1.15. *A metric outer measure μ_* on a metric space X is a Borel measure μ on X when restricted to $\mathcal{B}$.*

Again, the proof of this theorem is more involved but can be found in [7] and other similar texts. It is clear that it suffices to show that closed sets are measurable with respect to a metric outer measure, and that is exactly what most proofs of the theorem do.

2. Hausdorff Measure and Dimension

How many more intervals of length $\frac{1}{4}$ are needed to cover the unit interval $[0, 1]$ compared to intervals of length $\frac{1}{2}$? Very obviously, our instincts say $2 = 2^1$ times as many. Why? Because length is "1-dimensional."

How many more squares of side length $\frac{1}{4}$ are needed to cover the unit square $[0, 1]^2$ compared to intervals of side length $\frac{1}{2}$? Very obviously, our instincts say $4 = 2^2$ times as many. Why? Because area is "2-dimensional."

This idea of dimension comes up often in mathematics. For instance, topological dimension and fractal dimension are distinct ways of assigning dimension to sets of points. However, topological dimension is coarse, being restricted to integer values. Fractal dimension not only takes on positive real values but also is a more natural realization of the behavior observed with the motivating questions posed above.

2.1. **The "Right" Measure.** Fractal dimension, as originally developed by Hausdorff and Besicovitch on subsets of $\mathbb{R}^n$, can naturally be extended to arbitrary metric spaces. This more generalized theory is what we will work with throughout this paper.

We begin, rather surprisingly, with measure theory. To understand this necessity, if we take the rudimentary understanding of dimension outline by the motivating questions, the unit square $[0,1]^2$ has measure ∞ for a "1-dimensional" measure and measure 0 for a "3-dimensional" measure. In other words, it is much larger than $[0,1]$ but negligible compared to $[0,1]^3$. Only with a "2-dimensional" measure should we expect a positive finite measure for the unit square.

Definition 2.1. Let (X,d) be a metric space. Suppose $\delta > 0$ and $F \subseteq X$. A δ-cover of F is a collection $\mathcal{U}$ of subsets of X such that $\operatorname{diam}(U) \leq \delta$ for all $U \in \mathcal{U}$ and $\mathcal{U}$ covers F.

Throughout this paper, I will denote $\operatorname{diam}(U) = \sup\{d(a,b) : a,b \in U\}$ by $|U|$ for any subset U of a metric space X.

Definition 2.2. Fix $s > 0$ and $F \subseteq X$. For all $\delta > 0$, define

$$\mathcal{H}^s_\delta(F) = \inf\left\{ \sum_{i=1}^{\infty} |U_i|^s : U_1, U_2, ... \text{ is a countable } \delta\text{-cover of } F \right\}$$

Then, define the s-dimensional Hausdorff outer measure by

$$\mathcal{H}^s_*(F) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(F)$$

Note that for $\varepsilon > \delta$, there are fewer δ-covers than there are ε-covers. So, by the monotone convergence theorem, the limit in the definition of $\mathcal{H}^s_*(F)$ exists and can freely be replaced by a supremum over all $\delta > 0$.

Many alternative definitions of Hausdorff outer measure are equivalent: we can restrict ourselves to countable δ-covers of only closed sets at each stage, etc.

Proposition 2.3. *For all $s > 0$, $\mathcal{H}^s_*$ is a metric outer measure on X.*

Proof. $\mathcal{H}^s_*(\emptyset) = 0$ is obvious, by definition.

Let $E \subseteq F$. Any δ-cover of F is a δ-cover of E. So, there are more countable δ-covers of E than there are of F. Thus, $\mathcal{H}^s_\delta(E) \leq \mathcal{H}^s_\delta(F)$ as the former is an infimum over more covers. This is montonicity on E and F if we let $\delta \to 0^+$.

Let $E_1, E_2, ...$ be subsets of X, with E their union. If $\mathcal{U}_k$ is an arbitrary countable δ-cover of E_k for all natural numbers k, then

$$\mathcal{U} = \bigcup_{k=1}^{\infty} \mathcal{U}_k$$

is a countable δ-cover of E. So, letting $\mathcal{V}_k = \mathcal{U}_k - \mathcal{U}_1 - \mathcal{U}_2 - ... - \mathcal{U}_{k-1}$ for any k, we have

$$\begin{aligned}\mathcal{H}^s_\delta(E) \leq \sum_{U \in \mathcal{U}} |U|^s &= \sum_{U_1 \in \mathcal{V}_1} |U_1|^s + \sum_{U_2 \in \mathcal{V}_2} |U_2|^s + ... \\ &\leq \sum_{U_1 \in \mathcal{U}_1} |U_1|^s + \sum_{U_2 \in \mathcal{U}_2} |U_2|^s + ...\end{aligned}$$

Taking the infimum over all covers of E_k for each k, we have

$$\mathcal{H}^s_\delta(E) \leq \mathcal{H}^s_\delta(E_1) + \mathcal{H}^s_\delta(E_2) + ...$$

which is countable subadditivity on $E_1, E_2, ...$ if we let $\delta \to 0^+$.

For the metric condition, let $A, B \subseteq X$ be any pair of positively separated sets. Suppose that $d(A, B) = \delta$ for some $\delta > 0$. Fix $\epsilon < \delta$. If any ε-cover $\mathcal{U}$ of $A \cup B$ has a set U_B which intersects B, then there is a point p which exists in $U_B \cap B$. So, if $u \in U_B$ and $a \in A$ are arbitrary, we have

$$d(u, p) \leq |U_B| \leq \varepsilon < \delta = d(A, B) \leq d(a, p)$$

However, this means that no point of U_B is in A. The same is obviously symmetrically true for U_A and B if U_A was some set in $\mathcal{U}$ that intersected A. So, from $\mathcal{U}$, we may find disjoint ε-covers $\mathcal{U}_A$ of A and $\mathcal{U}_B$ of B so that

$$\sum_{U \in \mathcal{U}} |U|^s \geq \sum_{U_A \in \mathcal{U}_A} |U_A|^s + \sum_{U_B \in \mathcal{U}_B} |U_B|^s \geq \mathcal{H}^s_\varepsilon(A) + \mathcal{H}^s_\varepsilon(B)$$

Taking the infimum over all covers of $A \cup B$, we have $\mathcal{H}^s_\varepsilon(A \cup B) \geq \mathcal{H}^s_\varepsilon(A) + \mathcal{H}^s_\varepsilon(B)$, which is additivity on A and B if we let $\varepsilon \to 0^+$. □

Corollary 2.4. *Let $\mathcal{H}^s$ be $\mathcal{H}^s_*$ restricted to the Borel sets $\mathcal{B}$ of X. Then, $\mathcal{H}^s$ is a Borel measure.*

Proof. This follows immediately from Theorem 1.15 and Proposition 2.3. □

Throughout this paper, $\mathcal{H}^s$ will denote the s-dimensional Hausdorff outer measure, its subscript simply being dropped. It will be obvious when the properties of measure are being used on appropriate sets.

We now want to demonstrate that this theory of various Hausdorff measures has the desirable property of dimension outlined at the start of this subsection. That is, for any set $F \subseteq X$, there is some $s > 0$ such that $\mathcal{H}^s$ is the "right" measure for F in that $\mathcal{H}^t(F)$ is certainly trivial (either zero or infinity) for $t > s$ and $t < s$.

Proposition 2.5. *Let $F \subseteq X$. Suppose $\mathcal{H}^s(F) < \infty$ for some $s > 0$. Then, $\mathcal{H}^t(F) = 0$ for all $t > s$.*

Proof. Fix $t > s$. Let $\mathcal{U}$ be an arbitrary countable δ-cover of F. Then, we have

$$\sum_{U \in \mathcal{U}} |U|^t = \sum_{U \in \mathcal{U}} |U|^{t-s}|U|^s \leq \sum_{U \in \mathcal{U}} \delta^{t-s}|U|^s = \delta^{t-s} \sum_{U \in \mathcal{U}} |U|^s$$

Taking the infimum over all covers $\mathcal{U}$ of F, we have $\mathcal{H}^t_\delta(F) \leq \delta^{t-s}\mathcal{H}^s_\delta(F)$.

If we let $\delta \to 0^+$, we have $\mathcal{H}^t(F) \leq 0$ since δ^{t-s} approaches 0 while $\mathcal{H}^s_\delta(F)$ approaches a definite finite value. □

Proposition 2.6. *Let $F \subseteq X$. Suppose $\mathcal{H}^s(F) > 0$ for some $s > 0$. Then, $\mathcal{H}^t(F) = \infty$ for all $t < s$.*

Proof. Fix $t < s$. Let $\mathcal{U}$ be an arbitrary countable δ-cover of F. Then, we have

$$\sum_{U \in \mathcal{U}} |U|^s = \sum_{U \in \mathcal{U}} |U|^{s-t}|U|^t \leq \sum_{U \in \mathcal{U}} \delta^{s-t}|U|^t = \delta^{s-t} \sum_{U \in \mathcal{U}} |U|^t$$

Taking the infimum over all covers $\mathcal{U}$ of F, we have $\mathcal{H}^s_\delta(F) \leq \delta^{s-t}\mathcal{H}^t_\delta(F)$ or, equivalently, $\mathcal{H}^t_\delta(F) \geq \delta^{t-s}\mathcal{H}^s_\delta(F)$.

If we let $\delta \to 0^+$, we have $\mathcal{H}^t(F) \geq \infty$ since $\mathcal{H}^s(F)$ approaches a non-zero value while δ^{t-s} approaches ∞. □

With this property in mind, calling $\mathcal{H}^s$ the s-dimensional Hausdorff measure makes sense. We can thus use this theory to rigorously understand the notion of the "right" measure for a set: there is indeed only one such measure for a given set, and that idea is encapsulated by the following definition.

Definition 2.7. The Hausdorff dimension of a set $F \subseteq X$ is

$$\dim_{\mathrm{H}}(F) = \inf\{s > 0 : \mathcal{H}^s(F) = 0\} = \sup\{s > 0 : \mathcal{H}^s(F) = \infty\}$$

2.2. **Examples of Fractals.** It may be helpful to review some examples of sets with interesting Hausdorff dimension. In particular, Hausdorff dimension is used to quantify the relative size of fractals.

Example 2.8. The middle-thirds Cantor set $\mathcal{C} \subset [0,1]$ has Hausdorff dimension less than $s = \log_3(2)$. To see this, let C_k be the kth stage in the natural construction of $\mathcal{C}$. Since C_k is composed of 2^k disjoint intervals each of length 3^{-k}, we see

$$\mathcal{H}^s(\mathcal{C}) \leq \mathcal{H}^s(C_k) \leq 2^k(3^{-k})^s = 2^k(3^s)^{-k} = 2^k(2^{-k}) = 1$$

so that $\mathcal{H}^\infty(\mathcal{C}) < \infty$ and thus $s \geq \dim_{\mathrm{H}}(\mathcal{C})$.

In general, it is more complex to give a lower bound on dimension (or on some Hausdorff measure) than it is to find an upper bound. I omit that part of the proof in the dimension of the Cantor set above though the upper bound is a nice result in and of itself (i.e., there exist uncountable sets of points on the real line of dimension strictly less than 1).

There are several "classical" methods used to find lower bounds on the dimension of a set, such as through the use of the mass distribution principle or the potential theoretic method. These methods are discussed and demonstrated in [2] and [3], with interesting results. However, these are not the focus of this paper as algorithmic methods are.

If we want to do some basic discovery, we can avoid such rigor for now and focus on a heuristic method to determine the dimension of so-called self-similar fractals.

Property 2.9. If $S : \mathbb{R}^n \to \mathbb{R}^n$ is a similarity transformation with a scale factor $\lambda > 0$ (i.e., $S(x) = \lambda x + c$, where c depends on the center of dilation), then we have $\mathcal{H}^s(S(F)) = \lambda^s \mathcal{H}^s(F)$ for any $s > 0$.

Example 2.10. The Cantor set $\mathcal{C}$ has Hausdorff dimension $s = \log_3(2)$. Assuming that $0 < \mathcal{H}^s(\mathcal{C}) < \infty$, we note that the Cantor set is self-similar: that is, it is two copies of itself scaled by one-third. So, if S_1 scales F to the first copy with $S_1(F) \subset [0, 1/3]$ and S_2 scales F to the second copy with $S_2(F) \subset [2/3, 1]$, then

$$\begin{aligned}\mathcal{H}^s(\mathcal{C}) &= \mathcal{H}^s(S_1(\mathcal{C})) + \mathcal{H}^s(S_2(\mathcal{C})) = (1/3)^s\mathcal{H}^s(\mathcal{C}) + (1/3)^s\mathcal{H}^s(\mathcal{C}) \\ &= 2(1/3)^s\mathcal{H}^s(\mathcal{C}) \implies 1/2 = (1/3)^s \implies s = \log_3(2)\end{aligned}$$

Example 2.11. The Sierpinski triangle $\mathcal{T}$ has Hausdorff dimension $s = \log_2(3)$. Assuming that $0 < \mathcal{H}^s(\mathcal{T}) < \infty$, we notice that $\mathcal{T}$ is self-similar: that is, it is three copies of itself scaled by one-half. So, if S_i scales F to the ith copy (contained in any of the three triangular regions), then

$$\begin{aligned}\mathcal{H}^s(\mathcal{T}) &= \mathcal{H}^s(S_1(\mathcal{T})) + \mathcal{H}^s(S_2(\mathcal{T})) + \mathcal{H}^s(S_3(\mathcal{T})) \\ &= (1/2)^s\mathcal{H}^s(\mathcal{T}) + (1/2)^s\mathcal{H}^s(\mathcal{T}) + (1/2)^s\mathcal{H}^s(\mathcal{T}) \\ &= 3(1/2)^s\mathcal{H}^s(\mathcal{T}) \implies 1/3 = (1/2)^s \implies s = \log_2(3)\end{aligned}$$

2.3. **Gauge Families.** A common generalization of Hausdorff dimension exists when we look beyond the specific idea of dimension we may know and love. In some respect, our development of an s-dimensional measure through understanding the "right" measure is arbitrary. What if there was another class of functions, one for each dimension, that generalize the scaling property in our original motivating questions?

Definition 2.12. A gauge function $\varphi_0 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a continuous, non-decreasing function with $\varphi_0(\delta) \to 0$ only as $\delta \to 0$ (i.e., φ_0 vanishes only at 0).

A gauge family $\varphi = \{\varphi_s : s > 0\}$ is a one-parameter family of gauge functions with

$$\lim_{\delta \to 0^+} \frac{\varphi_t(\delta)}{\varphi_s(\delta)} = 0$$

if $s < t$.

There are many useful families, which can be used to study dimension that does conform to traditional understandings of "scaling" as ungauged Hausdorff dimension might. The theory of Brownian motion, for example, can be formalized in another way using gauge families.

However, there is a particular limitation to gauge families that we remedy with the following concept.

Definition 2.13. A precision sequence $\alpha_0 : \mathbb{N} \to \mathbb{R}_{>0}$ for a gauge function φ_0 is a function with $\alpha_0(r) \to 0$ only as $r \to \infty$ and $\varphi_0(\alpha_0(r)) \leq O(\varphi_0(\alpha_0(r+1))$ for large enough r.

A precision family $\alpha = \{\alpha_s : s > 0\}$ is a one-parameter family of precision sequences for a gauge family φ with

$$\sum_{r=1}^{\infty} \frac{\varphi_t(\alpha_s(r))}{\varphi_s(\alpha_s(r))} < \infty$$

if $s < t$.

A precision family generalizes the concept of being able to look at countably many points leading up to a limit, which we often tend to do with ungauged Hausdorff measure and dimension. The next proposition encapsulates this concept.

Proposition 2.14. *Let $\theta = \{\theta_s : s > 0\}$ be family of functions given by $\theta_s(\delta) = \delta^s$, and let $\alpha = \{\alpha_s : s > 0\}$ be the family of functions given by $\alpha_s(r) = 2^{-sr}$. Then, θ is a gauge family, and α is a precision family for θ.*

Definition 2.15. Let φ be a gauge family (not necessarily equipped with a precision family). Fix $s > 0$ and $F \subseteq X$. For all $\delta > 0$, define

$$\mathcal{H}^{\varphi_s}_{\delta}(F) = \inf \left\{ \sum_{i=1}^{\infty} \varphi_s(|U_i|) : U_1, U_2, ... \text{ is a countable } \delta\text{-cover of } F \right\}$$

Then, define the φ-gauged s-dimensional Hausdorff outer measure by

$$\mathcal{H}^{\varphi_s}_{*}(F) = \lim_{\delta \to 0^+} \mathcal{H}^{\varphi_s}_{\delta}(F)$$

Similar results to Propositions 2.3, 2.5, and 2.6 can be proven to yield a cohesive definition of gauged Hausdorff dimension, which is the rigorous generalization we wanted.

Definition 2.16. Let φ be a gauge family. The φ-gauged Hausdorff dimension of a set $F \subseteq X$ is

$$\dim_{\mathrm{H}}^{\varphi}(F) = \inf\{s > 0 : \mathcal{H}^{\varphi_s}(F) = 0\} = \sup\{s > 0 : \mathcal{H}^{\varphi_s}(F) = \infty\}$$

3. More Definitions of Dimension

3.1. **Minkowski Dimension.** Hausdorff dimension is not the only viable definition of dimension of subsets of a metric space X. Counting boxes offers a way to understand dimension without measure theory.

Definition 3.1. The upper and lower Minkowski dimensions of a set $F \subseteq X$ are respectively

$$\overline{\dim}_{\mathrm{B}}(F) = \limsup_{n\to\infty} \frac{N_\delta(F)}{\log(1/\delta)}$$
$$\underline{\dim}_{\mathrm{B}}(F) = \liminf_{n\to\infty} \frac{N_\delta(F)}{\log(1/\delta)}$$

where $N_\delta(F)$ is the smallest number of sets of diameter at most δ needed to cover F. If these quantities agree, we denote the common value by $\dim_{\mathrm{B}}(F)$ and call it the Minkowski dimension of F.

The exact definition of $N_\delta(F)$ can vary depending on what we need to use: see Chapter 3 of [2] for examples. However, all definitions lead to equivalent values of the Minkowski dimensions. We use one of these alternative definitions in the gauged versions of Minkowski dimension.

Definition 3.2. Let φ be a gauge family. The φ-gauged upper and lower Minkowski dimensions of a set $F \subseteq X$ are respectively

$$\overline{\dim}_{\mathrm{B}}^{\varphi}(F) = \inf\{s > 0 : \liminf_{\delta\to 0^+} N_\delta(F)\varphi_s(\delta) = 0\}$$
$$\underline{\dim}_{\mathrm{B}}^{\varphi}(F) = \inf\{s > 0 : \limsup_{\delta\to 0^+} N_\delta(F)\varphi_s(\delta) = 0\}$$

where $N_\delta(F)$ is the smallest number of balls of radius δ with centers in F needed to cover F. If these quantities agree, we denote the common value by $\dim_{\mathrm{B}}^{\varphi}(F)$ and call it the φ-gauged Minkowski dimension of F.

Taking θ to be the canonical gauge family (with its standard precision family) in Proposition 2.14, we can formulate the coherence of such a generalization with the following.

Proposition 3.3. $\overline{\dim}_{\mathrm{B}}^{\theta}(F) = \overline{\dim}_{\mathrm{B}}(F)$ *and* $\underline{\dim}_{\mathrm{B}}^{\theta}(F) = \underline{\dim}_{\mathrm{B}}(F)$ *for all* $F \subseteq X$.

3.2. **Packing Dimension.** If we return to measure theory, developing a dual to coverings may be helpful to formulate another definition of dimension. Naturally, the dual of coverings could be considered packings, which are initially tricky to define correctly due to the requirements that we would like for a measure.

Definition 3.4. Let (X, d) be a metric space. Suppose $\delta > 0$ and $F \subseteq X$. A δ-packing of F is a pairwise disjoint collection $\mathcal{U}$ of open balls with centers in F such that $\mathrm{diam}(B) \leq \delta$ for all $B \in \mathcal{U}$.

Definition 3.5. Let φ be a gauge family. Fix $s > 0$ and $F \subseteq X$. For all $\delta > 0$, define

$$\mathcal{P}_{\delta}^{\varphi_s}(F) = \sup\left\{ \sum_{i=1}^{\infty} \varphi_s(|U_i|) : U_1, U_2, ... \text{ is a countable } \delta\text{-packing of } F \right\}$$

Then, define the s-dimensional packing premeasure by

$$\mathcal{P}_{0}^{\varphi_s}(F) = \lim_{\delta \to 0^+} \mathcal{P}_{\delta}^{\varphi_s}(F)$$

Note that for $\varepsilon > \delta$, there are fewer δ-packings than there are ε-covers. So, by the monotone convergence theorem, the limit in the definition of $\mathcal{P}_0^s(F)$ exists and can be freely replaced by an infimum over all $\delta > 0$.

$\mathcal{P}_0^s$ is not a (outer) measure because it violates countable (sub)additivity. We need one more step to complete this to a useful tool.

Definition 3.6. Let φ be a gauge family. Fix $s > 0$ and $F \subseteq X$. Define the φ-gauged s-dimensional packing outer measure by

$$\mathcal{P}_{*}^{\varphi_s}(F) = \inf\left\{ \sum_{i=1}^{\infty} \mathcal{P}_{0}^{\varphi_s}(U_i) : U_1, U_2, ... \text{ is any countable cover of } F \right\}$$

Similar results to Propositions 2.3, 2.5, and 2.6 can be proven to yield a cohesive definition of gauged packing dimension.

Definition 3.7. Let φ be a gauge family. The φ-gauged packing dimension of a set $F \subseteq X$ is

$$\dim_{\mathrm{P}}^{\varphi}(F) = \inf\{s > 0 : \mathcal{P}^{\varphi_s}(F) = 0\} = \sup\{s > 0 : \mathcal{P}^{\varphi_s}(F) = \infty\}$$

This extra step of a premeasure in developing the theory of packing dimension often proves to be annoying to deal with in practice. In this sense, while coverings and packings are dual to each other, Hausdorff and packing dimension don't necessarily appear to be precisely dual to one another.

The fact that they are duals of one another, however, is easily demonstrated by non-classical methods of understanding dimension. The algorithmic methods provide a unique perspective on dimension that captures this duality amongst many other useful characterizations of dimension.

4. Kolmogorov Complexity

We now turn to a different method by which to compute the dimension of sets. The algorithmic method employed in this paper has its roots with a prototypical version discussed by Downey and Hirschfeldt in [1]. This paper, however, focuses on specific algorithmic methods first employed by Lutz and co-authors in [5] and [6].

Kolmogorov complexity is the first step in understanding this fascinating connection. It is a purely computability-theoretic concept on its own, so this section will make no rigorous attempt to unite it with the classical mathematics of fractal dimension discussed prior.

4.1. **Preliminaries.** We begin with an overview theory of computation. Most of what is necessary for applying algorithmic complexity to geometric measure theory is articulated in this section, but more information can be found in introductory texts like Sipser's *Introduction to the Theory of Computation* or the like.

Definition 4.1. A string x is a finite sequence of 0s and 1s, which are called the bits of the string. The length of a string x is the number of bits in the string, denoted by $|x|$.

For any natural number n, the set of all strings of length n is denoted $\{0,1\}^n$. The set of all strings of any length is then

$$\{0,1\}^* = \bigcup_{k=0}^{\infty} \{0,1\}^k$$

with the empty string λ being the unique element of $\{0,1\}^0$.

Definition 4.2. A Turing machine is a two-way infinite tape of cells (each of which contains a 0, a 1, or a special blank symbol #) and a head that points to one of the cells. It operates under a finite and non-empty set of states Q and a finite instruction set that is encoded by $\delta : Q \times \{0,1,\#\} \to Q \times \{0,1\} \times \{L,S,R\}$.

At any given step of the computation, the machine is in some state with its head pointing to a symbol in a cell. Depending on this state and this symbol in the current cell, the machine can enter a new state; write a new symbol to the current symbol; and have the head move left, stay stationary, or move right (L, S, or R). This process is clearly encoded in the domain and the codomain of δ.

One of the states in Q is called the halting state. The machine terminates computation after entering the halting state: it stops using δ after that point and is considered to have halted. A Turing machine M is equipped with an input string w, where the head points to the first symbol in w and blank symbols are padded on either end of w: if the machine halts, its output $M(w)$ is said to be the string of non-blank symbols written on the tape while in the halting state.

The concatenation of strings x and y is denoted xy (with x^n representing x concatenated with itself n times). However, concatenating 0 and 11 is not distinct from concatenating 01 and 1, which leads to a question about the abilities of the Turing machine. What if we want two distinct strings as input to a machine? The ability to delimit strings solves this issue.

Definition 4.3. Let $\langle \cdot, \cdot \rangle : \{0,1\}^* \times \{0,1\}^* \to \{0,1\}^*$ be a joining function given by

$$0^{|[x]|}1[x]xy$$

where $[x]$ is taken to be the binary form of the natural number $|x|$.

Strings also appear limited as inputs. What if we want a number to be factored or the configuration of a boolean circuit to be evaluated? These are tasks we should expect a model of computation to be able to perform: thankfully, the Turing machine can.

Example 4.4. Countable sets can be enumerated in a suitable order and thus encoded in binary form. Examples of such sets include:

- The natural numbers $\mathbb{N}$ (each of which can be expressed as a binary number)

- The set of all graphs (with an n-vertex graph represented by a n-by-n adjacency matrix)
- The enumeration of all Turing machines (as the number of valid transition functions based on n states is finite)

We often denote the binary form of any of the above (or any other suitable encoding of an item) with the $\langle\cdot\rangle$ function (e.g., a graph G that can be used as input to a Turing machine is denoted $\langle G\rangle$).

Finally, oracles enrich the Turing machine model in a highly theoretical fashion. Oracles modify the capability of Turing machines as a model of computation beyond what we would normally observe by real practical computers.

Definition 4.5. An oracle Turing machine M is a Turing machine that is given access to a subset A of $\{0,1\}^*$ called an oracle. At any point in its computation, the machine may enter a query state in which it is immediately able determine whether the string that is currently written on the tape belongs to the oracle.

We denote M^A to be such a machine equipped specifically with oracle A and $M^A(w)$ to be its output if w is its input.

4.2. **Definitions.** We may then use the theory of computation to understand the complexity of strings x. In general, we ask what is the length of the shortest program p such $M(p) = x$ for some fixed Turing machine M?

Definition 4.6. Fix an oracle Turing machine M, an oracle A, and strings x, y. Then, the Kolmogorov complexity of x given y relative to A with respect to M is

$$C_M^A(x \,|\, y) = \min\{|p| : M^A \text{ halts on } \langle y, p\rangle \text{ and } M^A(y,p) = x\}$$

where the minimum is considered ∞ if no such p exists.

Definition 4.7. Fix an oracle Turing machine M, an oracle A, and a string x. Then, the plain Kolmogorov complexity of x relative to A with respect to M is $C_M^A(x) = C_M^A(x \,|\, \lambda)$.

We may also de-relativize these definitions by taking the oracle A to be the empty set $\emptyset$. A Turing machine by default already knows whether a string is in the empty set (in that it never is), so an oracle Turing machine with such an oracle is a Turing machine. When referring to de-relativized Kolmogorov complexity, we write $C_M(x \,|\, y)$ or $C_M(x)$.

A property that we would like Kolmogorov complexity to have is *universality*. Certainly, for each string x, we can always find a Turing machine M_x such that $M_x(\lambda) = x$ so that $C_{M_x}(x) = 0$. However, this isn't very useful for a universal notion of string complexity: we would like to fix some reasonable Turing machine U so that we can let $C_U^A(x \,|\, y)$ simply be Kolmogorov complexity.

Another property that we would like Kolmogorov complexity to have is *optimality*. For the universal notion of string complexity, we'd like to have a constant so that $C_U^A(x \,|\, y) \leq C_M^A(x \,|\, y) + O(1)$ for any strings x and y and Turing machine M. It turns out that there is such a notion which mostly fulfills these properties.

Theorem 4.8. *There exists a universal optimal oracle Turing machine U such that for all Turing machines M, there exists a constant $c_M > 0$ such that for all strings x and y, $C_U^A(x \,|\, y) \leq C_M^A(x \,|\, y) + c_M$.*

Proof. Let U be the oracle Turing machine that, on input $\langle\langle M\rangle, w\rangle$ where M is an oracle Turing machine and w is a string, simulates M^A on $\langle y, w\rangle$ and halts with the output $M^A(\langle y, w\rangle)$ if M halts on w.

Let p be the program testifying to $C^A_M(x \,|\, y)$: that is, p is the shortest program for which $M^A(\langle y, p\rangle) = x$. So, by the properties of the joining function in Definition 4.3, we have

$$C^A_U(x \,|\, y) \leq |\langle\langle M\rangle, w\rangle| = 2\log|\langle M\rangle| + 1 + |\langle M\rangle| + |w| = c_M + C^A_M(x \,|\, y)$$

if we let $c_M = 2\log|\langle M\rangle| + 1 + |\langle M\rangle|$. □

We fix such a universal optimal oracle Turing machine U for the rest of the paper. We drop the subscript of U when referring to Kolmogorov complexity relative to A on its own: we write $C^A(x \,|\, y)$ or $C^A(x)$.

I stated that this is a notion which *mostly* fulfills these properties that we want. That is, this is the best possible solution: we cannot get an additive constant in the upper bound that is independent of the choice of Turing machine M, as we will see with the following theorem.

Theorem 4.9. *For all Turing machines N and constants $c > 0$, there exist a Turning machine M and a string x such that $C_N(x) > C_M(x) + c$.*

Proof. Let x be a string for which $C_N(x) \geq c + 1$. Such a string exists as there are only $2^0 + 2^1 + ... + 2^c = 2^{c+1} - 1$ programs of length at most c. So, there must be some string x whose shortest program p for which $U(p) = x$ has length at least $c + 1$.

Let M be a Turing machine for which $M(x) = \lambda$. A Turing machine is based on a finite instruction set, and so one can obviously be hardcoded to output the string x on an empty input. Then, we have $C_N(x) \geq c + 1 > c = C_M(x) + c$. □

4.3. **Properties.** With this definition of Kolmogorov complexity, we have very many interesting properties that can be discerned.

Proposition 4.10. *There exists a constant $c > 0$ such that for all strings x and y, we have $C(x) \leq |x| + c$ and $C(x \,|\, y) \leq C(x) + c$.*

Proof. Let T be the identity Turing machine, which immediately halts on any input and leaves the tape written with the input. Then, $T(x) = x$ for any string x. So, we have

$$C(x) \leq C_T(x) + c \leq |x| + c$$

for some $c > 0$ dependent only on T.

Let V be the Turing machine, which ignores its first input and simulates U on the second input for the joined input $\langle y, w\rangle$. Then, $V(\langle y, p\rangle) = x$ for any strings x and y and some program p for which $U(p) = x$. So, if p testifies to $C(x)$, we have

$$C(x \,|\, y) \leq C_V(x \,|\, y) \leq |p| + c = C(x) + c$$

for some $c > 0$ dependent only on V. □

We often write $C(x, y)$ for $C(\langle x, y\rangle)$. This notation becomes rather convenient when we deal with binary expansions of Euclidean points in $\mathbb{R}^n$ for $n > 1$.

Proposition 4.11. *For all strings x and y, let $C = \min\{C(x), C(y)\}$. Then, we have $C(x, y) \leq C(x) + C(y) + O(\log C)$, where the implicit constant is independent of x and y.*

This result is obvious when considering programs that separate compute x and y. More astounding is the following theorem, called the symmetry of information, which strengthens the above result by also asserting equality.

Theorem 4.12. *For all strings x and y, we have*

$$C(x, y) = C(x) + C(y \,|\, x) + O(\log C(x, y))$$

where the implicit constant is independent of x andd y.

More properties follow from techniques similar to the proofs of Theorem 4.8 and Proposition 4.10. A comprehensive overview can be found in Chapter 2 of [4].

5. From String to Points

5.1. **Algorithmic Complexity.** We want to extend the notion of the complexity of strings to the complexity of points in a possibly uncountable metric space. It is pertinent that the space have a metric equipped since we eventually want to connect these ideas to Hausdorff dimension.

However, we need another requirement. Turing machines operate on a countable domain space (strings). It is not possible to extend the notion of complexity to any arbitrary metric space: for instance, spaces endowed with the discrete metric cannot be described using Turing machines. The condition of separability in the metric space allows us to have a connection to countable subsets that can be sufficient for describing a space.

Remark 5.1. Let X be a separable metric space. Letting $Y \subseteq X$ be the countable dense set, we fix a one-to-one and onto function $f : \{0,1\}^* \to Y$ to be the enumeration of elements in Y.

This idea relates to Example 4.4, where I claimed that countable sets can be used as inputs to Turing machines based on such enumerations. For instance, there exist an obvious enumeration of the rational-numbered points $\mathbb{Q}^n$ of $\mathbb{R}^n$ based on increasing precision in the binary expansions of the numbers.

Usually, I will say that we can "describe" members of such countable sets like rational-numbered points using a Turing machine, just as we can "describe" strings. This terminology is useful, and the formal use of the enumeration f is often dropped in common practice.

From this point on, all metric spaces X are assumed to be separable. With this, we define the complexity of points based on precision.

Definition 5.2. Let A be an oracle, $x \in X$, and $\delta > 0$. Then, the Kolmogorov complexity of x relative to A at precision δ is

$$C^A_\delta(x) = \min\{C^A(q) : q = f(w) \text{ for some string } w \text{ and } d(q, x) < \delta\}$$

We require the following important lemma to assert that many of the useful properties of Kolmogorov complexity of strings can be suitably translated into useful properties of Kolmogorov complexity of points (in Euclidean space, at least).

Lemma 5.3. *Fix $r \in \mathbb{N}$ and $x = (x_1, ..., x_n) \in \mathbb{R}^n$. Let $x^{:r} = \langle x_1^{:r}, ..., x_n^{:r} \rangle$ be the string such that $x_i^{:r}$ is the binary expansion of x_i up to r bits past the decimal point. Then, for any oracle A, we have*

$$C^A_{2^{-r}}(x) = C^A(x^{:r}) + O(\log r)$$

where the implicit constant depends only on n and $||x||$.

Proof. Let d_x be the dyadic rational-numbered point in $\mathbb{R}^n$ at the corner of the 2^{-r-1}-mesh cube which contains x.

It is obvious that $C^A(x{:}r) = C^A(d_x) + O(1)$: we can describe the coordinates of x up to r bits after the decimal point using d_x, and vice versa, assuming the suitable enumeration of the rational numbers mentioned prior. So, it suffices to show $C^A_{2^{-r}}(x) = C^A(d_x) + O(\log r)$.

For one direction, we know that $d(d_x, x) \leq 2^{-r-1}\sqrt{2} < 2^{-r}$. So, by definition, we have $C^A_{2^{-r}}(x) \leq C^A(d_x)$.

For the other direction, let q be the rational-numbered point in $\mathbb{R}^n$ testifying to $C^A_{2^{-r}}(x)$. Then, q is contained in one of at most 5^n total 2^{-r-1}-mesh cubes that may intersect $B(x, 2^{-r})$. Thus, if a Turing machine is allowed to use n along with $||x||$ to account for bits before each coordinate's decimal point, we can describe d_x using q using r: that is, $C^A(d_x \,|\, q) = O(\log r)$. So, we have

$$\begin{aligned} C^A(d_x) &\leq C^A(d_x \,|\, q) + C^A(q) + 2\log(C^A(d_x \,|\, q)) + O(1) \\ &= O(\log r) + C^A_\delta(x) + 2\log(O(\log r)) = C^A_\delta(x) + O(\log r) \end{aligned}$$

as desired. □

Aside from being able to use this to show analog properties of the complexity of points, it also helps prove the following lemma, stating that growth rate of the complexity of a point in $\mathbb{R}^n$ is dependent on the ambient dimension n of the space.

Lemma 5.4. *Fix an oracle A, two natural number r and s with $r \geq s$, and a point $x \in \mathbb{R}^n$. Then, we have*

$$C^A_{2^{-r}}(x) \leq C^A_{2^{-s}}(x) + n(r-s) + O(\log r)$$

where the implicit constant depends only on n and $||x|$.

Proof. We apply Lemma 5.3 as intended so that it suffices to show

$$C^A(x{:}r) \leq C^A(x{:}s) + n(r-s) + O(\log r)$$

Let M be the oracle Turing machine which, when given a natural number m and a string $y{:}s$, uses m to pad $y{:}s$ into $y{:}r$. With this construction, m can be at most $2^{n(r-s)}$ to account for all possibilities of each coordinate's $(s+1)$th, ... rth digits of the binary expansion past the decimal point.

Let p testify to $C^A(x{:}s)$. So, we have

$$\begin{aligned} C^A(x{:}r) &\leq C^A_M(x{:}r) + O(1) = |\langle\langle m\rangle, p\rangle| + O(1) \\ &\leq |p| + |\langle m\rangle| + O(\log|\langle m\rangle|) \\ &\leq C^A(x{:}s) + \log m + O(\log\log m) \\ &\leq C^A(x{:}s) + n(r-s) + O(\log r) \end{aligned}$$

as desired. □

5.2. **Algorithmic Dimension.** The complexity of points at various precisions on their own is not particularly useful. It is worth analyzing the limiting behavior of the complexity of points at arbitrarily high precisions.

Definition 5.5. Let A be an oracle and let $x \in X$. Then, the algorithmic dimension of x relative to A is

$$\dim^A(x) = \liminf_{\delta \to 0^+} \frac{C^A_\delta(x)}{\log(1/\delta)}$$

Moreover, the strong algorithmic dimension of x relative to A is

$$\operatorname{Dim}^A(x) = \limsup_{\delta \to 0^+} \frac{C^A_\delta(x)}{\log(1/\delta)}$$

We have the following result for points in Euclidean space that follows from a previous lemma. Though basic, it is important in establishing algorthimic dimension as a viable understanding of some kind of dimension due to the fact that it is bounded in a meaningful space by the ambient dimension of the space.

Theorem 5.6. *Let $x \in \mathbb{R}^n$. Then, $0 \leq \dim^A(x) \leq \operatorname{Dim}^A(x) \leq n$.*

Proof. The first inequality is obvious since Kolmogorov complexity is non-negative and $\log(1/\delta)$ is non-negative too for sufficiently small δ. The second inequality is also obvious by the definition of limit inferior and limit superior.

The third inequality makes use of Lemma 5.4. Suppose δ is sufficiently small, and let r be the natural number for which $2^{-r} \leq \delta < 2^{-r+1}$. Since a ball of radius δ is less precise than one of radius 2^{-r}, we have $C^A_\delta(x) \leq C^A_{2^{-r}}(x)$. Thus, we see

$$\begin{aligned} C^A_\delta(x) &\leq C^A_{2^{-r}}(x) \leq C^A_1(x) + nr + O(\log r) \\ &\leq C^A_1(x) + n(\log(1/\delta) + 1) + O(\log\log(1/\delta)) \\ &= n\log(1/\delta) + n + C^A_1(x) + o(\log(1/\delta)) \end{aligned}$$

Dividing by $\log(1/\delta)$ and taking the limit superior, we have $\operatorname{Dim}^A(x) \leq n$ as desired. □

Moreover, we can extend the notion of algorithmic dimension to gauged algorithmic dimension. This will then offer a more direct look into how these concepts relate to the classical study of fractal dimension.

Definition 5.7. Let A be an oracle, φ be a gauge family, and $x \in X$. Then, the φ-gauged algorithmic dimension of x relative to A is

$$\dim^{\varphi,A}(x) = \inf\left\{s > 0 : \liminf_{\delta \to 0^+} 2^{C^A_\delta(x)}\varphi_s(x) = 0\right\}$$

Moreover, the strong φ-gauged algorithmic dimension of x relative to A is

$$\operatorname{Dim}^{\varphi,A}(x) = \inf\left\{s > 0 : \limsup_{\delta \to 0^+} 2^{C^A_\delta(x)}\varphi_s(x) = 0\right\}$$

Similar results to Propositions 2.5 and 2.6 can be proven to yield a cohesive definition of gauged algorithmic dimension.

Taking θ to be the canonical gauge family (with its standard precision family) in Proposition 2.14, we can demonstrate the coherence of this generalization.

Proposition 5.8. $\dim^{\theta,A}(x) = \dim^A(x)$ *and* $\operatorname{Dim}^{\theta,A}(x) = \operatorname{Dim}^A(x)$ *for all* $x \in X$.

Two important results are that there cannot be too many points of low dimension and, for Euclidean space, even that most points have high dimension.

Theorem 5.9. *Let A be an oracle, φ be a gauge family with a precision family α, and n be a natural number. Define the set*

$$\dim_{\leq s}^{\varphi,A} = \{x \in X : \dim^{\varphi,A}(x) \leq s\}$$

Then, $\dim_{\mathrm{H}}(\dim_{\leq s}^{\varphi,A}) \leq s$.

Proof. Let $u > t > s$. Let C_t be the constant such that $\varphi_t(\alpha_t(r)) \leq C_t\varphi_t(\alpha_t(r+1))$ for sufficiently large r.

Then, for each natural number r, define the collection of balls

$$\mathcal{U}_r = \left\{ B(f(w), \alpha_t(r)) : C^A(w) \leq \log \frac{C_t}{\varphi_t(\alpha_t(r))} \right\}$$

Notice that $\mathcal{U}_r$ is a finite collection since there are only finitely many strings w that can be described by a program $C^A(w)$ shorter than the value of the expression on the right. In particular, we have

$$|\mathcal{U}_r| \leq 2^{\log \frac{C_t}{\varphi_t(\alpha_t(r))}+1} - 1 \leq \frac{2C_t}{\varphi_t(\alpha_t(r))}$$

With that, for each natural number r, define the countable collection of balls

$$\mathcal{W}_r = \bigcup_{k=r}^{\infty} \mathcal{U}_k$$

Fix $x \in \dim_{\leq s}^{\varphi,A}$. As $\dim^{\varphi,A}(x) < t$, there exists a sequence $\{\delta_k\}_{k=1}^{\infty}$ with $\delta_k \to 0$ as $k \to \infty$ such that for sufficiently large k

$$2^{C^A_{\delta_k}(x)}\varphi_t(\delta_k) \leq 1$$

Then, since α_t vanishes, there corresponds to each δ_k a natural number r_k such that for all $r > r_k$, $\alpha_t(r) < \delta_k \leq \alpha_t(r_k)$.

Since a ball of radius $\alpha_t(r_k)$ is less precise than one of radius δ_k, we have $C^A_{\alpha_t(r_k)}(x) \leq C^A_{\delta_k}(x)$. Moreover, since α_t is a precision sequence, we have

$$\varphi_t(\alpha_t(r_k)) \leq C_t\varphi(\alpha_t(r_k+1)) \leq C_t\varphi_t(\delta_k)$$

Thus, there exists a sequence $\{r_k\}_{k=1}^{\infty}$ with $r_k \to \infty$ as $k \to \infty$ such that for sufficiently large k

$$2^{C^A_{\alpha_t(r_k)}(x)}\varphi_t(\alpha_t(r_k)) \leq C_t \;\Leftrightarrow\; C^A_{\alpha_t(r_k)}(x) \leq \log \frac{C_t}{\varphi_t(\alpha_t(r_k))}$$

That is, there are infinitely many k for which $x \in B \in \mathcal{U}_{r_k}$. In other words, $x \in B \in \mathcal{W}_r$ for all r.

Fixing r, let $r_0 \geq r$ be a natural number for which $\alpha_t(r') \leq \alpha(r)$ whenever $r' \geq r_0$. So, $\mathcal{W}_{r_0}$ is a countable $\alpha_t(r)$-cover of the set $\dim_{\leq s}^{\varphi,A}$. We note that, though the radius of each ball in $\mathcal{W}_{r_0}$ is $\alpha_t(r)$, we could easily divide each ball into a fixed and finite number of sets of diameter $\alpha_t(r)$.

Then, based on the radii on the balls, we have

$$\begin{aligned}\mathcal{H}^{\varphi_u}_{\alpha_t(r)}(\dim^{\varphi,A}_{\leq s}) &\leq \sum_{U\in\mathcal{W}_{r_0}} \varphi_u(\frac{|U|}{2}) = \sum_{k=r_0}^{\infty}\sum_{U\in\mathcal{U}_k} \varphi_u(\frac{|U|}{2}) \\ &\leq \sum_{k=r_0}^{\infty}\sum_{U\in\mathcal{U}_k} \varphi_u(\alpha_t(k)) \leq \sum_{k=r_0}^{\infty} \frac{2C_t}{\varphi_t(\alpha_t(r))}\cdot\varphi_u(\alpha_t(k)) \\ &= 2C_t\sum_{k=r_0}^{\infty}\frac{\varphi_u(\alpha_t(k))}{\varphi_t(\alpha_t(r))} \leq 2C_t\sum_{k=1}^{\infty}\frac{\varphi_u(\alpha_t(k))}{\varphi_t(\alpha_t(r))}\end{aligned}$$

Taking $r\to\infty$, we use that α is a precision family to see

$$\mathcal{H}^{\varphi_u}(\dim^{\varphi,A}_{\leq s}) \leq 2C_t\sum_{k=1}^{\infty}\frac{\varphi_u(\alpha_t(k))}{\varphi_t(\alpha_t(r))} < \infty$$

So, $\dim^{\varphi}_{\mathrm{H}}(\dim^{\varphi,A}_{\leq s}) \leq u$. As u was arbitrary, this completes the proof. □

Corollary 5.10. *Almost all points in $\mathbb{R}^n$ have algorithmic dimension n.*

Proof. Defining $\dim^{\varphi,A}_{<n} = \{x\in X : \dim^{\varphi,A}(x) < n\}$ and $\dim^{\varphi,A}_{\leq s}$ for all $s\geq 0$ as above, we note that

$$\dim^{\varphi,A}_{<n} = \bigcup_{k=1}^{\infty}\dim^{\varphi,A}_{\leq(n-1/k)}$$

By Theorem 5.9, we have that $\mathcal{H}^n(\dim^{\varphi,A}_{\leq(n-1/k)}) = 0$. So, $\mathcal{H}^n(\dim^{\varphi,A}_{<n}) = 0$, by countable additivity. The desired result follows by recognizing that $\mathcal{H}^n$ is also the n-dimensional Lebesgue measure. □

6. The Point-to-Set Principle

The final step in connecting Hausdorff's theory of fractal dimension to Kolmogorov's algorithmic complexity theory is to see if we can describe entire sets using the algorithmic information of points. It turns out that there is a rather concise statement that captures this.

Theorem 6.1 (Point-to-Set Principle)**.** *Let $E\subset X$ be an arbitrary (not necessarily Borel or analytic) set. Then, we have*

$$\begin{aligned}\dim_{\mathrm{H}}(E) &= \min_{A\subseteq\{0,1\}^*}\sup_{x\in E}\dim^A(x) \\ \dim_{\mathrm{P}}(E) &= \min_{A\subseteq\{0,1\}^*}\sup_{x\in E}\mathrm{Dim}^A(x)\end{aligned}$$

This principle was first discovered by Lutz in [5], where it was used to prove the Kakeya set conjecture in $\mathbb{R}^2$ using these algorithmic methods. Later, the principle was generalized to gauged dimension.

Theorem 6.2 (Extended Point-to-Set Principle)**.** *Let φ be a gauge family. Let $E\subset X$ be an arbitrary (not necessarily Borel or analytic) set. Then, we have*

$$\begin{aligned}\dim^{\varphi}_{\mathrm{H}}(E) &\geq \min_{A\subseteq\{0,1\}^*}\sup_{x\in E}\dim^{\varphi,A}(x) \\ \dim^{\varphi}_{\mathrm{P}}(E) &\geq \min_{A\subseteq\{0,1\}^*}\sup_{x\in E}\mathrm{Dim}^{\varphi,A}(x)\end{aligned}$$

Equality holds if there is a precision family for φ.

Proof. The proof of the first part of this theorem can be found in [6].

Assuming there is a precision family, we can demonstrate the second part of the theorem for gauged Hausdorff dimension using Theorem 5.9. Fixing an oracle A and letting $s = \sup_{x\in E} \dim^{\varphi,A}(x)$, it is clear $E \subseteq \dim^{\varphi,A}_{\leq s}$. So, we have

$$\dim^{\varphi}_{\mathrm{H}}(E) \leq \dim^{\varphi}_{\mathrm{H}}(\dim^{\varphi,A}_{\leq s}) \leq s = \sup_{x\in E} \dim^{\varphi,A}(x)$$

for any oracle A, as desired. □

6.1. **Problems of the Field.** There are several applications of the point-to-set principle. For instance, it is often used to generalize theorems relating to geometric measure theory since the principle (and the lemmas it depends on for its complete proof) do not depend on the Borelness of the set.

However, for the rest of this paper, I will focus on an overview of its potential application to the Kakeya set conjecture. This offers a fascinating look at how algorithmic methods provide an alternative route to solving more pertinent and classic open problems.

Definition 6.3. A Kakeya set K is a subset of $\mathbb{R}^n$ for which there exists a family of lines $\mathcal{L}$ in $\mathbb{R}^n$, consisting of a line in every direction, such that $K \cap l$ contains a unit line segment for all $l \in \mathcal{L}$.

Conjecture 6.4. *Let $K \subseteq \mathbb{R}^n$ be Kakeya. Then, $\dim_{\mathrm{H}}(K) = n$.*

A characterization of the algorithmic dimension of points on lines seems necessary to see if the algorithmic route to understanding the conjecture is viable. After all, the definition of a Kakeya set directly involves lines with a specified dimension (i.e., a unit line segment clearly has dimension 1, wherever it exists).

Theorem 6.5. *Let A be an oracle, and let $x, a, b \in \mathbb{R}$.. Then*

$$\dim^{A}(x, ax+b) \geq \dim^{A,a,b}(x) + \min\{\dim^{A}(a,b), \dim^{A,a,b}(x)\}$$

Notice that this theorem has our first reference to the encoding of Euclidean points as oracles: we join the points a and b as oracles themselves to the original oracle A. This is an important concept to be able to apply, but the details of how the encoding works are largely unimportant: essentially, an oracle Turing machine M should have $M^y(r) = y{:}r$ for any natural number r if y is to be treated as an oracle (i.e., y is said to be computable relative to such an oracle).

A deeper discussion of Theorem 6.5 is found in [5], where it is used to prove the Kakeya set conjecture in $\mathbb{R}^2$. The proof of the conjecture in $\mathbb{R}^2$ is relatively simple using this theorem, which is much more difficult to prove. With such a relative gap in proof difficulty, it is clear that the point-to-set principle helps in completely reformulating the approach to geometric measure theory.

It is actually only required that the theorem only be true when $\dim^{A,a,b}(x) = 1$. The following conjecture would generalize this notion to lines existing in higher-dimensional Euclidean spaces.

Conjecture 6.6. *Let A be an oracle. Let $a \in S^{n-1}$ be a point directing the slope of a line in $\mathbb{R}^n$ and $b \in \mathbb{R}^n$ be a point on the line. Then, for any $t \in \mathbb{R}$ with $\dim^{A,a,b}(t) = 1$, we have*

$$\dim^{A}(at+b) \geq \min\{1 + \dim^{A}(a,b), n\}$$

Proof of Conjecture 6.4 using Conjecture 6.6. Let A be a Hausdorff oracle for K. That is, A is the oracle for which $\dim_{\mathrm{H}}(K) = \sup_{x\in K} \dim^A(x)$ since Theorem 6.1 states that it is a minimum and not simply an infimum.

Fix $\varepsilon > 0$. So, there is a family of lines $\mathcal{L}$, consisting of a line in every direction, such that $K \cap l$ contains a unit line segment for all $l \in \mathcal{L}$. By any reasonable metric on the space of all lines in $\mathbb{R}^n$, this means

$$\dim_{\mathrm{H}}(\mathcal{L}) \geq \dim_{\mathrm{H}}(S^{n-1}) \geq n - 1$$

With that, there exists a line l that we identify in such a metric with the ordered pair $(a, b) \in S^{n-1} \times \mathbb{R}^n$ such that $\dim^A(a, b) \geq n - 1 - \varepsilon$.

Define the set $S = \{t \in \mathbb{R} : (at + b) \in K\}$. Thus, we have

$$\dim_{\mathrm{H}}(K) = \sup_{x\in K} \dim^A(x) \geq \sup_{x\in K\cap l} \dim^A(x) = \sup_{t\in S} \dim^A(at + b)$$

Since $K \cap l$ contains a unit line segment, clearly $\dim_{\mathrm{H}}(K \cap l) = 1$. As almost all $t \in \mathbb{R}$ have $\dim^{A,a,b}(t) = 1$ by Corollary 5.10, there exists a $t_0 \in S$ with such a property. With that, we have

$$\begin{aligned}\dim^A(at_0 + b) &\geq \min\{1 + \dim^A(a, b), n\} \\ &\geq \min\{1 + (n - 1 - \varepsilon), n\} = n - \varepsilon\end{aligned}$$

So, $\dim_{\mathrm{H}}(K) \geq \dim^A(at_0 + b) \geq n - \varepsilon$. As ε was arbitrary, $\dim_{\mathrm{H}}(K) \geq n$. The other direction of inequality is obvious by the monotonicity of Hausdorff dimension. □

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