FREE BOUNDARY REGULARITY FOR THE CLASSICAL OBSTACLE PROBLEM

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Abstract. We first present some of the foundational results in the regularity theory of elliptic PDE: namely, the Schauder & Calderón-Zygmund estimates. We then explore the classical theory surrounding the obstacle problem including optimal regularity, the classification of blowups, and the regularity of the free boundary. We assume the reader has familiarity with the contents of Chapters 2/5/8 of [1] or alternatively, Chapter 1 of [2].

Contents

1. Introduction

An important class of second-order partial differential equations (PDEs) is the family of *elliptic* equations. Morally, these equations resemble the Dirichlet problem

$$\begin{cases} \Delta u = f \text{ in } \Omega \\ u = g \text{ on } \partial\Omega \end{cases} \tag{1.1}$$

for some domain Ω. Elliptic equations frequently arise in physics and are often associated with the conservation of certain quantities, such as energy.

We are primarily interested in a related class of nonlinear PDEs known as *free boundary problems*. In these equations, the domain itself is not predetermined but

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is instead part of the solution to the PDE. A canonical example of a free boundary problem is known as the *Stefan problem* and has a clear physical interpretation (see [4] for the explicit formulation). Consider a tank containing a mixture of ice and water. Intuitively, we know that the ice will eventually melt entirely, leaving only water. We can model how exactly this melting process occurs with a PDE. Specifically, in the interior of the region of water (i.e., positive temperature), the temperature distribution will obey the usual heat equation. Meanwhile, on the interior of the region of ice (i.e., zero temperature), the temperature remains constant. In particular, the PDE governing the system is defined piecewise. In this context, the surface along which the water and ice meet is the free boundary which also evolves over time. Therefore, studying the free boundary would allow us to answer interesting questions such as whether the process of ice melting in water (or a snowflake melting in air) produces fractal surfaces.

However, we will primarily focus on another elliptic free boundary problem called the *obstacle problem*, which also has a physical origin. This problem involves minimizing the elastic potential energy of a thin membrane constrained above an obstacle. We can therefore formulate the problem as minimizing the functional

$$I[u] = \frac{1}{2}\int_\Omega |\nabla u|^2.$$

over a bounded domain $\Omega \subset \mathbb{R}^d$ and functions u above the obstacle φ (i.e., $u \geq \varphi$) with some fixed boundary data $u\big|_{\partial\Omega} = f$. In particular, the solution can be thought of as an elastic membrane attached to some boundary wire that we let fall subject to gravity and constrain above some fixed surface (the obstacle). Then, the free boundary $\Gamma = \partial\{u > \varphi\} \cap \Omega$ is the set where the membrane touches the constraint which makes sense to study. We will study this minimization problem in more depth in Section 3 and derive many interesting properties of minimizers and the free boundary Γ. A plot of a solution (see Example 3.18) in two dimensions is below.

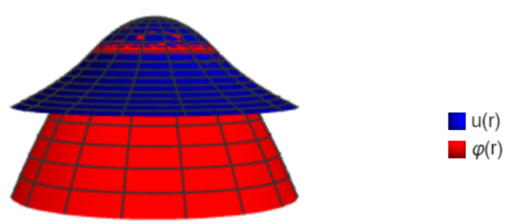

FIGURE 1. Plot of Radial Solution to Obstacle Problem in 2D

One can also interpret the obstacle problem as the linearization of the minimal surface (area-minimizing) problem. In particular, one can see that minimizing the functional

$$J[u] = \int_\Omega \sqrt{1 + |\nabla u|^2}$$

is approximately the same as minimizing $I[\cdot]$ whenever the fluctuations in u (i.e, $|\nabla u|$) are small via Taylor expansion.

One final consideration relates to Newton's shell theorem which states that a spherical shell exerts no gravitational force in its interior. Then, a natural question to ask is whether there are other shapes for which the shell theorem holds. It turns out that answering this question is closely related to the global obstacle problem. In particular, the shell theorem holds for a geometry if and only if it is the free boundary of a solution to the global obstacle problem (see [8] for a complete characterization). Of course, in this context, understanding the behavior of the free boundary Γ is crucial.

For now, the important fact about the obstacle problem is that it is elliptic. Given this, it seems reasonable to first build some of the regularity theory of elliptic equations so that we can eventually apply it to the obstacle problem. As alluded to earlier, the simplest elliptic operator is the Laplacian Δ. Naturally, we begin by considering the PDE

$$\Delta u = f \text{ in } B_1 \tag{1.2}$$

where $u \in H^1(B_1)$ is a bounded weak solution. We are interested in the behavior of u on the interior of the domain B_1, so we do not impose any boundary conditions. In that vein, one might ask how the regularity of f affects the regularity of the solution u. Schauder theory and the Calderón-Zygmund estimates give us insight into the regularity of u in the cases where $f \in \mathcal{C}^\alpha(B_1)$ or just $f \in L^\infty(B_1)$.

2. Elliptic Regularity

2.1. Overview.

We begin by stating the two previously mentioned elliptic regularity results.

Theorem 2.1. *(Schauder for Δ)*
Suppose $u \in H^1(B_1)$ is a bounded weak solution to

$$\Delta u = f \in \mathcal{C}^\alpha(B_1) \text{ in } B_1 \tag{2.2}$$

for some $0 < \alpha < 1$. Then,

$$\|u\|_{\mathcal{C}^{2,\alpha}\left(B_{1/2}\right)} \leq C\Big(\|f\|_{\mathcal{C}^\alpha(B_1)} + \|u\|_{L^\infty(B_1)}\Big) \tag{2.3}$$

for some absolute constant $C = C(\alpha, d) > 0$. In particular, we see that $u \in \mathcal{C}^{2,\alpha}\left(B_{1/2}\right)$.

Remark 2.4. The above theorem is stated specifically for the Laplace equation. It should be noted that such an estimate holds for more general elliptic equations (see Theorem 1.9 of [5]).

Theorem 2.5. *(Calderón-Zygmund)*
Suppose $u \in H^1(B_1)$ is a bounded weak solution to

$$\Delta u = f \in L^\infty(B_1) \text{ in } B_1. \tag{2.6}$$

Then, for each $0 < \varepsilon < 1$, there exists a constant $C_\varepsilon > 0$ so that

$$\|u\|_{\mathcal{C}^{1,1-\varepsilon}\left(B_{1/2}\right)} \leq C_\varepsilon\Big(\|f\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\Big). \tag{2.7}$$

This means that $u \in \mathcal{C}^{1,1-\varepsilon}\left(B_{1/2}\right)$ for each $0 < \varepsilon < 1$.

Remark 2.8. The above theorem is more often stated in the form that $f \in L^p$ implies $u \in W^{2,p}$. This implies our version of Calderón-Zygmund via Morrey's inequality (see Section 5.6.2 of [1]). We prove the $p = 2$ case of this more general version in Proposition 2.29. Also, note the theme in both theorems 2.1 and 2.5 that given a regularity constraint on f, the solution u gains two more derivatives.

A curious reader may wonder whether these estimates are the best we can do. In general, the answer is yes – the estimates do not hold at the endpoints $\alpha, \varepsilon = 0$. To see this, consider the functions $u_1(x, y) = (x^2 - y^2)\log(x^2 + y^2)$ and $u_2(x, y) = (x^2 - y^2)\log|\log(x^2 + y^2)|$ defined on B_1. Then, one may compute

$$\Delta u_1 = \frac{8\big(x^2 - y^2\big)}{x^2 + y^2} = 8\cos(2\theta) \in L^\infty(B_1)$$

and

$$\Delta u_2 = \frac{4\big(x^2 - y^2\big)\big(2\log\big(x^2 + y^2\big) - 1\big)}{(x^2 + y^2)\log^2(x^2 + y^2)} = \frac{4\cos(2\theta)(4\log r - 1)}{4\log^2 r} \in \mathcal{C}(B_1)$$

where both expressions are written in polar coordinates. For both u_1 and u_2, the second partials are unbounded, so we have $u_1 \notin \mathcal{C}^{1,1}(B_1)$ and $u_2 \notin \mathcal{C}^2(B_1)$ thus providing counterexamples for both theorems at the endpoints. However, in the case of the obstacle problem, we will exploit the particular structure of the right-hand side of the equation to improve the regularity to $\mathcal{C}^{1,1}$.

For our purposes, we only prove the Calderon-Zygmund estimates. The proof of Schauder is similar in nature and can be found in [5] or Simon Seignourel's paper. We take a more modern approach (due to Caffarelli) than found in [3] without relying on as many hard analysis estimates. The general method is often referred to as an *improvement of flatness*. To proceed, we first establish some helpful properties of Hölder functions.

2.2. Properties of Hölder Functions.

We first show a form of an interpolation inequality for $\mathcal{C}^{1,\alpha}$ functions. In particular, one only needs to control the L^∞ norm of the function and the Hölder seminorm of the derivative to place a function in $\mathcal{C}^{1,\alpha}$. A similar statement holds for higher-order spaces.

Lemma 2.9. *(Interpolation Inequality for $\mathcal{C}^{1,\alpha}$ functions) Let $u \in \mathcal{C}^{1,\alpha}(B_1)$. Then, we have that*

$$\|\nabla u\|_\infty \le C(\|u\|_\infty + [\nabla u]_\alpha).$$

Proof. We may choose an $x_0 \in B_{1/2}$ so that $|\nabla u(x_0)| \ge \frac{1}{2}\|\nabla u\|_{L^\infty(B_{1/2})}$ by continuity. Taking $\nu = \frac{\nabla u(x_0)}{|\nabla u(x_0)|}$, we may write

$$|\nabla u(x_0)| = \nabla u(x_0) \cdot \nu = (\nabla u(x_0) - \nabla u(x_0 + s\nu)) \cdot \nu + \nabla u(x_0 + s\nu) \cdot \nu$$

for $0 < s < \frac{1}{2}$. Then, we may write the above as the average integral over $s \in \big[0, \frac{1}{4}\big]$:

$$\begin{aligned} |\nabla u(x_0)| &= 4\int_0^{\frac{1}{4}} (\nabla u(x_0) - \nabla u(x_0 + s\nu)) \cdot \nu \,\mathrm{d}s + 4\int_0^{\frac{1}{4}} \nabla u(x_0 + s\nu) \cdot \nu \\ &\le [\nabla u]_\alpha 4^{1-\alpha} + 8\|u\|_\infty \end{aligned}$$

recognizing that the integrand in the second integral is a total derivative. As for dealing with the $B_1 \setminus B_{\frac{1}{2}}$ case, use the previous analysis:

$$|\nabla u(y)| \leq |\nabla u(y) - \nabla u(x_0)| + |\nabla u(x_0)|$$

where the first term is controlled by the Holder seminorm and the second by the estimate on $B_{\frac{1}{2}}$. □

Remark 2.10. The utility in this statement is that it gives us an equivalent norm on $\mathcal{C}^{1,\alpha}$. That is, we can take

$$\|f\|_{\mathcal{C}^{1,\alpha}} = \|f\|_\infty + [\nabla f]_\alpha \tag{2.11}$$

without loss of generality.

We finally establish the key characterization of Hölder continuous functions. In particular, we show that the rate of convergence of the Taylor polynomial of a Hölder continuous function converges at a faster rate than for ordinary differentiable functions. We prove the result for $\mathcal{C}^{1,\alpha}$ functions, but one should be able to convince oneself that similar results hold for higher-order spaces.

Proposition 2.12. *Let $f \in \mathcal{C}(\Omega)$. Assume there exists a constant $C > 0$ such that for any ball $B_r(x) \subset \Omega$, there exists a linear function $\ell(x) = a \cdot x + b$ for which*

$$\sup_{B_r(x)} |f(y) - \ell(y)| \leq Cr^{1+\alpha}. \tag{2.13}$$

Then $f \in \mathcal{C}^{1,\alpha}(\Omega)$. Moreover, there exists a constant $C_0 > 0$ so that

$$\frac{1}{C_0}[\nabla f]_\alpha \leq C \leq C_0[\nabla f]_\alpha.$$

Proof. We may first show that f is Lipschitz:

$$|f(x) - f(y)| \leq |f(x) - \ell(x)| + |f(y) - \ell(y)| + |\ell(y) - \ell(x)| \leq C|x - y|$$

setting $r = |x - y|$ and using the fact that Ω is bounded. Therefore, by Taylor's theorem, we have that

$$\ell = \ell_x(y) = f(x) + \nabla f(x) \cdot (y - x)$$

since the remainder term vanishes linearly. Now, set $|y - x| = |y - z| = r$. Then, expand f at z about x, at x about y, and at y about x. Manipulating, we find that

$$[\nabla f(x) - \nabla f(y)] \cdot (z - y) = o\big(r^{1+\alpha}\big) \tag{2.14}$$

or

$$\nabla f(x) = \nabla f(y) + o(r^\alpha). \tag{2.15}$$

This implies $\nabla f \in C^\alpha$, so $f \in \mathcal{C}^{1,\alpha}$ by Lemma 2.9. Then, Taylor expanding and applying the mean value theorem, we see that the best possible constant in (2.13) is comparable to $[\nabla f]_\alpha$. □

Remark 2.16. In fact, the condition of Lemma 2.12 is equivalent to $f \in \mathcal{C}^{1,\alpha}$ as can be seen by Taylor expansion. This result will be pivotal in establishing the Calderón-Zygmund estimates.

2.3. **Calderón-Zygmund Estimates.**
As previously stated, we will only prove the Calderón-Zygmund estimates as in Theorem 2.5. As such, we fix notation in this subsection and assume $u \in H^1(B_1)$ is a bounded weak solution to (2.6). We first recall a useful bound on weak solutions to the Dirichlet problem.

Proposition 2.17. *Let $v \in H^1(B_1)$ be a weak solution of*

$$\begin{cases} \Delta v = f & \text{in } B_1, \\ v = g & \text{on } \partial B_1. \end{cases} \tag{2.18}$$

Then, we have that

$$\|v\|_{L^\infty(B_1)} \le C(\|f\|_{L^\infty(B_1)} + \|g\|_{L^\infty(\partial B_1)}) \tag{2.19}$$

where C is an absolute constant.

Proof. See Lemma 1.14 of [2]. □

Remark 2.20. A similar result will hold on other bounded domains Ω, but the constant C will depend on $\operatorname{diam}(\Omega)$.

Keeping in mind Proposition 2.12, our goal should be to show that there exists some linear function $\ell(x) = a \cdot x + b$ so that

$$\sup_{B_r(x)} |u(y) - \ell(y)| \le Cr^{2-\varepsilon}$$

for any ball $B_r(x) \subset B_{1/2}$. We also note that since f is uniformly bounded, one can zoom in by defining $u_r(x) = u(rx)$ so that $|\Delta u_r| = r^2 |f(rx)| \le Cr^2$ meaning that u_r looks more and more like a harmonic function as $r \downarrow 0$.

Moreover, without loss of generality, we may assume that $\|u\|_\infty \le \frac{1}{2}$ and $\|f\|_\infty \le \delta$ for any fixed $\delta > 0$ by replacing u with itself divided by a factor $\|u\|_\infty + \frac{\|f\|_\infty}{\delta}$. We choose the bound of $\frac{1}{2}$ so that $\operatorname{osc}_{B_1} u \le 1$. We also define the **harmonic replacement** w of u by the solution to

$$\begin{cases} \Delta w = 0 & \text{in } B_1, \\ w = u & \text{on } \partial B_1 \end{cases} \tag{2.21}$$

and write $u = v + w$ so that v solves

$$\begin{cases} \Delta v = f & \text{in } B_1, \\ v = 0 & \text{on } \partial B_1. \end{cases} \tag{2.22}$$

Now, we state the main claim used in the proof of Theorem 2.5. Our approach here follows [5] closely.

Proposition 2.23. *Let $0 < \varepsilon < 1$. Then, there exists a $\delta > 0$ and $r_0 \in (0,1)$ so that if $osc_{B_1} u \le 1$ and $\|f\|_{L^\infty(B_1)} < \delta$, there is some $b \in \mathbb{R}^d$ such that*

$$osc_{B_{r_0}} [u - b \cdot x] \le r_0^{2-\varepsilon}. \tag{2.24}$$

Proof. We first recall that the harmonic replacement w of u is harmonic and therefore $\mathcal{C}^\infty$. In particular, $w \in \mathcal{C}^2$. Thus, we may apply Taylor's theorem to estimate the remainder

$$|w(x) - w(0) - x \cdot \nabla w(0)| \le Cr^2$$

when $x \in B_r$ and C depends on $\|D^2 w\|_\infty$. Moreover, we should check that this C is in fact absolute. That is, $\|D^2 w\|_\infty$ should depend on only $\operatorname{osc}_{B_1} u$. Defining $\tilde{u} = u - \inf_{B_1} u$, one sees that $D^2\tilde{w} = D^2 w$ and that by derivative estimates for harmonic functions, that

$$\|D^2 w\|_{L^\infty(B_{1/2})} \leq C\|\tilde{w}\|_{L^\infty(B_1)} = C\|\tilde{u}\|_{L^\infty(B_1)} = C\operatorname{osc}_{B_1} u, \tag{2.25}$$

using the maximum principle. Now, use the fact that $u = v + w$ to write

$$|u(x) - v(x) - w(0) - x \cdot \nabla w(0)| \leq Cr^2.$$

Therefore, by the triangle inequality, and using the fact that $w(0)$ is some constant, we see that

$$\operatorname{osc}_{B_r}(u(x) - x \cdot \nabla w(0)) \leq 2\big[Cr^2 + \operatorname{osc}_{B_r} v\big]. \tag{2.26}$$

Hence, it suffices to choose some r_0 so that the right-hand side of (2.26) is bounded above by $r_0^{2-\varepsilon}$.
Using Proposition 2.17 with $g \equiv 0$, we note that

$$\operatorname{osc}_{B_r} v \leq C'\|f\|_{L^\infty(B_1)} \leq C'\delta.$$

Hence,

$$2\big[Cr^2 + \operatorname{osc}_{B_r} v\big] \leq C''\big(r^2 + \delta\big).$$

Then, we may choose $\delta = r_0^2$ and $r = r_0 \in (0,1)$ so that $r_0^\varepsilon \leq \frac{1}{2C''}$ and thus,

$$C''\big(r_0^2 + \delta\big) = 2C''r_0^2 \leq r_0^{2-\varepsilon}.$$

Hence, we can conclude the claim with $b = \nabla w(0)$. □

Now, we are finally ready to prove Theorem 2.5.

Proof. (Calderón-Zygmund)
By Proposition 2.3, we have some $b_1 \in \mathbb{R}^d$ so that

$$\operatorname{osc}_{B_{r_0}}[u - b_1 \cdot x] \leq r_0^{2-\varepsilon}.$$

Now, if we let $u_1(x) = r_0^{-(2-\varepsilon)}[u(r_0 x) - b_1 \cdot x]$, we see $|\Delta u_1| = r_0^\varepsilon |f| \leq |f|$ and that $\operatorname{osc}_{B_1} u_1 \leq 1$. Thus, we may iterate and reapply Proposition 2.3. Inductively, we may define $u_k(x) = r_0^{-k(2-\varepsilon)}\big[u\big(r_0^k x\big) - b_k \cdot r^k x\big]$. Unraveling definitions, we see that having $\operatorname{osc}_{B_{r_0}}\big[u_k - b'_{k+1} \cdot x\big] \leq r_0^{2-\varepsilon}$ means that

$$\operatorname{osc}_{B_{r_0}}\Big[u\big(r_0^k x\big) - b_k \cdot r_0^k x - b'_{k+1} \cdot r_0^{k(2-\varepsilon)} x\Big] \leq r_0^{(k+1)(2-\varepsilon)}.$$

Therefore, after rescaling, we see that

$$\operatorname{osc}_{B_{r_0^{k+1}}}\Big[u(x) - \Big(b_k + r_0^{k(1-\varepsilon)} b'_{k+1}\Big) \cdot x\Big] \leq r_0^{(k+1)(2-\varepsilon)},$$

so we take the term in parentheses to be b_{k+1}. Hence, we have vectors $b_k \in \mathbb{R}^d$ such that

$$\operatorname{osc}_{B_{r_0^k}} |u(x) - b_k \cdot x| \leq r_0^{k(2-\varepsilon)} \tag{2.27}$$

for each $k \in \mathbb{N}$. Thus, for any $r \in (0,1)$ not a power of r_0, we can choose a k so that $r_0^{k+1} \leq r < r_0^k$ so that

$$\operatorname{osc}_{B_r} |u(x) - b_k \cdot x| \leq \operatorname{osc}_{B_{r_0^k}} |u(x) - b_k \cdot x| \leq r_0^{k(2-\varepsilon)} \leq Cr^{2-\varepsilon} \tag{2.28}$$

with C depending on r_0 and ε by (2.27). Therefore, by Proposition 2.12, we have that $u \in \mathcal{C}^{1,1-\varepsilon}$ at the origin. Then, a standard covering argument propagates $u \in \mathcal{C}^{1,1-\varepsilon}$ to all of $B_{\frac{1}{2}}$. $\square$

We also give a short proof of the alternate formulation of Calderón-Zygmund in the $p = 2$ case for completeness sake.

Proposition 2.29. *(Calderón-Zygmund for $p = 2$)*
Suppose $u \in H^1(B_1)$ solves

$$\Delta u = f \in L^2 \text{ in } B_1 \tag{2.30}$$

in the weak sense. Then, $u \in H^2(B_{1/2})$ with

$$\|u\|_{H^2(B_{1/2})} \leq C\Big(\|u\|_{L^2(B_1)} + \|f\|_{L^2(B_1)}\Big).$$

Proof. First, suppose that $u \in \mathcal{C}_c^\infty(B_1)$. Then, integrating by parts, we find that $\|D^2u\|_{L^2(B_1)} = \|f\|_{L^2(B_1)}$. Now, instead suppose that $f \in L^2(B_1)$ and mollify so that $\Delta u^\varepsilon = f^\varepsilon \in \mathcal{C}^\infty(B_1)$. Then, since $u \in H^1$, we see that $u^\varepsilon \to u$ in H^1 (see Thm. 6 of Appendix C.4 in [1]).

Let $\eta \in \mathcal{C}_c^\infty(B_1)$ be a fixed cutoff function with $\eta \geq 0$, $\eta \equiv 1$ in $B_{1/2}$ and $\eta \equiv 0$ in $B_1 \setminus B_{3/4}$. Then, for $v^\varepsilon = u^\varepsilon \eta \in \mathcal{C}_c^\infty(B_1)$, we have

$$\Delta v^\varepsilon = f^\varepsilon \eta + 2\nabla u^\varepsilon \cdot \nabla \eta + u^\varepsilon \Delta \eta.$$

Thus,

$$\|D^2u^\varepsilon\|_{L^2(B_{1/2})} \leq \|D^2v^\varepsilon\|_{L^2(B_1)} \leq C\Big(\|f^\varepsilon\|_{L^2(B_1)} + \|u^\varepsilon\|_{H^1(B_1)}\Big)$$

using the fact that $v^\varepsilon \in \mathcal{C}_c^\infty(B_1)$. Thus, interpolating the gradient and using the facts $f^\varepsilon \to f$ in L^2 and $u^\varepsilon \to u$ in H^1, it suffices to bound $\|D^2u\|_{L^2(B_{1/2})}$ in terms of $\|D^2u^\varepsilon\|_{L^2(B_{1/2})}$. Remarking that $\|D^2u^\varepsilon\|_{L^2(B_{1/2})}$ is uniformly bounded in ε and using Banach-Alaoglu, we may extract a weakly convergent subsequence $D^2u^\varepsilon \rightharpoonup w$ converging weakly in L^2. Then, one may integrate by parts to identify w with D^2u and use the weak lower semicontinuity of the norm to conclude the entire estimate. $\square$

3. The Obstacle Problem

3.1. Properties of the Minimizer.

We now move on to a particularly important free boundary problem known as the **obstacle problem**. As in the introduction, the variational formulation involves minimizing the functional

$$I[v] = \frac{1}{2}\int_\Omega |\nabla v|^2 \tag{3.1}$$

over the set of functions $v \geq \varphi$ with some fixed boundary condition $v = g$ on $\partial\Omega$ where φ is some smooth function and $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain. The Lagrangian here is the same as for the Laplace equation, except our set of admissible functions is restricted by φ. This key difference introduces a singular element and means the usual tools of the calculus of variations may not fully apply.

As for the name, φ is called the **obstacle** and $I[\cdot]$ can be thought of as the Dirichlet energy. As discussed in Section 1, a common physical interpretation of the problem is to minimize the elastic potential of a thin membrane that lies above some obstacle. For our purposes, we generally consider a shifted version of the obstacle problem known as the *zero-obstacle problem* by defining $u = v - \varphi$ so that

$$I[u+\varphi] = \frac{1}{2}\int_\Omega |\nabla u|^2 + |\nabla \varphi|^2 + 2\nabla u\cdot\nabla\varphi = \frac{1}{2}\int_\Omega |\nabla u|^2 + |\nabla\varphi|^2 - 2u\Delta\varphi + \int_{\partial\Omega} g\varphi_\nu.$$

Thus, ignoring the constant terms involving g and φ, we seek to minimize

$$J[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 + fu \tag{3.2}$$

with $f = -\Delta\varphi$ among all functions $u \geq 0$ with the boundary condition $u = g - \varphi$ on $\partial\Omega$. We will assume that $f \geq 0$ throughout and follow parts of [2] closely. In fact, we can reframe this constrained minimization problem as a minimization problem with a nonsmooth term.

Proposition 3.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Suppose $g\colon \partial\Omega \to \mathbb{R}$ is a suitable boundary condition so that the set of admissible functions*

$$\mathcal{A} = \{u \in H^1(\Omega),\ u \geq 0 \text{ in } \Omega,\ u = g \text{ on } \partial\Omega\}$$

is nonempty. Then, minimizing the functional in (3.2) is equivalent to minimizing

$$E[u] := \int_\Omega \frac{1}{2}|\nabla u|^2 + fu_+ \tag{3.4}$$

where $u_+ := \max\{u, 0\}$.

Proof. We only need to consider the case where $u \leq 0$ since otherwise $u_+ = u$ and the functionals are the same. We use the identity $|\nabla u|^2 = |\nabla u_+|^2 + |\nabla u_-|^2$ and the fact that $u_\pm \in H^1$ (i.e., $u_+ \in \mathcal{A}$) to argue that

$$\int_\Omega \frac{1}{2}|\nabla u_+|^2 + fu_+ \leq \int_\Omega \frac{1}{2}|\nabla u|^2 + fu_+.$$

If $u \neq u_+$ on a set of positive measure, then the inequality is strict and so, u would not be a minimizer of E as $E[u_+] < E[u]$. Therefore, we must have $u \geq 0$, and so both formulations are equivalent. □

We can derive the corresponding Euler-Lagrange equations by perturbing by test functions.

Proposition 3.5. *(Euler Lagrange Equations)*
Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $f \in \mathcal{C}^\infty(\Omega)$, and $u \in H^1(\Omega)$ be a minimizer as in (3.4) with boundary condition $u\big|_{\partial\Omega} = g$. Then, u solves

$$\Delta u = f\chi_{\{u>0\}} \tag{3.6}$$

in the weak sense.

Proof. Let $\eta \in H^1_0(\Omega)$ be a test function and $\varepsilon > 0$. Since u is a minimizer, we have

$$E[u] \leq E[u + \varepsilon\eta].$$

Then, looking at the derivative at $\varepsilon = 0$ from above, we observe that

$$0 \leq \lim_{\varepsilon\to 0^+} \frac{E[u+\varepsilon\eta] - E[u]}{\varepsilon} = \int_\Omega \nabla u\cdot\nabla\eta + \lim_{\varepsilon\to 0^+}\int_\Omega \frac{(u+\varepsilon\eta)^+ - u^+}{\varepsilon} f$$
$$= \int_\Omega \nabla u\cdot\nabla\eta + f\big(\eta\chi_{\{u>0\}} + \eta^+\chi_{\{u=0\}}\big).$$

If $\eta \geq 0$ (that is, $\eta = \eta^+$), we have that

$$\int_\Omega \nabla u \cdot \nabla \eta + f\eta \geq 0$$

for each $\eta \in H^1_0(\Omega)$. In other words, $\Delta u \leq f$ in the weak sense. If instead $\eta \leq 0$, we have that

$$\int_\Omega \nabla u \cdot \nabla \eta + f\eta\chi_{\{u>0\}} \geq 0$$

so that $\Delta u \geq f\chi_{\{u>0\}}$ in the weak sense. Remarking that f is smooth, we have that $f \in L^\infty_{\text{loc}}$. Applying Calderón-Zygmund estimates (Theorem 2.5), we find that $u \in \mathcal{C}^{1,1-\varepsilon}_{\text{loc}}$ for each $0 < \varepsilon < 1$. Moreover, since $f \in L^\infty_{\text{loc}} \subset L^2_{\text{loc}}$, we may use the alternate formulation of Calderón-Zygmund (Proposition 2.29) to conclude that $u \in H^2_{\text{loc}}$. Thus, we know that $\Delta u = 0$ almost everywhere in the set $\{u = 0\}$ (see chapter 1 of [2]). Hence, we have that $\Delta u = f\chi_{\{u>0\}}$. □

Definition 3.7. The **free boundary** is the set $\Gamma = \partial\{u > 0\} \cap \Omega$. This is precisely where the solution touches the obstacle for the first time on the interior of the contact set.

Remark 3.8. Note that we showed $u \in \mathcal{C}^1_{\text{loc}}$, so we have that the gradient vanishes continuously at Γ along with u itself.

We have deduced some properties of the minimizer to (3.4), but we have not yet shown such a minimizer exists. Fortunately for us, existence and uniqueness follow from standard techniques of the calculus of variations.

Proposition 3.9. *(Existence and Uniqueness of Minimizer)*
Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Suppose $g\colon \partial\Omega \to \mathbb{R}$ is a suitable boundary condition so that the set of admissible functions

$$\mathcal{A} = \{u \in H^1(\Omega),\ u \geq 0 \ \textit{in}\ \Omega,\ u = g \ \textit{on}\ \partial\Omega\}$$

is nonempty. Then, for a given $f \in L^2(\Omega)$, there exists a unique minimizer $u \in \mathcal{A}$ of (3.2).

Proof. See Proposition 5.1 in [2]. □

3.2. Optimal Regularity via Convexity.

We now establish the optimal regularity of u near the free boundary. We previously showed that $u \in \mathcal{C}^{1,1-\varepsilon}$ for each $0 < \varepsilon < 1$. Now, we show that in fact, $u \in \mathcal{C}^{1,1}$. Our approach relies on convexity, but there are numerous ways to prove this result. For instance, see Theorem 7.6 of [5] or Theorems 5.5 and 5.13 of [2]. We first show a key lemma.

Lemma 3.10. *Let $u : B_1 \to \mathbb{R}$ be a smooth superharmonic function, i.e., $\Delta u \leq 0$. Assume there exists $N > 0$ such that for all $x \in B_1$, there exists $b \in \mathbb{R}^d$ such that*

$$u(y) \geq u(x) + b \cdot (y - x) - N|y - x|^2 \qquad \textit{for all } y \in B_1.$$

Then, $u \in \mathcal{C}^{1,1}$ with $\|D^2u\|_\infty \leq 2(d + 1)N$.

Proof. We begin by showing that $u(\cdot) + N|\cdot|^2$ is convex. Fix $x, y \in B_1$ so that

$$u(y) + N|y - x|^2 \geq u(x) + b \cdot (y - x)$$

by the given inequality. Expanding the quadratic on the left-hand side and regrouping, we have

$$u(y) + N|y|^2 \geq u(x) + N|x|^2 + (b + 2Nx) \cdot (y - x),$$

so $v(\cdot) = u(\cdot) + N|\cdot|^2$ is convex. Thus, u is semiconvex. We note that $\Delta v = \Delta u + 2dN \leq 2dN$, so we have that v is convex and has bounded Laplacian. Then, since u is smooth, we have that $D^2 v$ is positive semidefinite. In particular, the eigenvalues (all of which are nonnegative) of $D^2 v$ sum to a nonnegative constant $\Delta v \leq 2dN$, so each eigenvalue itself is bounded $0 \leq \lambda_i \leq 2dN$. Hence, the matrix entries can be bounded:

$$|\partial_{ij} v| = |e_j \cdot D^2 v e_i| \leq 2dN$$

using Cauchy-Schwarz and the bound on the eigenvalues. Thus, we have $D^2 v \in L^\infty$ and therefore $v \in \mathcal{C}^{1,1}$. Furthermore, we have that $|y|^2 \in \mathcal{C}^{1,1}$ since it has bounded second derivative. Therefore, $u = v - N|\cdot|^2 \in \mathcal{C}^{1,1}$ as well. Moreover, $\|D^2 u\|_\infty \leq \|D^2 v\|_\infty + 2N \leq 2(d+1)N$. □

Remark 3.11. The notion of semiconvexity is closely related to the uniform ellipticity of the Laplacian. Suppose u is superharmonic and semiconvex. Then, semiconvexity gives a lower bound on the eigenvalues of the Hessian while superharmonicity gives an upper bound on their sum. Thus, the eigenvalues of $D^2 u$ are bounded. Note that the superharmonicity condition could be modified for a more general uniformly elliptic operator for which the conclusion would still hold.

Throughout the rest of this subsection, we will consider the following formulation of the obstacle problem

$$\begin{cases} \Delta u \leq 0 & \text{on } B_1 \\ u \geq \varphi & \text{on } B_1 \\ \Delta u = 0 \text{ when } u > \varphi \\ u = g > \varphi \text{ on } \partial B_1. \end{cases} \tag{3.12}$$

where $\varphi \in \mathcal{C}^{1,1}$ and g is assumed to be smooth. Note that in this formulation, since $g > \varphi$ on ∂B_1, the fixed boundary and the free boundary do not touch, so u is smooth near ∂B_1. Away from the boundary (in particular the obstacle), we have that $\|u\|_{\mathcal{C}^{1,1}(B_1 \setminus B_{1-2\delta})} < \infty$ for some $\delta > 0$ since u is smooth in this region. We now turn to a useful characterization of a solution to the obstacle problem known as the *least supersolution principle*.

Lemma 3.13. *(Least Supersolution Principle)*
Suppose u solves (3.12). Then,

$$u = \inf\{v \ : \ \Delta v \leq 0, v \geq \varphi \text{ on } B_1, v \geq g \text{ on } \partial B_1.\}$$

That is, the solution to the obstacle problem is the least supersolution above the obstacle and boundary data.

Proof. Suppose w also satisfies (3.12). Then, $u - w$ is superharmonic on the set $B_1 \cap \{u > \varphi\}$ and thus obeys a minimum principle. Since $w - u \geq 0$ on $\partial\{B_1 \cap \{u > \varphi\}\}$ (as $u = \varphi$ there), we have $w \geq u$ on the entire set. □

Now, take $N > \max\{\|u\|_{\mathcal{C}^{1,1}(B_1 \setminus B_{1-2\delta})}, \|\varphi\|_{\mathcal{C}^{1,1}(B_1)}\}$. We establish the following proposition in the hope of obtaining the hypothesis of Lemma 3.10.

Proposition 3.14. *For $|h| < \delta$, set*

$$v_h(x) \coloneqq \frac{u(x+h)+u(x-h)}{2} + N|h|^2.$$

Then, $v_h \geq \varphi$ in $B_{1-\delta}$ and $v_h \geq u$ on $\partial B_{1-\delta}$. Thus, v_h is superharmonic and so, $v_h \geq u$ on $B_{1-\delta}$. Therefore, we have that $\partial_{ee} u \geq -2N$ in $B_{1-\delta}$ for any unit vector e (i.e., u is locally semi-convex).

Proof. To establish the first two estimates, we Taylor expand at x. On $B_{1-\delta}$, we have

$$\varphi(x \pm h) \geq \varphi(x) \pm \nabla\varphi(x) \cdot h - \|\varphi\|_{\mathcal{C}^{1,1}(B_1)}|h|^2$$

after Taylor expanding the φ terms and estimating the remainder. Thus, we may conclude

$$v_h(x) \geq \frac{\varphi(x+h)+\varphi(x-h)}{2} + \|\varphi\|_{\mathcal{C}^{1,1}(B_1)}|h|^2 \geq \varphi(x)$$

on $B_{1-\delta}$. Similarly, on $\partial B_{1-\delta}$, one can show that $v_h \geq u$ via Taylor expansion. To show that v_h is superharmonic, we compute the Laplacian as

$$\Delta v_h(x) = \frac{\Delta u(x+h) + \Delta u(x-h)}{2} \leq 0$$

on $B_{1-\delta}$. Applying the least supersolution principle (Lemma 3.13), we obtain that $v_h \geq u$ on $B_{1-\delta}$. Finally, we have that by definition,

$$\partial_{ee} u(x) = \lim_{h\to 0} \frac{u(x+he)+u(x-he)-2u(x)}{|h|^2}.$$

Using the fact that $v_h \geq u$ on $B_{1-\delta}$, we see that

$$u(x+h)+u(x-h) \geq 2u(x) - 2N|h|^2$$

so

$$\partial_{ee} u(x) \geq \lim_{h\to 0} \frac{-2N|h|^2}{|h|^2} = -2N,$$

which is the claim. □

Now, we can finally show the optimal regularity result.

Theorem 3.15. *(Optimal Regularity)*
Suppose u satisfies (3.12). Then, $u \in \mathcal{C}^{1,1}(B_1)$ with

$$\|u\|_{\mathcal{C}^{1,1}(B_{1/2})} \lesssim \|u\|_{L^\infty(B_1)} + \|\varphi\|_{\mathcal{C}^{1,1}(B_1)}.$$

Proof. Taylor expanding and using the local semiconvexity shown in Proposition 3.14, we have that

$$u(y) \leq u(x) + \nabla u(x) \cdot (y-x) - N|y-x|^2$$

for $x, y \in B_{1-\delta}$. Thus, by Lemma 3.10, we have that $u \in \mathcal{C}^{1,1}(B_{1-\delta})$. Moreover, we already know that $u \in \mathcal{C}^{1,1}(B_1 \setminus B_{1-2\delta})$, so combining these two results gives the claim $u \in \mathcal{C}^{1,1}(B_1)$. The estimate on $\|u\|_{\mathcal{C}^{1,1}(B_{1/2})}$ also follows from Lemma 3.10. □

Remark 3.16. We know that $u \notin \mathcal{C}^2$ since Δu is discontinuous across the free boundary, so $u \in \mathcal{C}^{1,1}$ is the optimal regularity result.

Remark 3.17. In the zero-obstacle case (as in (3.6)), we obtain a similar estimate to Theorem 3.15. According to Theorem 5.13 of [2], we can write

$$\|u\|_{\mathcal{C}^{1,1}(B_{1/2})} \lesssim \|u\|_{L^\infty(B_1)}$$

when $f \equiv 1$.

We now give an example of a concrete solution to the obstacle problem in 2 dimensions.

Example 3.18. We are interested in computing the solution u to a radial obstacle problem with obstacle $\varphi(r) = 1 - r^2$ on $B_1 \subset \mathbb{R}^2$. We fix boundary data $u\big|_{\partial B_1} = c$ for some constant $0 \le c < 1$. By symmetry, the solution must be radial. In particular, the solution u must agree with φ for $0 \le r \le r_0$ up to a critical value. This critical value is characterized by the conditions $u(r_0) = \varphi(r_0)$ and $\nabla u(r_0) = \nabla\varphi(r_0)$. Moreover, we have the condition $\Delta u(r) = 0$ for $r > r_0$ and $u(1) = c$.

Therefore, by examining the harmonicity condition, we find that $u_{rr} + \frac{u_r}{r} = \frac{1}{r}\partial_r(r\partial_r u) = 0$ for $r > r_0$. This implies that

$$u(r) = A \ln r + B$$

for $r > r_0$. In particular, checking the boundary condition, we have

$$u(r) = A \ln r + c$$

for $r > r_0$. Then, checking continuity tells us that

$$u(r_0) = 1 - r_0^2 = A \ln r_0 + c$$

while checking the continuity of the gradient tells us that

$$\frac{A}{r_0} = -2r_0.$$

Thus, we find that

$$1 - r_0^2 = -2r_0^2 \ln r_0 + c.$$

One can numerically solve for the critical radius r_0 in terms of c. Hence, we obtain a solution of the form

$$\begin{cases} u(r) = \varphi(r) = 1 - r^2 & \text{for } 0 \le r \le r_0 \\ u(r) = -2r_0^2 \ln r + c & \text{for } r_0 \le r < 1. \end{cases}$$

See Figure 1 for a plot (from Mathematica) of the solution in the case $c = \frac{1}{2}$.

3.3. Nondegeneracy.

We now investigate how the solution behaves near the free boundary Γ. The goal of this subsection is to establish the *nondegeneracy* of the solution and its gradient near Γ. In this context, we mean that the solution grows quadratically away from Γ and ∇u grows linearly near Γ. In contrast with the previous subsection, we reduce to the case of the zero obstacle problem (i.e., as in (3.6)). We begin with a variant of the Harnack inequality.

Lemma 3.19. *Let $u\colon B_1 \to \mathbb{R}$ be a nonnegative solution to $\Delta u = f \in L^\infty(B_1)$. Assume u is continuous on ∂B_1, so that we can uniquely decompose $u = v + w$ where*

$$\begin{cases} \Delta v = 0 & \textit{in } B_1, \\ v = u & \textit{on } \partial B_1 \end{cases}$$

and

$$\begin{cases} \Delta w = f & \text{in } B_1, \\ w = 0 & \text{on } \partial B_1 \end{cases}.$$

Then, $\max_{B_1} |w| \le \dfrac{\|f\|_\infty}{2d}$*. Also, if* $f \le 0$*, then* $0 \le v, w \le u$*.*

Proof. Consider the auxiliary function

$$\tilde{w} = w + \frac{|x|^2}{2d}\|f\|_\infty$$

so that $\Delta \tilde{w} \ge 0$. Then, $\tilde{w}$ achieves its maximum on the boundary. Therefore, we have the estimate

$$\max_{B_1} |w| \le \frac{\|f\|_\infty}{2d}$$

since $w = 0$ on ∂B_1. Now, suppose $f \le 0$. Since v is harmonic, it achieves its minimum on the boundary, so $v \ge 0$ since $u \ge 0$. Therefore, we find that $w \le u$. Finally, we use the fact that w is superharmonic to conclude that $w \ge 0$ (by the minimum principle) so that $v \le u$. In sum, we have $0 \le v, w \le u$. □

Proposition 3.20. *(Harnack with right-hand side)*

Let $u\colon B_r \to \mathbb{R}$ *be a nonnegative function satisfying* $\Delta u = f$ *in* B_{4r}*. Then,*

$$\sup_{B_r} u \le C\left(\inf_{B_r} u + r^2\|f\|_\infty\right)$$

for some $C = C(d)$*.*

Proof. We use the decomposition $u = v + w$ prescribed in Lemma 3.19. Since v is harmonic, it obeys the ordinary Harnack inequality (Sec. 2.2 Thm. 11 in [1]):

$$\sup_{B_r} v \le C \inf_{B_r} v.$$

Then, we have

$$\sup_{B_r} u \le \sup_{B_r} v + \sup_{B_r} w \le C \inf_{B_r} v + \frac{\|f\|r^2}{2d} = C'\left(\inf_{B_r} u + r^2\|f\|_\infty\right)$$

using the minimum principle for the final equality and a scaled version of the bound on w from Lemma 3.19. □

We are now ready to establish the nondegeneracy of u.

Proposition 3.21. *(Nondegeneracy)*

Let u *be a solution to*

$$\begin{cases} \Delta u = f\chi_{\{u>0\}} \text{ in } B_1, \\ u \ge 0 \text{ in } B_1. \end{cases}$$

Suppose f *is bounded away from zero. That is,* $f \ge c > 0$*. Then, for free boundary points* $x_0 \in \partial\{u > 0\} \cap B_{\frac{1}{2}}$*, we have*

$$0 < cr^2 \le \sup_{B_r(x_0)} u < Cr^2$$

for each $r \in \left(0, \frac{1}{2}\right)$*.*

Proof. The upper bound is immediate from Proposition 3.20 since

$$\sup_{B_r(x_0)} u \leq C\left(\inf_{B_r(x_0)} u + r^2\|f\chi_{\{u>0\}}\|_\infty\right) = C'r^2$$

as $u(x_0) = 0$ implies the first term vanishes. As for the lower bound, let $x_1 \in \{u > 0\}$. Then, consider the auxiliary function

$$v(x) = u(x) - \frac{c}{2d}|x - x_1|^2$$

so that $\Delta v \leq 0$ in $\{u > 0\} \cap B_r(x_1)$. Note that this set is open since we previously showed u is continuous in Proposition 3.5. Also, recognize that $v(x_1) > 0$. We may apply the maximum principle to conclude that v attains a positive maximum on $\partial\{\{u > 0\} \cap B_r(x_1)\}$. However, on the free boundary $\partial\{u > 0\}$, we have that $u = 0$, so $v < 0$. Thus, we have

$$0 < \sup_{\partial B_r(x_1)} v = \sup_{\partial B_r(x_1)} u(x) - \frac{c}{2d}r^2$$

or

$$c'r^2 < \sup_{\partial B_r(x_1)} u.$$

Letting $x_1 \to x_0$, we obtain the lower bound and thus conclude the proof. □

We may now show the nondegeneracy of the gradient using Proposition 3.21.

Lemma 3.22. *(Nondegeneracy of Gradient)*
Let u solve

$$\begin{cases} \Delta u = \chi_{\{u>0\}} \text{ in } B_1, \\ u \geq 0 \text{ in } B_1. \end{cases}$$

Then, the gradient is nondegenerate. That is, for any point $x_0 \in \Gamma$,

$$0 < c_0 r \leq \sup_{B_r(x_0)} |\nabla u| \leq c_1 r$$

where $c_0, c_1 > 0$ are absolute constants.

Proof. We establish the lower bound by appealing to the nondegeneracy of u (Proposition 3.21). We have

$$cr^2 \leq \sup_{B_r(x_0)} u \leq Cr^2.$$

Choose some point $\xi \in B_r(x_0)$ so that $u(\xi) \geq cr^2$. Note that since $x_0 \in \Gamma$, we know that $u(x_0) = 0$. Then, by the mean value theorem, we have

$$cr \leq \frac{|u(\xi) - u(x_0)|}{|\xi - x_0|} \leq \|\nabla u\|_{L^\infty(B_r(x_0))}.$$

The upper bound follows from the fact that ∇u is Lipschitz and that ∇u vanishes at $x_0 \in \Gamma$. In more detail, for any $z \in B_r(x_0)$, we have

$$|\nabla u(z)| = |\nabla u(z) - \nabla u(x_0)| \leq \|\nabla u\|_{\mathcal{C}^{0,1}} r.$$

Thus, the gradient is nondegenerate. □

3.4. **Rectifiability of the Free Boundary.**

Convention 3.1. In this section, we use $|\cdot|$ to denote Lebesgue measure.

The goal of this subsection is to establish the $\mathcal{H}^{n-1}$ rectifiability of the free boundary Γ in the sense that Γ has finite $\mathcal{H}^{n-1}$ measure. For the remainder of the paper, we assume that u solves

$$\begin{cases} \Delta u = \chi_{\{u>0\}} \text{ in } B_1 \\ u \geq 0 \text{ in } B_1. \end{cases} \tag{3.23}$$

disregarding boundary values. We are taking $f \equiv 1$ in (3.6) for simplicity's sake. We also continue to assume that $0 \in \Gamma$.

We now introduce a useful definition and investigate how it relates to Γ.

Definition 3.24. A set E is said to be **δ-porous** for some $\delta \in (0,1)$ if for every $x \in E$ and $B = B_r(x)$, there exists $B_{\delta r}(y) \subset B_r(x) \setminus E$.

Proposition 3.25. *(Porosity)*
Any measurable δ-porous set in $\mathbb{R}^n$ has $\mathcal{H}^n$-measure zero. Moreover, the free boundary Γ is locally porous, i.e., for each compact subset [1] *$K \subset\subset B_1$, there exists $\delta \in (0,1)$ such that $K \cap \Gamma$ is δ-porous.*

Proof. Let $E \subset \mathbb{R}^n$ be δ-porous. Then, we have that

$$\lim_{r\to 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} \leq 1 - \delta^n < 1. \tag{3.26}$$

Since E is measurable, the Lebesgue density theorem tells us the density (the limit in (3.26)) at points in E must be 1 almost everywhere. Therefore, E must have $\mathcal{H}^n$ measure zero.

For the obstacle problem, we wish to show that Γ is locally porous. Let $K \subset\subset B_1$ be compact. Fix an $x_0 \in K \cap \Gamma$ and consider some ball $B_r(x_0)$. Then, by the nondegeneracy of the gradient, we can choose some point $y \in B_{r/2}(x_0)$ with $|\nabla u(y)| = cr$. Then, since the gradient is Lipschitz, we may choose some δ so that for any $z \in B_{\delta r}(y)$, we have that $|\nabla u(z)| > 0$ meaning $z \notin \Gamma$ since the gradient vanishes there. Moreover, we choose $\delta < \frac{1}{2}$ so that $B_{\delta r}(y) \subset B_r(x_0)$ implying that $K \cap \Gamma$ is δ-porous. □

Remark 3.27. A useful consequence of Proposition 3.25 is that after taking a countable union, we see that Γ also has $\mathcal{H}^n$ measure zero.

Proposition 3.28. *Let u solve (3.23). Then, $|\nabla u(x_0)| \leq Cu(x_0)^{1/2}$ for all $x_0 \in B_{1/2}$.*

Proof. Let $u(x_0) = a > 0$ (otherwise the inequality is trivially satisfied as both sides vanish). Then, by nondegeneracy of u (Proposition 3.21), we have that $a \leq C\text{dist}(x_0, \Gamma)^2$, so we have some ball $B_{M\sqrt{a}}(x_0) \subset \{u > 0\}$. In particular, choose the factor M appropriately so that $B_{4M\sqrt{a}}(x_0) \subset \{u > 0\}$ as well. Then, consider the auxiliary function

$$w = u + \frac{M^2 a - |x - x_0|^2}{2d}.$$

[1]We use the notation $A \subset\subset B$ to mean that A is compactly embedded into B. This means that the inclusion map is a compact operator (i.e., any bounded sequence in A has a convergent subsequence in B).

Then, we have that $w \geq 0$ and $\Delta w = 0$ (since $\Delta u = 1$) on $B_{M\sqrt{a}}(x_0)$. We may now use derivative estimates (Sec. 2.2 Thm. 7 of [1]) to say that

$$|\nabla u(x_0)| = |\nabla w(x_0)| \leq \frac{C}{M\sqrt{a}} \sup_{B_{M\sqrt{a}}(x_0)} |w|$$

where the first equality comes from the fact that the gradient of the quadratic term vanishes at x_0. Since $w \geq 0$ and w is harmonic, we know that $w = |w|$ attains its maximum on the boundary. In particular, it suffices to bound just u. We then see that

$$\sup_{B_{M\sqrt{a}}(x_0)} |w| = \sup_{\partial B_{M\sqrt{a}}(x_0)} |u| \leq C\big(a + M^2 a\big) = C'a$$

via Proposition 3.20.

Thus, we have

$$|\nabla u(x_0)| \leq \frac{C}{M\sqrt{a}} C'a = C''\sqrt{a} = C'' u(x_0)^{\frac{1}{2}}$$

concluding the proof. □

Now, we show two key lemmas and then proceed with the $\mathcal{H}^{n-1}$ rectifiability result.

Lemma 3.29. *Set $u_e = \partial_e u$ for some direction $e \in \mathbb{S}^{n-1}$. Then*

$$\int_{B_r \cap \{0 \leq u_e \leq h\}} |\nabla u_e|^2 \leq Chr^{n-1}.$$

Proof. We first remark that it suffices to prove the $r = 1$ case as the rescaling $u_r(x) = \frac{u(rx)}{r^2}$ also solves the obstacle problem. Now, to prove the $r = 1$ case, fix a direction $e \in \mathbb{S}^{n-1}$ and set $\tilde{u}_e = \min\{(u_e - \varepsilon)_+, h\}$, a shifted truncation of u_e above at h and below at $\varepsilon > 0$. Then, by Green's identity,

$$\int_{B_1} \nabla \tilde{u}_e \cdot \nabla u_e + \tilde{u}_e \Delta u_e = \int_{\partial B_1} \tilde{u}_e \partial_\nu u_e.$$

We note that $\Delta u_e = 0$ where $u_e > \varepsilon$ because the gradient of u vanishes at the free boundary and, in the region $\{u > 0\}$, the Laplacian of u is constant. Then, by the definition of $\tilde{u}_e$, we have that $\tilde{u}_e = 0$ when $u_e \leq \varepsilon$. Thus, the second term disappears entirely. Therefore, we have

$$\int_{B_1} \nabla \tilde{u}_e \cdot \nabla u_e = \int_{\partial B_1} \tilde{u}_e \partial_\nu u_e \leq Ch$$

where the constant depends on $\|u\|_{\mathcal{C}^{1,1}}$. Now, taking the limit $\varepsilon \to 0$ yields:

$$\int_{B_1 \cap \{0 \leq u_e \leq h\}} |\nabla u_e|^2 \leq Ch.$$

Note that the interchange of the limit and integral is justified by the dominated convergence theorem since $u \in \mathcal{C}^{1,1}$. This proves the lemma. □

We now use Lemma 3.29 to prove another lemma.

Lemma 3.30. *Let $S_h = \{0 < u < h^2\}$. Then*

$$|S_h \cap B_r| \leq Chr^{n-1}.$$

Proof. We may similarly remark that it suffices to prove the $r = 1$ case. Noting that $u_r(x) \coloneqq \frac{u(rx)}{r^2}$ solves the obstacle problem, we have

$$|S_h \cap B_r| = r^n |\{x \in B_1 : 0 < u_r(x) < (h/r)^2\}| \leq Chr^{n-1}$$

assuming the $r = 1$ case, so it suffices to show just that. Moreover, we also note that

$$S_h \subset \{|\nabla u| < ch\} \subset \bigcap \{u_{\pm e_n} < ch\}$$

for some absolute $c > 0$ using Proposition 3.28 for the first inclusion and the definition of ∇u for the second. Now, we estimate $|S_h \cap B_1|$ noting that $\Delta u = 1$ there as

$$|S_h \cap B_1| = \int_{S_h \cap B_1} \Delta u \leq c \int_{S_h \cap B_1} |D^2 u| \leq c|S_h \cap B_1|^{1/2} \left[\int_{S_h \cap B_1} |D^2 u|^2 \right]^{1/2}$$

using Cauchy-Schwarz. Dividing over the $|S_h \cap B_1|^{1/2}$ term and squaring both sides, it suffices to bound the term in the brackets. We have

$$\int_{S_h \cap B_1} |D^2 u|^2 \leq \int_{B_1 \cap \bigcap \{u_{\pm e_n} < ch\}} |D^2 u|^2 \leq \sum_{e = e_k} \int_{B_1 \cap \bigcap \{u_{\pm e_n} < ch\}} |\nabla \partial_e u|^2 \leq Ch$$

using the $r = 1$ case of Lemma 3.29. □

Remark 3.31. From the proof of porosity (Proposition 3.25), we have that there exists a δ so that $B_{\delta r}(y) \subset B_r(x_0) \setminus \Gamma$ for some $y \in \Omega = \{u > 0\}$. Thus,

$$|B_r(x_0) \cap \Omega| \geq \delta^n |B_r| = c|B_r|.$$

for small enough $r > 0$ and each $x_0 \in \Gamma$.

We are now ready to prove the final $\mathcal{H}^{n-1}$ rectifiability result. We use a Besicovitch covering theorem argument.

Theorem 3.32. *($\mathcal{H}^{n-1}$ Rectifiability of Γ)*
For every compact $K \subset\subset B_1$,

$$\mathcal{H}^{n-1}(\Gamma \cap K) < \infty.$$

Proof. For our purposes, we may as well just assume $\Gamma \subset\subset B_1$ and disregard K. This can be done when $u > 0$ and is continuous on the boundary.

Begin by setting $E_\varepsilon \coloneqq \{|\nabla u| \leq \varepsilon\}$. We have that $E_\varepsilon \subset S_{M\varepsilon}$ with M depending on $\|u\|_{\mathcal{C}^{1,1}}$. Thus, by Lemma 3.30, we see that

$$|B_{1/2} \cap E_\varepsilon| \leq |B_{1/2} \cap S_{M\varepsilon}| \leq C\varepsilon M.$$

Then, if B is a ball of radius ε with center in $\Gamma \cap B_{1/2}$, we see that $B \cap \Omega \subset E_{M\varepsilon}$ by nondegeneracy of the gradient (Lemma 3.22) with M depending on $\|u\|_{\mathcal{C}^{1,1}}$. Moreover, we note that Remark 3.31 implies

$$|B| \leq \frac{1}{\mu} |B \cap \Omega|.$$

Now, let $\{B_i\}$ be any disjoint covering of Γ consisting of balls of radius ε centered on $\Gamma \cap B_{1/2}$. We also impose the condition that at most N balls overlap. Then,

$$\sum |B_i| \leq \frac{1}{\mu} \sum |B_i \cap \Omega| \leq \frac{1}{\mu} |B_i \cap E_{M\varepsilon}| \leq \frac{N}{\mu} |B_{\frac{1}{2}} \cap E_{M\varepsilon}| \lesssim \varepsilon$$

combining our observations from the previous paragraph. Then, for a general covering of Γ, we can reduce to the above case by the Besicovitch Covering Theorem. Hence, after taking $\varepsilon \to 0$, we have the $\mathcal{H}^{n-1}$ rectifiability of Γ.

□

3.5. **Classification of Blowups.**

As we have already established some geometric properties of Γ and the optimal regularity of the solution u, we now turn our attention to the regularity of the free boundary. The most famous such result (due to Caffarelli as in [6]) states that Γ is $\mathcal{C}^\infty$ near regular points.

In general, we split Γ into *singular points* and *regular points*. By singular, we mean points where the set $\{u = 0\}$ has zero density in the Lebesgue sense. In fact, it can be shown that the set of singular points lies in a $(d-1)$-dimensional $\mathcal{C}^1$ manifold. The classic "dumbbell" picture (taken from [2]) illustrating the difference between regular and singular points is included below.

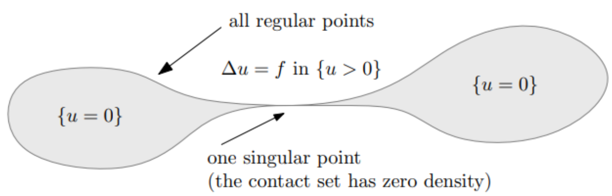

FIGURE 2. Dumbbell Diagram of Regular and Singular Points of Γ ([2])

Along the thin line, we see that $u > 0$ to both the left and right of the singular point. Then, zooming in, by nondegeneracy and quadratic growth, one might expect the formula for the blow-up given in Theorem 3.42. Similarly, at regular points along the handles, we see that u grows quadratically on one side while $u \equiv 0$ on the other. This too resembles the formula in condition (i.) of the blow-up in Theorem 3.42.

To examine the free boundary in more detail, we zoom in using *blow-ups*. For a solution u to

$$\begin{cases} u \in \mathcal{C}^{1,1}(B_1), \\ u \geq 0 \text{ in } B_1, \\ \Delta u = 1 \text{ in } \{u > 0\}, 0 \in \Gamma, \end{cases} \tag{3.33}$$

and a free boundary point $x_0 \in \Gamma$, we look at $u_r(x) = \frac{u(x_0+rx)}{r^2}$ which has L^∞ norm of the order 1 by nondegeneracy. Taking a sequence $r_k \to 0$ of these rescalings allows us to extract a subsequence converging in the $\mathcal{C}^1_{\text{loc}}$ norm. This is because u_{r_k} is bounded in the $L^\infty(B_1)$ norm by quadratic growth coupled with the fact that $\|D^2 u_r\|_{L^\infty(B_{1/2r})} = \|D^2 u\|_{L^\infty(B_{1/2})} \leq C$ by $\mathcal{C}^{1,1}$ regularity. Thus, the existence of such a sequence $u_{r_k} \to u_0$ converging in the $\mathcal{C}^1_{\text{loc}}$ norm is guaranteed by the compact embedding $\mathcal{C}^{1,1} \subset\subset \mathcal{C}^1$. It is worth noting that a priori, blow-ups are not unique.

In particular, the domain of the u_{r_k}'s expands to $\mathbb{R}^n$ as $r_k \to 0$. Therefore, the blow-ups are actually global solutions to the obstacle problem. Moreover, the nondegeneracy of u (Proposition 3.21) tells us that free boundary points remain free boundary points after blowing up. In more detail, any neighborhood of a free boundary point will still contain points z with $u_0(z) > 0$ after blowing up by nondegeneracy and uniform convergence.

To establish regularity results for Γ, the first key step is to classify blow-ups at both regular and singular points. This classification affords us insight into the behavior of Γ near regular points which we will use to deduce that the free boundary is Lipschitz. This fulfills the key hypothesis of Boundary-Harnack (see Appendix A) which gives us $\mathcal{C}^{1,\alpha}$ more or less immediately. One may then use a higher-order version of Boundary-Harnack allowing us to bootstrap to Caffarelli's celebrated result: Γ is $\mathcal{C}^\infty$ near regular points.

Example 3.34. We classify blowups in 1 dimension by hand to give some intuition for the formulas in Theorem 3.42. Let u solve (3.33) and $u_0(x)$ be a blowup of u at zero along some subsequence r_k. Since $0 \in \Gamma$, there must exist a sequence of points $x_k \to 0$ with $u_0(x_k) > 0$. After possibly taking a subsequence, suppose the x_k's approach from the right. Note that the equation for u is also satisfied by u_0. Then, using the fact that $u_0''(x_k) = 1$ and the continuity of u_0, u_0', we see that $u_0(x) = \frac{1}{2}x^2$ for $x > 0$. Similarly, if such a sequence exists from the left, we have $u_0(x) = \frac{1}{2}x^2$ for $x < 0$. Hence, if sequences exist from both sides, then, $u_0(x) = \frac{1}{2}x^2$ for all $x \in \mathbb{R}$ while if a sequence only exists from one side, we have $u_0 \equiv 0$ on the other side by continuity. Note that if sequences exist from both the left and right, then 0 is a singular point and a regular point otherwise. Observe that in Figure 3, the density of the contact set at the origin behaves as expected. Namely, in Figure 3a, the contact set has density $\frac{1}{2}$ while in Figure 3b, the contact set has zero density.

(A) Origin is a regular point: $u_0(x) = \frac{1}{2}x_+^2$ (B) Origin is a singular point: $u_0(x) = \frac{1}{2}x^2$

FIGURE 3. 1-D Classification of Blow-Ups

We begin by deducing some important properties of blow-ups in hopes of classifying them. First, we show the homogeneity of blow-ups. To do so, we state a monotonicity formula (a common theme across free boundary problems).

Proposition 3.35. *(Weiss' Monotonicity Formula)*
Let u solve (3.33) Then, the quantity

$$W_u(r) := r^{-(n+2)} \int_{B_r} \frac{1}{2}|\nabla u|^2 + u - r^{-(n+3)} \int_{\partial B_r} u^2 \tag{3.36}$$

is monotone increasing in r. In other words,

$$\frac{\mathrm{d}W_u(r)}{\mathrm{d}r} = r^{-(n+4)} \int_{\partial B_r} (x \cdot \nabla u - 2u)^2 \geq 0 \tag{3.37}$$

for $r \in (0,1)$.

Proof. The proof is by direct computation. First, note that $W_u(r) = W_{u_r}(1)$. Then, one may integrate by parts, use the equation, and note the identity $\frac{\mathrm{d}}{\mathrm{d}r}u_r = \frac{1}{r}(x \cdot \nabla u_r - 2u_r)$ to conclude the result. For the details of the computation, refer to Theorem 5.18 of [2]. □

Lemma 3.38. *(Homogeneity of Blowups)*
If u solves (3.33), then any blow-up of u at zero is homogeneous of degree 2. That is, $u_0(\lambda x) = \lambda^2 u_0(x)$.

Proof. This proof highlights the utility of the monotonicity formula (Proposition 3.35). One can verify that $x \cdot \nabla u_0 \equiv 2u_0$ implies 2-homogeneity of u, so it suffices to show that $W_{u_0}(r)$ is constant in r. Define the rescaling $u_r(x) = \frac{u(rx)}{r^2}$ and consider a blow-up $u_{r_k} \to u_0$ converging in the $\mathcal{C}^1_{\text{loc}}$ norm along some sequence $r_k \downarrow 0$. Noting the identity $W_{u_{r_k}}(r) = W_u(r_k r)$, we may write

$$W_{u_0}(r) = \lim_{r_k \to 0} W_{u_{r_k}}(r) = \lim_{r_k \to 0} W_u(r_k r) = W_u\big(0^+\big).$$

We remark that the limit must exist by monotonicity of W_u and using the fact that W_u is bounded below. Thus, we have that u_0 is 2-homogeneous.

□

We now show the convexity of global solutions. In particular, at free boundary points, u is touching down at zero, so we should expect it to be more and more convex after zooming in.

Proposition 3.39. *(Convexity of Global Solutions)*
Let u be a global solution of $\Delta u = \chi_{\{u>0\}}$, $u \geq 0$. Then, u is convex.

Proof. Suppose $m = \inf_{\{u>0\}} \partial_{ee} u < 0$. Without loss of generality, suppose $e = e_n$. We may now extract a subsequence x_j attaining the infimum, and set $u_j = \frac{u(d_j x + x_j)}{d_j^2}$ where $d_j = \operatorname{dist}(x_j, \partial\{u > 0\})$. Then, we observe that

$$D^2 u_j = D^2 u(d_j x + x_j),$$

but we know $u \in \mathcal{C}^{1,1}$, so we have that $D^2 u(d_j x + x_j)$ is bounded. Hence, u_j is bounded in $\mathcal{C}^{1,1}(B_R)$ for each $R > 0$. By the compact embedding $\mathcal{C}^{1,1} \subset\subset \mathcal{C}^{1,\alpha}$, we may extract a $\mathcal{C}^{1,\alpha}_{\text{loc}}$ subsequence converging to some u_0. By the definition of u_j and d_j, we know that $u_j > 0$ on B_1. We may assume convergence in $\mathcal{C}^2_{\text{loc}}(B_1)$ of $u_j \to u_0$ because we have $\Delta u_j = 1$ since $u_j > 0$ and thus $u_j \in \mathcal{C}^{2,\alpha} \subset\subset \mathcal{C}^2$ by the Schauder estimates. Hence, we may assume convergence of a subsequence in this norm as well.

Next, the $\mathcal{C}^2_{\text{loc}}(B_1)$ convergence coupled with the fact that $\Delta u_j = 1$ for each j tells us that $\Delta u_0 = 1$ as well. We also have that

$$\partial_{nn} u_j = \partial_{nn} u(d_j x + x_j) \geq m$$

since m is the infimum and taking limits, we obtain the same for u_0. Similarly,

$$\partial_{nn} u_0(0) = \lim_j \partial_{nn} u_j(0) = \lim_j \partial_{nn} u(x_j) = m.$$

Therefore, we use the fact that $\partial_{nn} u$ is harmonic and the maximum principle to conclude that $\partial_{nn} u \equiv m$.

Let $x = (x', x_n)$. Starting at $(x', 0) \in B_1$, move in the direction e_n until the path crosses the free boundary at, say, $\xi(x') = x_n$. Integrating in the nth coordinate, we find that $\partial_n u_0(x) = g(x') + mx_n$ where g is some arbitrary function of the other coordinates (i.e., the constant of integration). At the free boundary, the gradient vanishes, so we have that

$$0 = \partial_n u_0(x) = g(x') + mx_n = g(x') + m\xi(x')$$

so that $\xi(x') = \frac{-g(x')}{m}$. Thus, we have

$$\partial_n u_0(x) = m(x_n - \xi(x'))$$

or after integrating with respect to x_n, we have

$$u_0(x) = \frac{m}{2}(x_n - \xi(x'))^2.$$

Note that the boundary condition at Γ assures us that the constant of integration vanishes. However, since $m < 0$, we see that $u_0 \leq 0$ implying $u \equiv 0$, a clear contradiction. This implies the claim. □

Remark 3.40. Since blow-ups are global solutions, we have that blow-ups are also convex.

It is now possible to fully classify blow-ups leveraging convexity, homogeneity, and (3.33). We first establish a lemma used in the classification.

Lemma 3.41. *Let $u \in \mathcal{C}^1(\mathbb{R}^n)$ and suppose every derivative $\partial_e u$ has a sign. Then there exists a monotone function $\varphi \in \mathcal{C}^1(\mathbb{R})$ and a direction $e \in \mathbb{S}^{n-1}$ such that $u(x) = \varphi(x \cdot e)$.*

Proof. We may assume that u is not constant. Then, there exists a point $z \in \mathbb{R}^n$ so that $\nabla u(z) \neq 0$. Now, define $e_0 = \frac{\nabla u(z)}{|\nabla u(z)|} \in \mathbb{S}^{n-1}$. We note that $e \cdot e_0 > 0$ is equivalent to $\partial_e u(z) > 0$ by definition and similarly, $e \cdot e_0 < 0$ is equivalent to $\partial_e u(z) < 0$. Since each derivative maintains a sign, we have that $e \cdot e_0$ implies $\partial_e u \equiv 0$. Thus, $u(x)$ only depends on the projection of x onto the line spanned by e_0 (i.e., on $x \cdot e_0$), so we can write $u(x) = \varphi(x \cdot e_0)$ with φ monotone since derivatives maintain sign.

□

We are now ready to classify blow-ups. We divide the proof into two cases depending on the type of free boundary point we are looking at. In particular, the case for regular points uses Boundary-Harnack which was mentioned previously.

Theorem 3.42. *(Classification of Blow-ups)*

Suppose u solves (3.33) and u_0 is a blowup of u at the origin. Then, there are exactly two possibilities:

(i.) $u_0(x) = \frac{1}{2}(x \cdot e)_+^2$ *for some* $e \in \mathbb{S}^{n-1}$

(ii.) $u_0(x) = \frac{1}{2}x \cdot Ax$ *for some matrix* $A \geq 0$ *(in the sense of symmetric matrices) with unit trace.*

These two cases correspond to when $\Lambda = \{u_0 = 0\}$ has nonempty and empty interior, respectively.

Proof. We split the proof into two cases based on whether Λ has nonempty interior. We primarily focus on the nonempty interior case because we are more interested in regular points.

Suppose that Λ has nonempty interior. We then remark that Λ is a closed cone by continuity and homogeneity of u_0. Moreover, it is convex since $\Lambda = \{u_0 = 0\} =$

$\{u_0 \geq 0\}$ and u_0 is convex (Proposition 3.39). Now, pick a direction $-e \in \Lambda^o$. Then, for any $x \in \mathbb{R}^n$, we eventually have $x + se \in \Lambda^o$ for $s \ll 0$. Note that $\partial_e u_0(x + se) = 0$ for such s and of course, this is increasing in s by convexity, so $\partial_e u_0 \geq 0$ on all of $\mathbb{R}^n$.

Now, set $S = \{v \ : \ \partial_v u_0 = 0\}$. Notice we don't require v to be a unit vector so that S is a linear subspace of $\mathbb{R}^n$. We claim that S has codimension 1. Indeed, suppose $e_1, e_2 \in S^\perp$, and set $w_i = \partial_{e_i} u_0$.

We would like to eventually apply Boundary Harnack. First, we investigate some properties of w_i as to show the hypotheses of Boundary Harnack. We have that $\Delta w_i = \partial_{e_i} \Delta u_0 = 0$ on $\{u_0 > 0\}$, so w_i is harmonic on $\{u_0 > 0\}$. Also, by the 2-homogeneity of u_0, we have that w_i is 1-homogeneous. Finally, recall that $u_0 \in \mathcal{C}^{1,1}_{\text{loc}}$ and that the gradient vanishes continuously at Γ. Thus, w_i does too. Moreover, since $e_i \in S^\perp$, w_i is harmonic, we have that $w_i > 0$ in $\{u > 0\}$ by the maximum principle. We then recognize that $\{u > 0\}$ is a cone by homogeneity and that its complement, Λ, is a closed, convex cone. This immediately implies that $\partial\Lambda$ is Lipschitz, fulfilling the final hypothesis of Boundary-Harnack.

Applying Boundary-Harnack (see Appendix A), we find that $\frac{w_1}{w_2}$ is bounded, nonzero, and $\mathcal{C}^\alpha$. In particular, by the 1-homogeneity of the w_i's we see that the quotient $\frac{w_1}{w_2}$ is constant along each ray emanating from the origin. The only way for the quotient to be $\mathcal{C}^\alpha$ and constant along each ray is if $\frac{w_1}{w_2} = \lambda$ (some fixed value). Thus, we have that $w_1 = \lambda w_2$ and so, $S^\perp$ has dimension one. Therefore, S has codimension one.

Now, we may use Lemma 3.41 to write $u_0(x) = \varphi(x \cdot e)$ for some $e \in \mathbb{S}^{n-1}$ and $\varphi \in \mathcal{C}^{1,1}$ monotonic. Then, taking derivatives, we see that

$$\varphi''(x \cdot e) = \Delta u_0(x) = \chi_{\{u_0 > 0\}} = \chi_{\{\varphi(x \cdot e) > 0\}}.$$

When $x \cdot e > 0$, we have that $\varphi(x \cdot e) > 0$, so $\varphi''(x \cdot e) = 1$ for $x \cdot e > 0$. Similarly, one can conclude that $\varphi(x \cdot e) = 0$ for $x \cdot e \leq 0$. Therefore, we have that condition (i.) holds:

$$u_0(x) = \frac{1}{2}(x \cdot e)_+^2.$$

If, instead, the set Λ has empty interior, we argue by convexity that Λ is contained in a hyperplane. If this weren't the case, we could change coordinates so that Λ contains the origin and n basis vectors $e_1, \cdots, e_n$. Then, by convexity, Λ contains the interior of the n-simplex contradicting the fact that it has empty interior. Moreover, we can use Lemma 5.27 of [2] to conclude that $\Delta u_0 = 1$ in $\mathbb{R}^n$. Then, $\mathcal{C}^{1,1}$ regularity of u_0 tells us that second derivatives of u are bounded and harmonic and thus constant by Liouville's theorem. Hence, we can conclude that u_0 is a quadratic polynomial and so, condition (ii.) holds. □

Remark 3.43. The behavior of blowups differs at regular and singular points of Γ. In fact, condition (i.) of Theorem 3.42 corresponds to regular points, and condition (ii.) corresponds to blowups at singular points. Roughly speaking, this is a consequence of the fact that blowups are "one-dimensional" near regular points and (3.33).

Remark 3.44. The nonempty/empty interior cases for $\Lambda = \{u_0 = 0\}$ correspond exactly to the nonempty/empty interior cases for the contact set $\Omega = \{u = 0\}$ (see Lemma 5.29 of [2]).

3.6. **Regularity of the Free Boundary.**
We now intend to leverage the classification of blowups established in Section 3.5 to establish regularity results about Γ. In particular, our first step is to show that the free boundary Γ is Lipschitz. We combine the approaches of [2] and [7]. We continue to work with a solution u of (3.33) and will assume that 0 is a regular point with corresponding blow-up $u_0 = \frac{1}{2}(x \cdot e)_+^2$ along some particular subsequence $r_k \to 0$. Eventually, we would like to show that u_{r_0} is monotone in a cone of directions for some small $r_0 > 0$ from which we will deduce that Γ is Lipschitz near the origin. We begin with an *improvement of minimum* lemma.

Lemma 3.45. *(Improvement of Minimum for Supersolutions)*
Let v solve (3.33) on B_1 and h be harmonic on $B_1 \cap \{u > 0\}$. Suppose $h \geq 0$ on $\Gamma \cap B_1$ and $h - v \geq -\varepsilon_0$ on B_1. Then, if we take ε_0 small enough, $h - v \geq 0$ on $B_{1/2}$.

Proof. Suppose not. Then, there exists some $x_0 \in B_{1/2}$ with $h(x_0) - v(x_0) < 0$. Now, consider the auxiliary function

$$w := h - v + \frac{|x - x_0|^2}{2n}.$$

We see that w is harmonic in $\{u > 0\}$, $w(x_0) < 0$, and $w \geq 0$ on $\Gamma \cap B_{1/2}(x_0)$. Thus, by the minimum principle, w achieves a negative minimum on $\partial B_{1/2}(x_0)$. In particular,

$$-\varepsilon_0 \leq \inf_{\partial B_{1/2}(x_0)} (h - v) \leq \inf_{\partial B_{1/2}(x_0)} \left(w - \frac{|x - x_0|^2}{2n} \right) \leq \frac{-1}{8n},$$

a contradiction after taking ε_0 small enough. □

We now apply the lemma to prove the directional monotonicity result.

Proposition 3.46. *(Directional Monotonicity)*
There exists some small $r_0 > 0$ so that $\partial_\tau u_{r_0} \geq 0$ in $B_{1/2}$ for all directions $\tau \in \mathbb{S}^{n-1}$ with $\tau \cdot e \geq \frac{1}{2}$.

Proof. First, note that $\partial_\tau u_0(x) - u_0(x) = (x \cdot e)_+ \left(\tau \cdot e - \frac{1}{2}\right) \geq 0$ in B_1 when $\tau \cdot e \geq \frac{1}{2}$. Also, by the $\mathcal{C}^1$ convergence $u_{r_k} \to u_0$, for any $\varepsilon_0 > 0$, we can choose a small $r_0 > 0$ with $\partial_\tau u_{r_0} - u_{r_0} \geq -\varepsilon_0$ in B_1. Thus, applying Lemma 3.45 with $h := \partial_\tau u_{r_0}$ and $v := u_{r_0}$, we conclude that $h = \partial_\tau u_{r_0} \geq 0$ on $B_{1/2}$ whenever $\tau \cdot e \geq \frac{1}{2}$. □

We may finally conclude that Γ is Lipschitz.

Proposition 3.47. *(Lipschitz Regularity of Γ)*
There exists some $r_0 > 0$ so that the free boundary $\partial\{u_{r_0} > 0\}$ is Lipschitz in $B_{1/2}$. Therefore, the free boundary $\partial\{u > 0\}$ is Lipschitz in $B_{r_0/2}$.

Proof. In short, this follows from Proposition 3.46. This is because u_{r_0} obeys a cone condition equivalent to the free boundary being Lipschitz. In particular, let $x_0 \in B_{1/2} \cap \partial\{u_{r_0} > 0\}$ be a free boundary point and set

$$\Sigma_1 = \left\{ x \in B_{1/2} \colon x = x_0 - t\tau, t > 0, \tau \cdot e > \frac{1}{2} \right\}$$

and

$$\Sigma_2 = \left\{ x \in B_{1/2} \colon x = x_0 + t\tau, t > 0, \tau \cdot e > \frac{1}{2} \right\}.$$

Then, we claim that

$$\begin{cases} u_{r_0} & = 0 \text{ in } \Sigma_1, \\ u_{r_0} & > 0 \text{ in } \Sigma_2. \end{cases}$$

First, we note that for $\tau \cdot e > \frac{1}{2}$, we have $\partial_{-\tau} u_{r_0} = -\partial_\tau u_{r_0} \leq 0$. Then, for any $x \in \Sigma_1$, we have that $u_{r_0}(x) = 0$ since $u_{r_0}(x_0) = 0$. If instead $x \in \Sigma_2$ with $u_{r_0}(x) = 0$, we may look at the backwards cone emanating from x which contains x_0. In this cone, we would also have $u_{r_0} \equiv 0$ meaning that x_0 is not a free boundary point, a contradiction. □

Therefore, we can set $e = e_n$ without loss of generality and write $x = (x', x_n)$. Then, the free boundary is the graph of a Lipschitz function: $B_{1/2} \cap \partial\{u_{r_0} > 0\} = \{x_n = f(x')\} \cap B_{1/2}$. We have now fulfilled the key hypothesis of Boundary-Harnack (see Appendix A) and may proceed by showing that Γ is $\mathcal{C}^{1,\alpha}$ for some small $\alpha > 0$.

Theorem 3.48. *($\mathcal{C}^{1,\alpha}$ Regularity of Γ)*
The free boundary $\partial\{u_{r_0} > 0\}$ is $\mathcal{C}^{1,\alpha}$ in $B_{\frac{1}{4}}$. Moreover, the free boundary of u, $\partial\{u > 0\}$, is $\mathcal{C}^{1,\alpha}$ in $B_{\frac{r_0}{4}}$.

Proof. Define $w_1 := \partial_{e_i} u_{r_0} + \partial_{e_n} u_{r_0}$ and $w_2 := \partial_{e_n} u_{r_0}$. Denote $\Omega_{r_0} = \{u_{r_0} > 0\}$ so that w_1, w_2 are positive and harmonic on $\Omega_{r_0} \cap B_{1/2}$ and vanish on the boundary $\partial\Omega_{r_0} \cap B_{1/2}$. The domain is Lipschitz, so we may apply Boundary-Harnack (Appendix A) to argue that $\|\frac{w_1}{w_2}\|_{\mathcal{C}^\alpha\left(B_{1/4} \cap \overline{\Omega_{r_0}}\right)}$ is finite. Therefore, by the formulas for w_1 and w_2, we have that $\|\frac{\partial_{e_i} u_{r_0}}{\partial_{e_n} u_{r_0}}\|_{\mathcal{C}^\alpha\left(B_{1/4} \cap \overline{\Omega_{r_0}}\right)}$ is also finite.

To show that the free boundary is $\mathcal{C}^{1,\alpha}$, it suffices to show that the normal vector is a $\mathcal{C}^\alpha$ function. Writing down the expression for the normal vector to the level set $\{u_{r_0} = t\}$ where $u_{r_0}(x) = t > 0$, we have that the i'th component is

$$\nu^i(x) = \frac{\partial_{e_i} u_{r_0}(x)}{|\nabla u_{r_0}(x)|} = \frac{\partial_{e_i} u_{r_0}(x) / \partial_{e_n} u_{r_0}(x)}{\sqrt{1 + \sum_{k=1}^{n-1} (\partial_{e_k} u_{r_0}(x) / \partial_{e_n} u_{r_0}(x))^2}}.$$

This is a $\mathcal{C}^\alpha$ function since the map $x \mapsto \frac{x}{\sqrt{1+x^2}}$ is Lipschitz continuous and the composition of Lipschitz and α-Hölder continuous functions is $\mathcal{C}^\alpha$. Thus, taking $t \to 0^+$, we see that ν is $\mathcal{C}^\alpha$ at the free boundary. Hence, Γ is $\mathcal{C}^{1,\alpha}$.

□

Remark 3.49. One can now use a higher-order Boundary-Harnack result to leverage $\mathcal{C}^{1,\alpha}$ regularity of Γ into $\mathcal{C}^\infty$ regularity. See Theorem 5.38 of [2].

Appendix A. Boundary-Harnack

We would now like to outline a proof of Boundary-Harnack. Morally, this theorem tells us that positive harmonic functions which vanish at the boundary of a Lipschitz domain do so at the same rate. We precisely state the theorem as follows. We specialize in the case with Lipschitz constant 1 for simplicity (see [2] for further references).

Theorem A.1. *(Boundary-Harnack)*
Suppose Ω is a Lipschitz domain with Lipschitz constant 1 and u is a positive harmonic function on $\partial\Omega \cap B_1$. Normalize so that $u(e_n/2) = 1$. Then, we have that

$$\sup_{\Omega \cap B_1} u \leq C_1(\Omega)$$

for some $C_1 > 0$. Moreover, if v is a function satisfying the same conditions, then there exist constants $C_2(\Omega), C_3(\Omega) > 0$ such that

$$\frac{1}{C_2} \leq \frac{u}{v} \leq C_2$$

and

$$\|u/v\|_{\mathcal{C}^\alpha\left(B_{1/2}\cap\bar{\Omega}\right)} \leq C_3$$

for some small $\alpha > 0$.

The first step in the proof is establishing a result known as the *weak Harnack* inequality for subsolutions.

Proposition A.2. *(Weak Harnack)*
Suppose $u \leq 1$ is a subsolution (i.e., $\Delta u \geq 0$) and there is some $\mu > 0$ with $|\{u \leq 0\} \cap B_1| > \mu|B_1|$. Then, there exists some $\theta = \theta(\mu, n)$ such that $u \leq 1 - \theta$ on $B_{1/2}$.

Proof. See Simon Seignourel's paper or Appendix B of [2]. □

Remark A.3. This result says that subsolutions enjoy an *improvement of maximum* of the interior. In simpler terms, since $u \leq 0$ on a significant portion of the ball, it cannot shoot up to 1 on the half-ball.

We now state an immediate corollary of Weak Harnack. We will use a scaled version of the following later in the proof.

Corollary A.4. *Suppose u is a subsolution and there is some point $x_0 \in B_{1/2}$ where $u(x_0) \geq 1$. Also assume that $|\{u \leq 0\} \cap B_1| > \mu|B_1|$ for some $\mu > 0$. Then, there exists some $\theta = \theta(\mu, n)$ such that $\max_{B_{1/2}} u > 1 + \theta$.*

Proof. Apply Proposition A.2 to $v := \frac{u}{\max_{B_{1/2}} u}$. □

There are two more steps to prove the result. The first is called the *Carleson estimate* which gives a uniform L^∞ estimate up to the boundary in terms of the value at a single point and uses the weak Harnack inequality (Proposition A.2). We will then state a *positivity lemma* which is best interpreted through the lens of Brownian motion.

Lemma A.5. *(Carleson Estimate)*
Let u and Ω be as in the setting of Theorem A.1. Then, there is some absolute constant C such that $\sup_{\Omega\cap B_{1/2}} u \leq Cu(e_n/2)$.

Proof. We begin by extending u by zero on $\Omega^{\complement}$ so that u is subharmonic since it touches down to zero at $\partial\Omega$. Now apply a scaled version of weak Harnack (Proposition A.2) so that $\max_{B_{r/2}\cap\Omega} u \leq (1-\theta)\max_{B_r\cap\Omega} u$ for any ball B_r centered on $\partial\Omega$. Note that the hypothesis of weak Harnack is fulfilled since Ω is 1-Lipschitz.

Now, define $\Omega_\delta := \{x \in B_1 \cap \Omega \colon d\left(x, \Omega^{\complement}\right) > \delta\}$ so that $u \leq C$ on Ω_δ by the ordinary Harnack inequality. Moreover, we know that the cone $\Sigma = \{x_n > |x'|\} \subset \Omega$ since Ω is 1-Lipschitz where $x = (x', x_n)$. We would like to ascertain a bound on u for each line through the origin in Σ and then recenter to extend to $B_{1/2} \cap \partial\Omega$.

Then, let $y = (0, \varepsilon)$. We wish to connect y to points on Ω_δ using a Harnack chain of balls. First note that $d(y, \partial\Sigma) = \frac{\varepsilon}{\sqrt{2}}$ by the Pythagorean theorem so that $d(y, \Omega) \gtrsim \varepsilon$. Similarly, $z = \left(0, \delta\sqrt{2}\right) \in \Omega_\delta$ since $d(z, \partial\Sigma) = \delta$. Begin by applying

Harnack on $B_{\delta/2}(z)$ and then $B_{\delta/4}\big(z - \frac{\delta}{4}e_n\big)$ and so on until $\delta 2^{-k} < \varepsilon \leq \delta 2^{-k-1}$. This tells us that $u(y) \lesssim d(y, \partial\Omega)^{-q}$ for $q > 0$ depending on dimension.

Next, we set $A = u(x_0) = \max_{B_{1/2}} u \geq 1$ and $r = 2d\Big(x_0, \Omega^{\complement}\Big)$. Using that Ω is 1-Lipschitz, we apply weak Harnack (Corollary A.4) to choose some $x_1 \in B_{2r}(x_0)$ with $u(x_1) \geq (1+\theta)A$ and $|x_1 - x_0| \leq 2r$. We then iterate to inductively choose a point x_k with $u(x_k) \geq (1+\theta)^k A$ with $|x_k - x_{k-1}| \leq 4d\Big(x_{k-1}, \Omega^{\complement}\Big)$. Then, applying our previous power-bound, we have that

$$|x_{k+1} - x_k| \lesssim d\Big(x_k, \Omega^{\complement}\Big) \lesssim (1+\theta)^{-k/q} A^{-1/q}.$$

Taking A large enough by rescaling u (recall we normalized $u(e_n/2) = 1$), we can ensure the convergence of the x_k's occurs in $B_{2/3}$. However, this implies that $u(x_k) \to \infty$ in $B_{2/3}$ since $u(x_k) \geq (1+\theta)^k A$ with $\theta > 0$ absolute. This contradicts the fact that any smooth superharmonic function is bounded on a bounded domain. □

We now turn to the positivity lemma which characterizes the comparability of positive harmonic functions vanishing at the boundary. We do not prove it but instead, give some insight into why we expect it to be true through the Brownian motion interpretation and discuss its consequences.

Lemma A.6. *There exist constants ε, δ so that the following holds. Suppose that u is harmonic in $B_1 \cap \Omega$, $u \geq 1$ in Ω_δ, and that $u \geq -\varepsilon$ in $B_1 \cap \Omega$. Then, $u \geq 0$ in $B_{1/2}$.*

Proof. See Appendix B of [2]. □

Remark A.7. For a harmonic function u with boundary data $u\big|_{\partial\Omega} = f$, we can write $u(x) = \mathbb{E}[f(B^x_\tau)]$ where B^x_τ is the first place that a Brownian motion hits the boundary and τ is the hitting time. In the context of the lemma, we can think of the domain as containing a long thin strip oriented horizontally where $u \geq 1$ above, $u = 0$ below, and $u \geq -\varepsilon$ on each of the sides as in Figure 4.

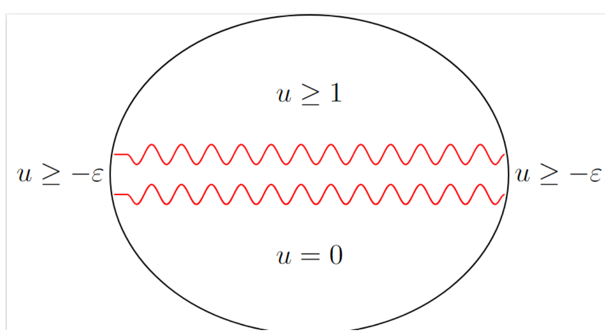

FIGURE 4. Brownian Motion Interpretation Diagram

Thus, for $x \in B_{1/2}$, we have that

$$u(x) \geq \mathbb{P}(B^x_\tau \text{ hits top first}) - \varepsilon\mathbb{P}(B^x_\tau \text{ hits either side first}) > 0$$

since the strip is very thin (i.e., it is much more likely to hit the top than the sides).

Moreover, this lemma implies that the quotient u/v will remain bounded up to the boundary. This is because v is bounded on $B_{1/2}$ by Lemma A.5. Then, for $\varepsilon_0 > 0$ small enough, $u - \varepsilon_0 v \geq -\varepsilon$ in Ω and $u - \varepsilon_0 v \geq \frac{1}{2}$ in Ω_δ. Thus, we would have $u - \varepsilon_0 v \geq 0$ in $B_{1/2}$ by the lemma.

We also omit the proof of Hölder continuity of the quotient as in the statement of the theorem. See Appendix B of [2] for this proof.

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