AN ELEMENTARY TREATMENT OF CHERN CLASSES

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ABSTRACT. Adapting the axiomatic approach of Grothendieck, we introduce and study the Chern classes of a smooth complex vector bundle. We conclude by showing that the Chern classes are the only cohomological invariant of complex vector bundles in a precise sense. By working in the de Rham cohomology and restricting to vector bundles over manifolds of finite type, we are able to make our exposition entirely elementary.

CONTENTS

1. INTRODUCTION

Characteristic classes are global invariants of topological spaces which unify algebraic topology, differential geometry, and algebraic geometry. In this paper, we introduce and study the *Chern classes* of a complex vector bundle. In a rough sense, these measure the extent to which a bundle is twisted, i.e. how far it deviates from the trivial bundle. Our main source will be Raoul Bott and Loring Tu's *Differential Forms in Algebraic Topology* [2]. In contrast to the approach employed herein, a purely geometric approach using connections and curvature is possible (and takes historical precedence). See John Milnor and Jim Stasheff's *Characteristic Classes* [6, Appendix C] for details.

We briefly describe the general approach: we explicitly construct the first Euler class of a line bundle $L \to M$ using the data of a partition of unity on M and the

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transition functions on L. This allows us to define the Chern class of an arbitrary bundle using the Leray-Hirsch theorem. We then introduce a useful computational tool: the splitting principle, which tool allows us to compute the cohomology ring of the complex Grassmannian in terms of the Chern classes of one of its subbundles. Then, using the fact that the Grassmannian is the classifying space for complex vector bundles, we use the computation of the cohomology ring of the Grassmannian to show that the Chern classes are the only cohomological invariants of a smooth complex vector bundle in a precise sense.

The main prerequisite for this paper is a familiarity with the general theory of fiber and vector bundles, though anything more than the basic definitions can be taken on faith by the reader without loss to their understanding of the general picture. We recommend Dale Husemoller's *Fibre Bundles* [5] and, for a concise treatment of the necessities, Michael Atiyah's *K-theory* [1]. In addition, familiarity with differential topology and the basics of de Rham cohomology at the level of Warner [8] will be essential.

2. Preliminaries

In this section, we fix some notation, recall the basics of vector bundles (including the Leray-Hirsch theorem), and define the first Euler class of an oriented rank 2 real bundle.

Following the excellent textbook *Differential Forms in Algebraic Topology* by Raoul Bott and Loring Tu [2], we make the following simplifying assumption: we work in the category of C^∞ manifolds of finite type, i.e. smooth n-manifolds equipped with a finite open cover $\{U_n\}_{n=1}^N$ such that all non-empty intersection $U_{i_1} \cap \cdots \cap U_{i_k}$ are diffeomorphic to $\mathbb{R}^n$. Thus, in the absence of qualification, a map is a C^∞ map of manifolds and $H^*(M)$ denotes the De-Rham cohomology, most often viewed as a graded algebra. Nearly every result herein is true in much greater generality than we state it, often with the same proof. With a few exceptions, we will not comment on this to keep the exposition mercifully brief.

To fix notation, we begin by briefly recalling the basics of vector bundles. The familiar reader should skip ahead to Section 2.1 and refer back as needed.

Definition 2.1. Let G be a Lie group which acts faithfully on a vector space V on the left, and let $\pi : E \to M$ be a surjective map between manifolds E and M such that each fiber $E_x := \pi^{-1}(x)$ is isomorphic to V. We will call π a C^∞ *real (resp. complex) vector bundle* of *rank* n with structure group G if $V \cong \mathbb{R}^n$ (resp. $V \cong \mathbb{C}^n$) and if there exists an open cover $\{U_\alpha\}$ of M and fiber-preserving diffeomorphisms

$$\varphi_\alpha : E\mid_{U_\alpha} = \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times V$$

such that the transition functions are smooth maps with values in G:

$$g_{\alpha\beta}(x) = \varphi_\alpha \circ \varphi_\beta^{-1}\mid_{\{x\}\times V} \in G.$$

Note that the transition functions satisfy the cocycle condition

$$g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma} \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma.$$

Unless otherwise stated, the structure group of a real vector bundle of rank n should be assumed to be $GL_n(\mathbb{R})$, and for complex bundles $GL_n(\mathbb{C})$.

We will have occasion to use the more general notion of *fiber bundle*, where the vector space V is replaced by a smooth manifold F and the fibers are required to be diffeomorphic to F. There are many other constructions in the general theory of fiber bundles which we will use and which do not appear in this collection of preliminary results. Notable among them include the definition of a pullback bundle, operations on bundles, orientations of bundles, reduction of structure group, and the existence of a metric on a bundle. For a complete treatments of these and more see [5].

We also recall the basic language of sections.

Definition 2.2. Let U be an open subset of M. A map $s : U \to E$ is a *section* of the vector bundle E over U if $\pi \circ s$ is the identity on U. When $U = M$, we call s a *global section*. A collection of sections $s_1, \dots, s_n$ over an open set U form a *frame* on U if for every $x \in U$, $\{s_1(x), \dots, s_n(x)\}$ is a basis for the vector space E_x.

We will make extensive use of the following basic invariant.

Definition 2.3. For a smooth manifold M, we define its *Poincaré series* to be the polynomial

$$P_t(M) := \sum_{k=0}^{\infty} \dim H^k(M) t^k.$$

We will need two standard theorems describing the cohomology of fiber bundles which we state without proof. The following version of the Leray-Hirsch theorem will suffice for our purposes. For a concise proof of a slightly more general version, see [5, Ch. 16, Theorem 1.1, pp. 231].

Theorem 2.4 (Leray-Hirsch)**.** *Let E be a fiber bundle over M with fiber F, where M is a smooth manifold of finite type. If there exist global cohomology classes $e_1, \dots, e_r$ on E which, when restricted to each fiber, freely generate the cohomology of the fiber, then $H^*(E)$ is a free module over $H^*(M)$ with basis given by $\{e_1, \dots, e_r\}$. That is*

$$H^*(E) \cong H^*(M) \otimes \mathbb{R}\{e_1, \dots, e_r\} \cong H^*(M) \otimes H^*(F)$$

The second is the Gysin sequence, which first requires the definition of the Euler class.

2.1. **The first Euler class.** We now construct the Euler class of an oriented rank 2 real vector bundle, following the explicit geometric approach of [2]. For a succinct algebraic construction via the Thom Isomorphism theorem, see [6, Section 9, pp. 98].

Let E be a rank 2 real vector bundle over M, E^0 the complement of the zero section in E, and $\{U_\alpha\}$ be an open cover of M. Endow E with a Riemannian structure so we have a radius function r on E and angular coordinates θ_α on each $E|_{U_\alpha}$ and so that we can choose an orthonormal frame on each U_α. This defines polar coordinates r_α, θ_α on $E^0|_{U_\alpha}$ in the following sense: if $x_1, \dots, x_n$ are coordinates on U_α, then $\pi^* x_1, \dots, \pi^* x_n, r_\alpha, \theta_\alpha$ are coordinates on $E^0|_{U_\alpha}$. On the overlaps $U_\alpha \cap U_\beta$, we see that r_α and r_β agree, whereas the angular coordinates θ_α and θ_β agree up to a rotation. Since E is oriented with fiber $\mathbb{R}^2$, we may speak of the counterclockwise direction in each fiber, which allows us to unambiguously define the functions $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{R}$ up to a multiple of 2π as the angle of rotation in the counterclockwise direction from the α-coordinate system to the β-coordinate system:

$$\pi^* \varphi_{\alpha\beta} := \theta_\beta - \theta_\alpha.$$

Though the functions $\{\varphi_\alpha \beta\}$ fail to satisfy the cocycle condition $\varphi_{\alpha\beta} + \varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$, by definition we have

$$\varphi_{\alpha\beta} + \varphi_{\beta\gamma} - \varphi_{\alpha\gamma} \in 2\pi\mathbb{Z} \tag{2.1.1}$$

The following lemma shows that the 1-forms $\{d\varphi_{\alpha\beta}\}$ in fact do satisfy this condition.

Lemma 2.2. *There exists one forms ξ_α on U_α for each α such that*

$$\frac{1}{2\pi} d\varphi_{\alpha\beta} = \xi_\beta - \xi_\alpha. \tag{2.1.3}$$

Proof. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to the cover $\{U_\alpha\}$. Now let

$$\xi_\alpha := \frac{1}{2\pi} \sum_\gamma \rho_\gamma d\varphi_{\gamma\alpha}.$$

Then by (2.1.1) and the symmetry $\varphi_{\alpha\beta} = -\varphi_{\beta\alpha}$, we have

$$\begin{aligned}
\xi_\beta - \xi_\alpha &= \frac{1}{2\pi} \sum_\gamma \rho_\gamma d(\varphi_{\gamma\beta} - \varphi_{\gamma\alpha}) \\
&= \frac{1}{2\pi} \sum_\gamma \rho_\gamma d(-\varphi_{\beta\alpha}) \\
&= \frac{1}{2\pi} \sum_\gamma \rho_\gamma d\varphi_{\alpha\beta} \\
&= \frac{1}{2\pi} d\varphi_{\alpha\beta}
\end{aligned}$$

as required. □

Taking the exterior derivative of both sides of (2.1.3) shows that we have an equality of 2-forms $\xi_\alpha = \xi_\beta$ on the overlaps $U_\alpha \cap U_\beta$. Hence, taking a partition of unity $\{\psi_\alpha\}$ subordinate to $\{U_\alpha\}$ allows us to glue these 2-forms together to obtain a global 2-form $e := \sum_\alpha \psi_\alpha d\xi_\alpha$ on M. e is closed, but not necessarily exact since exactness would require the 1-forms ξ_α to agree on overlaps, which does not necessarily (or usually) occur. One easily verifies that e is independent of the choice of ξ_α, so the following definition is justified.

Definition 2.4. The *Euler class* of a rank 2 oriented real vector bundle over a manifold M is the cohomology class of $e \in H^2(M)$. We may sometimes write $e(E)$ instead of e.

We will need two properties of the Euler class. The first is naturality.

Proposition 2.5. *The Euler class is functorial, i.e. if $f : N \to M$ is a C^∞ map of manifolds and E is a rank 2 oriented real vector bundle over M, then*

$$e(f^{-1}E) = f^* e(E).$$

Proof. We derive an explicit formula for $e(E)$ on each U_α in terms of the transition functions. Let $g_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(2)$ be the transition functions of E (we may take the structure group to be $SO(2)$ by the standard reduction argument as in [5]). Using the identification $SO(2) \simeq S^1$ by $\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \cos\theta & \sin\theta \end{smallmatrix}\right) \mapsto e^{i\theta}$, $g_{\alpha\beta}$ may be though of as complex valued functions and thus the angle betwee the β-coordinate system and the α-coordinate system is $-i \log g_{\alpha\beta}$. Thus

$$\theta_\alpha - \theta_\beta = \pi^*(-i \log g_{\alpha\beta})$$

and

$$\pi^*\varphi_{\alpha\beta} = -\pi^*(-i\log g_{\alpha\beta}).$$

Since π is a surjection and thus of maximal rank, π^* is injective and thus

$$\varphi_\alpha\beta = i\log g_{\alpha\beta}.$$

Now, let $\{\rho_\gamma\}$ be a partition of unity subordinate to $\{U_\gamma\}$. Then

$$\frac{1}{2\pi}d\varphi_{\alpha\beta} = \xi_\beta - \xi_\alpha$$

where

$$\xi_\alpha := \frac{1}{2\pi}\sum_\gamma \rho_\gamma d\varphi_{\gamma\alpha} = \frac{i}{2\pi}\sum_\gamma \rho_\gamma d\log g_{\alpha\beta}.$$

Therefore

$$e(E) = \frac{i}{2\pi}\sum_\gamma d(\rho_\gamma d\log g_{\alpha\beta}) \qquad \text{on} \quad U_\alpha. \tag{2.1.6}$$

Functoriality of the Euler class then follows immediately from (2.1.6). □

As a consequence of functoriality, we obtain the following useful fact.

Corollary 2.7. *If E is a trivial bundle over M, then $e(E) = 0$.*

Proof. Let X be a one-point space and let $f : M \to X$ be the unique map. Since the cohomology of X in nonzero dimensions is zero and E coincides with the pullback bundle $f^{-1}E$, the naturality of the first Euler class implies that

$$e(E) = e(f^{-1}E) = f^*(e(E)) = 0$$

as required. □

The following lemma is also very useful, especially in tandem with Corollary 2.7.

Lemma 2.8. *If a complex line bundle L has a nowhere vanishing section, then L is trivial*

Proof. If s is a nowhere vanishing section of L, then the map $(x, \lambda) \mapsto \lambda s(x)$ from $M \times \mathbb{C} \to L$ is a trivialization. □

As promised, we end the section by stating a standard result of algebraic topology: the Gysin sequence. We state a far less general form than this theorem is available. For a proof of a more general statement see Milnor-Stasheff [6, Theorem 12.2, pp. 143].

Theorem 2.9. *Let $\pi : E \to M$ be an oriented fibre bundle with fibre S^1. Then there is a long exact sequence*

$$\cdots \to H^k(E) \xrightarrow{\pi_*} H^{k-1}(M) \xrightarrow{\wedge e} H^{k+1}(M) \xrightarrow{\pi^*} H^{k+1}(E) \to \dots$$

where the maps $\pi_*, \wedge e$, and π^* are integration along the fiber, multiplication by the Euler class of E, and the pullback by π, respectively.

3. Construction and first properties of the Chern classes

Adapting the axiomatic approach of Grothendieck [3], we show that to define the Chern classes of an arbitrary finite rank complex vector bundle, it suffices to define the first Chern class of a complex line bundle. In particular, we will define the first Chern class of a complex line bundle as the Euler class of the underlying real bundle. Then, using the Leray-Hirsch Theorem 2.4, we compute the cohomology ring of the projectivization $P(E)$ of an arbitrary complex vector bundle E and define the Chern classes of E in terms of the ring structure of $H^*(P(E))$.

3.1. **The first Chern class of a line bundle.** By discarding the complex structure on the fiber, any complex vector bundle E of rank n has an underlying real vector bundle $E_{\mathbb{R}}$ of rank $2n$. It's of note that, in the case of complex line bundles, there is actually a bijection between complex line bundles and oriented rank 2 real vector bundles: this follows from a reduction of structure group argument hinging on the isomorphism $U(1) \cong SO(2)$. See [2, pp. 267] for details. With this identification, we may make the following crucial definition.

Definition 3.1. For a complex line bundle L over a manifold M, we define the *first Chern class* to be the Euler class of the underlying real vector bundle $L_{\mathbb{R}}$: $c_1(L) := e(L_{\mathbb{R}}) \in H^2(M)$.

We can see immediately that the first Chern class satisfies two nice properties. If L and L' are complex line bundles with transition functions $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$, then their tensor product $L \otimes L'$ is the complex line bundle with transition functions $\{g_{\alpha\beta} \cdot g'_{\alpha\beta}\}$. Thus the explicit formula (2.1.6) for the Euler class in terms of the transition functions gives us

$$c_1(L \otimes L') = c_1(L) + c_1(L'). \tag{3.1.2}$$

This identity shows that

$$c_1(L^*) = -c_1(L) \tag{3.1.3}$$

since $L \otimes L^* \cong End(L)$ has a nowhere vanishing section given by the identity map and thus by Lemma 2.8 and Corollary 2.7 has a trivial first Chern class. Therefore by (3.1.2) we have

$$0 = c_1(L \otimes L^*) = c_1(L^*) + c_1(L)$$

and thus (3.1.3) follows.

Definition 3.4. Let V be a complex vector space and $P(V)$ its *projectivization*:

$$P(V) := \{1 - \text{dimensional subspaces of } V\}.$$

$P(V)$ is readily seen to be a smooth manifold of finite type (in fact $P(V)$ is compact). On $P(V)$ there are three God-given bundles we will consider: the *product bundle* $\hat{V} := P(V) \times V$, the *universal subbundle* S defined by

$$S := \{(\ell, v) \in P(V) \times V \mid v \in \ell\},$$

and the *universal quotient bundle* Q defined by the exact sequence

$$0 \to S \to \hat{V} \to Q \to 0.$$

We will sometimes refer to the dual S^* as the hyperplane bundle.

Using the Gysin sequence Theorem 2.9, we can compute the cohomology ring of the projectivization of an n-dimensional complex vector space: by definition $\xi := e(S)$ is equal to the Euler class of S. Equip V with a Hermitian metric and let

$$E := \{(\ell, v) \in P(V) \times S^{2n-1} \mid v \in \ell\}$$

be the unit sphere bundle of S. Since the projection onto the second coordinate is a diffeomorphism $E \cong S^{2n-1}$, the Gysin sequence for E reduces to

$$0 = H^{k+1}(E) \to H^k(P(V)) \xrightarrow{\wedge \xi} H^{k+2}(P(V)) \to H^{k+2}(E) = 0$$

for $0 \leq k \leq 2n-2$. Thus the odd-dimensional groups vanish and

$$H^0(P(V)) \cong H^2(P(V)) \cong \ldots \cong H^{2n}(P(V)).$$

Since $P(V)$ is connected we have $H^0(P(V)) \cong \mathbb{R}$ and thus $H^{2k}(P(V)) \cong \mathbb{R}$ and generated by ξ^k for all $1 \leq k \leq n$. This shows that

$$H^*(P(V)) \cong \mathbb{R}[\xi]/(\xi^n). \tag{3.1.5}$$

From (3.1.5) we see that the Poincaré polynomial of $P(V)$ is

$$P_t(P(V)) = 1 + t^2 + t^4 + \ldots t^{2(n-1)} = \frac{1 - t^{2n}}{1 - t^2}.$$

This description of the ring structure of $H^*(P(V))$ lets us define Chern classes in the general case.

3.2. Construction of the Chern classes.

Henceforth, all vector bundles are complex and of arbitrary rank n. We are now equipped to define the Chern classes of such a bundle. We first need a few definitions.

Let $\rho : E \to M$ be a complex vector bundle of rank n with transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{C})$

Definition 3.1. The *projectivization* of E, $\pi : P(E) \to M$, is the vector bundle whose fiber at $p \in M$ is the projective space $P(E_p)$ and whose transition functions $\overline{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to PGL_n(\mathbb{C})$ are induced by $g_{\alpha\beta}$ (via the quotient $PGL_n(\mathbb{C}) := GL_n(\mathbb{C})/\{\text{scalar matrices}\}$).

Example 3.2. We consider three tautological bundles on $P(E)$: the pullback bundle $\pi^{-1}E$, the *universal subbundle* S, and the *universal quotient bundle* Q.

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r] & \pi^{-1}E \arrow[r] \arrow[d] & Q \arrow[r] & 0 \\
 & & P(E) \arrow[drr, "\pi"] & & E \arrow[d, "\rho"] \\
 & & & & M
\end{tikzcd}$$

The pullback bundle $\pi^{-1}E$, which is the vector bundle over E whose fiber at $\ell \in P(E_p) \subset P(E)$ is E_p. When restricted to the fiber $\pi^{-1}(p)$ it becomes the trivial bundle $P(E)_p \times E_p$ since $\rho : E_p \to \{p\}$ is trivial.

The universal subbundle S over $P(E)$ is given by

$$S := \{(\ell, v) \in \pi^{-1}E \mid v \in \ell\}$$

and the universal quotient bundle Q is the bundle determined by the exact sequence

$$0 \to S \to \pi^{-1}E \to Q \to 0.$$

To avoid confusion we may sometimes write S_E and Q_E to indicate the dependence on E.

Now we may define the Chern classes of E. Set $x := c_1(S^*) \in H^2(P(E))$. The universal subbundle $\tilde{S}$ of $P(E_p)$ coincides with the restriction of S to the fiber $P(E_p)$, so $c_1(\tilde{S})$ is the restriction of x to $P(E_p)$ by the naturality of the first Chern class. Hence, by (3.1.5), $\{1, x, \ldots, x^{n-1}\}$ is a set of global cohomology classes on $P(E)$ whose restriction to each fiber freely generates the cohomology of the fiber. Thus, by the Leray Hirsch theorem $H^*(P(E))$ is a free module over $H^*(M)$ with basis $\{1, x, \ldots, x^{n-1}\}$. In particular, the element $-x^n$ of $H^*(P(E))$ may be expressed uniquely as a linear combination of $\{1, x, \ldots, x^{n-1}\}$ with coefficients in $H^*(M)$. These coefficients are our Chern classes:

Definition 3.3. We define the *Chern classes* of the complex vector bundle E to be the unique cohomology classes $c_i(E) \in H^{2i}(E)$ such that

$$x^n + c_1(E)x^{n-1} + \cdots + c_n(E) = 0. \tag{3.2.4}$$

In this equation, by $c_i(E)$ we really mean $\pi^* c_i(E)$.[1] We call $c_i(E)$ the *ith Chern class* of E and the sum

$$c(E) = 1 + c_1(E) + \cdots + c_n(E) \in H^*(M)$$

the *total Chern class* of E. We define $c_i(E) := 0$ for all $i > n$.

We thus obtain the following:

Proposition 3.1. The ring structure of the cohomology of $P(E)$ is given by

$$H^*(P(E)) = H^*(M)[x]/(x^n + c_1(E)x^{n-1} + \cdots + c_n(E)) \tag{3.2.5}$$

where $x := c_1(S^*)$. The Poincaré polynomial of E is

$$P_t(P(E)) = P_t(M)\frac{1 - t^{2n}}{1 - t^2}. \tag{3.2.6}$$

The first fact is immediate from the Leray Hirsch theorem, and the second is a consequence of the Künneth formula. For the statement and proof of the Künneth formula, see [2, pp. 47]

We leave it to the reader to verify that the two definitions of the first Chern class of a line bundle coincide.

The following proposition summarizes some basic properties of the Chern classes

Proposition 3.7. *Let E and E' be complex vector bundles of rank n and m over a common base space M, let L be a complex line bundle, and let $f : N \to M$ be a map of manifolds.*

- *i) (Naturality) $c(f^{-1}E) = f^*c(E)$.*
- *ii) (Normalization) $c(L) = 1 + e(L_{\mathbb{R}})$.*
- *iii) (Whitney Product formula) $c(E \oplus E') = c(E)c(E')$.*
- *iv) If E has a nonvanishing section, then the top Chern class $c_n(E)$ vanishes.*

[1] We will have occasion to make this abuse of notation again, and we will make it clear when we do so.

It is a fact, which we will not prove, that properties i)-iii) uniquely characterize the Chern class (see Hirzebruch [4, pp. 58-61]). We have already seen property ii), property i) follows readily from the functoriality of each of the constructions involved in the definition of the Chern class, and property iii) will be a consequence of the techniques developed in Section 4, so we will supply an independent proof of the last property which introduces a useful technique. With the Whitney Product formula in hand, iv) is immediate.

Proof of iv). Let s be a nowhere vanishing section of E. s induces a section $\tilde{s}$ of $P(E)$ whose value at a point $p \in X$ is the unique line through $s(p)$ and the origin in E_p. Then $\tilde{s}^{-1}S$ is a line bundle with a nowhere vanishing section and thus by Lemma 2.8 is the trivial line bundle. Thus by the naturality of the Chern class $\tilde{s}^{-1}c_1(S_E) = 0$ so that $\tilde{s}^*x = 0$. Pulling back the equation

$$x^n + c_1(E)x^{n-1} + \cdots + c_n(E) = 0$$

along $\tilde{s}$ shows that $\tilde{s}^*c_n(E) = 0$. By our abuse of notation this means that $\tilde{s}^*\pi^*c_n(E) = 0$, so indeed $c_n(E) = 0$ as desired. $\square$

For several properties of the Chern classes not relevant to our purposes see [4].

4. The Splitting Principle and the Whitney product formula

In this section we prove the Whitney product formula and compute the cohomology ring of a flag manifold. The main tool will be the splitting principle.

4.1. The Splitting Principle. Given a vector bundle $\pi : E \to M$, we want to construct a manifold $F(E)$, which we will call a *split manifold* of E, and a map $\sigma : F(E) \to M$ with the following two properties:

i) the pullback of E to $F(E)$ splits as a direct sum of line bundles
ii) σ^* is an epimorphism and thus embeds $H^*(M)$ in $H^*(F(E))$.

We construct split manifolds in the rank 2 and 3 cases in order to illustrate the general construction. If E has rank 1, then $F(E) = E$ and $\sigma = \pi$ suffice. If E has rank 2, then we take $F(E) = P(E)$ and $\sigma : P(E) \to M$ to be the projection. On $P(E)$ we have the short exact sequence

$$0 \to S_E \to \sigma^{-1}E \to Q_E \to 0$$

and since an exact sequence of C^∞ complex vector bundles splits, the pullback $\pi^{-1}E = S_E \oplus Q_E$ splits as a direct sum of line bundles.

When E has rank 3, $\sigma^{-1}E$ splits as a direct sum of the line bundle S_E and the quotient bundle Q_E. Q_E has rank 2 over $P(E)$, so the pullback to $P(Q_E)$ splits as a direct sum of line bundles:

$$\begin{tikzcd}
 & & \beta^{-1}S_E \oplus S_{Q_E} \oplus Q_{Q_E} \arrow[d] \\
 & S_Q \oplus Q_E \arrow[d] & P(Q_E) \arrow[dl, "\beta"'] \\
E \arrow[d] & P(E) \arrow[dl, "\alpha"'] & \\
M & &
\end{tikzcd}$$

Thus we may take $F(E) := P(Q_E)$ and $\sigma := \alpha \circ \beta$.

Hopefully the general pattern is now clear: we split off one subbundle at a time by pulling back to the projectivization of a quotient bundle. For a vector bundle E with rank n, $F(E)$ will be obtained by a sequence of $n-1$ projectivizations:
(4.1.1)

$$\begin{tikzcd}
 & & & S_1 \oplus \cdots \oplus S_{n-1} \oplus Q_{n-1} \arrow[d] \\
 & & S_1 \oplus S_2 \oplus Q_2 \arrow[d] & P(Q_{n-2}) =: F(E) \arrow[dl, dashed] \\
 & S_1 \oplus Q_1 \arrow[d] & P(Q_1) \arrow[dl] & \\
E \arrow[d] & P(E) \arrow[dl] & & \\
M & & &
\end{tikzcd}$$

Thus for a vector bundle E of arbitrary finite rank, the split manifold $F(E)$ exists and is given by (4.1.1). We will obtain a more convenient algebraic description of $F(E)$ in Section 5.1, where it will also be made clear that $F(E)$ is indeed a manifold.

Remark 4.2. The cohomology ring $H^*(F(E))$ is a free $H^*(M)$-module with the basis given by

$$\{x_1^{\alpha_1} x_2^{\alpha_2} \dots x_{n-1}^{\alpha_{n-1}} \mid \alpha_i \in \mathbb{N}, \alpha_i \leq n-i\}. \tag{4.1.3}$$

This description of the vector space structure of $H^*(F(E))$ is refined by the more precise description of the ring structure proven independently of this fact in Proposition 5.5 , so we leave this as a remark without proof.

With the existence of split manifolds established, we may formulate the powerful Splitting Principle: *to prove a polynomial identity in the Chern class of a bundle E, it suffices to do so under the assumption that E is a direct sum of line bundles.*

4.2. **Proof of the Whitney product formula.** Recall the Whitney product formula of Proposition 3.7:

Theorem 4.1. *If E and E' are complex vector bundles, then $c(E \oplus E') = c(E)c(E')$.*

We first need a standard technical lemma. For a proof, see Warner [8, Lemmas 1.9 and 1.10, pp. 9-10].

Lemma 4.2. *Suppose M is a manifold with a finite open cover $\{U_i\}_{i \in I}$. Then there exists an open cover $\{V_i\}_{i \in I}$ of M such that $\overline{V_i} \subset U_i$ and there exist functions $\rho_i : M \to [0,1]$ which are identically 1 on $\overline{V_i}$ and which vanish outside of U_i.*

Of course, we first consider the case of a direct sum of line bundles.

Proposition 4.3. *If $E = L_1 \oplus \cdots \oplus L_n$ is a sum of line bundles, then*

$$c(E) = \prod_{i=1}^{n} c(L_i).$$

Proof. By abuse of notation we will write $\pi^{-1}E = L_1 \oplus \cdots \oplus L_n$ for the pullback of E to the projectivization $P(E)$. Let S be the universal subbundle of $\pi^{-1}E$. The following claim is the crux of the proof: there exists an open cover $\{V_i\}_{i=1}^n$ of $P(E)$ such that $c_1(S^* \otimes L_i)$ may be represented by a global form on $P(E)$ which vanishes on V_i. We will prove this claim after seeing how it proves the proposition. Since the V_i cover $P(E)$, we have $\prod_i c_1(S^* \otimes L_i) = 0$. By (3.1.2) $c_1(S^* \otimes L_i) = x + c_1(L_i)$, where $x := c_1(S^*)$. Thus

$$0 = \prod_{i=1}^{n}(x + c_1(L_i)) = x^n + s_1 x^{n-1} + \dots s_n = 0$$

where s_i is the ith elementary symmetric polynomial in $c_1(L_1), \dots, c_1(L_n)$. This is precisely the defining equation (3.2.4) for the Chern classes, so we must have $s_i = c_i(E)$ for each i by uniqueness and thus

$$c(E) = \prod_i (1 + c_1(L_i)) = \prod_i c(L_i)$$

so the Whitney product formula holds modulo the claim which we will now prove.

Let σ_i be the projection of S onto L_i. Then σ_i is a section of $\hom(S, L_i) \cong S^* \otimes L_i$ (recall the bundle of linear maps $\hom(E, F) \to M$ is the fiber bundle with fiber $\hom(E, F)_x := \hom(E_x, F_x)$). The projections $\sigma_1, \dots, \sigma_n$ cannot simultaneously vanish at a point $y \in P(E)$ since the fiber S_y is a 1-dimensional subspace of $(\pi^{-1}E)_y$, so the sets

$$U_i := \{y \in P(E) \mid \sigma_i(y) \neq 0\}$$

form an open cover of $P(E)$. When restricted to each U_i, the line bundle $S^* \otimes L_i$ has a nowhere-vanishing section given by σ_i, so by Lemma 2.8 $(S^* \otimes L_i)\,|_{U_i}$ is trivial. Let ξ_i be a global 2-form on $P(E)$ representing the cohomology class of $c_1(S^* \otimes L_i)$. Then $\xi_i\,|_{U_i} = d\omega_i$ for some 1-form ω_i on U_i. We wish to extend $\xi_i - d\omega_i$ to a global form on $P(E)$. Using the refinement $\{V_i\}$ and functions ρ_i furnished by Lemma 4.2, we see that $\rho_i\omega_i$ is a global form which agrees with ω_i on V_i, so $\xi_i - d(\rho_i\omega_i)$ is a global form representing the cohomology class $c_1(S^* \otimes L_i)$ and vanishing on V_i. This proves the claim and thus completes the proof of the proposition. □

Now, by applying the splitting principle, we may swiftly prove the Whitney product formula.

Proof of Theorem 4.1. Let E and E' be vector bundles of rank n and m respectively, and let $\pi : F(E) \to M$ and $\pi' : F(\pi^{-1}E') \to F(E)$ be the splitting construction. Both bundles split completely when pulled back to $F(\pi^{-1}E')$, as the diagram below illustrates.

$$\begin{tikzcd}
 & & & L_1 \oplus \dots L_n \oplus L_1' \oplus \cdots \oplus L_m' \arrow[d] \\
 & & L_1 \oplus \dots L_n \oplus \pi^{-1}E' \arrow[d] & F(\pi^{-1}E') \arrow[dl, "\pi'"'] \\
E \oplus E' \arrow[d] & & F(E) \arrow[dll, "\pi"'] & \\
M & & &
\end{tikzcd}$$

Set $\sigma := \pi' \circ \pi$. Then

$$\begin{aligned}
\sigma^* c(E \oplus E') &= c(\sigma^{-1}(E \oplus E')) \\
&= c(L_1 \oplus \dots L_n \oplus L'_1 \oplus \dots \oplus L'_m) \\
&= \left[\prod_{i=1}^{n} c(L_i)\right] \cdot \left[\prod_{j=1}^{m} c(L'_j)\right] \qquad \text{by Proposition 4.3} \\
&= \sigma^* c(E) \sigma^* c(E') \\
&= \sigma^*(c(E)c(E')).
\end{aligned}$$

Since π^* and π'^* are injective, σ^* is injective and thus $c(E \oplus E') = c(E)c(E')$ as desired. □

The splitting principle also enables one to compute, using the theory of symmetric functions, formulae for the Chern classes of tensor products, exterior products and duals. For these computations and several more see [2, pp.278-282].

5. Flag bundles and the Grassmannian

In this section, we introduce the notion of a flag bundle $Fl(E)$ associated with a vector bundle E and show that this construction coincides with the split manifold $F(E)$ of the previous section. Flag bundles allow us to obtain a more precise description of the ring structure of $H^*(P(E))$, and as a consequence we obtain a succinct description of $H^*(Fl(E))$. We then apply these results to study the cohomology ring of the complex Grassmannian.

5.1. **The cohomology of a flag bundle.**

Definition 5.1. Let V be a complex vector space of dimension n. A *flag* in V is a sequence of subspaces $A_1 \subset A_2 \subset \cdots \subset A_n = V$, $\dim_{\mathbb{C}} A_i = i$. We write $Fl(V)$ for the collection of all flags in V. We call $Fl(V)$ the flag manifold of V, a label which will be justified directly.

Thinking of the geometric picture, it's clear that $GL_n(\mathbb{C})$ acts transitively on $Fl(V)$. Moreover, it not difficult to see that the stabilizer at a flag is the closed subgroup T of invertible upper-triangular matrices. Thus, as sets we have $Fl(V) \cong GL_n(\mathbb{C})/T$, and since the quotient of a Lie group by a closed subgroup can be made into a Lie group ([8, Theorem 3.58, pp. 120]) $Fl(V)$ can be made into a manifold.

Similar to the projectivization of a bundle $E \to M$, we now describe a way to associate a fiber bundle $Fl(E)$ to E where each fiber is a flag manifold.

Definition 5.2. Given a vector bundle $E \to M$, its *associated flag bundle* $Fl(E)$ is obtained from E by taking $Fl(E)_p := Fl(E_p)$. The local trivializations $E\,|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{C}^n$ induce trivializations $Fl(E)\,|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times Fl(\mathbb{C}^n)$.

We now prove the promised equivalence.

Proposition 5.3. *The flag bundle $Fl(E)$ associated to a vector bundle E is the split manifold $F(E)$ constructed in Section 4.1.*

Proof. The split manifold $F(E)$ is obtained by a sequence of $n-1$ projectivizations as illustrated in (4.1.1). Recall that a point of $P(E)$ is a pair (p, ℓ) with $p \in M$ and $\ell \in E_p$. By introducing a Hermitian metric on E, we may regard the quotient bundles $Q_1, \ldots, Q_{n-1}$ in (4.1.1) as subbundles of E, where Q_1 is the orthogonal complement of S_1, Q_2 is the orthogonal complement of $S_1 \oplus S_2$, and so on. Thus a point of $P(Q_1)$ over $(p, \ell_1) \in P(E)$ is a triple (p, ℓ_1, ℓ_2) where ℓ_2 is a line in the orthogonal complement of ℓ_1 in E_p. Similarly, a point of $P(Q_2)$ over $(p, \ell_1, \ell_2) \in P(Q_1)$ is a 4-tuple $(p, \ell_1, \ell_2, \ell_3)$ where ℓ_3 is a line in the orthogonal complement of ℓ_1 and ℓ_2 in E_p. In general, a point in the split manifold $F(E) = P(Q_{n-1})$ may be identified with the flag

$$(p, \ell_1 \subset \{\ell_1, \ell_2\} \subset \{\ell_1, \ell_2, \ell_3\} \subset \cdots \subset E_p).$$

This shows that the split manifold $F(E)$ constructed in Section 4.1 coincides with the flag bundle $Fl(E)$ as desired. □

Henceforth, we will use the notation $Fl(E)$. A further remark on notation: if A is a graded ring and $a, b, f \in A$, then (a, b) denotes the ideal generated by A, while $(f = 0)$ denotes the ideal generated by the homogeneous components of f.

To compute the ring structure of $H^*(Fl(E))$, first recall the formula (3.2.5) for the cohomology ring of $P(E)$:

$$H^*(P(E)) = H^*(M)[x]/(x^n + c_1(E)x^{n-1} + \cdots + c_n(E))$$

where $x := c_1(S^*)$. We may recast this in a useful form. We write $H^*(M)[c(S), c(Q)]$ for $H^*(M)[c_1(S_{P(E)}), c_1(Q_{P(E)}), \ldots, c_{n-1}(Q_{P(E)})]$.

Lemma 5.4.

$$H^*(P(E)) = \frac{H^*(M)[c(S), c(Q)]}{(c(S)c(Q) = \pi^* c(E))}.$$

The proof is a purely formal exercise in generators and relations, so we omit the proof. With this lemma, we may compute $H^*(Fl(E))$.

Proposition 5.5. *Let E be a complex vector bundle of rank n and put $x_i = c_1(S_i)$ for $i = 1, \ldots, n-1$ and $x_n = c(Q_{n-1})$, where S_i and Q_i are as in (4.1.1). Then the cohomology ring of $Fl(E)$ is*

$$H^*(Fl(E)) = H^*(M)[x_1, \ldots, x_n] \Big/ \left(\prod_{i=1}^{n} (1 + x_i) = c(E) \right) \tag{5.1.6}$$

and it has Poincaré polynomial

$$P_t(Fl(E)) = P_t(M) \frac{(1-t^2)(1-t^4)\ldots(1-t^{2n})}{(1-t^2)(1-t^2)\ldots(1-t^2)}.$$

Proof. We obtain $Fl(E)$ by a sequence of $n-1$ projectivizations $P(Q_1), \ldots, P(Q_{n-1})$ as in (4.1.1). Applying Lemma 5.4 twice obtains

$$\begin{aligned} H^*(P(Q_1)) &= H^*(P(E))[c(S_2), c(Q_2)]/(c(S_2)c(Q_2) = c(Q_1)) \\ &= H^*(M)[c(S_1), c(S_2), c(Q_1), c(Q_2)]/(c(S_2)c(Q_2) = c(Q_1), c(S_1)c(Q_1) = c(E)) \\ &= H^*(M)[c(S_1), c(S_2), c(Q_2)]/(c(S_1)c(S_2)c(Q_2) = c(E)). \end{aligned}$$

By induction we obtain the formula

$$H^*(P(Q_{n-1})) = H^*(M)[c(S_1), \ldots, c(S_{n-1}), c(Q_{n-1})]/(c(S_1)c(S_{n-1})c(Q_{n-1} = c(E)).$$

Thus using the notation $x_i = c_1(S_i)$ for $i = 1, \ldots, n-1$ and $x_n = c(Q_{n-1})$, we have

$$H^*(Fl(E)) = H^*(M)[x_1, \ldots, x_n] \Big/ \left(\prod_{i=1}^{n} (1 + x_i) = c(E) \right)$$

as required. As for the Poincaré polynomial, (3.2.6) can be read as saying that each time we projectivize a rank k vector bundle, the Poincaré polynomial gains a $(1-t^{2k})/(1-t^2)$ factor. Thus

$$P_t(Fl(E)) = H^*(M)\frac{(1-t^{2n})}{1-t^2}\cdot\frac{(1-t^{(2n-2)})}{1-t^2}\cdots\frac{1-t^2}{1-t^2}$$

$\square$

The following useful corollary follows immediately by considering a vector space as the trivial bundle over a point.

Corollary 5.7. *If V is an n-dimensional complex vector space, we have*

$$H^*(Fl(V)) = \mathbb{R}[x_1,\ldots,x_n]\Big/\left(\prod_{i=1}^{n}(1+x_i)=1\right) \tag{5.1.8}$$

and

$$P_t(Fl(V)) = \frac{(1-t^2)(1-t^4)\ldots(1-t^{2n})}{(1-t^2)(1-t^2)\ldots(1-t^2)}. \tag{5.1.9}$$

5.2. **The cohomology of a complex Grassmannian.** We now introduce, the complex Grassmannian, a generalization of the projectivization of a vector space which is closely related to flag manifolds.

Let V be a complex vector space of dimension n. The *complex Grassmannian* $G_k(V)$ is the set of all subspaces of V of complex **codimension** k. Note that $G_{n-1}(V) = P(V)$. Equipping V with a Hermitian metric, we see that $U(n)$ acts transitively on the set of all $(n-k)$-dimensional subspaces of V. Moreover, a unitary matrix stabilizing an $(n-k)$-dimensional subspace must also fix its k-dimensional orthogonal complement, so the stabilizer of a point in $G_k(V)$ is $U(n-k)\times U(k)$ and thus we have the identification

$$G_k(V) \cong \frac{U(n)}{U(n-k)\times U(k)}$$

Since $U(n-k)\times U(k)$ is a closed subgroup of the Lie group $U(n)$, this shows that $G_k(V)$ is a smooth manifold.

Just as on projective space, there are a few tautological bundles over $G_k(V)$: the *universal subbundle* S, whose fiber at $\Lambda \in G_k(V)$ is Λ itself, the *product bundle* $\hat{V} = G_k(V)\times V$, and the *universal quotient bundle* Q defined by the exact sequence

$$0 \to S \to \hat{V} \to Q \to 0.$$

We observe that, over $G_k(V)$, the universal subbundle S has rank $n-k$ and the universal quotient bundle Q has rank k.

The following theorem will be crucial for the final section. We state it in multiple parts.

Theorem 5.1. *Let V be a complex vector space of dimension n.*

(a) The Poincaré polynomial of the complex Grassmannian $G_k(V)$ is given by

$$P_t(G_k(V)) = \frac{(1-t^2)\ldots(1-t^{2n})}{(1-t^2)\ldots(1-t^{2k})(1-t^2)\ldots(1-t^{2(n-k)})}; \tag{5.2.2}$$

(b) We have the following ring isomorphism:

$$H^*(G_k(V)) \cong \frac{\mathbb{R}[c_1(Q), \ldots, c_k(Q)]}{(c(S)c(Q) = 1)}; \tag{5.2.3}$$

(c) When k and i are fixed, there are no polynomial relations of degree i among the generators $c_1(Q), \ldots, c_k(Q)$ when n, the dimension of V, is sufficiently large.

We will need to following lemma. The proof is not trivial but, being as it is not terribly difficult and completely algebraic, we omit the proof and refer the reader to [2, pp. 294-297]. The Poincaré series of a graded algebra $A = \bigoplus_{i=1}^{\infty} A_i$ over a field k is by definition $P_t(A) = \sum_{i=1}^{\infty} (\dim_k A_i) t^i$.

Lemma 5.4. *If $A := \mathbb{R}[x_1, \ldots, x_{n-k}, y_1, \ldots, y_k]$ and I is the ideal generated by the homogeneous terms of*

$$(1 + x_1 + \cdots + x_{n-k})(1 + y_1 + \cdots + y_k) - 1$$

where $\deg x_i = \deg y_i = 2i$, then the Poincaré series of A/I is

$$P_t(A/I) = \frac{(1-t^2)\ldots(1-t^{2n})}{(1-t^2)\ldots(1-t^{2(n-k)})(1-t^2)\ldots(1-t^{2k})}.$$

We now prove our theorem by making the similarities between flag manifolds and Grassmannians precise.

Proof. The flag manifold $Fl(V)$ can be obtained from the Grassmannian $G_k(V)$ by two flag constructions as follows: let $\hat{Q}$ be the pullback of Q to the flag bundle $Fl(S)$. We claim that $Fl(\hat{Q})$ is the flag manifold $Fl(V)$, both considered over $G_k(V)$.

$$\begin{tikzcd}
 & \hat{Q} \arrow[dd] & \\
S \oplus Q \arrow[dd] & & Fl(\hat{Q}) \arrow[dl] \\
 & Fl(S) \arrow[dl] & \\
G_k(V) & &
\end{tikzcd}$$

A point of $Fl(S)$ is a pair $(\Lambda, A_1 \subset \cdots \subset A_{n-k-1} \subset \Lambda)$ consisting of an element $\Lambda \in G_k(V)$ and a flag in Λ. Thus a point of $Fl(\hat{Q})$ is a point in $Fl(S)$, i.e. a pair $(\Lambda, A_1 \subset \cdots \subset \Lambda)$, together with a flag in V/Λ. In other words, we may write a point in $Fl(\hat{Q})$ as a pair of the form $(\Lambda, A_1 \subset \ldots A_{n-k-1} \subset \Lambda \subset A_{n-k+1} \subset \cdots \subset V)$, which is exactly the form of an element of $Fl(V)$ as claimed. Hence, applying Proposition 5.5 twice obtains the relation

$$\begin{aligned}
P_t(Fl(V)) = P_t(Fl(\hat{Q})) = P_t(Fl(S)) &\frac{(1-t^2)\ldots(1-t^{2(n-k)})}{(1-t^2)\ldots(1-t^2)} \\
= P_t(G_k(V)) &\frac{(1-t^2)\ldots(1-t^{2(n-k)})(1-t^2)\ldots(1-t^{2k})}{(1-t^2)\ldots(1-t^2)(1-t^2)\ldots(1-t^2)}.
\end{aligned}$$

On the other hand, (5.1.9) gives a formula for $P_t(Fl(V))$ independent of $P_t(G_k(V))$, so by dividing through by that formula we obtain

$$P_t(G_k(V)) = \frac{(1-t^2)\dots(1-t^{2n})}{(1-t^2)\dots(1-t^{2k})(1-t^2)\dots(1-t^{2(n-k)})}$$

which verifies (5.2.2).

To establish (5.2.3), we obtain an injection and argue that it must be an isomorphism for reasons of dimension. By (5.1.6) the cohomology ring of the flag manifold over $G_k(V)$ is

$$H^*(Fl(V)) = \frac{H^*(G_k(V))[x_1,\dots,x_{n-k},y_1,\dots,y_k]}{(\prod_i(1+x_i) = c(S), \prod_i(1+y_i) = c(Q))}.$$

On the other hand, by (5.1.8), the cohomology of $Fl(V)$ is also given by

$$H^*(Fl(V)) = \frac{\mathbb{R}[x_1,\dots,x_{n-k},y_1,\dots,y_k]}{\left(\prod_i(1+x_i)\prod_j(1+y_j) = 1\right)}. \tag{5.2.5}$$

Thus, in $H^*(G_k(V))$, the Chern classes of S and Q can satisfy no relations aside from $c(S)c(Q) = 1$, for any relation would appear as a relation among the x_i's and y_i's in (5.2.5). Thus we have an injection

$$\frac{\mathbb{R}[c(S),c(Q)]}{(c(S)c(Q)=1)} \hookrightarrow H^*(G_k(V)) \tag{5.2.6}$$

By Lemma 5.4, the Poincaré series of $\mathbb{R}[x_1,\dots,x_{n-k},y_1,\dots,y_k]/(c(S)c(Q) = 1)$ is

$$P_t\left(\frac{\mathbb{R}[c(S),c(Q)]}{(c(S)c(Q)=1)}\right) = \frac{(1-t^2)\dots(1-t^{2n})}{(1-t^2)\dots(1-t^{2(n-k)})(1-t^2)\dots(1-t^{2k})}.$$

By (5.2.2) this is exactly the Poincaré polynomial of $G_k(V)$. Therefore the injection (5.2.6) is in fact an isomorphism. With the relation $c(S)c(Q) = 1$, we may write $c(S) = 1/c(Q)$, which allows us to eliminate the generators $c_1(S),\dots,c_{n-k}(S)$ and thus obtain (5.2.3).

The equation $c(S) = 1/c(Q)$ actually gives us a bit more. Looking at the homogeneous components provides polynomial relations of degrees $2(n-k+1),\dots,2n$ among the Chern classes $c_1(Q),\dots,c_k(Q)$. Thus when i is fixed, there are no polynomial relations of degree i among the Chern classes of Q if n is so large that $2(n-k+1) > i$. This establishes (c) and thus completes the proof. □

6. Chern classes are the only cohomological invariant

In this section, we prove that the Chern classes are the only cohomological invariant of a smooth complex vector bundle in a precise sense. First we describe the classification of complex vector bundles.

We need the following basic lemma.

Lemma 6.1. *If $E \to M$ is a rank k vector bundle over a manifold of finite type, then there exist finitely many smooth global sections on E which span the fiber at every point of M.*

Proof. Let $\{U_i\}_{i\in I}$ be a finite good cover for M. Since U_i is contractible, $E\,|_{U_i}$ is trivial, so we may find k sections $s_{i,1},\dots,s_{i,k}$ which form a basis for the fiber at every point of U_i. Using Lemma 4.2, we may refine $\{U_i\}$ and find cutoff functions

ρ_i such that $\{\rho_i s_{i,1}, \dots, \rho_i s_{i,k}\}_{i \in I}$ is a family of global sections of E which span the fiber at every point of M. This completes the proof. □

This theorem shows that the complex Grassmannian $G_k(\mathbb{C}^n)$ is universal in a sense which we will not make precise here. See [7] or [5] for the details and a far more elegant statement.

Theorem 6.2. *Suppose E is a rank k complex vector bundle over a manifold M of finite type. Suppose E is equipped with n global sections which span the fiber at every point. Then there is a map f from M into some Grassmannian $G_k(V)$ with $\dim_{\mathbb{C}} V = n$ such that E is the pullback under f of the universal quotient bundle Q.*

Proof. Let $s_1, \dots, s_n$ be the n spanning sections of E and let V be the complex vector space they span. For each $p \in M$, the evaluation map $\mathrm{ev}_p : V \to E_p$ given by $v \mapsto \sum_i v_i s_i(p)$ is surjective since the s_i are spanning sections. Thus $\ker \mathrm{ev}_p$ is a codimension k subspace of V and the fiber of the universal quotient bundle Q of $G_k(V)$ at $\ker \mathrm{ev}_p$ is $V/\ker \mathrm{ev}_p = E_p$. Thus, the map $f : M \to G_k(V)$ given by $p \mapsto \ker \mathrm{ev}_p$ is as desired: $E = f^{-1}Q$. □

The map $f : M \to G_k(V)$ is called the *classifying map* for the bundle E. It can be shown that the homotopy class of f is uniquely determined by E and that the set of isomorphism classes of rank k complex vector bundles are in bijection with homotopy classes of classifying maps (again see [7] or [5]).

We arrive at the most incredible result in this paper: the Chern classes are the only cohomological invariant of complex vector bundles. We need to fix some language and notation. Let $\mathrm{Vect}_k(M)$ denote the set of isomorphisms classes of rank k complex vector bundles over M. $\mathrm{Vect}_k(\cdot)$ and $H^*(\cdot)$ are functors from the category of C^∞ manifolds to the category of sets. Recall that a *natural transformation* between the functors $\mathrm{Vect}_k(\cdot)$ and $H^*(\cdot)$ is a family of maps $T_M : \mathrm{Vect}_k(M) \to H^*(M)$ such that the naturality diagrams

$$\begin{tikzcd} M \arrow[d, "f"'] & \mathrm{Vect}_k(M) \arrow[r, "T_M"] & H^*(M) \\ N & \mathrm{Vect}_k(N) \arrow[u, "f^{-1}"] \arrow[r, "T_N"'] & H^*(N) \arrow[u, "f^*"'] \end{tikzcd}$$

commute for every smooth manifold N. The Chern classes are examples of such natural transformations by Proposition 3.7.

Theorem 6.3. *Every natural transformation from the isomorphism classes of complex vector bundles $\mathrm{Vect}_k(\cdot)$ over a manifold of finite type to the de Rham cohomology ring $H^*(\cdot)$ can be given as a polynomial in the Chern classes.*

Proof. Let T be such a natural transformation. If E is any rank k complex vector bundle over M and $f : M \to G_k(V)$ is a classifying map for E, then by Theorem 6.2 and the naturality of T we have

$$T(E) = T(f^{-1}Q) = f^*T(Q)$$

where Q is the universal quotient bundle of $G_k(V)$. By Theorem 5.1, the cohomology of $G_k(V)$ is generated by the Chern classes of Q, so there exists a polynomial

P, depending on T, such that

$$T(Q) = P(c_1(Q), \ldots, c_k(Q)).$$

Combining our two equations obtains the result:

$$T(E) = f^*P(c_1(Q), \ldots, c_k(Q)) = P(f^*c_1(Q), \ldots, f^*c_k(Q)) = P(c_1(E), \ldots, c_k(E)).$$

□

We end by briefly discussing how these results extend to the case of an arbitrary smooth manifold. In order to remain elementary, we have restricted ourselves to smooth complex vector bundles over manifolds of finite type, but the finite type restriction can be shed in a somewhat straightforward manner. The classifying space is the infinite Grassmannian $G_k(\mathbb{C}^\infty)$ defined by taking the direct limit, and the universal quotient bundle Q is defined similarly. There is a countable analogue of Lemma 6.1 and this allows one to prove the analogue of Theorem 6.2 in much the same way as above. Based on Theorem 5.1, one may conjecture that the cohomology ring of the infinite Grassmannian $G_k(\mathbb{C}^\infty)$ is the free polynomial algebra

$$\mathbb{R}[c_1(Q), \ldots, c_k(Q)].$$

This is indeed the case, though the proof is quite different than in the finite type case. See [6, Theorem 14.5, pp. 161] or [5, Chapter 20, Theorem 3.2, pp. 297]. The proof of Theorem 6.3 goes through as above with no alteration.

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