RESIDUAL LIVES OF RENEWAL PROCESSES

ANTHONY LUO

Abstract. The goal of this paper is to explore the time-average residual life of a renewal process. Renewal processes are arrival processes in which the inter-arrival intervals are independent, and identically distributed random variables. First, we begin with some concepts and definitions to establish a foundation in probability and measure. Then, we will prove important theorems regarding the Strong Law of Large Numbers, arrival processes, and Poisson processes. Finally, we introduce the concept of a residual life and we characterize the limit of the time-average residual life as time approaches infinity.

Contents

1. Introduction

Renewal processes are a type of stochastic process which model the times at which random events occur. These processes are analyzed and implemented to provide a framework for a variety of scenarios and industries, including queueing systems, reliability engineering, inventory management, and insurance claim modeling. In this paper, we will explore several key theorems regarding renewal processes, including the Strong Law of Large Numbers for Renewal Processes and the time-average residual life of a renewal process. The paper will start by providing a background in probability theory before introducing the Strong Law of Large Numbers in section 3, which will be a vital tool in later proofs. Afterward, section 4 will discuss renewal processes, and section 5 will illustrate the time-average residual life for Poisson and general renewal processes.

2. Measure and Probability

We begin by introducing basic concepts regarding the probability space and random variables that will be crucial for contextualizing further theorems.

Definition 2.1. A σ-field $\mathcal{F}$ on a set Ω is a collection of subsets of Ω such that:

(1) The empty set $\emptyset$ is in $\mathcal{F}$;

(2) If the set X is in $\mathcal{F}$, then X^c is also in $\mathcal{F}$;
(3) If $X_1, X_2, \ldots$ is a countable collection of sets in $\mathcal{F}$, then $\bigcup_n X_n$ is in $\mathcal{F}$.

Definition 2.2. The Borel σ-field on a set Ω is the smallest σ-field containing all open sets of Ω. A Borel set is an element of the Borel σ-field.

Remark 2.3. The ordered pair, $(\Omega, \mathcal{F})$, is called a measurable space.

Definition 2.4. A measure is a function $\mu : \mathcal{F} \to \mathbb{R}$ such that:

(1) $\mu(\emptyset) = 0$;
(2) $\mu(X) \geq 0$ for all sets X in the σ-field $\mathcal{F}$;
(3) If $X_1, X_2, \ldots$ is a countable collection of mutually disjoint sets in $\mathcal{F}$, then $\sum\limits_n \mu(X_n) = \mu(\bigcup\limits_n X_n)$.

In particular, a probability measure is a positive measure such that the total measure equals 1.

Definition 2.5. A probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ where:

(1) Ω is a sample space, or a set of possible outcomes;
(2) $\mathcal{F}$ is a set of events where each event is a set of outcomes. Note that $\mathcal{F}$ is a σ-field on the set Ω.
(3) $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a probability measure that assigns each event a value from 0 to 1.

To illustrate how the probability space works, let's consider a simple scenario involving rolling a fair six-sided die. When rolling a die, the sample space consists of all possible outcomes: $\Omega = \{1, 2, 3, 4, 5, 6\}$. The events are subsets of the sample space. One possible event is $A = \{2, 4, 6\}$ (the event "rolling an even number"). Since the die is fair, each outcome is equally likely. The probability measure assigns probabilities to the events: $\mathbb{P}(\{i\}) = \frac{1}{6}$ for each $i \in \Omega$.

Definition 2.6. Conditional probability refers to the probability of an event occurring, given that another event has already occurred. It is equal to the probability of both events occurring together divided by the probability of the event that has already occurred. Mathematically, we have $\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$.

Definition 2.7. Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ be measurable spaces. A function $X : \Omega \to \Omega'$ is a measurable function if $X^{-1}(F) \in \mathcal{F}$ for every $F \in \mathcal{F}'$.

Definition 2.8. A function X is a random variable if it is a real-valued measurable function from Ω to $\mathbb{R}$.

Remark 2.9. If X and Y are random variables, then $Z = X + Y$ is a random variable. Similarly, $A = X - Y$ is also a random variable, where subtraction is merely adding the additive inverse. In fact, the random variables form a vector space over $\mathbb{R}$.

Notice that while in the dice example we were considering probabilities of discrete events, we will subsequently focus mainly on continuous probability, where we will consider a continuum of possible outcomes. Whereas with the discrete probability example we assigned probabilities to individual outcomes, in continuous probability, we will assign probabilities to intervals of outcomes.

Definition 2.10. The expectation of a random variable X in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is the Lebesgue integral of X with respect to a probability measure $\mathbb{P}$, written as $\mathbb{E}[X] = \int_\Omega X \, d\mathbb{P}$.[1]

The expectation (or expected value) of a random variable is a measure of the central tendency or average value that the random variable takes on.

Definition 2.11. The distribution of a random variable X in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a measure $\mathbb{P}_X$ such that $\mathbb{P}_X(B) = \mathbb{P}(X \in B)$ for all B, where B is a Borel set. Furthermore, the distribution function of X is the function $F_X : \mathbb{R} \to [0, 1]$ such that $F_X(x) = \mathbb{P}(X \leq x)$.

The distribution of X describes how the values of X are spread over the real line. It assigns probabilities to all possible Borel sets in the real line, encapsulating the likelihood of X taking values within any given set.

The distribution function, often referred to as the cumulative distribution function (CDF), provides a way to understand the distribution of a random variable X. It gives the probability that X takes a value less than or equal to a particular value.

Definition 2.12. If the measure $\mathbb{P}_X$ for a random variable X is absolutely continuous, then there exists a density function, f, such that the density of F is $f = F'$, where F' is the almost everywhere derivative of F.[2]

Remark 2.13. If X and Y are two independent random variables, then the density of $Z = X + Y$ is the convolution of the densities.

Example 2.14. Two examples of probability distributions and their densities are:

(1) The exponential distribution (which will be discussed in Remark 4.3);
(2) The standard normal distribution: the distribution with the density function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Definition 2.15. If $\mathbb{E}|X| < \infty$ and F_X has a density f_X, then $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx$.

Definition 2.16. The variance of a random variable is $\mathrm{Var}(X) = \mathbb{E}[X - \mathbb{E}[X]]^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2$. Additionally, the standard deviation of a random variable X, σ, is defined as $\sigma = \sqrt{\mathrm{Var}(X)}$.

Definition 2.17. We say that two events A and B are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

In probability theory, the concept of independence describes a situation where the occurrence of one event does not affect the occurrence of another. When two events are independent, knowing that one event has occurred gives no information about whether the other event will occur.

Definition 2.18. An independent and identically distributed (IID) collection of random variables is a collection of random variables such that each random variable has the same probability distribution as the rest and all of the random variables are mutually independent.

[1]Further reading on the Lebesgue measure can be found in Chapter 6 of [1].
[2]See [1] Chapter 14.

Theorem 2.19 (Markov Inequality)**.** *If X is a positive, continuous random variable and $a > 0$, then $\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}$.*

Proof. Suppose X is a positive, continuous random variable and $\alpha > 0$. Then by Definition 2.15,

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx \tag{2.20}$$

Since X is positive,

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_{0}^{\infty} x f_X(x) dx \tag{2.21}$$

Additionally,

$$\mathbb{E}[X] = \int_{0}^{\infty} x f_X(x) dx \geq \int_{a}^{\infty} x f_X(x) dx \geq \int_{0}^{\infty} a f_X(x) dx \tag{2.22}$$

because $x > a$ within the bounds of integration. Thus,

$$\mathbb{E}[X] \geq \int_{0}^{\infty} a f_X(x) dx = a \int_{0}^{\infty} f_X(x) dx = a\mathbb{P}(X \geq a) \tag{2.23}$$

so

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a} \tag{2.24}$$

□

3. Strong Law of Large Numbers

In this section, we will establish the Strong Law of Large Numbers. The Strong Law of Large Numbers (SLLN) states that the sample average of a sequence of independent and identically distributed (IID) random variables will almost surely converge to the expected value of the random variables as the sample size becomes infinitely large. Importantly, it formalizes the intuitive notion of the "law of averages," which is often used informally to suggest that outcomes will "even out" over time.

We will begin to prove the SLLN by first discussing convergence with probability 1.

Definition 3.1. Let $X_1, X_2, \ldots$ be a sequence of random variables in a sample space Ω and let X be another random variable in the sample space Ω. Then the sequence $\{X_n\}$ converges with probability 1 to X if

$$\mathbb{P}(\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)) = 1$$

.

Remark 3.2. Convergence with probability 1 is sometimes also referred as convergence almost surely or convergence almost everywhere.

For the purposes of simplifying notation and better understanding, we will define $Y_n = X_n - X$ for all n. Thus, the sequence $\{Y_n\}$ converges to 0 with probability 1 if and only if $\{X_n\}$ converges to X with probability 1.

Lemma 3.3. *Let X be a nonnegative random variable such that $\mathbb{E}[X] < \infty$. Then $\mathbb{P}(X < \infty) = 1$. In other words, X is finite almost surely.*

Proof. By Theorem 2.19, $\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a} < \infty$ for all $a > 0$. Thus, as $a \to \infty$,

$$\mathbb{P}(X = \infty) = \lim_{a\to\infty} \mathbb{P}(X \geq a) \leq \lim_{a\to\infty} \frac{\mathbb{E}[X]}{a} = 0 \tag{3.4}$$

Because $\mathbb{P}$ maps to values from 0 to 1, the rightmost inequality becomes an equality. Thus, we have

$$\mathbb{P}(X = \infty) = \lim_{a\to\infty} \mathbb{P}(X \geq a) = \lim_{a\to\infty} \frac{\mathbb{E}[X]}{a} = 0 \tag{3.5}$$

Thus, $\mathbb{P}(X < \infty) = 1$. □

We will now introduce a lemma that establishes some criteria for which convergence to 0 with probability 1 occurs. Later, this lemma will be used in our proof of the Strong Law of Large Numbers.

Lemma 3.6. *Let $Y_1, Y_2, \ldots$ be a sequence of random variables with finite expectation. If $\sum_{n=1}^{\infty} \mathbb{E}[|Y_n|] < \infty$, then $\mathbb{P}(\omega : \lim_{n\to\infty} Y_n(\omega) = 0) = 1$.*

Proof. Suppose $\sum_{n=1}^{\infty} \mathbb{E}[|Y_n|] < \infty$. Thus, by the Fubini-Tonelli Theorem, $\mathbb{E}[\sum_{n=1}^{\infty} |Y_n|] < \infty$.[3] Therefore, because $\mathbb{E}[\sum_{n=1}^{\infty} |Y_n|]$ is finite and $\sum_{n=1}^{\infty} |Y_n|$ is nonnegative, by Lemma 3.3, $\sum_{n=1}^{\infty} |Y_n|$ is finite with probability 1. Thus, $\sum_{n=1}^{\infty} Y_n$ is finite with probability 1, so $\lim_{n\to\infty} Y_n = 0$ with probability 1, so $\mathbb{P}(\omega : \lim_{n\to\infty} Y_n(\omega) = 0) = 1$. □

We now move onto proving the Strong Law of Large Numbers under the assumption that the fourth moment of the random variable is finite.

Theorem 3.7. *Strong Law of Large Numbers For all $n \geq 1$, define $S_n = X_1 + X_2 + \cdots + X_n$, where $X_1, X_2, \ldots$ are IID random variables such that $\mathbb{E}(|X_n|)$ is finite. Then*

$$\mathbb{P}\left(\omega : \lim_{n\to\infty} \frac{S_n(\omega)}{n} = \mathbb{E}[X]\right) = 1$$

Proof. We assume in this proof that $\mathbb{E}[X^4] < \infty$. Set $\mathbb{E}[X^4] = \gamma$. Without loss of generality, suppose that $\mathbb{E}[X] = 0$ (as otherwise we can consider a new random variable $Z = X - \mathbb{E}[X]$). We first prove that if $\mathbb{E}[X^4] < \infty$, then $\sigma^2 = \mathrm{Var}(X) = \mathbb{E}[X^2] < \infty$. Notice that for all real numbers x such that $|x| \leq 1$, then $x^2 \leq 1$. Additionally, if $|x| > 1$, then $x^2 < x^4$. Thus, for all real numbers x, $x^2 < 1 + x^4$. Therefore, $\sigma^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[X^2] \leq 1 + \mathbb{E}[X^4]$. Thus, if $\mathbb{E}[X^4] < \infty$, $\mathbb{E}[X^2] < \infty$.

[3] A proof of the Fubini-Tonelli theorem can be found in Chapter 11 of [1].

Notice that $S_n = X_1 + X_2 + \cdots + X_n$, so

$$\begin{aligned}\mathbb{E}[S_n^4] &= \mathbb{E}[(\sum_{i=1}^{n} X_i)(\sum_{j=1}^{n} X_j)(\sum_{k=1}^{n} X_k)(\sum_{l=1}^{n} X_l)] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} \mathbb{E}[X_i X_j X_k X_l]\end{aligned}$$

Thus, when we multiply out the sum, we will get terms in the form:

$$\mathbb{E}[X_i^3 X_j], \mathbb{E}[X_i^2 X_j X_k], \mathbb{E}[X_i X_j X_k X_l]$$

where i, j, k, l are all distinct. Notice that all terms in the forms above will equal 0, as $\mathbb{E}[X] = 0$ and all the random variables are mutually independent. Additionally, we will get n terms in the form:

$$\mathbb{E}[X_i X_j X_k X_l]$$

where $i = j = k = l$. These n terms all will equal $\mathbb{E}[X_i X_j X_k X_l] = \mathbb{E}[X^4] = \gamma$. Additionally, we will have $3n(n-1)$ terms in the form:

$$\mathbb{E}[X_i X_j X_k X_l] = \mathbb{E}[X_i^2 X_j^2]$$

where we have two pairs of equal indices. These $3n(n-1)$ will all equal σ^4. Thus, in total,

$$\mathbb{E}[S_n^4] = n\gamma + 3n(n-1)\sigma^4 \tag{3.8}$$

We now consider the sequence of random variables $\{\frac{S_n^4}{n^4}\}$. The sum of the expected value of this sequence then equals

$$\sum_{n=1}^{\infty} \mathbb{E}\left[\frac{S_n^4}{n^4}\right] = \sum_{n=1}^{\infty} \frac{n\gamma + 3n(n-1)\sigma^4}{n^4} \tag{3.9}$$

Because the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ both converge,

$$\sum_{n=1}^{\infty} \mathbb{E}\left[\frac{S_n^4}{n^4}\right] = \sum_{n=1}^{\infty} \frac{n\gamma + 3n(n-1)\sigma^4}{n^4} < \infty \tag{3.10}$$

Thus, by Lemma 3.6, $\lim_{n\to\infty} \frac{S_n^4}{n^4} = 0$ with probability 1. Thus, for all ω such that $\lim_{n\to\infty} \frac{S_n^4(\omega)}{n^4} = 0$, the non-negative fourth root of the sequence also approaches zero. Thus, $\lim_{n\to\infty} \frac{S_n}{n} = 0 = \mathbb{E}[X]$ with probability 1, so $\mathbb{P}\left(\omega : \lim_{n\to\infty} \frac{S_n(\omega)}{n} = \mathbb{E}[X]\right) = 1$. □

4. Arrival Processes

We will now begin to discuss arrival processes, mathematical models which are used to describe the occurrence of events over time, particularly in systems where items, people, or signals "arrive" randomly.

Definition 4.1. An arrival process is a sequence of increasing random variables such that $S_n < S_{n+1}$ and $S_{n+1} - S_n$ is a positive random variable.

The random variables $S_1, S_2, \dots$ are known as arrival epochs and indicate the times when a recurring incident takes place. This process begins at time 0 and simultaneous arrivals are either not possible or occur with probability zero.

Although they are denoted as arrival processes, they can also model departures or any sequence of events.

There are two other methods commonly used to discuss and interpret an arrival process. Firstly, we can think about an arrival process as a collection of interarrival times. That is, we have a sequence of random variables, $X_1, X_2, \dots$, such that X_i represents the amount of time between S_{i-1} and S_i. Thus, $X_1 = S_1, X_2 = S_2 - S_1$, and so on. Conversely, $S_2 = X_1 + X_2, S_3 = X_1 + X_2 + X_3$, and so on.

Secondly, we can consider this arrival process as a counting process. A counting process is a family of random variables, $\{N(t); t > 0\}$, where for each $t > 0$, $N(t)$ is a random variable that represents the number of arrivals that have occurred up until and including time t; that is, for an integer $n \geq 1$ and $t > 0$, $\{S_n \leq t\} = \{N(t) \geq n\}$, meaning that the event that the nth arrival occurs by time t is equal to the event that there are at least n arrivals at time t. Note that $N(t)$ is an integer and that $N(0) = 0$ with probability 1, as we are only considering arrivals that happen when $t > 0$. Additionally, for $\tau \geq t > 0$, $N(\tau) \geq N(t)$, so $\{N(t); t > 0\}$ is increasing.

Definition 4.2. A renewal process is an arrival process in which the sequence of interarrival times ($X_n = S_n - S_{n-1}$ for a natural number n) are independent and identically distributed random variables.

Remark 4.3. A renewal process to note is a Poisson process. A Poisson process is a renewal process in which the independent and identically distributed sequence of interarrival times are exponential random variables. That is, each X_i has a distribution function with the density $f(x) = \lambda e^{-\lambda x}$ for some $\lambda > 0$ for all $x > 0$.

For Poisson processes, λ is denoted as the arrival rate of the process.

Lemma 4.4. *Let $\{N(t); t > 0\}$ be a counting process and $\{X_n; n \geq 1\}$ be a sequence of interarrival random variables for a renewal process. Then $\lim_{t\to\infty} N(t) = \infty$ with probability 1 and $\lim_{t\to\infty} \mathbb{E}[N(t)] = \infty$.*

Proof. We begin by proving that $\lim_{t\to\infty} N(t) = \infty$ with probability 1. For all sample points ω, $N(t, \omega)$ is an increasing real-valued function, so it has either a finite or infinite limit. Thus, by Definition 4.1, the probability that the limit of $N(t) < n$ for some finite n is:

$$\lim_{t\to\infty} \mathbb{P}(N(t) < n) = \lim_{t\to\infty} \mathbb{P}(S_n > t) = 1 - \lim_{t\to\infty} \mathbb{P}(S_n \leq t) \tag{4.5}$$

Since all S_n are a sum of X_i's, which are random variables, S_n is a random variable for all n, so $\lim_{t\to\infty} \mathbb{P}(S_n \leq t) = 1$ for all n. Thus, $\lim_{t\to\infty} \mathbb{P}(N(t) < n) = 0$ for all n. Thus, the set of sample points such that $N(t, \omega)$ is finite has a probability of 0, so $\lim_{t\to\infty} N(t) = \infty$ with probability 1.

We now prove that $\lim_{t\to\infty} \mathbb{E}[N(t)] = \infty$. Notice that $\mathbb{E}[N(t)]$ is an increasing function, so $\lim_{t\to\infty} \mathbb{E}[N(t)]$ is either finite or infinite. However, for all n, there exists large enough t such that $\mathbb{P}(N(t) \geq n) \geq \frac{1}{2}$, so $\mathbb{E}[N(t)] \geq \frac{n}{2}$ for that t. Thus, $\lim_{t\to\infty} \mathbb{E}[N(t)]$ must be infinite. ∎

Lemma 4.6. *Let $Z_1, Z_2, \dots$ be a sequence of random variables such that $\lim_{n\to\infty} Z_n = \alpha$ with probability 1. Let f be a real-valued function of a real-valued variable that is continuous at α. Then $\lim_{n\to\infty} f(Z_n) = f(\alpha)$.*

Proof. Let $\epsilon > 0$. Because f is continuous at α, there exists $\delta > 0$ such that if $|z-\alpha| < \delta$, then $|f(z) - f(\alpha)| < \epsilon$. Additionally, because $\lim_{n\to\infty} Z_n = \alpha$, there exists N such that if $n \geq N$, $|Z_n - \alpha| < \delta$. Thus, for all $n \geq N$, $|f(Z_n) - f(\alpha)| < \epsilon$, so $\lim_{n\to\infty} f(Z_n) = f(\alpha)$. □

Theorem 4.7 (Strong Law for Renewal Processes)**.** *Let $X_1, X_2, \dots$ be interarrival random variables for a renewal process with a finite mean interarrival time (i.e. $\mathbb{E}[X] < \infty$). Then $\lim_{n\to\infty} \frac{N(t)}{t} = \frac{1}{\mathbb{E}[X]}$ with probability 1.*

Proof. Notice that $\frac{N(t)}{S_{N(t)}} \leq \frac{N(t)}{t} \leq \frac{N(t)}{S_{N(t)+1}}$. Thus, we will prove that $\lim_{n\to\infty} \frac{N(t)}{t} = \frac{1}{\mathbb{E}[X]}$ with probability 1 by proving that $\lim_{t\to\infty} \frac{N(t)}{S_{N(t)}} = \lim_{t\to\infty} \frac{N(t)}{S_{N(t)+1}} = \frac{1}{\mathbb{E}[X]}$ with probability 1. Because $\mathbb{P}(X > 0) = 1$, $\mathbb{E}[X] > 0$. Thus, because $\frac{1}{x}$ is continuous at $x = \mathbb{E}[X]$, by Lemma 4.6 and the Theorem 3.7,

$$\lim_{n\to\infty} \frac{n}{S_n} = \frac{1}{\mathbb{E}[X]} \text{with probability 1} \tag{4.8}$$

By Lemma 4.4, $\lim_{t\to\infty} N(t) = \infty$ with probability 1, so as t increases, $N(t)$ increases through all integers greater than or equal to 0. Thus, with (4.8), we have

$$\lim_{t\to\infty} \frac{N(t)}{S_{N(t)}} = \lim_{n\to\infty} \frac{n}{S_n} = \frac{1}{\mathbb{E}[X]} \tag{4.9}$$

Similarly,

$$\begin{aligned}
\lim_{t\to\infty} \frac{N(t)}{S_{N(t)+1}} &= \lim_{n\to\infty} \frac{n}{S_n + 1} \\
&= \lim_{n\to\infty} \frac{n}{S_n + 1} \frac{n+1}{n+1} \\
&= \lim_{n\to\infty} \frac{n}{n+1} \frac{n+1}{S_{n+1}} \\
&= \frac{1}{\mathbb{E}[X]} \text{ with probability 1.}
\end{aligned}$$

Thus, $\lim_{n\to\infty} \frac{N(t)}{t} = \frac{1}{\mathbb{E}[X]}$ with probability 1. □

5. Average Residual Life

Finally, we will investigate the time-average residual life of a renewal process for a Poisson process and a general renewal process.

Definition 5.1. Let $t > 0$. The residual life $Y(t)$ of a renewal counting process $\{N(t); t > 0\}$ is the period between the time t and the next arrival epoch. Thus, $Y(t) = S_{n+1} - t$, where $n = N(t)$.

Definition 5.2. The time-average residual life of a renewal counting process $\{N(t); t > 0\}$ over the interval $(0, t]$ is $\frac{1}{t}\int_0^t Y(\tau)d\tau$.

Our goal in this section is to prove the following theorem.

Theorem 5.3. *The time-average residual life $Y(t)$ as t approaches ∞ of an arrival process $\{N(t); t > 0\}$ is equal to $\frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]}$, where X is the random variable that describes the interarrival times.*

Before we prove Theorem 5.3, we will first show a simple argument in the case where the renewal process is a Poisson process; that is, when X is exponentially distributed. In this case, X has the memoryless property.

Definition 5.4. A positive random variable X has the memoryless property if for all $x \geq 0$ and $t \geq 0$, $\mathbb{P}(X > x + t \mid X > t) = \mathbb{P}(X > x)$. Equivalently, X has the memoryless property if $\mathbb{P}(X > x + t) = \mathbb{P}(X > x)\mathbb{P}(X > t)$.

Suppose X is a random variable that describes the waiting time for a bus. If X has the memoryless property, a person who has already waited for the bus for t minutes is no better off than a person who has just gotten to the bus stop. The distribution of X after already waiting t minutes is the same as the distribution after waiting 0 minutes.

Theorem 5.5. *A positive random variable is exponentially distributed if and only if it has the memoryless property.*

Proof. $\Rightarrow$) Let $x \geq 0$ and $t \geq 0$. Suppose X is a random variable that is exponentially distributed. Notice that if $X > x + t$, then $X > t$, so $\mathbb{P}(X > x + t \text{ and } X > t) = \mathbb{P}(X > x + t)$. Thus,

$$\begin{aligned}
\mathbb{P}(X > x + t \mid X > t) &= \frac{\mathbb{P}(X > x + t \text{ and } X > t)}{\mathbb{P}(X > x)} \\
&= \frac{\mathbb{P}(X > x + t)}{\mathbb{P}(X > x)} \\
&= \frac{e^{-\lambda(x+t)}}{e^{-\lambda x}} \\
&= e^{-\lambda x} \\
&= \mathbb{P}(X > x).
\end{aligned}$$

$\Leftarrow$) Let X be a positive random variable with the memoryless property. Define $g(x) = \mathbb{P}(X > x) = 1 - F_X(x)$. Thus, because X is memoryless, for all $x \geq 0$ and $t \geq 0$, $\mathbb{P}(X > x+t) = \mathbb{P}(X > x)\mathbb{P}(X > t)$, so $g(x+t) = g(x)g(t)$. Notice that when $x = t$,

$$g(2t) = g(t + t) = g(t)g(t) = g(t)^2 \tag{5.6}$$

Similarly, $g(3t) = g(t)^3$, so $g(kt) = g(t)^k$. Additionally,

$$g(\frac{t}{2}) = g(t)^{\frac{1}{2}}, \ldots, g(\frac{t}{k}) = g(t)^{\frac{1}{k}} \tag{5.7}$$

By combining the results of Equations (3.7) and (3.8), we get that

$$g(\frac{m}{n}t) = g(t)^{\frac{m}{n}} \tag{5.8}$$

Because every real number is the limit of a convergent sequence of rational numbers and g is continuous,

$$g(xt) = g(t)^x \text{ for all real } x > 0$$

Let $t = 1$. Then

$$g(x) = g(1)^x = e^{x \ln g(1)} = e^{-\lambda x} \text{ where } \lambda = -\ln g(1)\ . \tag{5.9}$$

Thus, $1 - F_X(x) = g(x) = e^{-\lambda x}$, so $F_X(x) = 1 - e^{-\lambda x}$, so $f(x) = F_X(x)' = \lambda e^{-\lambda x}$, so X is an exponentially distributed random variable.

□

Due to the memoryless property of the Poisson process, we can easily prove that the time-average residual life $Y(t)$ as t approaches ∞ of a Poisson process with an arrival rate λ is equal to $\frac{1}{\lambda} = \frac{\frac{2}{\lambda^2}}{2\frac{1}{\lambda}} = \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]}$.

Proof of Theorem 5.3 when X is exponentially distributed. Due to the memoryless property of the exponential distribution, the residual life after waiting 0 minutes is the same as the residual life after waiting t minutes and is equal to the expected value of the interarrival random variable: $\mathbb{E}[X] = \frac{1}{\lambda}$. Thus, we have $Y(t) = \frac{1}{\lambda}$, so $\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t Y(\tau)d\tau = \lim\limits_{t\to\infty} \frac{1}{t}\int_0^t \frac{1}{\lambda}d\tau = \frac{1}{\lambda}$. □

Finally, we present the general proof of Theorem 5.3.

Proof of Theorem 5.3. We begin by observing that in every interarrival interval, (S_{n-1}, S_n), $\lim\limits_{t\to S_{n-1}^+} Y(t) = X_n$ and $\lim\limits_{t\to S_n^-} Y(t) = 0$, with $Y(t)$ decreasing linearly with a slope of -1 between S_{n-1} and S_n. Thus, in each interval,

$$\int_{S_{n-1}}^{S_n} Y(t)dt = \frac{1}{2}X_n^2 \tag{5.10}$$

Therefore, the integral of $Y(t)$ is equal to the sum of $\frac{1}{2}X_n^2$ for all interarrival intervals up to $N(t)$ and the integral of $Y(t)$ from $S_{N(t)}$ to t. Thus, we have

$$\int_0^t Y(\tau)d\tau = \sum_{n=1}^{N(t)} \frac{1}{2}X_n^2 + \int_{S_{N(t)}}^t Y(\tau)d\tau \tag{5.11}$$

Based on the last term of equation (5.11), we can bound $\int_0^t Y(\tau)d\tau$, so we have

$$\sum_{n=1}^{N(t)} \frac{1}{2}X_n^2 \le \int_0^t Y(\tau)d\tau \le \sum_{n=1}^{N(t)+1} \frac{1}{2}X_n^2 \tag{5.12}$$

Therefore, the time-average residual life is also bounded as such:

$$\frac{1}{2t}\sum_{n=1}^{N(t)} X_n^2 \le \frac{1}{t}\int_0^t Y(\tau)d\tau \le \frac{1}{2t}\sum_{n=1}^{N(t)+1} X_n^2 \tag{5.13}$$

By manipulating the left side of (5.13) as t approaches ∞, we end up with

$$\lim_{t\to\infty} \frac{1}{2t}\sum_{n=1}^{N(t)} X_n^2 = \lim_{t\to\infty} \frac{\sum\limits_{n=1}^{N(t)} X_n^2}{2t} = \lim_{t\to\infty} \frac{\sum\limits_{n=1}^{N(t)} X_n^2}{N(t)} \frac{N(t)}{2t} \tag{5.14}$$

By Lemma 4.4, $\lim_{t\to\infty} N(t) = \infty$ with probability 1. Thus, with Theorem 3.7, we get

$$\lim_{t\to\infty} \frac{\sum_{n=1}^{N(t)} X_n^2}{N(t)} = \lim_{k\to\infty} \frac{\sum_{n=1}^{k} X_n^2}{k} = \mathbb{E}[X^2] \text{ with probability 1.} \tag{5.15}$$

Additionally, by Theorem 4.7,

$$\lim_{t\to\infty} \frac{N(t)}{2t} = \frac{1}{2\mathbb{E}[X]} \text{ with probability 1.} \tag{5.16}$$

By combining equations (5.14), (5.15) and (5.16), we see that

$$\lim_{t\to\infty} \frac{1}{2t} \sum_{n=1}^{N(t)} X_n^2 = \lim_{t\to\infty} \frac{\sum_{n=1}^{N(t)} X_n^2}{N(t)} \frac{N(t)}{2t} = \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]} \text{ with probability 1.} \tag{5.17}$$

Similarly, the right part of equation (5.14) simplifies to

$$\lim_{t\to\infty} \frac{1}{2t} \sum_{n=1}^{N(t)+1} X_n^2 = \lim_{t\to\infty} \frac{\sum_{n=1}^{N(t)+1} X_n^2}{N(t)+1} \frac{N(t)+1}{N(t)} \frac{N(t)}{2t} = \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]} \text{ with probability 1.} \tag{5.18}$$

Thus, combining (5.13), (5.17), and (5.18) we have that

$$\lim_{t\to\infty} \int_0^t Y(\tau) d\tau = \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]} \text{ with probability 1.} \tag{5.19}$$

□

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