The Four Color Theorem

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Abstract

We will outline some of the most important ideas used in K. Appel and W. Haken's proof [2] of the Four Color Theorem. The goal of this paper is to consolidate the work on the Four Color Theorem in one paper that is easy to follow and does not require the reader to translate between articles with different definitions and notations.

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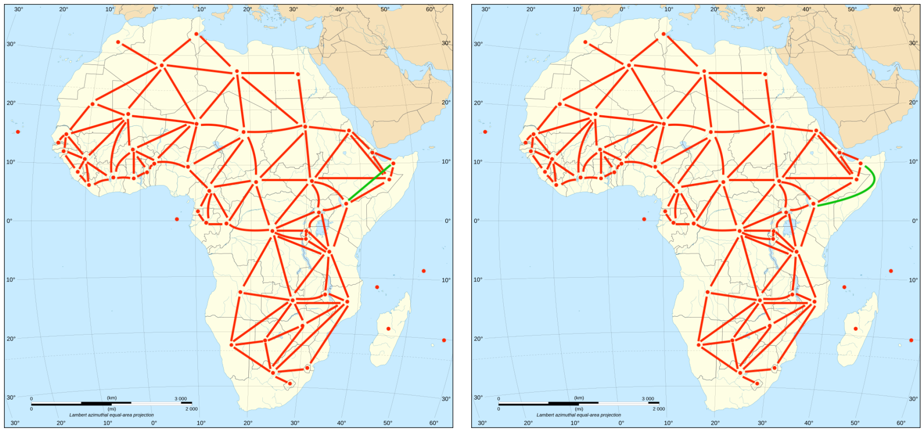
1 Introduction

Suppose we are given a map of Africa and we want to color it. The water is already colored blue, so we do not need to worry about it. Ideally, we want to color the map so that countries that border each other do not have the same color. How many colors do we need to do this? This is the question answered



(a) drawing vertices and edges (b) a plane graph

Figure 1: Two drawings of a planar graph representing the map of Africa. The map used in both images is made by Eric Gaba [5].

by the Four Color Theorem, which was left unverified for nearly a century after the conjecture was first formally proposed in 1852.

The path to a proof of the Four Color Theorem was not the smoothest. Multiple incorrect proofs have been proposed that were left unchallenged for over a decade. In 1879, Alfred Kempe published a paper [6] in which he believed he proved the Four Color Theorem. It stood until 1890 when P.J. Heawood discovered its flaw. In 1880, Peter Guthrie Tait claimed that he had proved the Four Color theorem with his eponymous conjecture [7]. This time, it took until 1946 for W.T. Tutte to refute Tait's conjecture by providing a counterexample [8]. The first correct proof of the Four Color Theorem did not come until 1977 when Kenneth Appel and Wolfgang Haken used a computer to verify it [2], which was controversial given that it was the first computer-assisted proof of a theorem.

We will outline one of the incorrect proofs of the theorem and then show how some of its ideas are useful in the Appel-Haken proof. We will also outline the most important ideas used in the Appel-Haken proof. Before that, we have to lay the groundwork for these ideas.

In order to frame the problem of map coloring more mathematically, we come up with a natural identification between maps and graphs. Consider the map of Africa. First, we can assign each country a vertex that lies inside its boundary. Then, if two countries border each other, we can draw an edge between their corresponding vertices. After that, we are left with the graph in Figure 1a.

If we are careful so that the edges only cross through the two countries they connect, none of the edges will intersect with each other except where they meet at a vertex. For example, we can redraw the edge connecting Somalia and Kenya so that we get such a drawing, as shown in Figure 1b. We call graphs

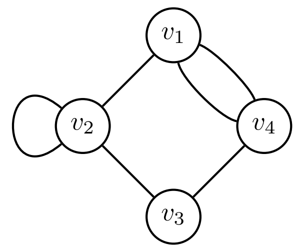


Figure 2: A graph with a loop at v_2 and parallel edges connecting v_1 and v_4. The graph is connected, but not simple.

that can be drawn in this way *planar* graphs.

Definition 1.1. A graph G is *planar* if it can be drawn in the plane such that no two edges intersect each other unless at a common vertex.

It is important that we distinguish between the graph and its drawing. Although only the drawing in Figure 1b does not have edges that intersect, both drawings represent the same graph. We have a name for the drawings we care about.

Definition 1.2. A *plane graph* is a drawing of a planar graph such that no two edges intersect each other unless at a common vertex.

Remark. A plane graph and a planar graph are not the same. A plane graph is not really a graph; it is a drawing. On the other hand, a planar graph is a graph.

Since it doesn't make sense for a country to border itself, every edge in the graph connects two different vertices. We call such graphs *loopless*.

Definition 1.3. A *loop* is an edge with identical vertices. (See Figure 2.)

We should also explicitly define a *proper coloring* and what it means to be *colorable*.

Definition 1.4. A *proper k-coloring* of a graph is an assignment of one of k colors to each of its vertices such that no two adjacent vertices have the same color.

Definition 1.5. A graph is *k-colorable* if there is a proper k-coloring of it.

Now, we are ready for a precise statement of the Four Color Theorem.

Theorem 1.6 (Four Color Theorem)**.** *Every loopless planar graph is* 4*-colorable.*

Although the Appel-Haken proof of the Four Color Theorem is too long for us to completely reconstruct without a computer, we can outline some of the most important ideas from the proof. Before we do that, we will tackle a much more manageable problem: showing every map can be colored with 5 colors. First, we will need a few more definitions and some notation.

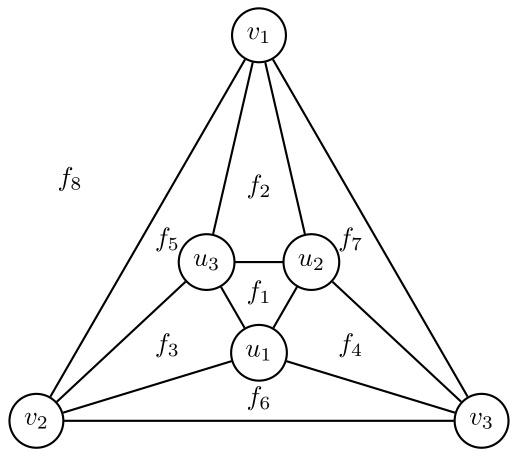


Figure 3: A plane graph with 8 faces, labeled $f_1, \ldots, f_8$. In this example, f_8 is the outer face. It is the exterior of the triangle $v_1v_2v_3$.

Definition 1.7. A *face* of a plane graph is a path-connected open set whose boundary consists of vertices and edges.

Definition 1.8. The *outer face* of a plane graph is its unbounded face.

For a plane graph $\tilde{G}$, we denote the set of vertices by $V = V(\tilde{G})$, the set of edges by $E = E(\tilde{G})$, and the set of faces by $F = F(\tilde{G})$. The sizes of each of these sets are denoted by $n = v(\tilde{G})$, $m = e(\tilde{G})$, and $f(\tilde{G})$, respectively. Likewise, for a planar graph G, we use V, E, m, and n for the same notation. However, we cannot talk about the faces of a planar graph in general, since they are unique to plane graphs.

In a planar graph, each vertex v has a *degree*, $d(v)$, which is the number of edges incident to it. We use δ to denote the smallest degree of any vertex in a graph. Likewise, in a plane graph, each face f has a degree, $d(f)$, which is the number of edges in its boundary. Now, we can define a few more terms we will need to talk about graphs.

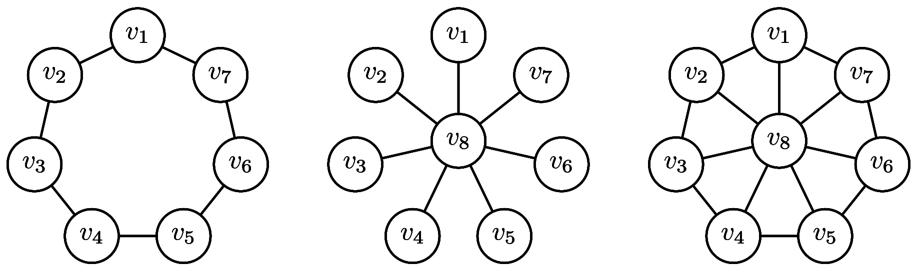
Definition 1.9. A graph is *connected* if there exists a sequence of edges connecting any two vertices in the graph. A graph that is not connected is *disconnected*.

Definition 1.10. Two or more edges are *parallel* if they share the same vertices. (See Figure 2.)

Definition 1.11. A graph is *simple* if it contains no loops and no parallel edges.

Definition 1.12. The *union* of two simple graphs G and H is the graph $G \cup H$ with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$.

The union of two graphs depends on the labeling of the vertices. If a vertex in G is labeled the same as a vertex in H, there exists a vertex in $G \cup H$



(a) C_7 (b) A star with 8 vertices (c) W_7

Figure 4: A cycle, a star, and a wheel. The union of the cycle and the star, with vertices labeled the same, is the wheel.

corresponding to these vertices that is incident to edges in G and edges in H. Alternatively, if none of the vertices of G are labeled the same as some vertex of H, then $G \cup H$ is a disconnected graph such that each of its connected components belongs entirely to G or H.

We will also sometimes subtract a vertex v from a graph G to get the graph $G - v$. This graph will be the graph of G with v and the edges incident to v removed.

Finally, we define a few special types of graphs. First, we define a *triangulation*, since we will use them frequently in our discussion of the Four Color Theorem.

Definition 1.13. A plane graph is a *triangulation* if it is simple, connected, and the degree of every face is 3.

Example 1.13.1. The plane graph in Figure 3 is a triangulation.

We will also frequently encounter *n-cycles*, denoted by C_n, and *wheels with n spokes*, denoted by W_n. We show examples of a 7-cycle and a 7-wheel in Figure 4a and Figure 4c respectively.

2 The Five Color Theorem

Before we consider the Four Color Theorem, it may be helpful for us to tackle an easier problem, namely how to color a loopless planar graph with 5 colors. We will first state a simple, but important, theorem.

Theorem 2.1 (Euler's Formula)**.** *For a connected plane graph $\tilde{G}$,*

$$v(\tilde{G}) - e(\tilde{G}) + f(\tilde{G}) = 2.$$

We leave the proof of Euler's Formula to Appendix A. Essentially, we proceed by induction on the number of faces, noting that the result is immediate in the case when the plane graph is a tree. Euler's Formula will not only be a part of

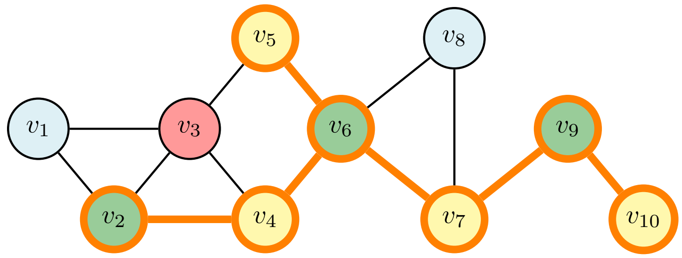


Figure 5: A Kempe chain, outlined in orange.

our proof of the Five Color Theorem, but it will also be essential to our approach to the Four Color Theorem. In addition, we will frequently use the corollaries that we derive from it. Some of them are left to Appendix A.

Corollary 2.2. *Let G be a simple planar graph on at least 3 vertices. Let m denote the number of edges in G and n the number of vertices. Then $m \leq 3n-6$. Furthermore, $m = 3n-6$ if and only if every plane graph representing G is a triangulation.*

Corollary 2.3. *Every simple planar graph has a vertex of degree at most* 5.

We will need two more important ideas before we prove the Five Color Theorem. The first is the idea of *Kempe interchange*.

Definition 2.4. A *Kempe chain* is a subgraph H in a graph with vertices colored either c_1 or c_2, none of which are adjacent to a vertex colored c_1 or c_2 not in H. The act of swapping the colors of each vertex in H is called *Kempe interchange*. (See Figure 5.)

Kempe interchange does not affect whether or not a graph is a proper coloring. We encapsulate this idea in a lemma, whose short proof we leave in Appendix B.

Lemma 2.5. *Let $f : V(G) \to C$ be a proper coloring of some graph G with a set of colors C. Let $g : V(G) \to C$ be the coloring we get after Kempe interchange on some Kempe chain H that comes from f. Then, g is a proper coloring of G.*

The second idea we need for the Five Color Theorem comes from topology, and we will not prove it here.

Theorem 2.6 (Jordan Curve Theorem)**.** *Any simple closed curve C in the plane partitions the rest of the plane into two disjoint path-connected open sets.*

In some of our later proofs, we may implicitly use the Jordan Curve Theorem without mention of it. However, for this proof we will explicitly state when we use it. Now, we are ready to prove the Five Color Theorem.

Theorem 2.7 (Five Color Theorem)**.** *Every loopless planar graph is* 5*-colorable.*

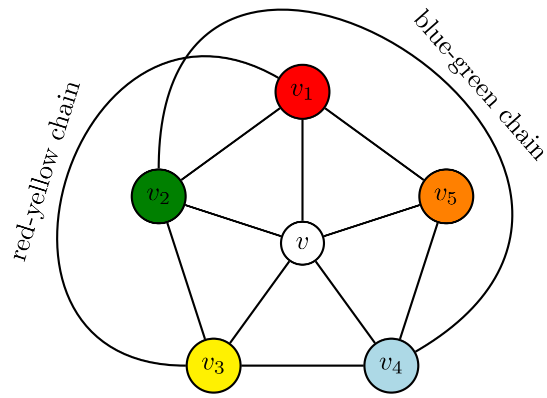


Figure 6: Kempe chains in the proof of the Five Color Theorem. We only consider the case when v_1 and v_3 are connected by a Kempe chain, H_1, since otherwise we can use Kempe interchange to color v_1 yellow and v red. Likewise, we assume there is a Kempe Chain, H_2, connecting v_2 and v_4. Then, by the Jordan Curve Theorem, H_1 and H_2 must intersect someplace. However, this contradicts the planarity of the graph. So, H_1 and H_2 cannot both exist, meaning we can use Kempe interchange to color the graph.

Proof. We proceed by induction on the number of vertices. Clearly, when a graph has one vertex, it is 5-colorable. Now, consider a graph G with n vertices, and assume that all graphs with $n-1$ vertices are 5-colorable. By Corollary 2.3 there is some vertex v in G with at most 5 neighbors. By the inductive hypothesis, $G-v$ has some 5-coloring. If v has degree less than 5, then we are done, since we can color v with one of the 5 colors not used by its neighbors in the coloring of $G-v$ to get a 5-coloring of G. By the same reasoning, if v has degree 5, and two of its neighbors share the same color, then G is 5-colorable. Therefore, we only need to consider when v has 5 neighbors each with a different color.

Let v_1, v_2, v_3, v_4, and v_5 be the vertices adjacent to v, labeled in cyclical order, and let them be colored red, green, yellow, blue, and orange, respectively, as in Figure 6. If v_1 is not adjacent to a yellow vertex, then we can color it yellow and color v red, completing the proof. So, we only have to consider when v_1 is adjacent to a yellow vertex—in other words, when there is some yellow-red Kempe chain H of at least 2 vertices containing v_1. If v_3 is not in H, we use Kempe interchange to color v_1 yellow. Since this does not affect the color of v_3, we can color v red, completing the proof. Therefore, we only consider when v_1 and v_3 lie in the same red-yellow Kempe chain. Likewise, we only consider when v_2 and v_4 lie in the same green-blue Kempe chain.

We will show that it is actually impossible for there to be two Kempe chains connecting v_1 and v_3 as well as v_2 and v_4. Let C_1 be the closed curve that comes from v, v_1, v_3 and the vertices of the Kempe chain connecting v_1 and

v_3. In addition, let C_2 be the closed curve that comes from v, v_2, v_4 and the the vertices of the Kempe chain connecting v_2 and v_4. Since v_1, v_2, v_3, and v_4 occur in cyclical order, either v_2 lies in the interior of C_1 and v_4 lies in the exterior of C_1 or vice versa. In either case, since C_1 is a closed curve, and C_2 connects v_2 to v_4, C_1 and C_2 must intersect someplace besides v by the Jordan Curve Theorem. Since C_1 and C_2 do not share any vertices with the same color, besides v, their edges must intersect each other. However, this contradicts the planarity of G. Therefore, both Kempe chains cannot exist. □

3 Kempe's (Incorrect) Proof of the Four Color Theorem

While the proof of the Five Color Theorem cleverly uses Kempe interchange to show that every loopless planar graph is 5-colorable, Kempe did not exactly discover the proof himself. Instead, he constructed an erroneous proof for the Four Color Theorem, and the ideas from it were reconstructed by Heawood to prove the Five Color Theorem after he recognized Kempe's mistake. We will reconstruct the erroneous proof and then show why it is flawed.

First, we will lay the groundwork for some of Kempe's ideas. We say that a graph G is *smaller* than H if $v(G) + e(G) < v(H) + e(H)$. In addition, we say that a graph is *minimal* given some constraints if there are not any other graphs smaller than it that satisfy those constraints. A key strategy to Kempe's proof is to assume for contradiction that there exists a minimal counterexample to the Four Color Theorem. Then, by using properties that a minimal counterexample must have, we come closer to a contradiction. Although Kempe's proof was incorrect, it is still helpful to consider the properties of a minimal counterexample, since this same technique was used by Appel and Haken in their correct proof of the Four Color Theorem. We will prove a few properties a minimal counterexample should have for now, and leave a couple for later. First, we need one definition.

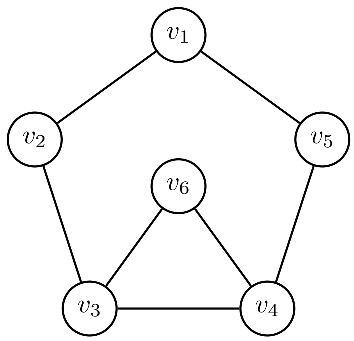
Definition 3.1. To *identify* non-adjacent vertices v_i and v_j is to replace them by a single vertex incident to all edges which were incident to v_i and v_j. (See Figure 7.)

If we get parallel edges after vertex identification, we will remove one of them since it does not affect the graph for our purposes. Now, we can proceed.

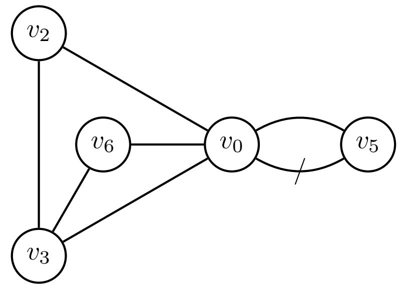
Proposition 3.2. *If $\tilde{G}$ is a minimal counterexample to the Four Color Theorem, then both the following two statements must be true:*

- *$\tilde{G}$ is a triangulation.*
- *$\tilde{G}$ has no vertex of degree less than* 5*.*

We leave the proof of Proposition 3.2 to Appendix C. Combining Proposition 3.2 and Corollary 2.3, we immediately get a nice corollary.



(a) A simple graph



(b) Identifying v_1 and v_4

Figure 7: Identifying vertices. One of the parallel edges between v_0 and v_5 is removed, since it does not affect the graph in any meaningful way.

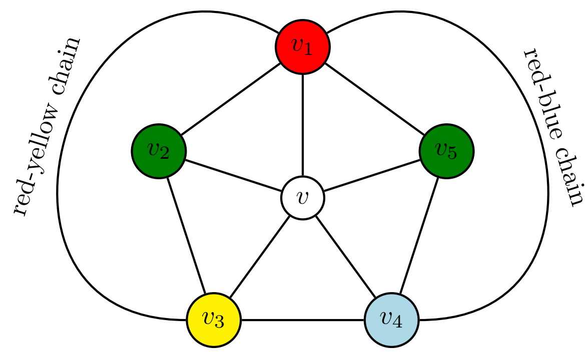


Figure 8: Kempe chains in Kempe's erroneous proof of the Four Color Theorem

Corollary 3.3. *If $\tilde{G}$ is a minimal counterexample to the Four Color Theorem, then $\tilde{G}$ contains the wheel W_5.*

Now, we can outline Kempe's erroneous proof. Assume for contradiction that $\tilde{G}$ is a minimal counterexample to the Four Color Theorem. Let v be the inner vertex of W_5 in $\tilde{G}$. Since $\tilde{G}$ is a minimal counterexample, $\tilde{G} - v$ is 4-colorable. We want to find a 4-coloring of $\tilde{G}$ using the 4-coloring on $\tilde{G} - v$.

As before, if v is not adjacent to vertices with 4 different colors, then we can color v with one of the 4 colors not used. Therefore, we essentially only have to consider the cases when the cycle $v_1v_2v_3v_4v_5$ that v is in has coloring $rgybg$ or

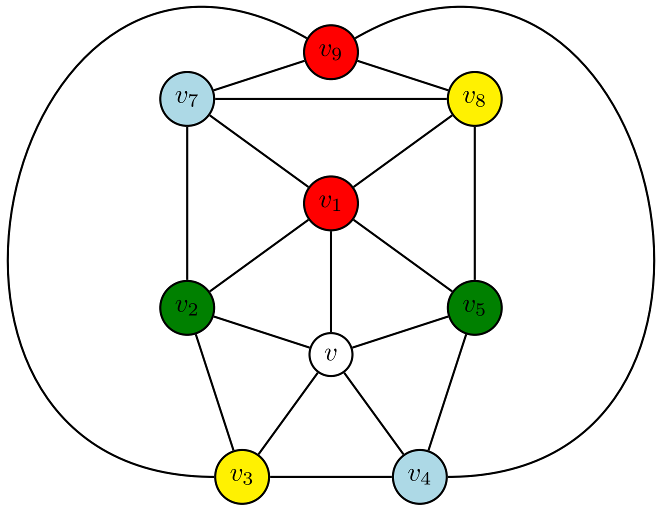


Figure 9: A graph showing the mistake in Kempe's proof [4]

rgyby[1,2]. We will only consider when the cycle has the first coloring, so that we get the picture in Figure 8, since the argument is the same in both cases.

Suppose there is not a red-yellow Kempe chain connecting vertices v_1 and v_3. Then, if v_3 is not adjacent to a red vertex, we can color v_3 red and v yellow. Otherwise, if v_3 is adjacent to a red vertex, we can swap the colors on the red-yellow Kempe chain containing v_3 to get *rgrbg* as a coloring for the cycle. This leaves yellow as a color for v, which implies that $\tilde{G}$ is 4-colorable, a contradiction. Therefore, we only have to consider the case when there is a red-yellow Kempe chain connecting v_1 and v_3. Likewise, we only have to consider the case when there is a red-blue Kempe chain connecting v_1 and v_4.

Now, if there is a green-yellow Kempe chain containing v_5, swapping its colors must not affect any other vertices, since this would require the chain to cross the red-blue Kempe chain connecting v_1 and v_4, which is impossible. Therefore, we can swap the colors along the green-yellow Kempe chain so that v_5 is yellow. Similarly, we can swap the colors along the green-blue Kempe chain, if there is one, containing v_2 so that v_2 is blue. Then, we are left with *rbyby* as a coloring for the cycle. This leaves green for v. Therefore, $\tilde{G}$ is 4-colorable, which is a contradiction. So, there does not exist a minimal counterexample to the Four Color Theorem.

At first glance, this proof seems completely acceptable. However, the mistake comes in the last part of the proof. Specifically, it is not true that swapping

[1]We use r, g, y, and b to denote red, green, yellow, and blue, respectively. We may concatenate these abbreviations to indicate a coloring of a cycle. For example, *rgbr* is a coloring of a 4-cycle.

[2]We say we "essentially" only have to consider *rgybg* or *rgyby* because while there are plenty of other valid colorings given the constraints, we can simply swap the names of colors or cyclically permute colors in these colorings to get either *rgybg* or *rgyby*.

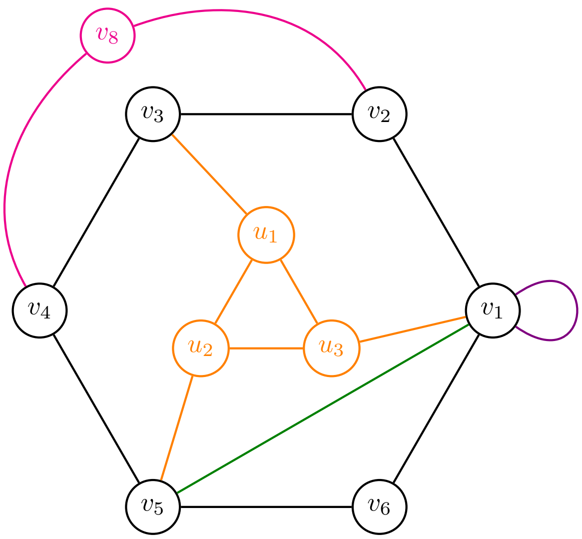


Figure 10: A 6-cycle (in black) with 4 bridges (in pink, orange, green, and purple). Although the vertices of attachment are not colored, they are a part of the bridges. For example, v_2, v_4, and v_8 are part of the pink bridge.

the colors in the Kempe-chains containing v_2 and v_5 does not affect the other vertices. Take the graph in Figure 9, for example. If we swap the colors in the green-blue Kempe-chain containing v_2 and the colors in the green-yellow Kempe chain containing v_5, we get two adjacent green vertices. This is because the red-yellow and red-blue Kempe chains containing v_1 and v_3 and v_1 and v_4, respectively, have a common vertex at v_1. This leaves room for the Kempe chain containing v_5 to connect to the Kempe chain containing v_2. Therefore, we cannot swap the colors in these Kempe chains without affecting other vertices.

4 A Different Strategy

While Kempe's proof was incorrect, we can still take some ideas from it to get closer to a correct proof. Specifically, we will use Kempe's strategy of assuming a minimal counterexample to the Four Color Theorem exists and then arriving at a contradiction. First, we need some more definitions.

Definition 4.1. A *bridge* of a cycle C is a connected graph such that

- one or more of its vertices, known as *vertices of attachment*, are in $V(C)$
- it remains connected after removing its vertices of attachment (assuming there are vertices left)
- besides the vertices of attachment, none of its vertices are adjacent to vertices not in the bridge

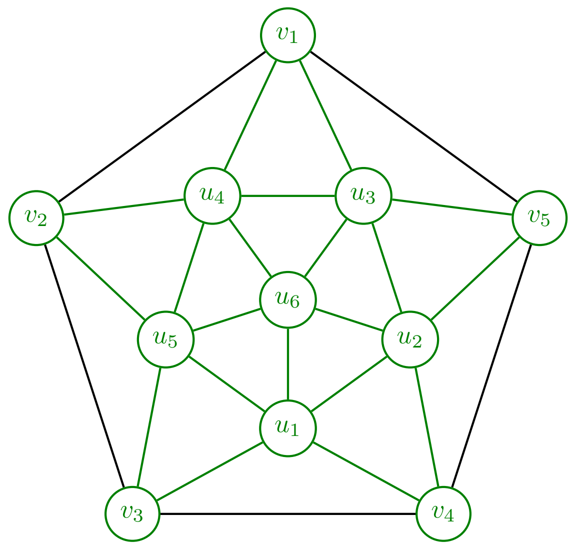


Figure 11: A configuration. The vertices $v_1, \ldots, v_5$ form a cycle, and the subgraph outlined in green is an inner bridge.

If a bridge lies in the interior of a cycle, we say it is an *inner bridge*. Otherwise, we say it is an *outer bridge*. It is possible for bridges to not have any vertices besides the vertices of attachment. For example, in Figure 10, the subgraphs outlined in green and purple, including the vertices of attachment, are bridges.

Definition 4.2. A *configuration* is a graph that can be represented as $B \cup C$ for some cycle C and an inner bridge B whose vertices of attachment are in C.

Configurations are essential to the proof of the Four Color Theorem, since they are the building blocks of any potential minimal counterexample to the Four Color Theorem. As we will see, *reducibility* is an important property of configurations.

Definition 4.3. A configuration is *reducible* if it cannot be a subgraph of a minimal counterexample to the Four Color Theorem.

Example 4.3.1. The configuration W_3 is reducible by Proposition 3.2 because if a graph contained W_3 then it would have a vertex of degree less than 5.

The proof of the Four Color Theorem involves finding an *unavoidable* set of configurations. To define unavoidability, we need a few more definitions and some more properties of a minimal counterexample to the Four Color Theorem.

Definition 4.4. A graph is k-connected if it has more than k vertices and remains connected after any combination of fewer than k vertices are removed.

Example 4.4.1. The configuration in Figure 11 is 5-connected. It is not 6-connected since removing $u_1, \ldots, u_6$ leaves u_6 disconnected from the rest of the graph.

Definition 4.5. A cycle C of a graph G is *separating* if $G - V(C)$ is disconnected.

Definition 4.6. A graph is *essentially k-connected* if it is $(k-1)$-connected and every separating $(k-1)$-cycle comes from neighbors of a vertex of degree $k-1$.

Example 4.6.1. The configuration in Figure 11 is essentially 6-connected.

Proposition 4.7. *If $\tilde{G}$ is a minimal counterexample to the Four Color Theorem, then $\tilde{G}$ contains no separating 4-cycle*

Corollary 4.8. *If $\tilde{G}$ is a minimal counterexample to the Four Color Theorem, then $\tilde{G}$ is essentially 6-connected.*

We prove Proposition 4.7 and Corollary 4.8 in Appendix C. Now, we can define what it means for a set to be unavoidable.

Definition 4.9. A set $\mathcal{U}$ of configurations is *unavoidable* if every essentially 6-connected triangulation necessarily contains at least one member of $\mathcal{U}$.

Example 4.9.1. The set $\{W_5\}$ is unavoidable since δ (the smallest degree of any vertex in a graph) is 5 for any essentially 6-connected triangulation

In his incorrect proof, Kempe attempted to show that W_5 is reducible. If we assume there exists a minimal counterexample to the Four Color Theorem and W_5 is reducible, then we quickly arrive at a contradiction. Since $\{W_5\}$ is an unavoidable set, W_5 must be in the minimal counterexample. However, since we assume it is reducible, W_5 cannot be in the minimal counterexample—a contradiction. Unfortunately, we do not have that W_5 is reducible as Kempe believed he proved. However, his incorrect proof gives us a strategy for how to construct a correct one. Namely, if we can find an unavoidable set of configurations such that each of them is reducible, we will have shown that a minimal counterexample to the Four Color Theorem cannot exist. In the next section, we will outline how to show that a configuration is reducible, and in the section after that we will briefly give an idea of how Appel and Haken went about finding an unavoidable set of configurations.

5 Reducibility

How do we check if a configuration is reducible? We will step through a couple of algorithms that do so. All of them require the use of a *reducer* at some step. We will define a few terms needed to understand reducers.

Definition 5.1. A cycle C is the *outer cycle* of a configuration G if C is the boundary of the outer face of G.

Definition 5.2. To *split* a vertex v is to replace it by two new vertices v' and v'' and to replace each edge incident to it by an edge incident to either v' or v''.

Example 5.2.1. If we split the vertex v_0 in Figure 7b, we can get a graph that looks like Figure 7a. So, splitting is essentially the reverse of vertex identification. Since we will not have parallel edges in the graphs we consider, we will often implicitly add one before splitting.

Definition 5.3. A graph H is a *reducer* of a configuration $\tilde{G}$ if it is smaller than $\tilde{G}$ and the outer cycles of H and $\tilde{G}$ have the same length, possibly after splitting one of its vertices.

Example 5.3.1. The graphs in both Figure 7b and Figure 7a (without parallel edges) are reducers for the configuration in Figure 11.

Definition 5.4. A proper coloring $f : V(G) \to C$ of some graph G, given a set of colors C, is *extensible* to H if there exists some proper coloring $g : V(G \cup H) \to C$ of $G \cup H$ such that $g = f$ on G.

Now, we can step through an algorithm [1] to check if a configuration $V \cup C_n$ is reducible. In the rest of this section, we will use V and U to denote the subgraphs that lie in the interior and exterior, respectively, of a cycle of a configuration. In addition, we will use V^- and U^- to denote V and U, respectively, with edges and/or vertices removed.

Algorithm 1 Simple but slow

```
Generate a reducer V⁻ ∪ C_n
Store all of the possible 4-colorings of C_n extensible to V⁻
for all such colorings do
    if the coloring is extensible to V, possibly after Kempe interchange[3] then
        Continue
    else
        V ∪ C_n is not reducible with the reducer V⁻ ∪ C_n
    end if
end for
V ∪ C_n is reducible
```

[3]Let v be a vertex in C_n, and let H be a Kempe chain in the exterior of C_n containing v. If we have a coloring for C_n, we can check if Kempe interchange on H will affect the other vertices in C_n. If not, we have a valid recoloring of C_n, and we can check if it is extensible to V. While it may seem computationally expensive to check all possible Kempe interchanges, the number of interchanges we could have is reasonably bounded. Since we can only guarantee that Kempe interchange on H will leave the other vertices in C_n unaffected when there is a Kempe chain in the two complementary colors connecting the neighbors of v in C_n, we only have finitely many Kempe interchanges to consider (corresponding to the finitely many number of such colorings on C_n).

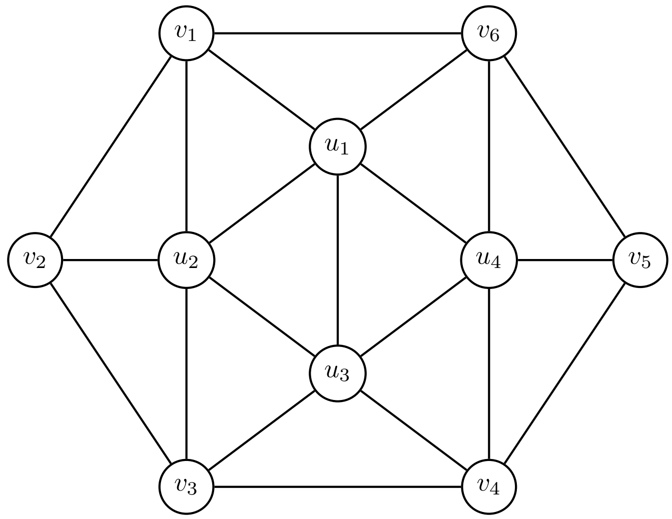


Figure 12: The Birkhoff Diamond

Let $\tilde{G}$ be an essentially 6-connected triangulation. We want to show that for any $\tilde{G}$, we have

$$\tilde{G} \text{ contains } V^- \cup C_n \text{ and is 4-colorable} \implies (\tilde{G} - V^-) \cup V \text{ is 4-colorable.} \quad (1)$$

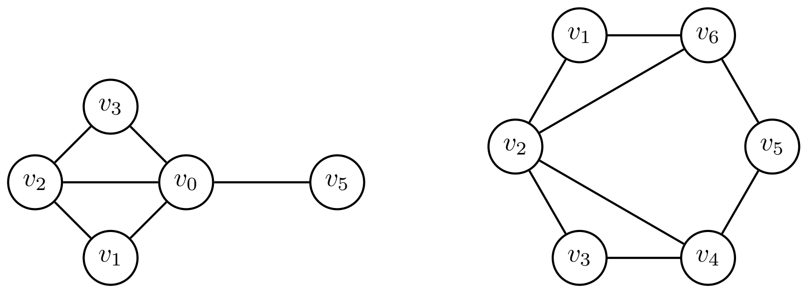
Once we establish this, the contrapositive gives us that $V \cup C_n$ is reducible. This is the strategy of Algorithm 1. Let S be the set of all 4-colorings of C_n extensible to V^-. Then, every coloring in S is extensible to V only if we have (1). So, for a given reducer, Algorithm 1 will check (1). We will step through a few examples of how the algorithm works for different configurations.

Example 5.5.1. Consider the configuration in Figure 11. We will show that it is reducible using Algorithm 1. Choose W_5 as a reducer. Since every vertex of C_5 is adjacent to the inner vertex in W_5, the cycle C_5 must be colored with 3 colors. In addition, since the cycle has 5 vertices, the only 4-colorings of C_5 extensible to W_5 are essentially

$$yrgrg, gyrgr, rgyrg, grgyr, \text{ and } rgrgy.$$

With the first coloring of the ring, we can choose *byrby* as a coloring for the cycle $u_1u_2u_3u_4u_5$. This leaves green for u_6, completing the 4-coloring of V. Since $V \cup C_n$ is rotationally symmetric, the other four colorings of C_5 are extensible to V as well. Therefore, $V \cup C_n$ is reducible.

Example 5.5.2 (Birkhoff Diamond)**.** Now, consider the slightly more complex configuration in Figure 12. It is also relatively straightforward to show that it is reducible using Algorithm 1. First, we choose the reducer in Figure 13a. It



(a) Reducer (b) Reducer after splitting

Figure 13: Reducer of the Birkhoff Diamond

is a reducer since v_0 can be split to create the graph in Figure 13b. It gives rise to 4-colorings on C_6 such that v_4 and v_6 have the same color, and v_4 and v_6 are adjacent to v_2. The only such valid colorings are essentially

$$gybrgr, ygyrgr, ygbrgr, ybgrgr, \text{ and } gygrgr.$$

The first four colorings can be easily extended to V without Kempe interchange. Given the first coloring, we can color u_1 yellow, u_2 red, u_3 green, and u_4 blue. Given the second coloring, we can color the cycle $u_1u_2u_3u_4$ with *grby*. Likewise, we can extend the third and fourth colorings of C_6 with *gryb*.

So, the only coloring left to consider is *gygrgr ryrygy*. Unfortunately, this coloring is not directly extensible to V, so we must look for alternative colorings that arise from Kempe interchange. First, if v_3 and v_5 are not connected by a green-blue Kempe chain, then green and blue can be swapped in the green-blue Kempe chain containing v_3 to produce a new coloring *gybrgr*, which has already been handled. So we will only consider the case in which v_3 and v_5 are connected by a green-blue Kempe chain. This Kempe chain must lie in the exterior of C_6 since V is not yet colorable.

Now, consider if there was a red-yellow Kempe chain lying in the exterior of C_6 connecting v_2 and v_4. Then, it would have to intersect the green-blue Kempe chain, which is impossible. Therefore, by Kempe interchange we can swap red and yellow in the Kempe chain containing v_2 to get the coloring *gygygr*. Then, $u_1u_2u_3u_4$ can be colored *ybrb*. Therefore, every coloring of the reducer leads to a coloring of the configuration, which implies that the configuration is reducible.

Although this algorithm works fine, it is more computationally expensive than we would like. This leads us to a second algorithm [1].

Algorithm 2 Faster

Store all possible 4-colorings of C_n in J_n and all possible 4-colorings of V
$K_0 \leftarrow$ all 4-colorings of C_n extensible to V
$S \leftarrow \{rrrrr\}$
$i \leftarrow 0$
while $K_i \neq J_n$ and $S \neq \emptyset$ **do**
  $S \leftarrow \emptyset$
  for all Kempe interchanges[4] **do**
    Add the coloring of C_n after interchange to S
  end for
  $i \leftarrow i + 1$
  $K_i \leftarrow K_{i-1} \cup S$
end while
if $K_i = J_n$ **then**
  Stop. $V \cup C_n$ is reducible
else
  search for a reducer $V^- \cup C_n$ such that the set of all 4-colorings of C_n extensible to V^- is a subset of K_i
  if such a reducer exists **then**
    $V \cup C_n$ is reducible
  else
    $V \cup C_n$ is not reducible by the algorithm
  end if
end if

Example 5.6.1. Consider the same configuration in Figure 11. We will show that it is reducible using Algorithm 2. First we find all possible 4-colorings of C_5 and store them in J_5. There are $4 \cdot 3^3 \cdot 2 = 216$ different 4-colorings of C_5. We will not list them all for now. In addition, there are $4 \cdot 3 \cdot 2^3 \cdot 1 = 144$ different 4-colorings of V. Let this set of colorings be I. Now, we want to determine which colorings from J_5 are extensible to V. Although there are 216 different colorings in J_5, there are essentially only 2 that we need to consider, and those are the ones in which $v_1v_2v_3v_4v_5$ is colored $rgryb$ or $rgrgy$. Every coloring in J_5 has two vertices that are the same color and that share a common neighbor. Then, either the other three vertices must be the 3 colors not used, or the common neighbor must be colored the same as one of the other vertices so that 3 total colors are used. Since the configuration $V \cup C_5$ is rotationally symmetric, the exact positions of these colors are not important.

When C_5 is colored $rgryb$, we can extend the coloring to V by coloring $u_1u_2u_3u_4u_5$ with $bybyg$ and u_6 with red. When C_5 is colored $rgrgy$, we can extend the coloring to V by coloring the inner cycle with $ybygb$ and u_6 with red. Therefore,

$$K_0 = \{rgryb, bybyg\} = S.$$

[4] We will search for Kempe interchanges in the same way we did in Algorithm 1.

So, $V \cup C_5$ is reducible.

Although Algorithm 2 seems to require more work than Algorithm 1, it is less computationally expensive. Luckily for us, in Example 5.5.1 we started with a reducer that worked for our purposes. However, we will not always be so lucky, and in cases we are not, we will need much more computation. That is why Algorithm 2 checks for reducers at the very end. (In fact, in Example 5.6.1, we did not even use a reducer.) Since this last step is difficult to handle efficiently, it requires us to pay attention to details that we will not discuss here.

As it turns out, every configuration with an outer cycle of length 5 and more than 1 vertex in its interior is reducible. This was proved by Birkhoff in 1913 [3]. We use this in our proof of Corollary 4.8. For brevity, we will show a similar result, which we use in our proof of Proposition 3.2.

Proposition 5.7. *Every configuration with an outer cycle of length* 4 *is reducible.*

The proof is left to Appendix D. The techniques used in it are the same as those used to prove that every configuration with an outer cycle of length 5 and more than 1 vertex in its interior is reducible.

6 Discharging

How do we find unavoidable sets? Trivially, the set of all possible configurations is an unavoidable set. However, this is not useful for us because the set is infinite. We have to somehow use the properties of essentially 6-connected triangulations to find other smaller unavoidable sets. By Corollary 2.2,

$$\tilde{G} \text{ is a triangulation } \iff m = 3n - 6.$$

Perhaps, we can use this property to discover unavoidable sets. First, we rewrite the equation in a way that suggests we use a specific technique.

$$6n - 2m = 12 \iff \sum_{v \in V} (6 - d(v)) = 12.$$

Let $q(v) = 6 - d(v)$. We will call $q(v)$ the *charge* of v. Then, the sum of all of the charges of each of the vertices, or the *total charge* of $\tilde{G}$ must be 12. So,

$$\tilde{G} \text{ is a triangulation } \iff \text{ the total charge of } \tilde{G} = 12.$$

If we take the contrapositive, we get that the total charge of a plane graph is not 12 if and only if the plane graph is not a triangulation. So, we can say that a set $\mathcal{U}$ is unavoidable if any essentially 6-connected triangulation that does not include some configuration in $\mathcal{U}$ does not have charge 12. If we assume that a 6-connected triangulation does not contain any configuration in U, we get a contradiction, since every essentially 6-connected triangulation must have charge 12. So, every 6-connected triangulation must contain a configuration in $\mathcal{U}$.

Sometimes, it may be difficult to determine the total charge of a graph. We may have too many charges to add, or we may not know the number of vertices in the graph. In this case, we can simply redistribute the charges of the vertices such that the total charge is maintained. If we pick a clever way to redistribute these charges, we may be able to get a reasonable estimate on the total charge—one which we did not have before. For example, if we can redistribute the charge of a graph such that every vertex has charge 0 or less, we know that graph has total charge less than 0, which implies it cannot be a triangulation. This method of redistributing charge is called *discharging*, and the algorithm we use to redistribute charge is called the *discharging algorithm*. If we can redistribute the charge of a graph such that each of its vertices has charge 0 or less, we say the graph is *dischargable*.

How does discharging help us find unavoidable sets? Well, if we find a set of configurations $\mathcal{U}$ such that every essentially 6-connected triangulation that does not contain a configuration in $\mathcal{U}$ is dischargable, we know that set is unavoidable. Alternatively, if we find a set of configurations that occur in every non-dischargable graph, we know that set is unavoidable. We will follow an example discharging algorithm from Bondy and Murty [4] to show how to find an unavoidable set. Define the algorithm as follows:

Algorithm 3 Simple discharging

```
for all vertices v ∈ V do
    if d(v) = 5 then
        Give each of v's neighbors charge 1/5, leaving no charge on v
    end if
end for
```

This algorithm preserves the total charge of any graph. Now, for any vertex v in $\tilde{G}$ the maximum charge it can have after the algorithm is

$$q_{\max}(v) = 6 - d(v) + \frac{1}{5}d(v).$$

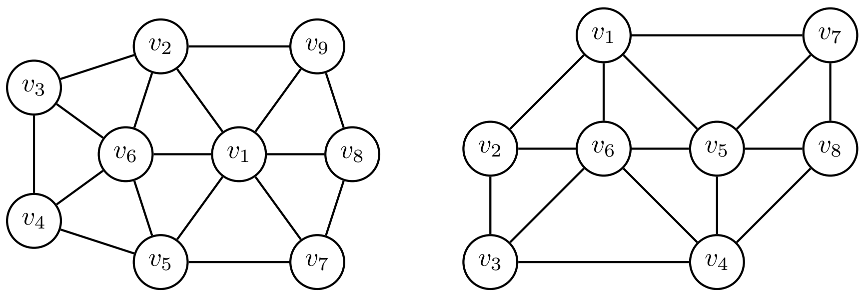
This happens when each of its neighbors has degree 5. Therefore, if $\tilde{G}$ contains a vertex v of degree 8 or more, that vertex will be discharged by the algorithm since

$$q_{\max}(v) = 6 - 8 + \frac{8}{5} = -\frac{2}{5} < 0.$$

Also, if $\tilde{G}$ contains a vertex v of degree 7 with at most 5 neighbors of degree 5 then

$$q_{\max}(v) = 6 - 7 + \frac{5}{5} = 0.$$

So, it is dischargable. Likewise, vertices of degree 5 and 6 without any neighbors of degree 5 are dischargable. Therefore, if $\tilde{G}$ is not dischargable, it



(a) Adjacent vertices with degrees 5 and 6 (b) Adjacent vertices both with degrees 5

Figure 14: An unavoidable set of configurations

must either contain a vertex of degree 5 or 6 with a neighbor of degree 5, or it must contain a vertex of degree 7 with 6 or 7 neighbors of degree 5. In the second case, we must have two vertices of degree 5 connected by an edge. So, $\tilde{G}$ must either contain two vertices of degree 5 connected by an edge, or a vertex of degree 5 and a vertex of degree 6 connected by an edge. In fact, since $\tilde{G}$ is a triangulation, these vertices must exist in the configurations in Figure 14a and Figure 14b. Therefore, the set of configurations containing the configuration in Figure 14a and the configuration in Figure 14b is unavoidable.

Unfortunately, both of these configurations cannot be shown to be reducible. In the original proof of the Four Color Theorem [2], the discharging algorithm was much more complex and ended up discovering a much larger unavoidable set of more than 1400 configurations.

Now, we have seen the most significant ideas in the original proof of the Four Color Theorem. We summarize these ideas in an implication diagram. Let $\mathcal{U}$ be a set of configurations. Then,

every non-dischargable graph contains some $G \in \mathcal{U}$

$\Downarrow$

$\mathcal{U}$ is unavoidable & every $G \in \mathcal{U}$ is reducible $\Longrightarrow$ a minimal counterexample does not exist.

A Euler's Formula and Its Corollaries

Throughout this section, we introduce theorems, corollaries, and proofs taken from Bondy and Murty [4]. Before we prove Euler's Formula, we must define the *dual* of a graph.

Definition A.1. A planar graph G^* is the *dual* of a plane graph $\tilde{G}$ if every vertex of G^* corresponds to a face of $\tilde{G}$, every edge of G^* corresponds to an

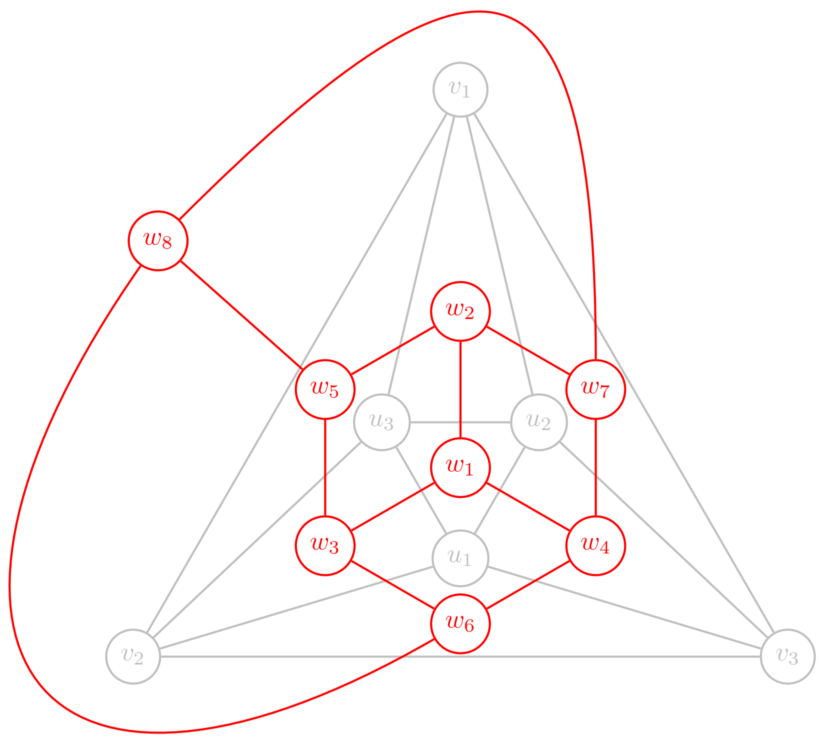


Figure 15: The plane graph in Figure 3 (in gray) and its dual (in red)

edge of $\tilde{G}$, and two vertices of G^* are joined by an edge in G^* if and only if their corresponding faces in $\tilde{G}$ are joined by the corresponding edge in $\tilde{G}$. (See Figure 15.)

With this definition, we can prove Euler's Formula.

Lemma A.2. *Let $\tilde{G}$ be a plane graph and m be its number of edges. Then,*

$$\sum_{f \in F} d(f) = 2m.$$

Proof. Consider the dual G^*. The degree of some vertex in G^* is the same as the degree of the corresponding face in $\tilde{G}$. Therefore, we must show

$$\sum_{v \in V(G^*)} d(v) = 2m.$$

If we sum each of the columns for a given row in the incidence matrix of G^*, we get the degree of the vertex in that row. Then summing over each of the rows we get $\sum_{v \in V} d(v)$. Alternatively if we sum each of the rows for a given column, we get 2 since each edge has two vertices. Then, summing over each column we get $2m$. □

Theorem 2.1 (Euler's Formula)**.** *For a connected plane graph $\tilde{G}$,*

$$v(\tilde{G}) - e(\tilde{G}) + f(\tilde{G}) = 2.$$

Proof. We perform induction on the number of faces. In the case when $f(\tilde{G}) = 1$, we have a tree, so $v(\tilde{G}) = e(\tilde{G}) + 1$, and

$$v(\tilde{G}) - e(\tilde{G}) + f(\tilde{G}) = 2.$$

Now, consider a plane graph $\tilde{G}$ with a different number of faces and assume that all plane graphs with one less face satisfy the formula. Then, there exists some edge e such that removing it from $\tilde{G}$ forms a graph with $f(\tilde{G}) - 1$ faces. Therefore,

$$\begin{aligned} v(\tilde{G}) - e(\tilde{G}) + f(\tilde{G}) &= v(\tilde{G}) - (e(\tilde{G}) - 1) + (f(\tilde{G}) - 1) \\ &= v(\tilde{G} \setminus e) - e(\tilde{G} \setminus e) + f(\tilde{G} \setminus e) \\ &= 2. \end{aligned}$$

□

Now, we are ready for the corollaries [4]. We use these in the proof of the Five Color Theorem and our discussion of the Four Color Theorem.

Corollary 2.2. *Let G be a simple planar graph on at least 3 vertices. Let m denote the number of edges in G and n the number of vertices. Then $m \leq 3n-6$. Furthermore, $m = 3n - 6$ if and only if every plane graph representing G is a triangulation.*

Proof. Let $\tilde{G}$ be a plane graph representing G. In a simple connected graph, it takes at least 3 edges to form the boundary for a face, so $d(f) \geq 3$ for all $f \in F$. Then, by Lemma A.2 and Euler's formula,

$$2m = \sum_{f \in F(\tilde{G})} d(f) \geq 3f(\tilde{G}) = 3(m - n + 2).$$

Rearranging gives us the desired result. □

Corollary 2.3. *Every simple planar graph has a vertex of degree at most* 5.

Proof. Since δ is less than or equal to the degree of each vertex, we have

$$\delta n \leq \sum_{v \in V} d(v) = 2m \leq 6n - 12.$$

This leaves

$$\delta \leq 6 - \frac{12}{n} \implies \delta \leq 5.$$

□

B Proof of Lemma 2.5

Lemma 2.5. *Let $f : V(G) \to C$ be a proper coloring of some graph G with a set of colors C. Let $g : V(G) \to C$ be the coloring we get after Kempe interchange on some Kempe chain H that comes from f. Then, g is a proper coloring of G.*

Proof. Assume for contradiction that g is not a proper coloring of G. Then, some vertex v in H is adjacent to a vertex v' of the same color with the coloring from g. This vertex v' must lie outside H. However, v' is colored with one of the two colors of H and adjacent to v in H with the coloring from f. This is a contradiction. So, g must be a proper coloring of G. □

C Properties of a Minimal Counterexample

Some of the proofs in this section come from Bondy and Murty [4].

Proposition 3.2. *If $\tilde{G}$ is a minimal counterexample to the Four Color Theorem, then both the following two statements must be true:*

- *$\tilde{G}$ is a triangulation.*
- *$\tilde{G}$ has no vertex of degree less than* 5.

Proof. First, we will show that $\tilde{G}$ is a triangulation. Assume for contradiction that $\tilde{G}$ contains a face bounded by an n-cycle C_n for some $n \geq 4$. Since $\tilde{G}$ is planar, two of the vertices of C_n are not adjacent. If we identify these vertices we get a new graph $\tilde{H}$ that is smaller than $\tilde{G}$. Then, if $\tilde{G}$ is not 4-colorable, neither is $\tilde{H}$ since the vertex in $\tilde{H}$ left after identification has the same neighbors as the two vertices in $\tilde{G}$ before identification. This is a contradiction, since $\tilde{G}$ is a minimal counterexample. Therefore, $\tilde{G}$ is a triangulation.

Now, we will show $\delta \geq 5$. We first show that it cannot be that $\delta \leq 3$. Let v be a vertex of degree δ. Since $\tilde{G}$ is a minimal counterexample, $\tilde{G} - v$ is 4-colorable. Since v has no more than 3 neighbors, we can color it with one of the colors not used by its neighbors to get a 4-coloring of $\tilde{G}$. This is a contradiction, so $\delta \geq 4$. To show that $\delta \neq 4$, we rely upon a proof we will show later, namely one that shows that every configuration with an outer cycle of length 4 is reducible (Proposition 5.7). □

Corollary 3.3. *If $\tilde{G}$ is a minimal counterexample to the Four Color Theorem, then $\tilde{G}$ contains the wheel W_5.*

Proof. By Corollary 2.3, $\delta \leq 5$, and by Proposition 3.2, $\delta \geq 5$. So, $\delta = 5$. Therefore, $\tilde{G}$ must contain a vertex of degree 5, and since $\tilde{G}$ is a triangulation, this implies that $\tilde{G}$ contains W_5. □

Proposition 4.7. *If $\tilde{G}$ is a minimal counterexample to the Four Color Theorem, then $\tilde{G}$ contains no separating 4-cycle*

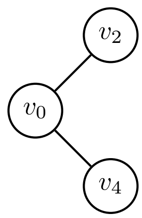
Proof. We use a result that we will prove later, namely that every configuration with an outer cycle of length 4 is reducible (Proposition 5.7). □

Corollary C.1. *If $\tilde{G}$ is a minimal counterexample to the Four Color Theorem, then $\tilde{G}$ is 5-connected.*

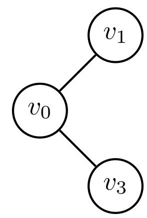
Proof. Suppose that there exists some set $S \subset V(\tilde{G})$ of 4 vertices such that $\tilde{G} - S$ is disconnected. If we can show that S must be the vertices of a cycle in $\tilde{G}$, we are done, since this implies S cannot exist by Proposition 4.7. Assume for contradiction that the vertices of S are not in a cycle. Let V be a connected component of $\tilde{G} - S$. Then, there exists a path (in the topological sense) in $\tilde{G}$ from a vertex v_i of V to another vertex v_j of some connected component U of $\tilde{G} - S$. Therefore, v_i and v_j are part of the boundary of the same face. Since $\tilde{G}$ is a triangulation, this implies that an edge connects v_i and v_j. This contradicts that $\tilde{G} - S$ is disconnected. Therefore, S must be the vertices of a cycle. This leads to another contradiction; so, $\tilde{G}$ is 5-connected. □

Corollary 4.8. *If $\tilde{G}$ is a minimal counterexample to the Four Color Theorem, then $\tilde{G}$ is essentially 6-connected.*

Proof. By Corollary C.1, $\tilde{G}$ is 5-connected. We use a result that we will discuss later, namely that every configuration with an outer cycle of length 5 is reducible if it has more than one vertex in its interior [3], to show that every separating 5-cycle is induced by neighbors of a vertex of degree 5. We will not prove this result, but the proof uses the same technique as the proof of Proposition 5.7. □



(a) $V^- \cup C_4$ and $C_4 \cup U^-$



(b) Alternative $V^- \cup C_4$ and $C_4 \cup U^-$

Figure 16: Subgraphs in the proof of Proposition 5.7

D Proof of Proposition 5.7

Proposition 5.7. *Every configuration with an outer cycle of length* 4 *is reducible.*

Proof. Since we are not given a configuration, it is impossible to determine whether a coloring from a reducer is extensible to a configuration. So, we cannot apply Algorithm 1. Likewise, we cannot use a straightforward application of Algorithm 2 either. Instead, we must use a more clever technique. Let $V \cup C_4$ denote some configuration, and let U be the rest of the graph, in the exterior of C_4. Then, let K_1 be the set of 4-colorings of C_4 that come from some choices of $C_4 \cup V^-$, and let K_2 be the set of 4-colorings of C_4 that come from some choices of $C_4 \cup U^-$[5]. Now, if we can show for any two colorings from K_1 and K_2, we can use Kempe interchange to match them, we are done.

To see why, suppose that our assumption is true. Then, for any choice of V, if a 4-coloring exists for $V \cup C_4 \cup U^-$, the coloring for C_4 is in K_2. Likewise, if a 4-coloring exists for $V^- \cup C_4 \cup U$, the coloring for C_4 is in K_1. Furthermore, we can use Kempe interchange to match these colorings. Therefore, if $V \cup C_4 \cup U^-$ and $V^- \cup C_4 \cup U$ are 4-colorable, so is $V \cup C_4 \cup U$ by superposition. If we take the contrapositive, we get that if $V \cup C_4 \cup U$ is not 4-colorable, either $V^- \cup C_4 \cup U$ or $V \cup C_4 \cup U^-$ is not either, which implies that $V \cup C_4$ is reducible. So we want to show that given any two colorings from K_1 and K_2 we can match them via Kempe interchange.

First, we will build K_2 by choosing U^-. Let $C_4 \cup U^-$ be the graph in Figure 16a, which comes from identifying v_1 and v_3 in C_4. This leads to 4-colorings on C_4 in which the first and third vertices have the same color. So, we essentially have

$$K_2 = \{rgrg, rgry\}.$$

Now, we build K_1. Our construction of K_1 will depend on V. First, if V is such that $rgrg$ is not a valid 4-coloring of C_4 in $V \cup C_4 \cup U^-$, we choose $V^- \cup C_4$ as represented in Figure 16a. This gives us

$$K_1 = \{rgrg, rgry\}.$$

[5]If $H = C_4 \cup U^-$, then we will use $C_4 \cup U^-$ to denote H or the graph that comes from identifying two vertices on C_4 in H.

If we pick U such that $rgry$ is the only valid 4-coloring of C_4 in $C_4 \cup U$ we are done, since $rgry$ is the only valid 4-coloring in C_2. Otherwise, we are left with

$$K_1 = \{rgrg, rgry\},$$

and

$$K_2 = \{rgry\}.$$

Now, if C_4 has coloring $rgrg$ in $V^- \cup C_4$, it is impossible for a red-blue Kempe chain to connect v_1 and v_3 and a green-yellow Kempe chain to connect v_2 and v_4. Therefore, either the green-yellow Kempe chain containing v_4 can be swapped so that v_4 becomes yellow, or the red-blue Kempe chain containing v_3 can be swapped so that v_3 becomes blue. In the first case, we can keep K_2 as is so that both colorings match. In the second case, we can build a new K_2 from the graph $C_4 \cup U^-$ in Figure 16b so that both colorings match.

Alternatively, if $rgrg$ is a valid 4-coloring of C_4 in $V \cup C_4 \cup U^-$, then we can either choose $V^- \cup C_4$ as shown in Figure 16a or as shown in Figure 16b $V^- \cup C_4$ to build K_1. We can use the same technique as before to choose which K_1 to build and to match each of the colorings in K_1 to $rgrg$. Therefore, since any coloring of K_1 can be matched with any coloring of K_2, every configuration with an outer cycle of length 4 is reducible. □

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