LINEAR INVISCID DAMPING FOR TWO-DIMENSIONAL HOMOGENEOUS EULER EQUATION

ZICHEN LU

Abstract. In 2018, Dongyi Wei, Zhifei Zhang and Weiren Zhao proved a linear inviscid damping result for a class of monotone shear flows in two-dimensional homogeneous Euler equation. We briefly introduce the inviscid damping, state this result and discuss some interesting points in the proof, such as Rayleigh operators, Rayleigh equations, and a duality argument to bound velocity by vorticity.

Contents

1. Introduction

In this paper, we consider an inviscid, incompressible and homogeneous fluid in a two-dimensional space. Here "homogeneous fluid" means that the density of the fluid is independent of time and space, hence a constant. We can normalize the equation so that the density is 1. Then the two-dimensional homogeneous Euler equation for such fluids says

$$\nabla \cdot u = 0 \tag{1.1}$$

$$\partial_t u + u \cdot \nabla u + \nabla p = 0, \tag{1.2}$$

where u is the velocity and p is the pressure. In this paper, the domain of fluids is the two-dimensional space $\mathbf{T} \times [0, 1]$ where $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$. We impose the boundary

Date: OCTOBER, 2024.

condition that

$$u^y(t,x,0) = u^y(t,x,1) = 0$$

where $u = (u^x, u^y)$.

Consider a fluid which is in a steady state, for example, a shear flow. If we give it a small perturbation, then the perturbation will decay over time. This phenomenon is referred to as inviscid damping. It is similar to the Landau damping, which predicts the rapid decay of the electric field of a small perturbation around an equilibrium. This phenomenon is interesting because we do not have any dissipative mechanism like viscosity.

To clarify the mechanism for inviscid damping and conduct mathematical analysis, we first introduce some physical quantities. We denote by ω the vorticity of the fluid. We view vorticity as a scalar. It is defined by

$$\omega = -\partial_y u^x + \partial_x u^y \tag{1.3}$$

where $u = (u^x, u^y)$ is the velocity. Physically, vorticity describes rotation of fluids. We introduce the stream function ψ which is also a scalar. The relations between velocity, vorticity and stream function are as follows,

$$\omega = -\Delta\psi \tag{1.4}$$

$$u = (\partial_y\psi, -\partial_x\psi). \tag{1.5}$$

One of the main mechanisms leading to inviscid damping is vorticity mixing. Therefore, we often use the vorticity formulation of (1.1) and (1.2), which is

$$\partial_t\omega + u\cdot\nabla\omega = 0. \tag{1.6}$$

For example,

$$\begin{cases} u_0 = (U(y), 0) \\ p_0 = C \end{cases} \tag{1.7}$$

is a solution to (1.1) and (1.2), where $U(y)$ is a C^1 function in y and C is a constant. In this case the vorticity Ω is $U'(y)$ by (1.3). This flow is an example of shear flows and is a steady state. In the paper we focus on this type of steady state and consider inviscid damping on it.

A small perturbation gives rise to an initial vorticity $\omega_0(x,y)$. Denote by u the total velocity and ω the total vorticity. We decompose them into

$$u = u_0 + u', \tag{1.8}$$

$$\omega = \Omega + \omega' \tag{1.9}$$

where u' and ω' are induced by the perturbation. We also have

$$\omega'(0,x,y) = \omega_0(x,y). \tag{1.10}$$

Moreover, the total velocity and vorticity satisfy (1.6). Therefore, (1.3)-(1.10) constitute the differential equations needed.

We give a description of what to expect according to inviscid damping. First, the velocity u which arises due to perturbation will converge to zero in some norm as time t tends to infinity. This corresponds to perturbation decay in physics. Second, the vorticity ω which also comes from the perturbation will converge in some norm as well as time t tends to infinity. This reflects vorticity mixing, one of the main mechanisms for inviscid damping. In the following sections we discuss how to study inviscid damping rigorously.

Mathematical rigor of inviscid damping has been a challenge for a long time. To make the problem easier, instead of studying (1.6) directly, mathematicians study the linearized version of (1.6) first. This research field is called "linear inviscid damping". Then naturally the research field of studying (1.6) directly is called "nonlinear inviscid damping".

For the linear case, we should see how to linearize (1.6) first. Use (1.8)-(1.9) to substitute ω and u in (1.6), expand all terms and notice that $u_0 \cdot \nabla u_0 = 0$, we will get

$$\partial_t \omega' + u_0 \cdot \nabla \omega' + u' \cdot \nabla \Omega + u' \cdot \nabla \omega' = 0. \tag{1.11}$$

In (1.11), the only nonlinear term is $u' \cdot \nabla \omega'$ since both u' and ω' are unknown. Then we drop this term. It is reasonable because u' and ω' are expected to be small. Hence the linearized equation is

$$\partial_t \omega' + u_0 \cdot \nabla \omega' + u' \cdot \nabla \Omega = 0. \tag{1.12}$$

Further, if we incorporate (1.4) and (1.5) into (1.12), we will get

$$\partial_t \omega' + \mathcal{L} \omega' = 0 \tag{1.13}$$

where $\mathcal{L}$ is the linear operator defined by

$$\mathcal{L} = U(y)\partial_x + U''(y)\partial_x(-\Delta)^{-1}. \tag{1.14}$$

(1.13),(1.14) and (1.10) are the linearized equations. Since the equations we study only involve the perturbation velocity and vorticity, for simplicity of notation, we drop the prime from now on.

In 2018, Wei et al. proved linear inviscid damping (1.13) for a class of monotone shear flows [1]. It is the main result of our paper. We will give the statement of this result in section 2 and then discuss some strategies of proof in section 3-5. In section 3 and 4, we discuss Rayleigh operators and Rayleigh equations. They apply to a wide variety of setups of linear inviscid damping. In section 5 we introduce a duality argument crucial to the estimate of velocity.

Remark 1.15. Here we have to explain what the inverse of the operator $-\Delta$ means. In fact, this remark not only applies to $-\Delta$ but also to any differential operator appeared in this paper whenever we encounter its inverse. Let X and Y be fixed function spaces based on context and D be a differential operator from X to Y. Let $f \in Y$. Then generally $D^{-1}f$ is the unique solution to the following equation

$$Dg = f$$

equipped with some fixed boundary condition. In this paper we adopt the Dirichlet boundary condition, in other words, vanishing on boundary.

Note that the operator $(-\Delta)^{-1}$ is applied to the vorticity ω to get the stream function. The stream function does vanish on boundary since fluids cannot cross the border. In later sections functions to which the inverse of differential operators apply all vanish on boundary. Hence imposing Dirichlet boundary condition is reasonable.

2. Statement of the Main Result

2.1. Preliminary.

To state the main result of this paper, we first introduce some definitions. Let $T : X \longrightarrow X$ be a linear operator on a Hilbert space X.

Definition 2.1. Define the *point spectrum*, *continuous spectrum*, *residue spectrum* and *spectrum* of T respectively by

$$\sigma_p(T) := \{\lambda \in \mathbf{C} :\ \mathrm{Ker}(\lambda \mathrm{Id} - T) \neq 0\},$$

$$\sigma_c(T) := \{\lambda \in \mathbf{C} :\ \mathrm{Ker}(\lambda \mathrm{Id} - T) = 0,\ \overline{\mathrm{R}(\lambda \mathrm{Id} - T)} = X,\ (\lambda \mathrm{Id} - T)^{-1} \notin \mathcal{L}(X)\},$$

$$\sigma_r(T) := \{\lambda \in \mathbf{C} :\ \mathrm{Ker}(\lambda \mathrm{Id} - T) = 0,\ \overline{\mathrm{R}(\lambda \mathrm{Id} - T)} \neq X\},$$

$$\sigma(T) := \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T).$$

We define the *discrete spectrum* contained in $\sigma_p(T)$ to be the set of eigenvalues which are isolated points in $\sigma(T)$ and have a finite-dimensional eigenspace. We denote the discrete spectrum by $\sigma_d(T)$. Moreover, if $\lambda \in \sigma_p(T)$ is not an isolated point in $\sigma(T)$, then λ is called an *embedded eigenvalue* of T.

We also introduce the operator $P_T : X \longrightarrow X$. It is called "*the spectral projection to* $\sigma_d(T)$". Let

$$H_0 := \mathrm{span}\{x \in H :\ \exists \lambda \in \sigma_d(T) \text{ s.t. } Tx = \lambda x\}.$$

Then we define P_T to be the orthogonal projection onto the closure of H_0.

Having introduced some definitions in spectral theory, we turn to the function spaces we are going to work with.

Definition 2.2. Let $f \in L^2(\mathbf{T} \times (0,1))$. Then we have the Fourier series of f, $f(x,y) = \sum_{\alpha \in \mathbf{Z}} e^{i\alpha x} f_\alpha(y)$. The Fourier series converges in L^2. For integers s_x and s_y, we define $H_x^{s_x} H_y^{s_y}$ norm of f to be

$$\|f\|_{H_x^{s_x} H_y^{s_y}} := \left(\sum_{\alpha \in \mathbf{Z}, \alpha \neq 0} |\alpha|^{2s_x} \|f_\alpha(y)\|^2_{H_y^{s_y}} \right)^{\frac{1}{2}}. \tag{2.3}$$

Then the space $H_x^{s_x} H_y^{s_y}$ is defined to be the subspace of $L^2(\mathbf{T} \times (0,1))$ consisting of functions having finite $H_x^{s_x} H_y^{s_y}$ norm.

2.2. **Main Result.** With the previous preparations, we can state the main result of this paper.

Theorem 2.4. *Suppose* $U \in C^4([0,1])$ *and there exists a positive constant* c_0 *such that* $U'(y) > c_0$ *for any* $y \in [0,1]$. *Suppose that the linearized operator* $\mathcal{L}$ *has no embedded eigenvalues. Assume that* $\int_{\mathbf{T}} \omega_0(x,y)dx = 0$ *and* $P_{\mathcal{L}}\omega_0 = 0$, *where* $P_{\mathcal{L}}$ *is the spectral projection to* $\sigma_d\big(\mathcal{L}\big)$. *Denote by* $\langle t \rangle := (1+t^2)^{\frac{1}{2}}$. *Then it holds that*

1. *if* $\omega_0 \in H_x^{-1} H_y^1$, *then* $\|u(t)\|_{L^2} \leq \frac{C}{\langle t \rangle} \|\omega_0\|_{H_x^{-1} H_y^1}$;
2. *if* $\omega_0(x,y) \in H_x^{-1} H_y^2$, *then* $\|u^y(t)\|_{L^2} \leq \frac{C}{\langle t \rangle^2} \|\omega_0\|_{H_x^{-1} H_y^2}$;
3. *if* $\omega_0(x,y) \in H_x^{-1} H_y^k$ *for* $k = 0,1$, *then there exists* $\omega_\infty(x,y) \in H_x^{-1} H_y^k$ *such that* $\|W(t) - \omega_\infty\|_{L^2} \longrightarrow 0 \quad as \quad t \to +\infty$. *Here we define* $W(t,x,y) := u(t, x + tU(y), y)$.

In the above theorem, the first and second conclusions correspond to perturbation decay. Besides, they give the decay rates. The third conclusion corresponds to vorticity mixing. Hence the theorem rigorously proves linear inviscid damping. We give several remarks on this result.

Remark 2.5. The condition "$\mathcal{L}$ has no embedded eigenvalue" does not impose too many restrictions on the background flow $(U(y), 0)$. For example, when $U(y)$ has no inflection point, $\mathcal{L}$ has no embedded eigenvalue. We refer to [6] for details of this example.

On the other hand, when $U''(y) = 0$ for any $y \in [0, 1]$, the case is simpler. For example, the Couette flow case $U(y) = y$ was addressed earlier by Lin and Zeng [8]. Hence we focus on the case where the term $U''(y)\partial_x(-\Delta)^{-1}$ in (1.14) does not vanish.

Remark 2.6. This remark discusses the condition $\int_{\mathbf{T}} \omega_0(x, y)dx = 0$. In fact, if we take Fourier series in x to obtain $\hat{\omega}(t, \alpha, y)$, then $\int_{\mathbf{T}} \omega_0(x, y)dx = 2\pi\hat{\omega}(0, 0, y)$.

The vorticity equation (1.13) implies

$$\partial_t\widehat{\omega} + i\alpha U\widehat{\omega} + i\alpha U''\widehat{\psi} = 0. \tag{2.7}$$

Hence when $\alpha = 0$, the value of $\hat{\omega}(t, 0, y)$ is independent of time t. Then the condition $\hat{\omega}(0, 0, y) = 0$ implies $\hat{\omega}(t, 0, y) = 0$ for any t. Physically, this means that the total vorticity (which means integration of vorticity along the circle $\mathbf{T}$) at any height y remains zero, which suggests some balance of the system.

Mathematically, this condition enables us to obtain estimates in $H_x^{s_x} H_y^{s_y}$ norms introduced in Definition 2.2. This is because we take the sum over $\mathbf{Z} - \{0\}$ in (2.3).

Remark 2.8. This remark discusses the reason for introducing $W(t, x, y) := \omega(t, x+tU(y), y)$. Mathematically, it is because we can combine the two terms $\partial_t\omega$ and $U(y)\partial_x\omega$ in (1.13) via $\partial_t W = (\partial_t + U(y)\partial_x)\omega$. Physically, the coordinate transform from (t, x, y) to $(t, x + tU(y), y)$ encodes physical information of the background flow $(U(y), 0)$. In other words, at time t, the background flow $(U(y), 0)$ takes a particle located initially at (x, y) to $(x + tU(y), y)$.

3. Spectrum of Rayleigh Operators

Now we begin to give a sketch of proof of Theorem 2.4. In terms of the stream function, (1.13) becomes

$$\partial_t\Delta\psi + U(y)\partial_x\Delta\psi - U''(y)\partial_x\psi = 0. \tag{3.1}$$

Since the domain of x-coordinate is periodic, we can take Fourier series in x to obtain

$$\partial_t\hat{\psi} = -i\alpha\mathcal{R}_\alpha\hat{\psi} \tag{3.2}$$

where

$$\mathcal{R}_\alpha = -\left(\frac{d^2}{dy^2} - \alpha^2\right)^{-1}\left(U''(y) - U(y)\left(\frac{d^2}{dy^2} - \alpha^2\right)\right). \tag{3.3}$$

The operators $\mathcal{R}_\alpha$ are called *Rayleigh operators*. Here we fix α and can assume $\alpha \neq 0$ by Remark 2.6. Rayleigh operators are widely studied in linear inviscid damping of different setups of fluids. Since the stream function ψ vanishes on boundaries, so does $\hat{\psi}$. Hence Remark 1.15 applies to $\mathcal{R}_\alpha$.

We can assume that $\mathcal{R}_\alpha$ is an operator from $L^2(0, 1)$ to itself. All derivatives are defined in the weak sense. In fact, local integrability is enough for a function to have weak derivatives. Our treatment of spectral properties of $\mathcal{R}_\alpha$ is based on [7].

3.1. **Spectral Properties.**

Lemma 3.4. *$\mathcal{R}_\alpha$ is a bounded operator and $\sigma(\mathcal{R}_\alpha)$ is compact.*

Proof. First we prove that $\mathcal{R}_\alpha$ is bounded. Let $\Phi \in C_c^\infty(0,1)$ and $f := \mathcal{R}_\alpha\Phi - U\Phi$. Then $f(0) = -U(0)\Phi(0) = 0$ and $f(1) = -U(1)\Phi(1) = 0$. We have the following equation

$$f'' - \alpha^2 f = -2(U'\Phi)'.$$

We construct Green's function to solve this equation

$$f = \int_0^1 \mathcal{G}(y,\eta)(-2)(U'\Phi)'(\eta)d\eta$$

where

$$\mathcal{G}(y,\eta) = \begin{cases} -\frac{\sinh(\alpha(1-y))\sinh(\alpha\eta)}{\alpha\sinh(\alpha)} & \text{for } y > \eta \\ -\frac{\sinh(\alpha y)\sinh(\alpha(1-\eta))}{\alpha\sinh(\alpha)} & \text{for } y < \eta \end{cases}$$

After integration by parts we obtain

$$f = \int_0^1 \mathcal{G}_1(y,\eta)(U'\Phi)(\eta)d\eta$$

where

$$\mathcal{G}_1(y,\eta) = \begin{cases} \frac{-2\sinh(\alpha(1-y))\cosh(\alpha\eta)}{\sinh(\alpha)} & \text{for } y > \eta \\ \frac{2\sinh(\alpha y)\cosh(\alpha(1-\eta))}{\sinh(\alpha)} & \text{for } y < \eta \end{cases} \tag{3.5}$$

Hence there exists a positive constant C such that $\|f\|_{L^2} \le C\|\Phi\|_{L^2}$. Since $C_c^\infty(0,1)$ is dense in $L^2(0,1)$, the above bound $\|f\|_{L^2} \le C\|\Phi\|_{L^2}$ holds for any $\Phi \in L^2(0,1)$.

Second we prove compactness. Denote by $r(\mathcal{R}_\alpha)$ the spectral radius of $\mathcal{R}_\alpha$. Then by a formula of Gelfand,

$$r(\mathcal{R}_\alpha) = \lim_{n\to+\infty} \|\mathcal{R}_\alpha^n\|^{\frac{1}{n}}.$$

The right hand side is less than or equal to the norm of $\mathcal{R}_\alpha$, hence $r(\mathcal{R}_\alpha)$ is finite and the spectrum is compact. □

Let c be a complex number. Let $\Phi, f \in L^2(0,1)$. Consider

$$(c\mathrm{Id} - \mathcal{R}_\alpha)\Phi = f. \tag{3.6}$$

It is equivalent to

$$(U-c)\Phi'' - ((U-c)\alpha^2 + U'')\Phi = (\alpha^2 - \partial_{yy})f. \tag{3.7}$$

(3.7) is an example of an *inhomogeneous Rayleigh equation*. Studying the solution to this equation is crucial to the study of spectrum of Rayleigh operators. A common approach to studying inhomogeneous Rayleigh equation is to study *homogeneous Rayleigh equation* first, which is of the form

$$(U-c)\Phi'' - ((U-c)\alpha^2 + U'')\Phi = 0. \tag{3.8}$$

Lemma 3.9. *(1) $\sigma_c(\mathcal{R}_\alpha) \cup \sigma_r(\mathcal{R}_\alpha) \subset [U(0), U(1)]$.*

(2) For every isolated eigenvalue $c \in \sigma_p(\mathcal{R}_\alpha)$ not in $[U(0), U(1)]$, the eigenspace of c is finite-dimensional.

Proof. We first prove (1). Suppose $c \notin [U(0), U(1)]$ and the operator $(c\mathrm{Id} - \mathcal{R}_\alpha)$ is injective. We only have to prove that $(c\mathrm{Id} - \mathcal{R}_\alpha)^{-1}$ is bounded. Since $\mathcal{R}_\alpha$ is bounded, by Banach Theorem, we only have to prove that for each function $f \in L^2(0,1)$, (3.7) has a solution.

Since $c \notin [U(0), U(1)]$, (3.7) does not have any singularity. By the proof of Lemma 3.4 and the fact that $\Phi(0) = \Phi(1) = 0$, (3.7) is equivalent to

$$\Phi(y) = (c - U(y))^{-1} f + \int_0^1 K(y,s)\Phi(s)ds \tag{3.10}$$

where

$$K(y,s) = (c - U(y))^{-1}\mathcal{G}_1(y,s)U'(s).$$

We refer to (3.5) for the definition of the function $\mathcal{G}_1$. Define a bounded linear operator $S : L^2(0,1) \longrightarrow L^2(0,1)$ via

$$S\Phi := \int_0^1 K(y,s)\Phi(s)ds.$$

It is well-known that S is a compact operator.

Then we rewrite (3.10) as

$$(\mathrm{Id} - S)\Phi = h. \tag{3.11}$$

Denote by S^* the adjoint of S. Since we assume at the beginning of the proof that $(c\mathrm{Id} - \mathcal{R}_\alpha)$ is injective, (3.8) has no nonzero solutions, hence the operator $S_1 := (\mathrm{Id} - S)$ is also injective. By Riesz–Fredholm Theorem, $\dim \mathrm{Ker}(S_1) = \dim \mathrm{Ker}(S_1^*) = 0$ and $\mathrm{R}(S_1) = \mathrm{Ker}(S_1^*)^\perp = L^2(0,1)$. Hence (3.11) has a solution and the proof of (1) is completed.

The claim (2) is due to the fact that compactness of S implies finite-dimensionality of $\mathrm{Ker}(S_1)$ by Riesz-Fredholm Theorem. □

Lemma 3.12. *Let $c \in [U(0), U(1)]$, then $c \in \sigma(\mathcal{R}_\alpha)$.*

Proof. Take $f = (c - U(y))^{-\frac{1}{8}} \in L^2(0,1)$. By the proof of Lemma 3.4, we know that (3.6) is equivalent to

$$\Phi = (c - U(y))^{-1}\int_0^1 \mathcal{G}_1(y,\eta)U'(\eta)\Phi(\eta)d\eta + (c - U(y))^{-\frac{9}{8}}. \tag{3.13}$$

If $\Phi \in L^2$, then $(c - U(y))^{\frac{9}{16}}\Phi \in L^2$. Multiply both sides of (3.13) by $(c - U(y))^{\frac{9}{16}}$ and we will get

$$(c - U(y))^{\frac{9}{16}}\Phi = (c - U(y))^{-\frac{7}{16}}\int_0^1 \mathcal{G}_1(y,\eta)U'(\eta)\Phi(\eta)d\eta + (c - U(y))^{-\frac{9}{16}}. \tag{3.14}$$

The right hand side of (3.14) is square integrable. Since $c - U(y) \sim y - y_c$, the term $(c - U(y))^{-\frac{7}{16}}\int_0^1 \mathcal{G}_1(y,\eta)U'(\eta)\Phi(\eta)d\eta$ is square integrable. Hence the term $(c - U(y))^{-\frac{9}{16}}$ should be square integrable by (3.14). But $(c - U(y))^{-\frac{9}{16}}$ is in fact not square integrable since $\frac{9}{16} > \frac{1}{2}$, which is a contradiction. Hence $(c - \mathcal{R}_\alpha)$ is never surjective. □

The lemma above also indicates that $\sigma_d(\mathcal{R}_\alpha) \subset \mathbf{C} - [U(0), U(1)]$ since every point in $[U(0), U(1)]$ is not an isolated point in $\sigma(\mathcal{R}_\alpha)$. The remaining task is to show that $(\mathbf{C} - [U(0), U(1)]) \cap \sigma(\mathcal{R}_\alpha) = \sigma_d(\mathcal{R}_\alpha)$.

Assume c is an eigenvalue of $\mathcal{R}_\alpha$ and Φ is a corresponding eigenfunction. We denote by $\mathcal{F}$ the Fourier transform in x and $\mathcal{F}^{-1}$ the inverse. We can verify

$$\mathcal{L}(-\Delta\circ\mathcal{F}^{-1}\Phi) = i\alpha c(-\Delta\circ\mathcal{F}^{-1}\Phi) \tag{3.15}$$

by taking Fourier transform in x on both sides. Hence $i\alpha c$ is an eigenvalue of $\mathcal{L}$ and $-\Delta\circ\mathcal{F}^{-1}\Phi$ is a corresponding eigenfunction. Conversely, if c is an eigenvalue of $\mathcal{L}$ with an eigenfunction Ψ, then by the same process, $\frac{c}{i\alpha}$ is an eigenvalue of $\mathcal{R}_\alpha$ with an eigenfunction $\mathcal{F}\circ(-\Delta)^{-1}\Psi$.

Therefore, $\mathcal{R}_\alpha$ has no embedded eigenvalue. By Lemma 3.9, we have proved that $(\mathbf{C}-[U(0),U(1)])\cap\sigma(\mathcal{R}_\alpha)=\sigma_d(\mathcal{R}_\alpha)$.

3.2. **An Application: Representation Formula of the Stream Function.** One of the main applications of the spectral properties is to study the representation formula of $\hat{\psi}$ based on (3.2).

The compactness of $\sigma(\mathcal{R}_\alpha)$ plays a role. We can choose a simply connected bounded domain Ω' that contains $\sigma(\mathcal{R}_\alpha)$. We can assume that the boundary of Ω' is a continuous Jordan curve in $\mathbf{C}$. Since $\mathcal{R}_\alpha$ is a bounded operator, we have the following representation formula

$$\hat{\psi}(t,\alpha,y) = \frac{1}{2\pi i}\int_{\partial\Omega'} e^{-i\alpha t c}(c-\mathcal{R}_\alpha)^{-1}\hat{\psi}(0,\alpha,y)dc \tag{3.16}$$

with $\hat{\psi}(0,\alpha,y)=(\alpha^2-\partial_{yy})^{-1}\hat{\omega}_0(\alpha,y)$. The right hand side of (3.16) is an example of a *Dunford's integral*. We refer to [2] for Dunford's integral and validity of (3.16).

But (3.16) is still difficult to compute because Ω' is arbitrary. We want to make it specific. The key is that only the discrete spectrum exists in $\mathbf{C}-[U(0),U(1)]$. Moreover, by (3.15), $P_{\mathcal{L}}\omega_0=0$ implies $P_{\mathcal{R}_\alpha}\hat{\psi}(0,\alpha,y)=0$. Therefore, the term $(c-\mathcal{R}_\alpha)^{-1}\hat{\psi}(0,\alpha,y)$ in (3.16) is holomorphic in $\mathbf{C}-[U(0),U(1)]$.

For each sufficiently small $\epsilon_0>0$, we construct a contour $\partial\Omega_{\epsilon_0}$ near $[U(0),U(1)]$, where $\Omega_{\epsilon_0}:=D_0\cup D_{\epsilon_0}\cup B^l_{\epsilon_0}\cup B^r_{\epsilon_0}$ is a domain. We define

$$\begin{aligned}
D_0 &:= [U(0),U(1)],\\
D_{\epsilon_0} &:= \big\{c=c_r+i\epsilon:\ c_r\in[U(0),U(1)],\ 0<|\epsilon|<\epsilon_0\big\},\\
B^l_{\epsilon_0} &:= \big\{c=U(0)+\epsilon e^{i\theta}:\ 0<\epsilon<\epsilon_0,\ \frac{\pi}{2}\le\theta\le\frac{3\pi}{2}\big\},\\
B^r_{\epsilon_0} &:= \big\{c=U(1)-\epsilon e^{i\theta}:\ 0<\epsilon<\epsilon_0,\ \frac{\pi}{2}\le\theta\le\frac{3\pi}{2}\big\}.
\end{aligned}$$

Therefore, by Cauchy integration formula, we have

$$\hat{\psi}(t,\alpha,y) = \frac{1}{2\pi i}\int_{\partial\Omega_{\epsilon_0}} e^{-i\alpha t c}(c-\mathcal{R}_\alpha)^{-1}\hat{\psi}(0,\alpha,y)dc.$$

However, the resolvent $(c-\mathcal{R}_\alpha)^{-1}$ is still not easy to compute. We let

$$(c-\mathcal{R}_\alpha)^{-1}\hat{\psi}(0,\alpha,y) = i\alpha\Phi(\alpha,y,c).$$

Then $\Phi(\alpha,y,c)$ satisfies the following inhomogeneous Rayleigh equation

$$\Phi'' - \left(\alpha^2+\frac{U''}{U-c}\right)\Phi = \frac{\hat{\omega}_0(\alpha,y)}{i\alpha(U-c)} \tag{3.17}$$

with the boundary condition

$$\Phi(0)=\Phi(1)=0. \tag{3.18}$$

Then we have

$$\hat{\psi}(t,\alpha,y)=\frac{1}{2\pi}\int_{\partial\Omega_{\epsilon_0}}e^{-i\alpha tc}\alpha\Phi(\alpha,y,c)dc. \tag{3.19}$$

Therefore, to study $\hat{\psi}$, it's important to analyze properties of the solution Φ which will be the topic of the next section.

4. Rayleigh Equations

We hope to get an explicit representation of the solution Φ to the inhomogeneous Rayleigh equation (3.17) with the boundary condition (3.18). Let

$$c_r=\begin{cases}Re(c) & \text{for } c\in D_0\cup D_{\epsilon_0}\\ U(0) & \text{for } c\in B^l_{\epsilon_0}\\ U(1) & \text{for } c\in B^r_{\epsilon_0}\end{cases} \tag{4.1}$$

Suppose ϕ is a nonzero solution to the homogeneous Rayleigh equation (3.8) and $c\in\Omega_{\epsilon_0}$. We can formally derive the representation formula of Φ:

$$\Phi=\phi\int_0^y\frac{1}{\phi(z,c)^2}\int_{y_c}^z\phi(y',c)f(y',c)dy'dz+\mu(c)\phi(y,c)\int_0^y\frac{1}{\phi(y',c)^2}dy' \tag{4.2}$$

where $y_c=U^{-1}(c_r)$, $f=\frac{\hat{\omega}_0(\alpha,y)}{i\alpha(U-c)}$ and

$$\mu(c)=-\frac{\int_0^1\frac{1}{\phi(z,c)^2}\int_{y_c}^z\phi f(y',c)dy'dz}{\int_0^1\frac{1}{\phi(y,c)^2}dy}. \tag{4.3}$$

We refer to Proposition 6.5 of [1] for the proof of (4.2). We should choose a good solution ϕ to ensure well-definedness of (4.2). To determine ϕ, we will choose a good initial condition of the homogeneous Rayleigh equation (3.8).

When $c\in D_0$, y_c is the singular point of the homogeneous Rayleigh equation (3.8). Take $y=y_c$ in (3.8) and we will get $\phi(y_c)=0$. Hence we extract the factor $(U(y)-c)$ from ϕ and let $\phi=(U-c)\phi_1$. We let ϕ_1 satisfy the initial condition $\phi_1(y_c)=1$ and $\phi_1'(y_c)=0$. Therefore, we complete the definition of ϕ when $c\in D_0$.

We should also construct ϕ when $c\in\Omega_{\epsilon_0}-D_0$. The idea is similar. We first specify a point $y_c'\in D_0$ and then choose the initial condition $\phi_1(y_c')=1$ and $\phi_1'(y_c')=0$. In fact,

$$y_c'=\begin{cases}y_c & \text{for } c\in D_{\epsilon_0}\\ 0 & \text{for } c\in B^l_{\epsilon_0}\\ 1 & \text{for } c\in B^r_{\epsilon_0}\end{cases} \tag{4.4}$$

4.1. **Homogeneous Rayleigh Equation.**

4.1.1. *Existence and Uniqueness.* The very first task is to prove existence and uniqueness of ϕ, which is the solution of the homogeneous Rayleigh equation (3.8) with the initial condition chosen above. This is usually achieved by transforming differential equations to integral equations and using contraction principle. Then (3.8) with the initial condition is transformed into

$$\phi_1=1+\alpha^2T\phi_1 \tag{4.5}$$

where T is an integral operator defined by

$$Tf := \int_{y_c}^{y} \frac{1}{(U(y')-c)^2} \int_{y_c}^{y'} f(z,c)(U(z)-c)^2 dz dy'. \tag{4.6}$$

The next critical step is to figure out the function space in which the equation (4.5) has a unique solution. The main component of the definition of a function space is the norm of functions. Therefore, we should specify a norm of functions such that $\alpha^2 T$ is a contraction. The proper idea is considering weighed L^∞ norms. Let γ be a positive function on $[0,1]\times\Omega_{\epsilon_0}$ to be determined later.

Consider the case $c \in D_0$. For each function f on $[0,1]\times D_0$, define

$$\|f\|_{X_0} := \|\gamma^{-1} f\|_{L^\infty([0,1]\times D_0)}.$$

Then since $|U(y')-c| \geq |U(z)-c|$ in (4.6),

$$\|Tf\|_{X_0} \leq \|\gamma^{-1} \int_{y_c}^{y}\int_{y_c}^{y'} \gamma(z,c) dz dy'\|_{L^\infty([0,1]\times D_0)} \|f\|_{X_0}.$$

We hope that $\|\gamma^{-1} \int_{y_c}^{y}\int_{y_c}^{y'} \gamma(z,c)dzdy'\|_{L^\infty([0,1]\times D_0)} < \frac{1}{\alpha^2}$. Hence we assume

$$\int_{y_c}^{y}\int_{y_c}^{y'} \gamma(z,c)dzdy' = \frac{\gamma(y,c)-\gamma(y_c,c)}{A^2} \tag{4.7}$$

where A is a positive constant greater than $C_0|\alpha|$ with $C_0 \geq 1$. C_0 only depends on c_0 and $\|U\|_{C^4}$ and will be determined later. We refer to Theorem 2.4 for the definition of the constant c_0. Solving the equation (4.7), we can take

$$\gamma = \cosh(A(y-y_c)).$$

Hence

$$\|f\|_{X_0} = \|\frac{f}{\cosh(A(y-y_c))}\|_{L^\infty([0,1]\times D_0)}. \tag{4.8}$$

We observe that γ is bounded away from 0. Let X_0 be the space of functions on $[0,1]\times D_0$ with finite X_0 norm. This X_0 norm is enough for $\alpha^2 T$ to be a contraction, and C_0 can be an arbitrary constant greater than 1. But it turns out that the X_0 norm is not strong enough since later we have to estimate the derivatives of ϕ. Thus we construct

$$\|f\|_{Y_0} := \sum_{k=0}^{2}\sum_{|\beta|=k} A^{-k}\|\partial^\beta f\|_{X_0} \tag{4.9}$$

and let Y_0 be the space of functions on $[0,1]\times D_0$ with finite Y_0 norms.

We refer to Definition 4.3 of [1] for the integral operators $T_{k,j}$ for non-negative integers k, j with $j \leq k+1$. These integral operators are useful because they appear once we take derivatives of Tf for each $f \in Y_0$. Then there exists a constant C independent of A such that for each $f \in Y_0$,

$$\|Tf\|_{Y_0} \leq \frac{C}{A^2}\|f\|_{Y_0}.$$

We refer to Lemma 4.4 of [1] for the proof. Since C also depends on c_0 and $\|U\|_{C^4}$, we can choose C_0 such that $C_0^2 > C$. Then $\alpha^2 T$ is still a contraction. Therefore, the equation (4.5) has a unique solution in the function space Y_0 when $c \in D_0$.

For $c \in D_{\epsilon_0}$, $c \in B^l_{\epsilon_0}$ and $c \in B^r_{\epsilon_0}$, we design function spaces separately. The idea is similar. We refer to Definition 4.1 and 4.2 of [1] and adopt notations in them.

Then we get existence and uniqueness of the solution to (4.5) in these regions respectively. We combine them to get existence of the solution ϕ in $[0,1]\times\Omega_{\epsilon_0}$. Uniqueness of ϕ results from uniqueness in these regions respectively. Therefore, we have finished the sketch of proof of the following proposition:

Proposition 4.10. *There exists a solution* $\phi(y,c)\in C^1\big([0,1]\times\Omega_{\epsilon_0}-D_0\big)\cap C\big([0,1]\times\Omega_{\epsilon_0}\big)$ *of* (4.5). *Moreover, there exists* $\epsilon_0>0$ *such that for any* $(y,c)\in[0,1]\times\Omega_{\epsilon_0}$,

$$|\phi_1(y,c)|\geq\frac{1}{2},\quad |\phi_1(y,c)-1|\leq C|U(y)-c|^2, \tag{4.11}$$

where $\phi_1(y,c)=\frac{\phi(y,c)}{U(y)-c}$ *and the constant* $C>0$ *may depend on* α.

We refer to Proposition 4.5 of [1] for details of the proof.

4.1.2. *Estimates of the Solution* ϕ. Getting estimates of the solution ϕ and its derivatives is crucial. It lays a solid foundation for our further estimates of the stream function, vorticity and so on. We refer to section 5 of [1] for all estimates needed. The most remarkable one is showing that $\phi(y,c)\sim\frac{\sinh(\alpha(y-y_c))}{\alpha}$ when $c\in D_0$. More precisely, we have

Proposition 4.12. *There exists a positive constant* C *independent of* α *such that for any* $(y,c)\in[0,1]\times D_0$,

$$\frac{\sinh(\alpha(y-y_c))}{C\alpha}\leq\phi(y,c)\leq\frac{C\sinh(\alpha(y-y_c))}{\alpha}. \tag{4.13}$$

Moreover, it holds that

$$|\partial_y^\beta\partial_c^\gamma\phi_1(y,c)|\leq C\alpha^{\beta+\gamma}\phi_1(y,c) \tag{4.14}$$

for $\beta+\gamma\leq 2$.

We remark that the function $\frac{\sinh(\alpha(y-y_c))}{\alpha}$ is the solution to the equation

$$w''-\alpha^2w=0$$

with the initial condition

$$w_1(y_c)=1,\quad w_1'(y_c)=0$$

where $w_1:=\frac{w}{y-y_c}$. The similarity between homogeneous Rayleigh equation (3.8) and the above differential equation indicates that ϕ may be close to $\frac{\sinh(\alpha(y-y_c))}{\alpha}$ because they share the same initial condition.

We now give a sketch of proof of Proposition 4.12. Let $\tilde\phi_1(y,c):=\frac{\alpha\phi(y,c)}{\sinh(\alpha(y-y_c))}$. Then $\tilde\phi_1(y,c)$ satisfies

$$\tilde\phi_1(y,c)=U'(y_c)+S\tilde\phi_1(y,c)$$

where the integral operator S is defined by

$$Sf:=\int_{y_c}^{y}\frac{dy'}{\sinh^2(\alpha(y'-y_c))}\int_{y_c}^{y'}\frac{\sinh^2(\alpha(z-y_c))U''(z)}{U(z)-U(y_c)}f(z,c)dz.$$

Direct computation shows that the integral operator S has better properties than T introduced in section 4.1.1. It is a bounded operator on the space $L^\infty([0,1]\times D_0)$. Besides, there exists a threshold $M_0>1$ such that for any $\alpha>M_0$, S is a contraction. Therefore, we have proved (4.13) for each $\alpha>M_0$. For $\alpha\leq M_0$, (4.13) and (4.14) all result from direct computations.

To finish the proof, we only have to prove (4.14) for $\alpha > M_0$. The main idea is considering the two cases, $|y-y_c| \geq \frac{1}{\alpha}$ and $|y-y_c| < \frac{1}{\alpha}$, respectively. Since y_c is the singular point of homogeneous Rayleigh equation, the first case is much easier and follows from direct computations. We deal with the second case. The integral operators $T_{k,j}$ introduced in section 4.1.1 will appear when taking derivatives of $S\tilde{\phi}_1$. Therefore, the main component of the proof is analyzing boundedness of $T_{k,j}$. We refer to Proposition 5.1 of [1] for more details.

4.2. **Inhomogeneous Rayleigh Equation.**

4.2.1. *Well-Definedness of* (4.2). Having chosen a good ϕ, we go back to analyze when the representation formula (4.2) is well-defined. To avoid singularity of the homogeneous Rayleigh equation (3.8), we first assume $c \in \Omega_{\epsilon_0} - D_0$. Then well-definedness is guaranteed if $\int_0^1 \frac{1}{\phi(y,c)^2} dy \neq 0$.

Let φ be the solution of (3.8) with initial condition $\varphi(0,c) = 0$ and $\varphi'(0,c) = 1$. Since (3.8) have no singularity, such φ exists and is unique. Besides, $\varphi(y,c)$ is also continuous in the c variable. We can directly check that

$$\varphi(y,c) = \phi(0,c)\phi(y,c) \int_0^y \frac{1}{\phi(z,c)^2} dz. \tag{4.15}$$

Hence

$$\varphi(1,c) = -\rho(c)\phi_1(0,c)\phi_1(1,c) \int_0^1 \frac{1}{\phi(z,c)^2} dz \tag{4.16}$$

where $\rho(c) = (U(1)-c)(c-U(0))$. By Proposition 4.10, $\int_0^1 \frac{1}{\phi(y,c)^2} dy \neq 0$ is equivalent to $\varphi(1,c) \neq 0$.

Since $\epsilon_0 > 0$ can be arbitrarily small, we can see if $\rho(c) \int_0^1 \frac{1}{\phi(z,c)^2} dz$ uniformly converges to a nonzero function on D_0 as ϵ_0 tends to zero. We consider the case where $c = c_r + i\epsilon$ such that $c_r \in [U(0), U(1)]$ and $\epsilon > 0$.

The idea of obtaining convergence results is repeatedly extracting terms without singularity based on estimates of ϕ_1. Since ϕ_1 is close to 1, we decompose

$$\begin{aligned} \rho(c) \int_0^1 \frac{1}{\phi(z,c)^2} dz = \rho(c) \int_0^1 \frac{1}{(U(z)-c)^2} \left(\frac{1}{\phi_1(z,c)^2} - 1 \right) dz \\ + \rho(c) \int_0^1 \frac{1}{(U(z)-c)^2} dz \end{aligned} \tag{4.17}$$

and extract the first term on the right hand side. Let

$$\Pi_3 := \int_0^1 \frac{1}{(U(z)-c_r)^2} \left(\frac{1}{\phi_1(z,c_r)^2} - 1 \right) dz.$$

Then by (4.11) and Lebesgue dominated convergence theorem,

$$\rho(c) \int_0^1 \frac{1}{(U(z)-c)^2} \left(\frac{1}{\phi_1(z,c)^2} - 1 \right) dz \longrightarrow \rho(c_r)\Pi_3.$$

as $\varepsilon \to 0$.

To analyse the second term on the right hand side of (4.17), we decompose

$$\rho(c) \int_0^1 \frac{dy}{(U(y)-c)^2} dy = \frac{I_1 - I_2}{U'(y_c)} \tag{4.18}$$

where

$$I_1 = \rho(c)\int_0^1 \frac{U'(y)dy}{(U(y)-c)^2}$$

and

$$I_2 = \rho(c)\int_0^1 \frac{U'(y)-U'(y_c)}{(U(y)-c)^2}dy.$$

By direct computations, $I_1 = U(0) - U(1)$.

The main reason for such a decomposition is that the integrand of I_2 has one order less singularity than $\frac{1}{(U(y)-c)^2}$, which is the integrand of the left hand side of (4.18). Since $U''(y_c)$ is not necessarily zero, the integrand of I_2 still has singularity of order one and isn't integrable. So we further decompose

$$I_2 = \rho(c)\int_0^1 \frac{g(y)}{(U(y)-c)^2}dy + \rho(c)\frac{U''(y_c)}{U'(y_c)^2}\int_0^1 \frac{U'(y)}{U(y)-c}dy \tag{4.19}$$

where

$$g(y) := U'(y) - U'(y_c) - \frac{U''(y_c)}{U'(y_c)^2}U'(y)(U(y)-c).$$

We remark that the fact $g''(y_c) = 0$ implies that the integrand $\frac{g(y)}{(U(y)-c)^2}$ has no singularity. Therefore,

$$\rho(c)\int_0^1 \frac{g(y)}{(U(y)-c)^2}dy \longrightarrow \rho(c_r)\int_0^1 \frac{g(y)}{(U(y)-c_r)^2}dy$$

as $\varepsilon \to 0$.

For the second term on the right hand side of (4.19), by direct computations, we have

$$\rho(c)\int_0^1 \frac{U'(y)}{(U(y)-c)}dy \longrightarrow \rho(c_r)\text{P.V.}\int_0^1 \frac{U'(y)}{(U(y)-c_r)}dy + i\pi\rho(c_r)$$

where we define the principal value of a singular integral as

$$\text{P.V.}\int_0^1 \frac{U'(y)}{(U(y)-c_r)}dy = \lim_{\epsilon\to 0^+}\int_{[0,1]\cap\{|y-U^{-1}(c_r)|>\epsilon\}} \frac{U'(y)}{(U(y)-c_r)}dy.$$

This definition allows us to perform cancellations when y approaches the singular point $U^{-1}(c_r)$. Similarly, we define

$$\text{P.V.}\int_0^1 \frac{U'(y)-U'(y_c)}{(U(y)-c_r)^2}dy = \lim_{\epsilon\to 0^+}\int_{[0,1]\cap\{|y-U^{-1}(c_r)|>\epsilon\}} \frac{U'(y)-U'(y_c)}{(U(y)-c_r)^2}dy.$$

Let

$$\Pi_2 := \text{P.V.}\int_0^1 \frac{U'(y)-U'(y_c)}{(U(y)-c_r)^2}dy,$$

$$A(c) := U(0) - U(1) - \rho(c)\Pi_2 + U'(y_c)\rho(c)\Pi_3,$$

$$B(c) := \pi\rho(c)\frac{U''(y_c)}{U'(y_c)^2}.$$

Then we have as $\epsilon \longrightarrow 0^+$,

$$\rho(c)\int_0^1 \frac{1}{\phi(y,c)^2}dy \longrightarrow \frac{1}{U'(y_c)}\big(A(c_r) - iB(c_r)\big). \tag{4.20}$$

Similarly, for $c = c_r - i\epsilon$ where $c_r \in [U(0), U(1)]$ and $\epsilon > 0$, we have as $\epsilon \longrightarrow 0^+$,

$$\rho(c) \int_0^1 \frac{1}{\phi(y,c)^2} dy \longrightarrow \frac{1}{U'(y_c)} \big(A(c_r) + iB(c_r) \big). \tag{4.21}$$

For $c \in B^l_{\epsilon_0} \cup B^r_{\epsilon_0}$, we have as $\epsilon \longrightarrow 0^+$,

$$\rho(c) \int_0^1 \frac{1}{\phi(y,c)^2} dy \longrightarrow \frac{U(0) - U(1)}{U'(y_c)}. \tag{4.22}$$

We refer to Lemma 6.3 of [1] for details of the proof.

If there exists $c_r \in [U(0), U(1)]$ such that $A(c_r) = B(c_r) = 0$, then $c_r \in (U(0), U(1))$ because $A(U(0)) = A(U(1)) = U(0) - U(1)$. Then $B(c_r) = 0$ implies $U''(c_r) = 0$. Through the singularity analysis of I_2 in (4.18), we know that if $U''(c_r) = 0$, then there is no singularity in the representation formula (4.15). Besides, there is no singularity in the homogeneous Rayleigh equation (3.8). Hence (4.15) still holds for c_r. From (4.16) we obtain

$$\varphi(1,c) = -\frac{\phi_1(0,c)\phi_1(1,c)A(c)}{U'(y_c)}. \tag{4.23}$$

Let $c = c_r$ in (4.23). Then $\varphi(1,c) = 0$. Hence $\varphi(\cdot, c)$ is a nonzero solution to the homogeneous Rayleigh equation (3.8) with boundary condition $\varphi(0,c) = \varphi(1,c) = 0$. This means that c_r is an eigenvalue. But the fact $c_r \in [U(0), U(1)]$ implies that c_r is not in the discrete spectrum. Hence c_r is an embedded eigenvalue of $\mathcal{R}_\alpha$, which is a contradiction. Hence $A(c_r)$ and $B(c_r)$ cannot be zero at the same time. By (4.20), (4.21), (4.22) and the fact that $\varphi(1,c)$ is continuous in c variable, there exists $\epsilon_0 > 0$ which is sufficiently small such that for any $c \in \Omega_{\epsilon_0}$, $\varphi(1,c) \neq 0$. Therefore we have proved well-definedness of the representation formula (4.2).

4.2.2. *Back to the Representation Formula of the Stream Function.* With the explicit representation formula of $\Phi(y,c)$ and various estimates of $\phi_1(y,c)$, we can now work on the convergence of $\Phi(y,c)$ as $c \longrightarrow c_r$. We refer to Proposition 6.7 of [1] for relevant results.

The reason for studying convergence results is to perform cancellation in the representation formula for $\hat{\psi}$ (3.19). In fact, there exists a function $\tilde{\Phi}(\alpha, y, c)$ defined on $\mathbb{Z} \times [0,1] \times [U(0), U(1)]$ such that

$$\hat{\psi}(t, \alpha, y) = \frac{1}{2\pi} \int_{U(0)}^{U(1)} \alpha \tilde{\Phi}(\alpha, y, c) e^{-i\alpha ct} dc. \tag{4.24}$$

$\tilde{\Phi}$ is obtained via convergence analysis of Φ and has better regularity than Φ.

5. A Duality Argument

In this section we introduce a duality argument used to estimate L^2 norm of velocity via vorticity. Denote by $\Omega_1 := \mathbf{T} \times (0,1)$. For any domain $\Omega_2 \subset \Omega_1$, we introduce the following subspaces of $L^2(\Omega_2)$:

$$H(\Omega_2) := \{u \in L^2(\Omega_2) : \ \nabla \cdot u = 0 \text{ and } u \cdot n = 0\},$$
$$G(\Omega_2) := \{u \in L^2(\Omega_2) : \ \exists q \in W^{1,2}_{loc}(\Omega_2) \text{ s.t. } u = \nabla q\}$$

where n is the outward-pointing normal vector of the boundary $\partial\Omega_2$. Here we do not distinguish vector-valued L^2 functions from scalar-valued L^2 functions in notations. When $\Omega_2 = \Omega_1$, we denote the corresponding subspaces by H and G. $H^\perp$ denotes the orthogonal complement of H in $L^2(\Omega_1)$. First we show that $G \subset H^{\perp}$. For each $u \in H$ and $q \in W^{1,2}_{loc}(\Omega_1)$, we have

$$\int_{\Omega_1} u \cdot \nabla q = -\int_{\Omega_1} (\nabla \cdot u) q + \int_{\partial\Omega_1} q u \cdot n$$

via integration by parts. Then by the definition of H we know that the right hand side is zero. Hence $G \subset H^{\perp}$.

In fact, we have the following lemma:

Lemma 5.1. *For any $u \in H^{\perp}$, there exists $q \in W^{1,2}_{loc}(\Omega_1)$ such that $u = \nabla q$.*

By the lemma, we know that $G = H^{\perp}$. Hence $L^2(\Omega_1)$ is the orthogonal direct sum of H and G. This is a partial result of *The Helmholtz–Weyl Decomposition*. We refer to chapter 3 of [4] for more comprehensive discussions.

To prove the lemma, we introduce the *standard mollifier* $\eta \in C^{\infty}(\mathbf{R}^2)$ and $\eta_\epsilon(x) := \epsilon^{-2}\eta(\epsilon^{-1}x)$ for each $\epsilon > 0$. We refer to Appendix C in [3] for definition details.

Proof. We split the proof into two steps.

Step 1: *prove the lemma when $u \in C^{\infty}(\Omega_1)$*

We first assume that $u \in C^{\infty}(\Omega_1) \cap H^{\perp}$. We show that there exists $q \in C^{\infty}(\Omega_1)$ such that $u = \nabla q$. In fact, we only have to prove that the integration of u along any piecewise smooth Jordan curve is zero. Suppose the curve l is represented by a piecewise smooth function $\gamma : [0,1] \longrightarrow \Omega_1$. Then

$$\int u \cdot dl = \int_0^1 u(\gamma(t)) \cdot \frac{d\gamma}{dt} dt.$$

For any sufficiently small $\epsilon > 0$, let $\Phi_\epsilon(x) := \int_0^1 \eta_\epsilon(x - \gamma(t)) \frac{d\gamma}{dt} dt$. Since the image of γ is a compact subset of Ω_1, the distance between the curve and the boundary of Ω_1 is positive. Hence as long as ϵ is sufficiently small, $\eta_\epsilon(x - \gamma(t))$ will vanish when $x \in \partial\Omega_1$ which shows that $\Phi_\epsilon \cdot n = 0$. We can check that $\Phi_\epsilon \in C_0^{\infty}(\Omega_1)$ and obtain $\nabla \cdot \Phi_\epsilon = 0$ almost everywhere through direct computations. Then $\Phi_\epsilon \in H$ and $\int_{\Omega_1} u \cdot \Phi_\epsilon = 0$.

Let $u_\epsilon := u * \eta_\epsilon$. By a direct computation we have $\int u_\epsilon \cdot dl = \int_{\Omega_1} u \cdot \Phi_\epsilon = 0$. Since $u \in C^{\infty}$, u_ϵ converges uniformly on compact subsets to u as $\epsilon \longrightarrow 0^+$. We refer to Appendix C of [3] for proof of this fact. Thus $\int u_\epsilon \cdot dl \longrightarrow \int u \cdot dl$ and $\int u \cdot dl = 0$. Therefore, there exists $q \in C^{\infty}$ such that $u = \nabla q$.

Step 2: *approximate an L^2 function by C^{∞} functions.*

Now we consider the case where $u \in H^{\perp}$. For any domain Ω_1' such that the closure of Ω_1' in Ω_1 is a compact subspace of Ω_1, there exists a domain Ω' containing Ω_1' whose closure is still compact in Ω_1.

We now show that for sufficiently small $\epsilon > 0$, $u_\epsilon \in H(\Omega')^{\perp}$, where the convolution is taken in Ω'. For any $w \in H(\Omega')$, denote by w_1 the function on Ω_1 obtained by w extending by zero. Then $w_\epsilon := w_1 * \eta_\epsilon$ lies in H, where the convolution is taken in Ω_1. Unwind the definition of convolution, we will obtain $w_\epsilon = w * \eta_\epsilon$, where the latter convolution is taken over Ω'. By Fubini's theorem, $\int_{\Omega'} u_\epsilon \cdot w = \int_{\Omega'} u \cdot w_\epsilon$. Since w_ϵ lies in H, $\int_{\Omega'} u_\epsilon \cdot w = 0$. Hence $u_\epsilon \in H(\Omega')^{\perp}$.

Since $u_\epsilon \in C^{\infty}(\Omega')$, there exists $q_\epsilon \in C^{\infty}(\Omega')$ such that $u_\epsilon = \nabla q_\epsilon$. Since $u_\epsilon \longrightarrow u$ in L^1 norm, $\{\nabla q_\epsilon\}$ is a Cauchy sequence in $L^1(\Omega')$, hence in $L^1(\Omega_1')$. Now q_ϵ

is bounded in the domain Ω_1' because of smoothness, hence $q_\epsilon \in L^1(\Omega_1')$. We can modify these q_ϵ such that $\int_{\Omega_1'} q_\epsilon = 0$ for any sufficiently small $\epsilon > 0$. By Poincaré inequality, $\{q_\epsilon\}$ is a Cauchy sequence in $W^{1,1}(\Omega_1')$. Therefore, there exists $q_1 \in W^{1,1}(\Omega_1')$ such that $\{q_\epsilon\}$ converges to q_1 in $W^{1,1}(\Omega_1')$ and $u = \nabla q_1$ in Ω_1'.

For each Ω_1' whose closure is compact in Ω_1, we obtain $q_1 \in W^{1,1}(\Omega_1')$ such that $u = \nabla q_1$ in Ω_1'. Then we can obtain $q \in W^{1,1}_{loc}(\Omega_1)$ such that $u = \nabla q$ in Ω_1 by gluing all of the q_1. Different q_1 are compatible by the construction procedure described above.

It remains to prove that $q \in W^{1,2}_{loc}(\Omega_1)$. For any domain Ω_1' whose closure is compact in Ω_1, $u = \nabla q \in L^2(\Omega_1')$. Then we obtain $q \in H^1(\Omega_1)$ by Poincaré inequality again. □

Now we can show the duality argument. Our treatment is based on [5]. Consider $V \in H$ and $\phi = \mathrm{curl}(V)$. Let $w \in L^2(\Omega_1)$. Suppose $\|w\|_{L^2} = 1$. By Helmholtz–Weyl decompostion, there exist $w_1 \in H$, $w_2 \in G$ such that $w = w_1 + w_2$. Then $\|w_1\|_{L^2} \le \|w\|_{L^2}$. Let $\psi, \Psi \in H_0^1$ be functions such that $V = (-\partial_y \psi, \partial_x \psi)$ and $w_1 = (-\partial_y \Psi, \partial_x \Psi)$. Since $\|\nabla \Psi\|_{L^2} = \|w_1\|_{L^2}$, by Poincaré inequality, $\|\Psi\|_{H^1} \le C$ for some positive constant C. We have

$$\begin{aligned}
\|V\|_{L^2} &= \sup_{w \in L^2, \|w\|_{L^2}=1} \int V \cdot w \\
&= \sup_{w \in L^2, \|w\|_{L^2}=1} \int V \cdot w_1 \\
&\le \sup_{w_1 \in H, \|w_1\|_{L^2} \le 1} \int V \cdot w_1 \\
&\le \sup_{\Psi \in H_0^1, \|\Psi\|_{H^1} \le C} \int \nabla\psi \cdot \nabla\Psi \\
&\le C \sup_{\Psi \in H_0^1, \|\Psi\|_{H^1}=1} \int (-\Delta\psi) \cdot \Psi \\
&\le C \sup_{\Psi \in H_0^1, \|\Psi\|_{H^1}=1} \int \phi \cdot \Psi.
\end{aligned}$$

Hence we deduce a way to estimate L^2 norm of a function $V \in H$ via H^{-1} norm of $\mathrm{curl}(V)$. This strategy is referred to as the "duality argument".

We notice that the velocity u in Theorem 2.4 lies in H and $\mathrm{curl}(u)$ is exactly the vorticity ω. Hence this duality argument enables us to estimate velocity using vorticity.

Finally we briefly introduce the remaining part of proof of Theorem 2.4. The third conclusion, which indicates vorticity mixing, is a corollary of the first two conclusions. So we focus on the estimate of velocity.

We go back to the representation formula (4.24) of the stream function. The most straightforward approach is to derive the $O(t^{-2})$ estimate of $\hat{\psi}$. Then we can derive the $O(t^{-1})$ estimate of velocity. The general approach to obtaining convergence rate is to apply integration by parts using $d(e^{-ikct}) = -ikce^{-ikct}dt$. However, $\tilde{\Phi}$ in (4.24) does not have the regularity needed. So instead of the stream function, we estimate vorticity first and then obtain velocity estimates using the duality argument.

Fortunately, $\hat{\psi}$ appears in the expression of $\hat{\omega}$ by (2.7). Further we have

$$e^{i\alpha t U(y)}\hat{\omega}(t,\alpha,y) = \hat{\omega}_0(\alpha,y) - i\alpha U''(y)\int_0^t e^{i\alpha\tau U(y)}\hat{\psi}(\tau,\alpha,y)d\tau.$$

Thus the properties of $\hat{\psi}$ can be used to estimate $\hat{\omega}$.

Acknowledgments

First and foremost, I would like to sincerely thank my mentor, Prof. Jincheng Yang, for agreeing to be my mentor, for introducing the interesting topic, inviscid damping, to me, for generously devoting her time to meeting with me and two other mentees, usually lasting up to four hours one time, for always being there to answer any questions from big pictures to details of proofs, for giving me problems to explore and being very open to discussions, for inviting me and two other mentees to dinner and sharing experiences with us, and for helping me prepare my participant talk and revising the draft of this paper. Her passion for mathematics and her kindness have left a lasting impression on me.

I also extend my thanks to Prof. Peter May for inviting me to this REU, for helping me connect with mentors, and for being patient enough to answer my questions and share ideas with me. Besides, I would like to thank Prof. Justin Campbell for spending nearly two hours introducing geometric representation theory to me during our first meeting, for his support during moments of uncertainty, and for encouraging me to follow my interests.

It is also a pleasure to thank my friends from Peking University, including Kailin Pan, Kaiyi Chen, Yike He, Yebo Peng and Yuyang Feng, for accompanying me during this REU and helping me quickly adjust to the new environment. I would like to thank Kailin Pan and Kaiyi Chen for their explanations of complex concepts and arguments in algebraic geometry and algebraic topology, Yike He for our discussions on geometry and Yuyang Feng for introducing some fields in probability.

Besides, I owe my deep thanks to my fellow mentees, Michael Lee and Minh Pham, for welcoming me to join their group meetings, for introducing me to the topics they work on this summer, and for discussing and answering questions that arise from their reports and readings. I also want to thank Dadu Chen, Zihao Jin and Cloudifold Zhang for including me in various activities.

Finally, I am deeply thankful to my family for their unwavering encouragement and support, both financial and emotional.

References

[1] Dongyi Wei, Zhifei Zhang and Weiren Zhao. Linear inviscid damping for a class of monotone shear flow in Sobolev spaces. Comm. Pure Appl. Math., 71(2018), 617-687.
[2] Kōsaku Yosida. Functional Analysis. Springer-Verlag Berlin Heidelberg, 1995.
[3] Lawrence C. Evans. Partial Differential Equations. American Mathematical Society, 1997.
[4] Giovanni P. Galdi. An introduction to the mathematical theory of the Navier-Stokes equation. Vol I, Springer-Verlag, 1994.
[5] Zhiwu Lin. Nonlinear instability of ideal plane flows. Int. Math. Res. Not., 41(2004), 2147–2178.
[6] Lord Rayleigh. On the stability, or instability, of certain fluid motions. Proc. London Math. Soc., S1-11 (1880), no. 1, 57–70.
[7] Steve I. Rosencrans and David H. Sattinger. On the spectrum of an operator occurring in the theory of hydrodynamic stability. J. Math. and Phys., 45 (1966), 289–300.
[8] Zhiwu Lin and Chongchun Zeng. Inviscid dynamic structures near Couette flow. Arch. Rat. Mech. Anal., 200(2011), 1075-1097.