HYPERBOLIC DYNAMICAL SYSTEMS AND STRUCTURAL STABILITY

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Abstract. This paper gives an introduction to discrete dynamical systems, the study of repeated iterations of a function on a space. We begin by giving an overview of dynamical systems and how we can learn about the system through patterns of recurrence. However, we can also learn about a system by studying how nearby dynamical systems behave. In particular, a dynamical system is *structurally stable* if nearby systems have the same structure. Hyperbolic dynamical systems have this property and will be the focus of this paper. We will then define hyperbolic sets or sets characterized by an invariant contraction and expansion. Afterwards, we prove that hyperbolic sets persist under small perturbation, leading up to proving the structural stability of hyperbolic dynamical systems.

Contents

1. Introduction

Every year, the city of Chicago hosts its annual duck derby where hundreds of rubber ducks are dumped in the Chicago river. Standing on the Riverwalk, you spot one of the sunglasses-wearing ducks and choose to follow it as it flows downstream. The duck's path is affected by where it is on the river. If our duck gets close to a boat with its engine on, ready to go, it will be repelled to the other side of the river. Alternatively, our duck could get stuck in a particular dock and remain stationary for the rest of the day, while other ducks nearby flow on down. If we take all the ducks and their paths down the river, we have what is called a *dynamical system*.

At a high level, a dynamical system describes the movement of points over time. We can break down the study of dynamical systems into discrete and continuous dynamics. The former would be akin to attaching an Airtag to each of our ducks and plotting out their journey on Apple maps as

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they flow in continuous time. In this paper, we will study *discrete dynamical systems*, which give us snapshots of the ducks' journeys per some discrete unit time. The study of discrete dynamical systems would be like studying the movement of our ducks through birds-eye photos of the river, taken every second.

One thing dynamicists look at when studying these systems are patterns of recurrence—does a particular subset of ducks stay in the same place from picture to picture? Do they ever return back to where they started? We are also interested in the system's *stability*—if I perturb my system in some smooth way or continuously change the shape of the river, will the movement of our ducks stay the same? What kind of behaviors are preserved?

This paper will give an introduction to topological dynamical systems and to *hyperbolicity*, a property we study because of its amazing stability. We begin by defining a dynamical system and invariance in §2. With that background, we delve into the definition and intuition behind hyperbolicity in §3. Next, in §4, we introduce one notion of stability for hyperbolic sets, *persistence*, or the notion that small perturbations of dynamical systems also have some copy of a hyperbolic set. Finally, we talk about structural stability in §5, which gives us an even stronger property of hyperbolic sets: small perturbations of a function are dynamically the same as the original function when restricted to their hyperbolic sets.

2. Basics of Dynamical Systems

In this section, we start by formalizing the notion of a discrete dynamical system as iterations of a homeomorphism. Then, we will describes some patterns of recurrence and invariance, as well as some examples of dynamical systems. Finally, we will define the notion of "dynamical sameness" through topological conjugacy.

2.1. **Definitions & Examples.** .

Definition 2.1 (Discrete Dynamical System)**.** Let M be a closed manifold, that is, a compact manifold without boundary. Let $f : M \to M$ be a homeomorphism. The set of homeomorphisms generated by the iterates of f, $\{f^n\}_{n=-\infty}^{\infty}$, where $f^n = f \circ \cdots \circ f$ is a discrete dynamical system.

Each iterate f^n represents one of the "snapshots" of our space, which in the introduction was the Chicago River. Since we want to be able to look at snapshots of our space in the past, we consider the backwards iterates f^{-n} and so require f it to be a homeomorphism.

One of the canonical examples of dynamical systems are circle homeomorphisms, which are simple to model and allow us to introduce some key notions of recurrence.

Example 2.2. Let $M = S^1$ and $f : S^1 \to S^1$ define a rotation of S^1 by a rational angle $\theta = \frac{\pi}{a}$, $a \in \mathbb{N}$. We can formally write this map as

$$f(x) = x + \alpha \mod 1, \qquad x \in [0, 1),\ \alpha \in \mathbb{Q}.$$

We are interested in how this dynamical system acts on a point $x \in S^1$ over time, which we can represent by the set of iterates $\{f^n x\}_{n=-\infty}^{\infty}$. This set is called the *orbit* of x under f and is denoted by $\mathrm{Orb}(x, f)$ or $\mathrm{Orb}(x)$.

A point $x \in M$ is called *periodic* if $f^m x = x$ for some $m \in \mathbb{N}$. In this example, all $x \in S^1$ are periodic, since after $m = 2a$ iterations, x would have made exactly one full rotation around the circle and returned to itself. What if, instead, α was irrational? Then, no $x \in S^1$ is periodic! In fact, for any point x on the circle, $\mathrm{Orb}(x)$ is dense in S^1. Since periodic orbits are finite, no $x \in S^1$ can be periodic.

We now introduce another example of a circle homeomorphism.

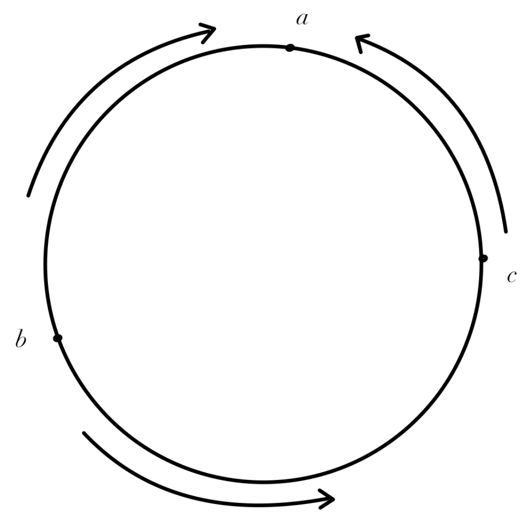

FIGURE 1. This is the system described in Example 2.3. Here, the arrows denote the direction that f contracts the open intervals between the fixed points.

Example 2.3. Let $M = S^1$ and f be a homeomorphism that fixes 3 points, a, b, and c (See Figure 1). The behavior around those points is as follows: a is a *sink* or an *attractor*, b is a *source*, c is a *saddle*.

A set Λ is invariant under f if $f(\Lambda) = \Lambda$. In this system, there are many invariant sets. For example, any open interval excluding the fixed points will be invariant. Of course, for any $x \in S^1$, $\text{Orb}(x)$ will be invariant. It actually turns out that a set Λ is invariant if, and only if, it is the union of orbits. The forward implication isn't too hard to see, and the backwards implication follows since

$$\Lambda = \bigcup_{x \in \Lambda} \text{Orb}(x).$$

Dynamical systems can also be quite complex and beautiful.

Example 2.4 (Logistic Map)**.** Let $M = [0, 1]$ and f be defined as $f(x) = rx(x + 1)$, $r \in [0, 4]$. f is called the *logistic map* since it can describe logistic population growth. What do the orbits of this map look like–in particular, what is their limiting behavior? Suppose $r < 1$. Then, for all $x \in [0, 1]$, $f^n x$ will approach 0 as n becomes large; our population dies off. However, if $r > 3$, as $n \to \infty$, $f^n x$ keeps oscillating in a non-periodic, non-recurrent way for all $x \in [0, 1]$. Still, the orbits of x have several sub-sequential limits. As r increases, so does the amount of subsequential limits of $\text{Orb}(x)$. We graph all the subsequential limits of x for each value of r in Figure 2.

2.2. **Stability.** With a few examples under our belt, what do dynamicists mean when we talk about *stability*? Why do we even care? In dynamics, we can learn a lot about a system by looking at its orbits and properties like periodicity and invariance. But, we can also learn about a system by looking at how nearby systems behave. Formally, if we perturb our system by adding some small smooth function, we ask, what does the orbit structure of our perturbed system look like?

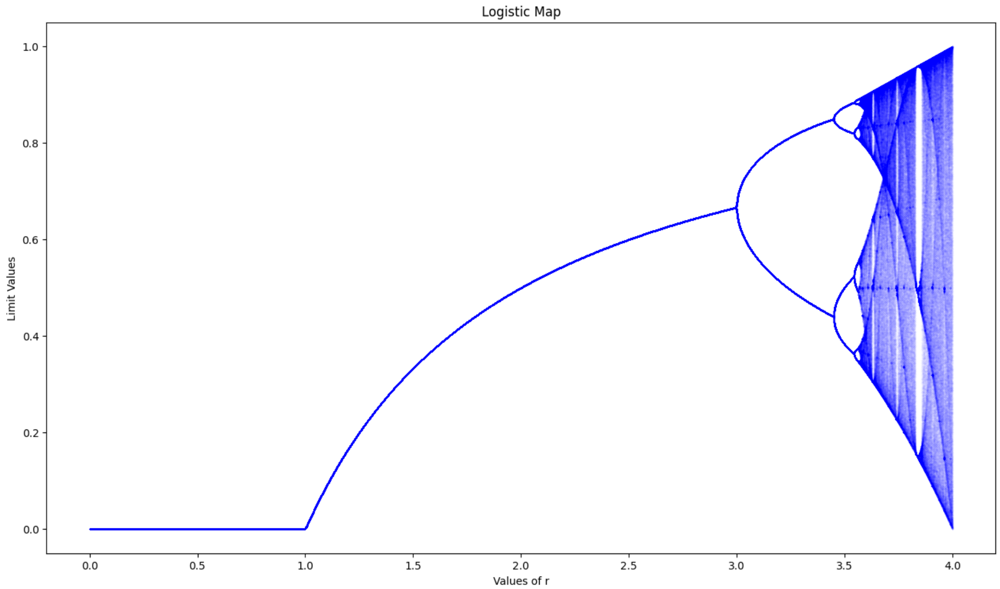

FIGURE 2. This graph plots the asymptotic limits of orbits under the Logistic Map for $r \in [0, 4]$. The code that generated this graph can be found in [5].

Recall Example 2.2. Let $f : S^1 \to S^1$ be the rational rotation

$$f(x) = x + \alpha \mod 1, \qquad \alpha = \frac{1}{2}.$$

Define ϕ to be the irrational rotation by some $\epsilon > 0$ small,

$$\phi(x) = x + \epsilon \mod 1.$$

If we *perturb* f by adding ϕ, $g = f + \phi$ will be an irrational rotation. Thus, we call g a *perturbation* of f. We know from Example 2.2 that all $x \in S^1$ are periodic under f; however, there are no periodic points under g! These are two very different orbit structures. Furthermore, no matter how small our perturbation ϕ is, we can always find some irrational rotation close to f whose orbit behavior is wildly different.

At a high level, a dynamical system is stable if for sufficiently small perturbations, orbit structures are the "same." So, for example, a system like a rotation of a circle would not be stable. This leads to another question: what's a sufficient notion of "sameness"? In topological dynamics on a smooth manifold, we care about preserving topology. So we define the dynamical notion of "sameness" between two functions through *topological conjugacy.*

Definition 2.5 (Topological Conjugacy)**.** Two homeomorphisms $f : X \to X$ and $g : X \to X$ are *topologically conjugate* if there exists a homeomorphism $h : X \to X$ that satisfies

$$hf = gh$$

or

$$g = hfh^{-1}. \tag{1}$$

Here $h : X \to X$ is called a *topological conjugacy.*

Equation (1) is reminiscent of the structure of similar matrices. In fact, if f, g, and h were all linear, h would simply be a change of basis matrix. Thus, we can think of topological conjugacy as a continuous change of coordinates.

Example 2.6. Two functions that are topological conjugates are the logistic map introduced in Example 2.4 and the *tent map*. Specifically, let $f : [0, 1] \to [0, 1]$ and $g : [0, 1] \to [0, 1]$ be defined by

$$f(x) = \begin{cases} 2x, & \text{if } x \in [0, 1/2] \\ -2x + 2, & \text{if } x \in (1/2, 1] \end{cases} \qquad g(x) = 4x(1 - x).$$

Our conjugacy would be $h : [0, 1] \to [0, 1]$, defined by $h(x) = \sin^2(\pi x/2)$. Verifying that this is a conjugacy just requires composing the functions and using some trigonometric identities; this computation is rigorously detailed in [6]. Graphing both maps in Figure 3, we can visualize the topological conjugacy.

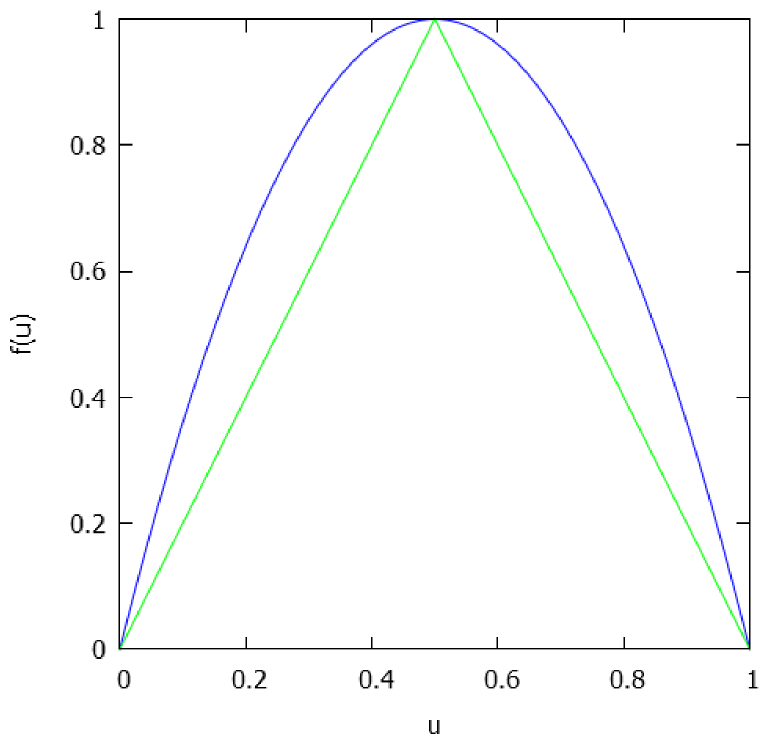

FIGURE 3. The tent and logistic map plotted on the same graph. If we "pull down" the logistic map, you can see we can get the tent map. This image is courtesy of [2].

With this notion of sameness, we define structural stability. Let $\text{Diff}^r(M)$ denote the set of C^r diffeomorphisms on a closed smooth manifold M. By C^r *diffeomorphism* we mean a continuously $r^{th}-$order differentiable function whose inverse is also continuously $r^{th}-$order differentiable. We define a topology on this space using charts to derive our open sets from $\mathbb{R}^n$. This topology is called the Whitney C^r topology, and its construction is detailed in [7]. When we reference a metric on this space, we will use the metric induced by the C^r norm, with the appropriate level of smoothness.

Definition 2.7 (Structural Stability)**.** For $r \geq 1$, $f \in \text{Diff}^r(M)$ is called *structurally stable* if there exists an open neighborhood $\mathcal{U} \subseteq \text{Diff}^r(M)$, where for all $g \in \mathcal{U}$, g is topologically conjugate to f.

Notice that our Definition 2.7 of stability requires that f be a diffeomorphism. Up until now, we've only required f to be a homeomorphism, so what's the reason for this additional condition? C^0 structural stability is vacant. For instance, consider the system in Example 2.3. Suppose that our fixed points were much closer together. Then, a small continuous perturbation could eliminate one of our fixed points, giving us a different orbit structure. Since continuous perturbations can so easily damage orbits, no nontrivial map would fit this criteria of stability.

We can get a stronger notion of stability by further restricting the topological conjugacy h.

Definition 2.8 (Strong Structural Stability)**.** If for any $\epsilon > 0$, f is structurally stable and we can choose a sufficiently small open neighborhood $\mathcal{U} \subseteq \text{Diff}^r(M)$ such that there exists a topological conjugacy $\epsilon-$close to the identity, f is strongly structurally stable.

By that we mean that

$$d(h, \text{id}) < \epsilon,$$

where d denotes the metric induced by the Riemannian norm.

Even in Definition 2.8, h need not be a diffeomorphism, even though f and g necessarily are. To show a reason why, suppose h is a diffeomorphism, and differentiate our conjugacy equation, applying the chain rule. Let x be a fixed point of f. Then,

$$Dh(f(x))Df(x) = Dg(h(x))Dh(x) \implies Df(x) = Dg(h(x)),$$

or the derivatives are preserved under conjugacy. However, as we will see in §4, orbits can be preserved so long as the derivatives are sufficiently close but not necessarily the same.

3. Hyperbolicity

A compelling reason we study hyperbolicity, in particular, hyperbolic sets, is because they are strongly structurally stable. At its core, hyperbolicity is a property that describes a uniform "stretch" and "squish"—it's akin to the saddle-like shape formed by hyperbolas, which makes this name apt. In this section, we first introduce the idea of a hyperbolic linear map. Then, we will use hyperbolic linear maps to define hyperbolic sets. For the remainder of this paper, M will denote a closed, smooth manifold and f a diffeomorphism of M.

3.1. Hyperbolic Linear Maps.

Definition 3.1 (Hyperbolic Linear Map)**.** Let E be a finite dimensional vector space and $A : E \to E$ a linear isomorphism. A is *hyperbolic* if it decomposes E into the direct sum of invariant contracting and expanding subspaces,

$$E = E^s \oplus E^u.$$

E^s and E^u are contracting and expanding, respectively, if they satisfy the properties below: there exists $C \geq 1$ and $0 < \lambda < 1$ such that

$$\forall v \in E^s, |A^n v| \leq C\lambda^n |v|, \forall n \geq 0$$
$$\forall v \in E^u, |A^{-n} v| \leq C\lambda^n |v|, \forall n \geq 0$$

A natural question is why are the superscripts s and u used to denote the contracting and expanding spaces, respectively? Here, these superscripts stand for "stable" and "unstable." Intuitively, this makes sense: if a space is expanding, it's running away to infinity, whereas a contracting space closes in on itself.

We will now introduce an example of a hyperbolic linear map.

Example 3.2 (Arnold's Cat Map)**.** Let $\mathbb{T}^2 = S^1 \times S^1$ be a 2D torus, and $f : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \mod 1.$$

Below, we graph the map acting on the unit square, and then mod out by 1. Superimposed on the unit square is a picture of a cat (as Arnold did in his original paper). Before we mod out the transformation, we can see that the cat got longer and thinner, the key characteristic of hyperbolicity.

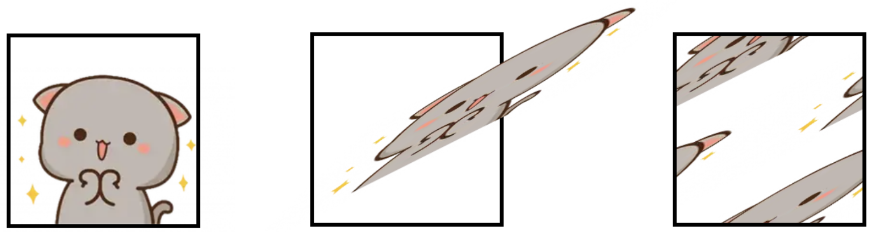

FIGURE 4. Here is an iteration of the Cat Map on the unit square over an illustration of a cat, inspired by Arnold's 1968 paper [10]. Image is courtesy of [8] and code can be found in [9].

Another way to define hyperbolicity is through the eigenvalues of the linear map, and this approach gives more intuition into the idea of the contracting and expanding space. If we look at the eigenvalues of the cat map, one is greater than 1 and the other is less than 1:

$$\frac{3+\sqrt{5}}{2} > 1 \qquad \frac{3-\sqrt{5}}{2} < 1.$$

These eigenvalues describe the behavior of every point on the torus since their corresponding eigenvectors span the space. Every point in our space can be decomposed into the linear combination of eigenvectors. Thus, the eigenspace corresponding to the eigenvalue less than 1 is the *contracting space*, and the eigenspace corresponding to the eigenvalue greater than 1 is the *expanding space*.

Remark 3.3. Notice that the expanding space is contracting if we move backwards in time; similarly, the contracting space is expanding if we move backwards in time. That means if we take the inverse of A, the expanding and contracting subspaces would be switched. Thus, if A is hyperbolic, A^{-1} must also be hyperbolic!

Example 3.4. Another example of a hyperbolic linear map is the one defined by the matrix

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}$$

Graphing this map and the movement at each point, it isn't hard to see that the contracting space is the $x-$axis and the expanding space is the $y-$axis. The direct sum of the two axes give us the plane.

3.2. **Hyperbolic Sets.** We now want to transfer this idea of a "contraction" and "expansion" to a set. We will do that through its tangent bundle. Recall that the *tangent space* at a point x, T_xM is a vector space consisting of all the derivations of $C^\infty(M)$ at x. The *tangent bundle* over a set $\Lambda \subseteq M$ is the disjoint union of the tangent spaces, $T_\Lambda M = \bigsqcup_{x\in\Lambda} T_xM$. We equip the tangent bundle with a *Riemannian norm* $|\cdot|$ induced by the Riemannian metric. Also recall that the *tangent mapping*, $T_xf : T_xM \to T_{fx}M$ is the analog to the derivative at x. Suppose there is a subspace $E(x) \subseteq T_xM$ for all $x \in \Lambda$. Then,

$$E = \bigsqcup_{x\in\Lambda} E(x)$$

is an $m-$dimensional *subbundle* of $T_\Lambda M$, if for every $x \in \Lambda$, there exists m-many C^r vector fields, $e_1(y), \dots, e_m(y)$ that span $E(y)$, for all y in an open neighborhood contained in Λ around x. We call the subspaces $E(x) \in E$ *fibers*. In this paper, we will only work with C^0 subbundles.

Definition 3.5 (Hyperbolic Set)**.** An invariant set $\Lambda \subseteq M$ of f is *hyperbolic,* if for each $x \in \Lambda$, T_xM splits into the direct sum of a contracting and expanding space.

$$T_xM = E^s(x) \oplus E^u(x)$$

where,

(1) E^s and E^u are invariant as a subbundle of $T_\Lambda M$ under the tangent mapping Tf. In particular,

$$Tf(E^s(x)) = E^s(f(x)) \qquad Tf(E^u(x)) = E^u(f(x)).$$

(2) There exists $C \geq 1$ and $0 < \lambda < 1$ such that

$$|Tf^n(v)| \leq C\lambda^n|v|, \forall x \in \Lambda, v \in E^s, n \geq 0$$

$$|Tf^{-n}(v)| \leq C\lambda^n|v|, \forall x \in \Lambda, v \in E^s, n \geq 0$$

We call Λ a *hyperbolic set* of f or under f, and (Λ, f) a hyperbolic dynamical system.

Remark 3.6. We can also define hyperbolicity as a splitting of the tangent bundle into the direct sum of subbundles instead of the splitting of a tangent space. In that case, we write

$$T_\Lambda M = E^s \oplus E^u.$$

Notation 3.7. For ease of reading, we drop the subscript x when referring to the tangent map. In particular, we write the tangent mapping applied to $v \in T_xM$ as

$$Tf(v) = T_xf(v) = D_xf(v),$$

where the tangent mapping over x is implied by the *bundle projection*, $\pi v = x$.

In a similar vein, we drop the subscript on our Riemannian norm. Here, the following notations are equivalent

$$|Tf(v)| = |Tf(v)|_x \qquad |v| = |v|_x,$$

where $\pi v = x$.

Notice that if Λ is a hyperbolic set of f, it will also be a hyperbolic set of f^{-1} with the expanding and contracting spaces flipped (See Remark 3.3). Another observation is that an invariant subset Δ of Λ will also be hyperbolic with the same splitting but restricted to $T_\Delta M$. Similarly, a finite union of hyperbolic sets is also hyperbolic; if we take the maximum of the constants, C and λ, the finite union will satisfy Definition 3.5. Note, however, that the infinite union of hyperbolic sets isn't necessarily hyperbolic. This is because our definition of hyperbolicity—in both the linear map and hyperbolic set cases—requires a uniform bound on expansion and contraction. This is why we require M to be a compact manifold; we seek a uniform control.

We now introduce equivalent characterizations of our expanding and contracting space, a proof of which can be found in [1] on page 79; this will be instrumental in understanding the intuition for Lemma 4.4.

Theorem 3.8 (Characterization of E^s and E^u)**.** *Let $\Lambda \subseteq M$ be a hyperbolic set of f with splitting $T_\Lambda M = E^s \oplus E^u$. For any $x \in \Lambda$, $E^s(x)$ is characterized by*

$$\begin{aligned} E^s(x) &= \{v \in T_xM : |Tf^n v| \to 0, n \to +\infty\} && (2)\\ &= \{v \in T_xM : \exists r > 0 \text{ such that } |Tf^n v| \le r, \forall n \ge 0\} && (3)\\ &= \{v \in T_xM : \exists \gamma > 0 \text{ such that } Tf^n v \in C_\gamma(E^s(f^n x)), \forall n \ge 0\}. && (4)\end{aligned}$$

Similarly,

$$\begin{aligned} E^u(x) &= \{v \in T_xM : |Tf^{-n} v| \to 0, n \to +\infty\} && (5)\\ &= \{v \in T_xM : \exists r > 0 \text{ such that } |Tf^{-n} v| \le r, \forall n \ge 0\} && (6)\\ &= \{v \in T_xM : \exists \gamma > 0 \text{ such that } Tf^{-n} v \in C_\gamma(E^u(f^n x)), \forall n \ge 0\}. && (7)\end{aligned}$$

Here $C_\gamma(E^s(x))$ and $C_\gamma(E^u(x))$ are the γ*-cones* at x about $E^s(x)$ and $E^u(x)$ defined by

$$C_\gamma(E^s(x)) = \{v \in T_xM : |v_u| \le \gamma |v_s|\} \qquad C_\gamma(E^u(x)) = \{v \in T_xM : |v_s| \le \gamma |v_u|\}.$$

Let's go over some examples of hyperbolic sets

Example 3.9 (Arnold's Cat Map)**.** Here—as with any hyperbolic linear map, like Example 3.2—the space itself is a hyperbolic set! As with the linear map case, the tangent space admits the same splitting as in the Example 3.2. We call systems where the entire manifold is hyperbolic *Anosov.*

Moreover, as mentioned earlier, any invariant subset of a hyperbolic set is hyperbolic. Here, our origin, a fixed point, is invariant, and similarly, its tangent mapping satisfies the properties above.

Example 3.10 (Smale Horseshoe)**.** A more complex and beautiful example is the *Smale horseshoe.* In the 1960s, it was proved that structurally stable maps on the disc were very simple: they had at most finitely many singularities and periodic orbits, both hyperbolic; all points approach a singularity; there is no saddle [11]. However, on a more general space–even one as simple as the unit square in $\mathbb{R}^2$–structurally stable maps can be much more complex. The Smale horseshoe is one such system: it is stable with *uncountably many* hyperbolic points!

We define this system heuristically. Begin with the unit square, and stretch and compress the square by some uniform factor to get a long rectangle. Then, fold rectangle over itself to get a "horseshoe" shape (See Figure 5). Technically, this transformation is not a homeomorphism of the unit square, but if we add semicircles that capture the "curve" and "tail" of the horseshoe to the top and bottom edges of the cube, it will be.

The hyperbolic set of the horseshoe map would be the invariant set

$$\Lambda = \bigcap_{n=-\infty}^{\infty} f^n Q,$$

where Q is our unit square. What does Λ look like though? Graphing a few iterations of Q under the horseshoe map in Figure 6, we can see that the intersection of these finite iterations is simply a finite iteration of Cantor Dust, which is graphed in Figure 7. In fact, Λ is a Cantor set! To prove this fact, we use *symbolic dynamics*, coding our system as a shift map on a bi-infinite binary

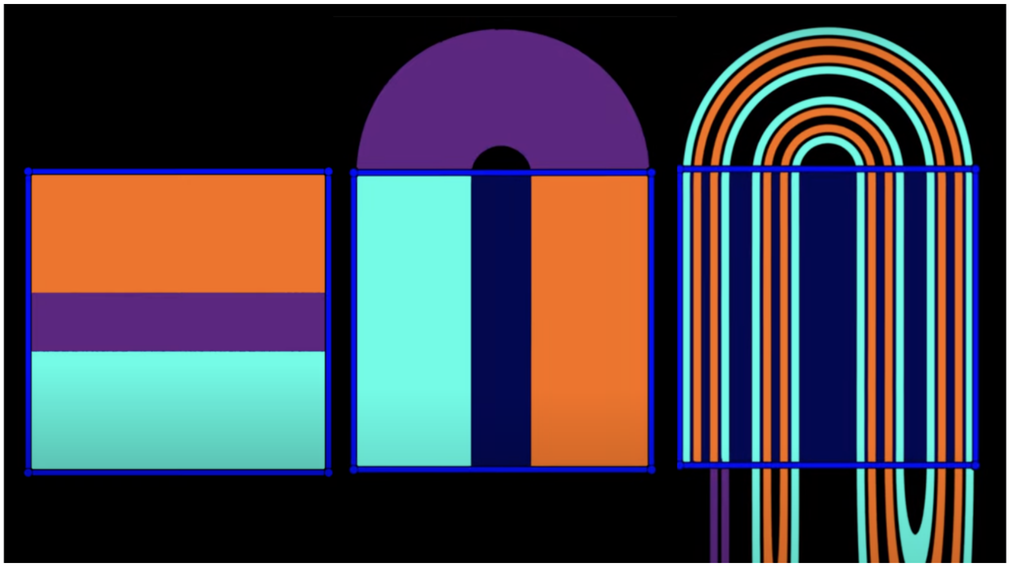

FIGURE 5. Mapped here are the 0th, 1st, and 4th iteration of the Smale horseshoe, with the colors highlighting how it stretches Q, outlined in dark blue. Images were taken from [12], which includes an animation of the transformation.

sequence. More information on symbolic dynamics and the horseshoe map can be found in [1] on page 57.

3.3. **Norms.** Throughout this paper, we work only with finite dimensional spaces. A theorem from linear algebra is that all norms on a finite dimensional space E are *equivalent*, meaning there exists some $K \in \mathbb{R}$ such that for any two norms, $|\cdot|, \|\cdot\|$

$$\frac{1}{K}\|v\| \leq |v| \leq K\|v\|, \qquad \forall v \in E.$$

As a result, we can work with a norm that has a uniform expansion or contraction, eliminating the constant C. Such a norm is called *adapted.* In particular, Theorem 3.11 states the existence of an adapted norm, and its proof constructs such a norm.

Theorem 3.11. *Let $\Lambda \subseteq M$ be a hyperbolic set with splitting $T_\Lambda M = E^s \oplus E^u$. There is a C^∞ Riemannian metric $\langle\langle \cdot, \cdot \rangle\rangle$ of M and a constant $0 < \tau < 1$ such that, with respect to the induced norm,*

$$\|Tf(v)\| \leq \tau\|v\|, \forall v \in E^s \tag{8}$$

$$\|Tf^{-1}(v)\| \leq \tau\|v\|, \forall v \in E^u. \tag{9}$$

Proof. Let N be sufficiently large such that $C\lambda^N < 1$, where C, λ are as in Definition 3.5.

Define our new Riemannian metric to be

$$\langle\langle u, v \rangle\rangle = \sum_{n=0}^{N-1} \langle Tf^n(v), Tf^n(u) \rangle, \qquad v, u \in T_x M, x \in M.$$

As a sum of Riemannian metrics, $\langle\langle \cdot, \cdot \rangle\rangle$ is a Riemannian metric. Now, we show that it satisfies the inequalities in equations 8 and 9. Here, $\|\cdot\|$ will denote the Riemannian norm corresponding to

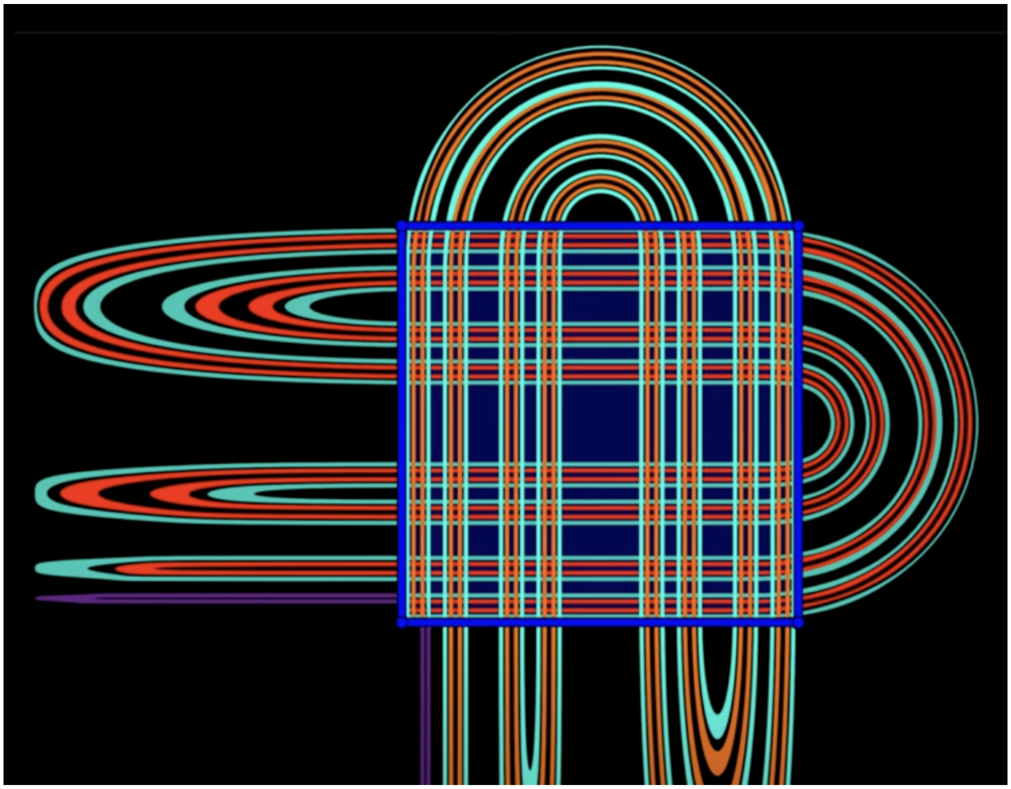

FIGURE 6. Here represents multiple iterations of the Smale horseshoe and how it affects vertical strips of Q, in addition to the horizontal strips, or how the map acts in reverse. The intersection of the two denote our invariant set. Notice that this looks like Cantor Dust in Figure 7. Image taken from [12].

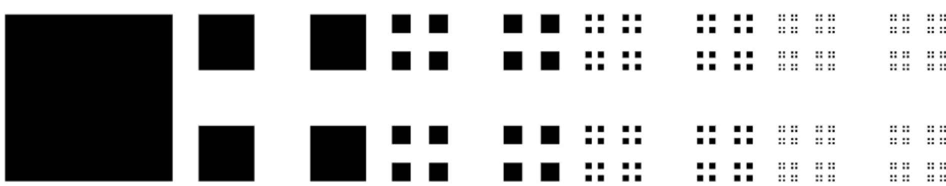

FIGURE 7. This is the Cartesian product of the Cantor set with itself, $\mathcal{C} \times \mathcal{C}$. A rigorous proof that the maximal invariant set of the horseshoe map is a Cantor Set can be found in [1] on page 58. Image taken from [13]

our new Riemannian metric, and $|\cdot|$ corresponds to the original Riemannian metric. Observe that for $v \in E^s$

$$\begin{aligned}\|v\|^2 &= \langle\langle v, v\rangle\rangle \\ &= \sum_{n=0}^{N-1} \langle Tf^n(v), Tf^n(v)\rangle = \sum_{n=0}^{N-1} |Tf^n v|^2 \\ &\leq \sum_{n=0}^{N-1} C^2\lambda^{2n}|v|^2\end{aligned}$$

Let $a = \sum_{n=0}^{N-1} C^2\lambda^{2n}$. Thus,

$$\|v\|^2 \leq a|v|^2 \qquad \forall v \in E^s.$$

Similarly for all $v \in E^u$,

$$\begin{aligned}\|v\|^2 &= \langle\langle v, v\rangle\rangle \\ &= \sum_{n=0}^{N-1} \langle Tf^n(v), Tf^n(v)\rangle = \sum_{n=0}^{N-1} |Tf^n v|^2 \\ &= \sum_{n=0}^{N-1} |Tf^{-n}(Tf^{N-1+n}v)|^2 \\ &\leq \sum_{n=0}^{N-1} C^2\lambda^{2n}|Tf^{N-1+n}v|^2 \\ &\leq \sum_{n=0}^{N-1} C^2\lambda^{2n}|Tf^{N-1}v|^2 = a|Tf^{N-1}v|^2.\end{aligned}$$

Then, for all $v \in E^s$

$$\begin{aligned}\|Tfv\|^2 \leq \sum_{n=1}^{N} |Tf^n v|^2 &= \|v\|^2 - |v|^2 + |Tf^N v|^2 \\ &\leq \|v\|^2 - (1 - C^2\lambda^{2n})|v|^2 \leq \|v\|^2 - a^{-1}(1 - C^2\lambda^{2N})\|v\|^2.\end{aligned}$$

Similarly, for all $v \in E^u$

$$\begin{aligned}\|Tf^{-1}v\|^2 = \sum_{n=0}^{N-2} |Tf^n v|^2 &= \|v\|^2 + |Tf^{-1}v|^2 - |Tf^{N-1}v|^2 \\ &\leq \|v\|^2 - (1 - C^2\lambda^{2n})|Tf^{N-1}v|^2 \leq \|v\|^2 - a^{-1}(1 - C^2\lambda^{2N})\|v\|^2.\end{aligned}$$

Let $\tau = \sqrt{1 - a^{-1}(1 - C^2\lambda^{2N})}$. Now, all we need to show is $\tau < 1$. Since $1 - C^2\lambda^{2N} \leq 1 \leq a$,

$$a^{-1}(1 - C^2\lambda^{2N}) \in (0, 1) \implies 1 - a^{-1}(1 - C^2\lambda^{2N}) \in (0, 1).$$

Thus, $\tau < 1$. □

This norm allows us to define the *skewness* of our hyperbolic set Λ, which describes the magnitude of the greatest contraction (forwards or backwards in time):

$$\tau(\Lambda) = \sup_{x\in\Lambda}\{\|Tf|_{E^s(x)}\|, \|Tf^{-1}|_{E^u(x)\|}\} < 1. \tag{10}$$

Finally, it is helpful if our norm fits into our hyperbolic splitting, that is, that it only considers the "stretch" or "squish," whichever is greater. Formally,

Definition 3.12 (Box-adjusted norm)**.** A norm $\|\cdot\|$ is called the *box-adjusted norm* of $|\cdot|$ with respect to a splitting of the tangent space $T_U M = E^1 \oplus E^2$ if

$$\|v\| = \max\{|v_1|, |v_2|\}, \qquad \forall v \in T_U M.$$

The box-adjusted norm of an adapted norm to Λ with respect to our hyperbolic splitting is adapted to Λ with the same skewness. This is the norm that we will use to prove the major theorems below.

4. Persistence of Hyperbolic Sets

We study hyperbolicity because hyperbolic sets persist. Let f be a diffeomorphism, Λ a hyperbolic set of f, and $\mathcal{U}$ an open neighborhood of f. Choose some $g \in \mathcal{U}$. Λ might not necessarily be a hyperbolic set of g—Λ might not even be g-invariant. But g isn't too far away from f, so what happens when we look at invariant sets in some small neighborhood of Λ? Here, we denote

$$B(\Lambda, a) = \bigcup_{x \in \Lambda} \{y \in M : d(x, y) < a\}$$

to be the union of balls of radius a around Λ. It turns out that we will find a compact $g-$invariant set $\Delta \subseteq B(\Lambda, a)$ that is hyperbolic, a "copy" of Λ. Not only that, every compact $g-$invariant set that is sufficiently close to Λ will also be hyperbolic. This idea is formalized in Theorem 4.1.

Theorem 4.1 (Persistence of Hyperbolic Sets)**.** *Let Λ, M, and f be as above. There exists a C^1 neighborhood $\mathcal{U} \subseteq Diff^1(M)$ of f and $a > 0$ such that for all $g \in \mathcal{U}$, every compact $g-$invariant set $\Delta \subseteq B(\Lambda, a)$ is hyperbolic.*

In other words, hyperbolicity of compact invariant sets is preserved under small C^1 perturbations. Before we go over the geometric intuition behind Theorem 4.1, we introduce the following notations.

Notation 4.2. Let $T_\Lambda M = E^1 \oplus E^2$ be a splitting of T_Λ, though not necessarily a hyperbolic splitting.

$$\pi_1 : T_\Lambda M \to E^1 \qquad \pi_2 : T_\Lambda M \to E^2,$$

denote the projection onto E^1 and E^2, respectively, where for all $v = v_1 + v_2 \in T_\Lambda M$,

$$\pi_1(v) = v_1 \qquad \pi_2(v) = v_2.$$

For a linear map $A : T_\Lambda M \to T_\Lambda M$, let

$$A_i = \pi_i \circ A \qquad A_{ij} = A_i|_{E^j}, \quad i = 1, 2.$$

Remark 4.3. If $E^1 \oplus E^2$ is any *invariant* splitting, for all $v \in T_\Lambda M$,

$$A_i v = A_i v_i = A_{ii} v_i, \qquad A_{ij} v, A_{ji} v = 0 \qquad i = 1, 2.$$

Why is this true? Without loss of generality suppose $i = 1$. Since E^1 is invariant, $Av_1 \in E^1$. So,

$$\begin{aligned}
A_1 v &= \pi_1(Av) \\
&= \pi_s(A(v_1 + v_2)) && \text{(By the splitting of } v\text{)} \\
&= \pi_s(Av_1 + Av_2) && \text{(Linearity)} \\
&= \pi_s(Av_1) + \pi_s(Av_2) \\
&= A_1 v_1 + 0 && (Av_2 \in E^2)
\end{aligned}$$

The last equality holds since $A_1 v_1 \in E^1$.

Let $A = Tf$ for ease and familiarity of notation. Recall Theorem 3.8 where equation 7 gives us that

$$E^u = \{v \in E : A^{-n} v \in C_\gamma(E^u), n \geq 0\}.$$

Consider $C_1(E^u) = \{v \in E : |v_s| \leq |v_u|\}$, where $|\cdot|$ is adapted to A with constant $0 < \tau < 1$. How does A act on $C_1(E^u)$? Well, A "contracts" $C_1(E^u)$ to the smaller cone $C_{\tau^2}(E^u)$. To see this, consider the opening of the cone or the set of $v \in E$ where $|v_s| = |v_u|$. Since

$$|A_s v| = |A_s v_s| \leq \tau |v_s| \qquad \text{(See Remark 4.3)}$$
$$|v_u| = |A^{-1}(A v_u)| \leq \tau |A v_u| = \tau |A_u v_u| = \tau |A_u v|,$$
$$|A_s v| \leq \tau |v_s| = \tau |v_u| \leq \tau^2 |A_u v|,$$

the new opening will correspond to that of $C_{\tau^2}(E^u)$

Geometrically, if we look back to the simple linear map case in Example 3.4, $C_1(E^u)$ would be the red-shaded area in Figure 4. $A(C_1(E^u))$ is the purple area, and notice that since the standard Euclidean norm is adapted to this map, that the boundary lines are $x \leq \frac{1}{4}|y|$ and $x \geq \frac{-1}{4}|y|$.

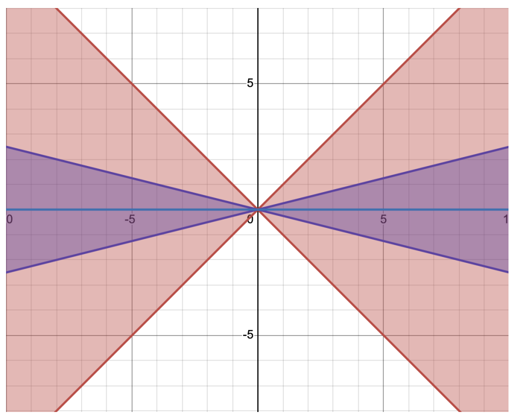

FIGURE 8. Unstable cones with E^u in blue.

If we keep iterating on $C_1(E^u)$, the iterates will eventually contract to the unstable space

$$E^u = \bigcap_{n=0}^{\infty} A^n(C_1(E^u)).$$

What if we take a linear map $B = A + \phi$—a small C^1 perturbation of A? Well, if we choose ϕ to be sufficiently small, B should similarly contract $C_1(E^u)$. Following the same reasoning, the unstable space corresponding of B should be

$$G^u = \bigcap_{n=0}^{\infty} B^n(C_1(E^u)),$$

Now, we need to show that G^u as defined above is in fact a $B-$invariant linear subspace of E. How do we do that? We consider the linear subspaces in $C_1(E^u)$ and then find the one that is invariant under B.

These linear subspaces can be represented as the graph of a linear map $P : E^s \to E^u, |P| \leq 1$, so that the graph of P $\text{gr}(P) \subseteq C_1(E^u)$. When B acts on $\text{gr}(P)$, it transforms it into another graph of a linear map. This is the intuition behind the *graph transform*, which is a function that formalizes this idea. If we're looking for a $B-$invariant subspace, all we need to do is find the fixed point of the graph transform. This is the heuristic behind the proof of Lemma 4.4.

Before we introduce Lemma 4.4, recall that a map $F : E^1 \to E^2$, where E^1, E^2 are C^0 subbundles of $T_\Lambda M$, is *fiber-preserving map* with respect to $f : M \to M$ if

$$\pi F = f\pi,$$

where $\pi : TM \to M$ is the bundle projection. In other words, F maps any fiber $E^1(x) \in E^1$ to $E^2(fx) \in E^2$, for all $x \in \Lambda$ as

$$\pi F(E^1(x)) = f\pi(E^1(x)) = fx.$$

If F is fiber-preserving with respect to f, we say that F *covers* f. A *bundle homomorphism* $P : E^1 \to E^2$ is a fiber-preserving map that is "pointwise" a linear homomorphism, meaning for all $x \in \Delta$, $P|_{E_1(x)}$ is a linear map.

Lemma 4.4. *Let $g : M \to M$ be a diffeomorphism and Δ an invariant set of g. Let $B : T_\Delta M \to T_\Delta M$ be a bounded C^0 bundle isomorphism over g, with respect to the direct sum $T_\Delta M = E_1 \oplus E_2$ as*

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

If there exists a norm $|\cdot|$ that is of box-type with respect to our splitting, and $\lambda, \epsilon > 0$ such that

$$\max\{|B_{11}^{-1}|, |B_{22}|\} < \lambda, \tag{11}$$

$$\max\{|B_{12}|, |B_{21}|\} < \epsilon, \tag{12}$$

$$\lambda + \epsilon < 1. \tag{13}$$

then, there exists a C^0 bundle homomorphism $P : E^1 \to E^2$ over the identity such that:

1. *$|P| < 1$, under our norm*
2. *the graph of P, $gr(P) = \{(x, Px) : x \in E^1\} \subseteq T_\Delta M$ is a $B-$invariant C^0 subbundle, with dimension $\dim E^1$, meaning*
$$B_x(gr(P_x)) = gr(P_{gx})$$
3. *$B_x|_{gr(P_x)}$ is expanding by a factor of $\lambda^{-1}+\epsilon$. Namely, for all $x \in \Delta$ and all $(v, P_x v) \in gr(P_x)$,*
$$(\lambda^{-1} + \epsilon)|(v, P_x v)| = (\lambda^{-1} + \epsilon)|v| \leq |B_x(v, P_x v)|$$

In this case, since P is a fiber-preserving map covering the identity, it maps each fiber to itself. This is incredibly useful since we can just work pointwise on an arbitrary fiber $E^1(x)$ and know that it generalizes to the whole bundle E^1.

Proof of Lemma 4.4. We will formally describe the graph transform and prove the existence of a linear map whose graph is invariant by finding its fixed point. To do so, we use the *contraction mapping theorem* which is instrumental in many proofs in dynamics.

Theorem 4.5 (Contraction Mapping Theorem)**.** *Let M be a complete metric space, and suppose that $f : M \to M$ is a contraction mapping, meaning for all $x, y \in M$, there exists $k < 1$ such that*

$$d(fx, fy) \leq kd(x, y).$$

Then, f has a unique fixed point.

A proof of this theorem can be found in [3] on page 240. Next, we will prove that the graph transform is expanding using our assumptions in (11) and (12) and that our norm is of box-type.

We begin deriving the graph transform. Let $P : E^1 \to E^2$ be a C^0 bundle homomorphism over the identity with $|P| < 1$. We first want to show that for any $x \in \Delta$, $B_x(\mathrm{gr}(P_x)) \subseteq \mathrm{gr}(P_{gx})$. Observe that for this inclusion to hold, for all v such that $\pi v = x$,

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}_x \begin{bmatrix} v \\ P_x v \end{bmatrix} = \begin{bmatrix} (B_{11})_x v + (B_{12})_x P_x v \\ (B_{21})_x v + (B_{22})_x P_x v \end{bmatrix} = \begin{bmatrix} (B_{11})_x v + (B_{12})_x P_x v \\ P_{gx}[(B_{11})_x v + (B_{12})_x P_x v] \end{bmatrix}.$$

Thus, we want to solve the following equation

$$(B_{21})_x + (B_{22})_x P_x = P_{gx}[(B_{11})_x + (B_{12})_x P_x].$$

First, we need to invert the rightmost term to write the equation in terms of P. To do so, we will use the *Lipschitz inverse function theorem.*

Theorem 4.6 (Lipschitz Inverse Function Theorem)**.** *Let $A : E \to E'$ be a linear isomorphism and $\phi : E \to E'$ be Lipschitz. If*

$$Lip\phi < m(A), \tag{14}$$

$A + \phi$ is a lipeomorphism and

$$Lip((A+\phi)^{-1}) < \frac{1}{m(A) - Lip\phi}.$$

Here, $m(A)$ denotes the conorm of A,

$$m(A) = \inf\{|Av|' : v \in E, |v| = 1\}$$

and Lip the minimal Lipschitz constant

$$Lip\phi = \min\{k \in \mathbb{R} : |\phi x - \phi y| \leq k|x - y|\}.$$

The proof of Theorem 4.6 can be found in [1] on page 35.

We now prove that $(B_{11})_x + (B_{12})_x P_x$ satisfies (14). Observe that the *conorm* of an invertible linear map B, is the inverse of the operator norm of B^{-1}

$$m(B) = \frac{1}{|B^{-1}|},$$

where $|\cdot|$ denotes the operator norm. So

$$m(B_{11}) \geq \lambda^{-1}.$$

Moreover, since the Lipschitz constant of a continuous map is bounded above by the operator norm of its derivative,

$$\mathrm{Lip} f \leq |Df|,$$
$$\mathrm{Lip}((B_{12})_x P_x) \leq |(B_{12})_x||P_x| \leq \epsilon.$$

Since $\lambda^{-1} > \epsilon$ as implied by (13), by the Lipschitz Inverse Function Theorem, $(B_{11})_x + (B_{12})_x P_x$ is invertible and Lipschitz. Thus, we can write

$$P_{gx} = ((B_{21})_x + (B_{22})_x P_x)((B_{11})_x + (B_{12})_x P_x)^{-1}, \forall x \in \Delta.$$

We define the *graph transform* $T_B(P) : L(E^1, E^2; \mathrm{id})(1) \to L(E^1, E^2; \mathrm{id})$ as

$$T(P_{gx}) = ((B_{21})_x + (B_{22})_x P_x)((B_{11})_x + (B_{12})_x P_x)^{-1}.$$

Here, $L(E^1, E^2; \mathrm{id})(1)$ denotes the space of linear bundle homomorphisms over the identity with operator norm less than 1.

Now, we need to prove that T maps $L(E^1, E^2; \mathrm{id})(1)$ into itself by showing that $|TP| < 1$, for all $P \in L(E^1, E^2; \mathrm{id})(1)$. With (11), (12), and Theorem 4.6, we know that for any P_{gx},

$$\begin{aligned} |T(P_{gx})| &\leq (|(B_{21})_x| + |(B_{22})_x P_x|)\,|(B_{11})_x + (B_{12})_x P_x)^{-1}| \\ &< (\epsilon + \lambda)\frac{1}{m((B_{11})_x) - \mathrm{Lip}((B_{12})_x P_x)} \\ &\leq \frac{\epsilon + \lambda}{\lambda^{-1} - \epsilon} < 1. \end{aligned}$$

Now, we prove that the graph transform is a contraction. Observe that by multiplying both sides of the graph transform by $(B_{11})_x + (B_{12})_x P_x$, we get that

$$(T(P))_{gx}((B_{11})_x + (B_{12})_x P_x) = (B_{21})_x + (B_{22})_x P_x, \tag{15}$$

$$(T(P'))_{gx}((B_{11})_x + (B_{12})_x P'_x) = (B_{21})_x + (B_{22})_x P'_x. \tag{16}$$

Subtracting (15) and (16),

$$\left[(T(P))_{gx} - (T(P'))_{gx}\right](B_{11})_x + (T(P))_{gx}(B_{12})_x P_x - (T(P'))_{gx}(B_{12})_x P'_x = (B_{22})_x\left[P_x - P'_x\right].$$

We want to further factor our expression to isolate $(T(P))_{gx} - (T(P'))_{gx}$. We can do this by factoring out a $((B_{11})_x + (B_{12})_x P_x)$ term which we know is invertible. To do so, we add and subtract $(T(P'))_{gx}(B_{12})_x P_x$. Factoring,

$$\begin{aligned} &\left[(T(P))_{gx} - (T(P'))_{gx}\right](B_{11})_x + \\ &(T(P))_{gx}(B_{12})_x P_x - (T(P'))_{gx}(B_{12})_x P_x + \\ &(T(P'))_{gx}(B_{12})_x P_x - (T(P'))_{gx}(B_{12})_x P'_x = (B_{22})_x\left[P_x - P'_x\right] \end{aligned}$$

we get

$$\left[(T(P))_{gx} - (T(P'))_{gx}\right]\left[(B_{11})_x + (B_{12})_x P_x\right] + (T(P'))_{gx}(B_{12})_x\left[P_x - P'_x\right] = (B_{22})_x\left[P_x - P'_x\right].$$

Thus, for all $x \in \Lambda$,

$$\left[(T(P))_{gx} - (T(P'))_{gx}\right] = \left[(B_{22})_x - (T(P'))_{gx}(B_{12})_x\right]\left[P_x - P'_x\right]\left[(B_{11})_x + (B_{12})_x P_x\right]^{-1}. \tag{17}$$

Since the estimate in (17) is uniform,

$$\begin{aligned} |T(P) - T(P')| &\leq |B_{22} - T(P')B_{12}||P_x - P'_x||\left[(B_{11})_x + (B_{12})_x P_x\right]^{-1}| \\ &\leq \left(|B_{22}| + |T(P')B_{12}|\right)|P_x - P'_x||\left[(B_{11})_x + (B_{12})_x P_x\right]^{-1}| \\ &\leq \frac{\lambda + \epsilon}{\lambda^{-1} - \epsilon}|P_x - P'_x|. \end{aligned}$$

Finally, since $\frac{\lambda+\epsilon}{\lambda^{-1}-\epsilon} < 1$, T must is a contraction mapping.

Thus, by Theorem 4.5, there exists a unique P_x such that the $B_x(\mathrm{gr}(P_x)) \subseteq \mathrm{gr}(P_{gx})$. Since the tangent mapping is linear bundle isomorphism, it is onto, meaning that $\mathrm{gr}(P_{gx}) \subseteq B_x(\mathrm{gr}(P_x))$. Thus, the fixed point of our graph transform, P_x, is $B-$invariant. Moreover, since the dimension of $\mathrm{gr}(P_{gx})$ is just the dimension of its domain, E^1, $\dim \mathrm{gr}(P) = \dim E^1$.

Next, we prove that B_x restricted to the graph of P is expanding. Here, we use our assumption that our norm is of box-type with respect to this splitting and that $|P_x| < 1$ to get that the norm of B_x must be the norm of the first coordinate:

$$|B_x(v, P_x v)| \geq |(B_{11})_x v + (B_{12})_x P_x v| \geq (\lambda^{-1} - \epsilon)|v|.$$

□

Lemma 4.4 will give us the existence of an expanding subspace in $T_\Lambda M$ for $B = Tf + \phi$. Since an expanding space of a map B is the contracting space of its inverse B^{-1}, if the inverse of B satisfies the hypothesis in Lemma 4.4, we have a contracting space with respect to B. Since these two spaces are of complementary dimensions, we have found a hyperbolic splitting of B with respect to an invariant set. This is a high-level sketch of the proof for Theorem 4.1.

Proof of Theorem 4.1. Let $T_\Lambda M = E^s \oplus E^u$ be the hyperbolic splitting of Λ, and let $U \subseteq M$ be an open neighborhood of Λ.Without loss of generality, suppose the Riemannian norm $|\cdot|$ is adapted to Λ. We can extend our splitting of $T_\Lambda M$ to the C^0 splitting

$$T_U M = G^s \oplus G^u.$$

By extending our splitting, we mean that for all $x \in \Lambda \subseteq U$, $E^s(x) = G^s(x)$ and $E^u(x) = G^u(x)$. We can do this by extending the domain of the local C^0 vector fields that define E^u and E^s to a neighborhood of U. Note, however, that this splitting is not necessarily invariant under the tangent map. We similarly extend $|\cdot|$ to the Riemannian norm $\|\cdot\|$, which is of box-type and adjusted to $G^s \oplus G^u$.

Let τ be the skewness of Λ with respect to $\|\cdot\|$, which is the same as the skewness of Λ with respect to $|\cdot|$. Since our splitting $T_\Lambda M = E^s \oplus E^u$ is invariant, the tangent map Tf on $T_\Lambda M$ with respect to this splitting, can be represented as

$$\begin{bmatrix} (Tf)_{uu} & (Tf)_{us} \\ (Tf)_{su} & (Tf)_{ss} \end{bmatrix} = \begin{bmatrix} (Tf)_{uu} & 0 \\ 0 & (Tf)_{ss} \end{bmatrix},$$

where by (10) and Remark 4.3,

$$\|(Tf)_{uu}^{-1}\| < \tau, \qquad \|(Tf)_{ss}\| < \tau, \qquad (Tf)_{us}, (Tf)_{su} = 0.$$

Observe that for a diffeomorphism g and any $g-$invariant set Δ, the tangent map $Tg : T_\Delta M \to T_\Delta M$ is a C^0 bundle isomorphism. Let $\tau < \lambda < 1$ and $\epsilon > 0$ such that $\lambda + \epsilon < 1$. If we choose a sufficiently small neighborhood $\mathcal{U} \subseteq \mathrm{Diff}^1(M)$ of f and $a > 0$ such that $B(\Lambda, a) \subseteq U$, for all $g \in \mathcal{U}$ and all $g-$invariant sets $\Delta \subseteq B(\Lambda, a)$, Tg will satisfy the hypothesis of Lemma 4.3 with respect to the splitting $T_\Delta = G^s|_\Delta \oplus G^u|_\Delta$ and the C^0 norm $\|\cdot\|$. For any invertible linear map A and B, the identity

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$$

tells us that if B is near A, then B^{-1} is near A^{-1}. This implies that if $\mathcal{U}$ is sufficiently small, Tg^{-1} will also satisfy the hypothesis of Lemma 4.3 with respect to $T_\Delta = G^s|_\Delta \oplus G^u|_\Delta$ and $\|\cdot\|$.

Applying Lemma 4.3 to Tg and Tg^{-1}, we get $P \in L(G^u|_\Delta, G^s|_\Delta)(1)$ and $Q \in L(G^s|_\Delta, G^u|_\Delta)(1)$ such that $\mathrm{gr}(P)$ and $\mathrm{gr}(Q)$ are expanding and contracting with respect to Tg, respectively. Since $\mathrm{gr}(P)$ is expanding and $\mathrm{gr}(Q)$ is contracting, $\mathrm{gr}(P) \cap \mathrm{gr}(Q) = \{0\}$. Moreover, since $\dim(\mathrm{gr}(P)) = \dim G^u|_\Delta$ and $\dim(\mathrm{gr}(Q)) = \dim G^s|_\Delta$, the two subspaces have complementary dimensions. Thus, Δ is hyperbolic with splitting

$$T_\Delta M = \mathrm{gr}(Q) \oplus \mathrm{gr}(P).$$

□

Finally, we state an extension of Theorem 4.1 that gives us information as to how the two hyperbolic sets relate with respect to norms.

Theorem 4.7. *Let $\Lambda \subseteq M$ be a compact hyperbolic set of f. Also let $|\cdot|_\Lambda$ be the box-adjusted norm with respect to the hyperbolic splitting of Λ and $\tau(\Lambda)$ be the skewness of Λ with respect to $|\cdot|_\Lambda$. Then, there exists a C^1 neighborhood $\mathcal{U}_0 \subseteq \mathrm{Diff}^1(M)$, $a_0 > 0$, and $K \geq 1$ such that:*

(1) For all $g \in \mathcal{U}_0$, every compact $g-$invariant set $\Delta \subseteq B(\Lambda, a)$ is hyperbolic (Theorem 4.1),

(2) The box-adjusted norm $|\cdot|_\Delta$ with respect to the hyperbolic splitting of Δ is equivalent to $|\cdot|_\Lambda$ with constant K

(3) Shrinking $\mathcal{U}_0$ and $a > 0$ further, the skewness of Δ with respect to $|\cdot|_\Lambda$ $\tau(\Delta)$ is arbitrarily close to $\tau(\Lambda)$.

More specifically, for any $\epsilon > 0$, there exists a C^1 neighborhood $\mathcal{U} \subseteq \mathcal{U}_0$ of f and $0 < a < a_0$, for all $g \in \mathcal{U}$, the skewness of every compact $g-$invariant set $\Delta \subseteq B(\Lambda, a)$ with respect to $|\cdot|_\Lambda$, $\tau(\Delta)$ satisfies

$$\tau(\Delta) \leq \tau(\Lambda) + \epsilon.$$

A proof of this extension can be found in [1] on page 89.

5. Structural Stability of Hyperbolic Sets

In §4, we proved that a function near f will have a hyperbolic set near Λ. We can do better than this. Since g and f are hyperbolic on Λ and Δ, respectively, they behave similarly on those sets. What we will show in this section is that their behavior is dynamically the same. In other words, if Λ is an f-hyperbolic set, f is strongly structurally stable on Λ as in Definition 2.8.

Theorem 5.1 (Structural Stability of Isolated Hyperbolic Sets)**.** *Let Λ be an isolated hyperbolic set of $f : M \to M$ with an isolating neighborhood U. For any $\epsilon > 0$, there is a C^1 neighborhood $\mathcal{U}$ of f such that, for any $g \in \mathcal{U}$, the maximal invariant set Γ of g in U is isolated in U and $g|_\gamma$ is $\epsilon-$conjugate to $f|_\Lambda$.*

Observe that the statement of Theorem 5.1 is only for *isolated* hyperbolic sets. By an *isolated* set, we mean a compact invariant set Λ of f such that

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U),$$

for an open neighborhood $\Lambda \subseteq U \subseteq M$. Isolated sets are also called *locally maximal* sets, since they are the maximal (largest possible) invariant set in U. It is important to note that there are non-isolated invariant sets which can also be hyperbolic, even though the examples presented in §3 such as the Smale Horseshoe and a hyperbolic fixed point are isolated. This is an important distintion since not all hyperbolic sets can be extended to an isolated hyperbolic set.

Why is the assumption that Λ be isolated necessary? In Theorem 5.1, our goal is to prove that f and g restricted to their maximal hyperbolic sets are invariant. The way we ensure that these maximal hyperbolic sets are close by is through dictating that Λ is isolated.

There are multiple ways to prove this theorem. One approach uses the shadowing lemma and expansivity to characterize stability and is detailed in [4]. However, we will use a functional analysis approach, where we find the topological conjugacy by determining finding the fixed point of a contraction—just like with the graph transform in the proof of Lemma 4.4!

To prove Theorem 5.1, we first need to prove this weaker result.

Theorem 5.2 (Embedding Stability)**.** *Let Λ be a hyperbolic set of f. There is a C^1 neighborhood U of f and two numbers $a > 0$ and $\epsilon > 0$ such that for any $g, g' \in \mathcal{U}_0$ and any compact invariant set $\Delta \subseteq B(\Lambda, a_0)$ of g, there exists a unique continuous map $h : \Delta \to M$ such that*

$$hg = g'h \tag{18}$$

and $d(h, id) \leq \epsilon_0$.

.

If we take $g' = f$, we see that we can map a $g-$hyperbolic set Δ into f. Where this theorem is weaker than 5.1, is that it does not guarantee topological conjugacy: h doesn't have to be onto.

Notice that in the proof of Lemma 4.4, we used the idea of fiber-preserving maps to work pointwise between tangent spaces. This worked well for that proof because we were ultimately working over a linear space. However, h is a continuous function on our manifold, which doesn't always have these nice linear properties. Still, our goals are the same as in Lemma 4.4: define a contraction mapping that solves (18) and prove that it's a contraction to retrieve h. Thus, it would be useful if we could have some notion of invariance that connects points on our manifold to vectors in our tangent space.

Recall that a *section* or a *vector field* is a map $\gamma : \Lambda \to T_\Lambda M$ such that $\gamma(x) \in T_x M$. At its core, it attaches a point on the manifold to a vector from its tangent space; in a way, it's the "opposite" of the bundle projection. Let $\Gamma^0(T_\Lambda M)$ denote the set of continuous section on $T_\Lambda M$. If we're working with a fiber-preserving map between tangent bundles $F : T_\Lambda M \to T_\Lambda M$, we call a section $\gamma \in \Gamma^0(T_\Lambda M)$ an *invariant section of* F if for all $x \in \Lambda$

$$F(\gamma(x)) = \gamma(f(x)).$$

In particular, let g and f be diffeomorphisms such that for $x, y \in M$ $d(x, y) \le r$, $d(fx, gy) \le \rho$. We want to find an invariant section of the *lifting* $F_f^g : TM(r) \to TM$ of g over f defined by

$$F_f^g(v) = \exp_{f(\pi v)}^{-1} g \exp_{\pi v}(v). \tag{19}$$

What does this lift do? First, recall that $\exp_x : T_x M(r) \to M$ refers to the *exponential map*, which maps a vector $v \in T_x M$ to the end point of a geodesic on the manifold with velocity v. Visually, we can think of the exponential map taking a vector and pulling it down along the curves of our manifold. With this image, what the map above is doing is taking a vector in our tangent space, pulling it down to a point on our manifold. Then, we let g act on that point, and then lift it up to the tangent space of the image of our original point. It's in the same spirit as Anosov's shadowing lemma (Theorem 4.24 in [1]): the tangent space corresponding to the $f-$orbit of x are being "shadowed" by the $g-$orbit of a point near x.

Another concept that is instrumental to this proof is *expansivity*, which is a property of diffeomorphisms that states if the orbits of two points stay close together, they must be the same point. Formally:

Definition 5.3. A homeomorphism $f : X \to X$ is *expansive* if there is a constant $r > 0$ such that, for every pair of points $x \neq y \in X$, there is an integer m such that $d(f^m(x), f^m(y)) \ge r$, where $r > 0$ is the *expansive constant*.

Equivalently, f is expansive if

$$d(f^n(x), f^n(y)) \le r, \forall n \in \mathbb{Z},$$

where r is the expansive constant, implies that $x = y$.

We will use expansivity to prove that h is injective. Expansivity is intuitively similar to injectivity: if two points are always close under iterations of f, they must be the same. We state the following result for hyperbolic sets, which is proved in [1] on page 103:

Theorem 5.4 (Uniform Expansivity of Hyperbolic Sets)**.** *Let $\Lambda \subseteq M$ be a hyperbolic set of f. Then $f|_\Lambda$ is expansive. In fact, there is a C^1 neighborhood of f $\mathcal{U}$ and two numbers $a_0 > 0$ and $r_0 > 0$ such that every compact invariant set $\Delta \subseteq B(\Lambda, a_0)$ of every $g \in \mathcal{U}_0$ is r_0-expansive.*

Finally, we state the following lemma, which asserts that a Lipschitz perturbation of the tangent map has a unique invariant section, if the Lipschitz constant is sufficiently small:

Lemma 5.5. *Let $\Lambda \subseteq M$ be a hyperbolic set of f with splitting $T_\Lambda M = E^s \oplus E^u$, and let $|\cdot|$ be an adapted and box-type norm with respect to our splitting. Let $0 < \tau < 1$ be the skewness of Λ. Let $r > 0$. If $\phi : T_\Lambda M(r) \to T_\Lambda M$ is a*

(1) continuous,
(2) fiber-preserving,
(3) fiber Lipschitz with

$$Lip_2\phi < 1 - \tau,$$

then $Tf + \phi$ has at most one invariant section $\gamma_\phi \in \Gamma^0(T_\Lambda M)(r)$. If ϕ also satisfies

(4)

$$|\phi(0_\Lambda)| = \sup_{x\in\Lambda} |\phi(0_x)| \leq (1 - \tau - Lip_2\phi)r,$$

then there exists a (unique) invariant section of $Tf + \phi$.

The proof of Lemma 5.5 can be found in [1] on page 118.

Here, *fiber-Lipschitz* means that for every $x \in \Lambda$, the fiber $\phi_x : T_x M(r) \to T_\Lambda M$ is Lipschitz. We denote the Lipschitz constant of ϕ,

$$\mathrm{Lip}_2\phi = \sup_{x\in\Lambda} \mathrm{Lip}(\phi|_{T_x M(r)}). \tag{20}$$

The subscript "2" in (20) refers to how we can think of ϕ coordinate-wise, where the first coordinate is the base point that corresponds to some open set in $\mathbb{R}^n$ and the second coordinate is corresponds to the tangent space at that point. Similarly, $D_2\phi$ refers to the partial derivative of ϕ with respect to the fiber component.

With this background, we begin our proof.

Proof of Theorem 5.2. We first begin with the lifting defined in (19), and we prove the claim that $\phi = F_g^{g'} - Tg$ for g, g' in a sufficiently small neighborhood of f, has a small Lipschitz constant by getting a bound on the derivative of the map. Next, we use this claim to show that ϕ satisfies hypotheses (1) - (3) of Lemma 5.5 to get that $F_g^{g'}$ has at most one invariant section. Then, we will prove that ϕ also satisfies hypothesis (4) of Lemma 5.5 to prove the existence of a (unique) invariant section. Finally, we prove injectivity using expansivity.

Let $|\cdot|$ be a Riemannian norm of M that is adapted to Λ. Take sufficiently small U_0 and r_0 so that $F_g^{g'} : TM(r_0) \to TM$ can be defined for all $g, g' \in \mathcal{U}$. Let ϕ denote the Lipschitz perturbation

$$\phi = \phi_{g,g'} = F_g^{g'} - Tg : TM(r) \to TM.$$

Since the sum of two fiber-preserving functions is fiber-preserving, ϕ must be.

We claim that for any $\epsilon > 0$, we can find a C^1 neighborhood $\mathcal{U}$ of f such that, for any $g, g' \in \mathcal{U}$, $\mathrm{Lip}_2\phi < \epsilon$ on $TM(r)$. This will allow us to choose a neighborhood that appropriately bounds the Lipschitz constant of our perturbation so that we can bound it with respect to our a new adapted norm.

We control $|D_2(F_g)(v) - Tg(v)|$ by choosing a sufficiently small C^1 neighborhood of f such that for all $g \in U$,

$$|D_2(F_g)(v) - Tg(v)| < \epsilon/2.$$

By applying the chain rule,

$$D_2(F_g^{g'})(v) = D\exp_{gx}^{-1}(g'(\exp_x v)) \circ Dg'(\exp_x v) \circ D(\exp_x)(v).$$

So,

$$|D_2(F_g^{g'})(v) - D_2(F_g)(v)| =$$
$$|D\exp_{gx}^{-1} \circ [(g'(\exp_x v)) \circ Dg'(\exp_x v) - (g(\exp_x v)) \circ Dg(\exp_x v)] \circ D\exp_x(v)| < \epsilon/2,$$

if we choose a small enough $\mathcal{U}$. Then,

$$|D_2\phi(v)| = |D_2(F_g^{g'})(v) - Tg(v)| \le |D_2(F_g^{g'})(v) - D_2(F_g)(v)| + |D_2(F_g)(v) - Tg(v)| < \epsilon.$$

Applying the Generalized Mean Value Theorem (See [1] on page 30),

$$\mathrm{Lip}_2\phi < \epsilon$$

on $TM(r)$.

Let $\Delta \subseteq B(\Lambda, a_0)$ be a compact $g-$invariant hyperbolic set, guaranteed by Theorem 4.1 for all g in a sufficiently small neighborhood $\mathcal{U}_0$ of f and for sufficiently small $a_0 > 0$. By the extension Theorem 4.7, we know that the box-adjusted norm $\|\cdot\|$ with respect to the hyperbolic splitting of Δ is equivalent to the box-adjusted norm $|\cdot|$ with respect to the hyperbolic splitting of Λ with constant K. Additionally, we know that $\tau(\Delta) \le \tau(\Lambda)$. Fix τ_0 such that

$$\tau(\Delta) \le \tau(\Lambda) < \tau_0 < \lambda < 1.$$

Applying our claim above, shrink $\mathcal{U}_0$ and $r_0 > 0$ enough such that for any $g, g' \in \mathcal{U}$

$$\mathrm{Lip}_{2,|\cdot|}\phi \le K^{-2}(\lambda - \tau_0),$$

on $TM(K^2 r_0; |\cdot|)$. Then, in our new norm,

$$\mathrm{Lip}_{2,\|\cdot\|}\phi \le \lambda - \tau_0$$

on $T_\Delta M(Kr_0; \|\cdot\|)$. Now we further shrink $\mathcal{U}_0, a_0$, and r_0 so that any $g-$invariant compact set Δ is r_0-expansive for $g \in \mathcal{U}_0$. Let $\epsilon_0 = r_0/2$. Then,

$$\mathrm{Lip}_{2,\|\cdot\|}\phi \le \lambda - \tau_0 \le \lambda - \tau(\Delta)$$

on $T_\Delta M(K\epsilon_0; \|\cdot\|)$. Thus, applying Lemma 5.5, $F_g^{g'}$ has at most one invariant section in $\gamma \in \Gamma^0(T_\Delta M)(K\epsilon_0; \|\cdot\|)$ and therefore at most one invariant section with respect to our original norm.

Now, we prove that there exists an invariant section of $F_g^{g'}$ We take a C^1 neighborhood of f $\mathcal{U} \subseteq \mathcal{U}_0$ such that for every $g \in \mathcal{U}$,

$$\sup_{x \in M} d(fx, gx) < \frac{(1-\lambda)\epsilon}{2K^2}.$$

Then,

$$\begin{aligned}
\|\phi(0_\Delta)\|_\Delta &= \sup_{x\in\Delta} \|\phi(0_x)\|_\Delta = \sup_{x\in\Delta} \|F_g^{g'}(0_x) - Tg(0_x)\| \\
&= \sup_{x\in\Delta} \|\exp_{gx}^{-1} g' \exp_x(0_x) - 0\| = \sup_{x\in\Delta} \|\exp_{gx}^{-1}(g'x)\| \\
&= K \sup_{x\in\Delta} |\exp_{gx}^{-1}(g'x)| = K \sup_{x\in\Delta} d(gx, g'x) \\
&\le K \cdot \frac{(1-\lambda)\epsilon}{2K^2} = (1-\lambda)K^{-1}\epsilon.
\end{aligned}$$

Thus, ϕ satisfies hypothesis (4), guaranteeing the existence of the unique invariant section γ. Then γ satisfies the following equation:

$$\exp_{gx}^{-1} g' \exp_x \gamma(x) = \gamma(gx), \qquad \forall x \in \Delta.$$

We now define $h(x) = \exp_x \gamma(x)$, which gives us a unique continuous map $h : \Delta \to M$ such that $d(h, \mathrm{id}) \le \epsilon_0$ and

$$g'h(x) = h(g(x)), \forall x \in \Delta.$$

Finally, we prove h is injective. Suppose there are $x, y \in \Delta$ with $h(x) = h(y)$. Then, for any $n \in \mathbb{Z}$

$$\begin{aligned} d(g^n x, g^n y) &\le d(g^n x, +h(g^n x)) + (h(g^n x), h(g^n y)) + d(h(g^n y), g^n y) \\ &= d(g^n x, +h(g^n x)) + (g^{n'}(hx), g^{n'}(hy)) + d(h(g^n y), g^n y) \\ &\le \epsilon + 0 + \epsilon \le 2\epsilon_0 = r_0. \end{aligned}$$

This implies that $x = y$ since $g|_\Delta$ is r_0-expansive. Thus, h is injective. □

Before proving Theorem 5.1, we prove this brief but important lemma on isolated sets.

Lemma 5.6. *Let $f : X \to X$ be a homeomorphism and $\Lambda \subseteq X$ an isolated invariant set of f with isolating neighborhood U. For any $a > 0$, there is a C^0 neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$, the isolated set of U with respect to g is contained in $B(\Lambda, a)$.*

Proof. By definition,

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U).$$

This implies that for sufficiently large $N \in \mathbb{N}$,

$$\bigcap_{n=-N}^{N} f^N(U) \subseteq B(\Lambda, a/2).$$

Choose a sufficiently small C^0 neighborhood $\mathcal{U}$ of f such that, for any $g \in \mathcal{U}$,

$$\bigcap_{n=-N}^{N} g^n(U) \subseteq B(\Lambda, a).$$

Then,

$$\bigcap_{n=-\infty}^{\infty} g^n(U) \subseteq \bigcap_{n=-N}^{N} g^n(U) \subseteq B(\Lambda, a).$$

□

Proof of Theorem 5.1. From Theorem 5.2, there exists a C^1 neighborhood $\mathcal{U}$ of f and two numbers $a_0 > 0$ and $\epsilon_0 > 0$ such that, for any $g, g' \in \mathcal{U}$ and any compact $g-$invariant set $\Delta \subseteq B(\Lambda, a_0)$ there exists a unique continuous map $h : \Delta \to M$ such that

$$hg = g'h \qquad d(h, \mathrm{id}) \le \epsilon_0. \tag{21}$$

By Lemma 5.6, there exists a neighborhood $\mathcal{U}_1$ of f such that for any $g \in \mathcal{U}_1$, we have that $\Gamma \subseteq B(\Lambda, a_0)$, where Γ is the maximal $g-$invariant set. Let $\epsilon > 0$ be given and suppose that $\epsilon \le \epsilon_0/2$. Shrink $\mathcal{U}$ such that

$$\mathcal{U} \subseteq \mathcal{U}_0 \cap \mathcal{U}_1,$$

and applying Theorem 5.2, for any $g, g' \in \mathcal{U}$ and any compact invariant set $\Delta \subseteq B(\Lambda, a_0)$ of g, there is at least one continuous injective map $h : \Delta \to M$ such that

$$hg = g'h, \qquad d(h, \mathrm{id}) \le \epsilon$$

Thus, since $\Gamma \subseteq B(\Lambda, a_0)$, for any $g, g' \in \mathcal{U}$, there exists a continuous injective map $h : \Gamma \to M$ onto its image satisfying

$$hg = g'h, \qquad d(h, \mathrm{id}) \leq \epsilon$$

on Γ. Similarly, for any g', there exists a continuous injective map $h' : \Gamma' \to M$ satisfying

$$h'g' = gh' \qquad d(h', \mathrm{id}) \leq \epsilon,$$

on Γ'.

Note that $h(\Gamma) \subseteq B(\Gamma, \epsilon) \subseteq B(\Lambda, a_0 + \epsilon) \subseteq U$, where the first equality follows from the fact that $d(h, \mathrm{id}) \leq \epsilon$. Since $h(\Gamma)$ is $g'-$invariant as $h(\Gamma) = h(g(\Gamma)) = g'(h(\Gamma))$ and Γ' is the maximal invariant set of g' in U,

$$h(\Gamma) \subseteq \Gamma'. \tag{22}$$

Similarly,

$$h'(\Gamma') \subseteq \Gamma. \tag{23}$$

Note that we use our assumption that Λ is isolated here. Otherwise, these inclusions need not hold.

If we compose h and h', cancel out the inverses, and substitute into (21), we get $h'h : \Gamma \to \Gamma$ such that

$$\begin{aligned} h'hg &= h'g'h && \text{(Composing on the left with } h'\text{)} \\ (h'h)g &= (gh')h && \text{(Theorem 5.2 applied to } g'\text{)} \\ (h'h)g &= g(h'h), \end{aligned}$$

with $d(h'h, \mathrm{id}|_\Gamma) \leq \epsilon + \epsilon \leq \epsilon_0$ on Γ. Note that this composition is well-defined on Γ because of (22) and (23). However, since we have that h is unique and the identity map restricted to g is an ϵ_0-conjugacy between $g|_\Gamma$ and itself,

$$h'h = \mathrm{id}|_\Gamma \qquad \text{and} \qquad hh' = \mathrm{id}|_{\Gamma'}.$$

Thus, $h(\Gamma) = \Gamma'$, meaning that $g|_\Gamma$ and $g'|_{\Gamma'}$ are $\epsilon-$conjugate. Letting $g' = f$ and $\Gamma' = \Lambda$, we conclude our proof. □

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References

[1] Lan Wen. Differentiable Dynamical Systems: An Introduction to Structural Stability and Hyperbolicity. The American Mathematical Society. 2016.
[2] Luís Lopes, Clara Garcio, and Sara Fernandes. Window of Chaotic Delayed Synchronization. Journal of Physics: Conference Series.2016.
[3] Charles C. Pugh. Real Mathematical Analysis. The American Mathematical Society. 2002.
[4] Anatole Katok and Boris Hasselblatt. Introduction to The Modern Theory of Dynamical Systems. Cambridge University Press. 1995. Reprinted 1999.
[5] Wikipedia, The Free Encyclopedia. Logistic Map. Wikimedia Foundation, Inc. 2024. URL: https://en.wikipedia.org/wiki/Logistic_map#Graphical_representation.
[6] A Blumenthal. https://math.stackexchange.com/users/54337/a-blumenthal. Proving that the tent map is topologically conjugate to the quadratic map. URL: https://math.stackexchange.com/q/2175569
[7] PlanetMath. C^r Topologies. 2013. URL: https://planetmath.org/mathcalcrtopologies.

[8] SigSticks. Cat. URL: https://www.sigstick.com/pack/HGJhPG4Y1iRCiGPLavnL
[9] Wikipedia, The Free Encyclopedia. Arnold's Cat Map. Wikimedia Foundation, Inc. May 12, 2024. URL: https://en.wikipedia.org/wiki/Arnold%27s_cat_map.
[10] Vladimir I. Arnold and André Avez. Ergodic Problems of Classical Mechanics. Benjamin. 1968.
[11] Maurício M. Peixoto. Structural stability on two dimensional manifolds. Topology 1. 1962.
[12] Prof Ghrist Math. AppDynSys : Smale Horseshoe : Iteration. YouTube. 2020. URL: https://www.youtube.com/watch?v=SrJm6bkLuPs
[13] Wolfram MathWorld. Cantor Dust. Wolfram Research, Inc. URL: https://mathworld.wolfram.com/CantorDust.html