EXT AT ODD PRIMES OF THE A_1 SUBALGEBRA OF THE STEENROD ALGEBRA

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Abstract. In this paper I examine the A_1 sub-algebra of the Steenrod Algebra. In particular, the cohomology groups $\mathrm{H}^{*,*}(A_1)$ over odd prime fields are calculated. This paper gives a modern treatment of the computation and elaborates on other computations which are concentrated at the odd prime 3. The main computation tool used for the paper will be [6].

Contents

1. Introduction

Much of algebraic topology is concentrated at the prime 2. This also extends to computations involving the Steenrod Algebra and its subalgebras. Let A be the mod p Steenrod algebra for a prime p. For odd p, let A_1 be the subalgebra of the Steenrod algebra generated by $\{\beta, P^1\}$.

Considerable computation has been done for the equivalent notion at the prime 2. Recently, there has been some interest in computations of A_1 at the prime 3, as well as higher odd primes. In particular, Culver gives a computation using Massey products for the prime 3 in [1], while Hill gives a computation for the same object in [2]. However, both references do not continue to higher odd primes, but restrict themselves to the prime 3.

To compute the necessary algebras and coalgebras in order to completely determine the cohomology of A_1 at all odd primes, several preliminary results must be recalled.

2. Background on Lie Algebras

All of the computational tools used in this paper are found in [6]. Some of the key results are summarized below.

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I will focus on describing the algebra and coalgebra structure necessary for constructing the spectral sequences used to compute the homology of A_1 over the field of characteristic p. The coalgebra structure is then integral in the calculations that follow.

First, before creating a resolution of restricted Lie algebras, I will review the resolution of non-restricted Lie algebras. Let L be any bigraded Lie algebra over a field K, of prime characteristic p, with universal enveloping algebra $U(L)$. Then, a $U(L)$-free resolution of K as a trivial right $U(L)$-module will be constructed. As a K module, the resolution will be given as $Y(L) = \bar{Y}(L) \otimes U(L)$ where $\bar{Y}(L) = \Gamma(sL^-) \otimes E(sL^+)$. Here, L^- denotes the odd degree elements in the Lie algebra, L^+ the even degree elements, Γ the standard divided polynomial algebra, E the standard exterior algebra, and sL indicates a copy of L where all elements are given a new, homological, degree of 1. The degree of an element will be given as the total degree, i.e. the sum of the homological degree and the degree inherited from the original grading.

If one gives $Y(L)$ a structure of an algebra over the Hopf algebra $U(L)$, then $Y(L)$, as a $U(L)$-module, can be defined as the semi-tensor product, $\bar{Y}(L) \odot U(L)$, which is originally defined by Massey and Peterson in [3]. Most important for our calculation however, is the algebra, coalgebra, and differential structure that can be created on $Y(L)$.

Lemma 2.1. [6] *$Y(L)$ may be given an algebra structure by requiring the product to agree with the natural one on $\bar{Y}(L)$ and on $U(L)$ and to satisfy the relations:*

$$u\langle y\rangle = (-1)^{deg\ u}\langle y\rangle u + (-1)^{deg\ u} s[u, y],$$
$$u \in L, \quad \langle y\rangle = sy \in sL^+.$$
$$u\gamma_r(x) = \gamma_r(x)u + (-1)^{deg\ u}\gamma_{r-1}(x)s[u, x],$$
$$u \in L, \quad \gamma_1(x) = sx \in sL^-.$$

Y(L) may be given a Hopf algebra structure with coproduct D by requiring D to be a morphism of algebras and to agree with the natural coproduct on $\bar{Y}(L)$ and on $U(L)$. In particular, the coalgebra structure on generators is given as:

$$D\langle y\rangle = \langle y\rangle \otimes 1 + 1 \otimes \langle y\rangle$$
$$D\gamma_r(x) = \sum_{i+j=r} \gamma_i(x) \otimes \gamma_j(x)$$

Additionally, $Y(L)$ can also be given a differential:

Theorem 2.2. [6] *Define a differential d on $Y(L)$ by*

$$d(ab) = d(a)b + (-1)^{deg\ a} ad(b), \quad a, b \in Y(L)$$

and

$$d(u) = 0, \quad d\langle y\rangle = y, \quad d\gamma_r(x) = \gamma_{r-1}(x)x - \frac{1}{2}\gamma_{r-2}(x)\langle [x, x]\rangle,$$

where $u \in U(L)$, $\langle y\rangle \in sL^+$, $\gamma_1(x) \in sL^-$, and $\gamma_{-1}(x) = 0$. Then, $Y(L)$ is a $U(L)$-free resolution of K, and is also a differential coalgebra over $U(L)$ with differential defined dually.

3. Background on Restricted Lie Algebras

Now, to continue to restricted Lie algebras, some additional constructions are necessary. Again, K is a field of characteristic p, this time over F_p for some prime p, L is again the (restricted) Lie algebra, and $V(L)$ is the universal enveloping algebra of L. Let $W(L) = \bar{Y}(L) \odot V(L)$ and $X(L) = \Gamma(s^2\pi L^+) \otimes W(L)$ as a K-module. Here, $s^2\pi L$ denotes a copy of L with all degrees multiplied by p and with all elements having a homological degree of 2. After $X(L)$ is endowed with the additional structure described below, it will become the desired resolution.

The additional structure twists both the coproduct and the differential within $X(L)$. Most important for us is the differential twisting. First, we need some preliminary definitions.

Definition 3.1. [6] We give $R = Hom(\Gamma(s^2\pi L^+), Y(L^+))$ a structure of a differential algebra with differential δ and product $\cup$. The differential is defined by

$$\delta(r)(b) = d(r(b)) + (-1)^{\deg r+1} r(d(b))$$

for $b \in \Gamma(s^2\pi L^+)$ and $r \in R$. Here, d is the differential on the appropriate algebra. Additionally, the product is defined by

$$(r \cup r')(b) = \pi(r \otimes r')D(b),$$

where π is the product on $Y(L^+)$, $r, r' \in R$, and D is the coproduct on $\Gamma(s^2\pi L^+)$.

Then, we introduce some additional structure.

Definition 3.2. [6] We give $\Gamma(s^2\pi L^+) \otimes W(L)$ the structure of a left differential $R = Hom(\Gamma(s^2\pi L^+), Y(L^+))$-module with module product $\cap$. This is defined by

$$r \cap (g \otimes w) = (1 \otimes \sigma)(1 \otimes r \otimes 1)(D \otimes 1)(g \otimes w).$$

In the equation above, $r \in R$, $g \in \Gamma(s^2\pi L^+)$, $w \in W(L)$, D is the coproduct on $\Gamma(s^2\pi L^+)$, and σ is determined by the epimorphism of differential algebras, $Y(L) \to W(L)$, and the algebra structure of W(L).

Then, the differential above can be twisted by a homomorphism t.

Definition 3.3. [6] For $t \in R^1$, with $t_n : \Gamma(s^2\pi L^+)_n \to Y(L^+)_{n-1}, n \geq 1$, we may define $d_t : \Gamma(s^2\pi L^+) \otimes W(L) \to \Gamma(s^2\pi L^+) \otimes W(L)$ by

$$d_t(g \otimes w) = d(g \otimes w) + t \cap (g \otimes w)$$

From this definition, a simple calculation yields $d_t^2(b \otimes f) = (\delta(t) + t \cup t) \cap (b \otimes f)$.

Definition 3.4. [6] t is said to be a twisting cochain if $\delta(t) + t \cup t = 0$ and $\epsilon t_1 = 0$, where ϵ is the counit $Y(L^+) \to K$. In particular, t is a twisting cochain if,

$$dt_n + t_{n-1}d + \sum_{i=1}^{n-1} t_i \cup t_{n-i} = 0, \quad n > 1.$$

If this map t is a twisting cochain, then d_t is a differential for $\Gamma(s^2\pi L^+) \otimes W(L)$. For the next calculations, we would like to choose t to have a specific property. In particular, we can (and will) choose the homomorphism t such that $t_2 : \Gamma(s^2\pi L^+)_2 \to Y(L^+)_1$ satisfies

$$t_2(\gamma_1(\tilde{y})) = \langle y \rangle y^{p-1} - \langle \xi(y) \rangle.$$

In the equation above, $\tilde{y}$ denotes the copy of y in $\Gamma(s^2\pi L^+)$ and ξ is the restriction.

Finally, let us prove a short Lemma on the general structure of A_1.

Lemma 3.5. *(Structure of A_1) A_1 is a primitively generated Hopf algebra with its restriction being identically zero. Hence, it is isomorphic to the universal restricted enveloping algebra of the restricted Lie algebra of its primitive elements, i.e. $A_1 = V(PA_1)$.*

Proof. A_1 is the subalgebra of the Steenrod algebra generated by the first Steenrod operation, P^1, and the Bockstein homomorphism, $\beta = Q_0$. According to Milnor [4], the coproduct on generating elements is given by

$$D(Q_0) = Q_0 \otimes 1 + 1 \otimes Q_0$$
$$D(P) = \sum_{i+j=1} P^i \otimes P^j = P^1 \otimes 1 + 1 \otimes P^1$$

indicating Q_0 and P^1 are both primitive. From the structure of the coproduct we also see that $Q_1 = [P^1, Q_0]$ is also primitive. Since both P^1 and Q_0 are primitive, A_1 is primitively generated and thus, $A_1 = V(PA_1)$ by a theorem of Milnor and Moore [5], where PA_1 is the restricted Lie algebra of primitive elements. Also note, since p is odd, the restriction is not defined on odd degree elements. □

4. Calculation for Unrestricted Lie Algebra

With this background, we can now proceed to a preliminary calculation of the cohomology of A_1 without considering its restriction. In this case, the differential graded algebra (DGA) we are considering to compute the cohomology of $U(PA_1)$ is $Y(L)^* = (\bar{Y}(L) \otimes U(L))^* = ((\Gamma(sL^-) \otimes E(sL^+)) \otimes U(L))^*$

Theorem 4.1. *If $p \geq 3$, then, when A_1 is considered as an unrestricted Lie algebra,*

$$H^{*,*}(U(PA_1)) = P[a_0, a_1] \otimes E[\gamma_j | 0 \leq j \leq p-1]$$

modulo the relations

$$a_0\gamma_j = 0 \text{ for } j \leq p-2 \text{ and } \gamma_i\gamma_j = 0 \text{ for all } i \text{ and } j.$$

where the elements $a_0, a_1, and \gamma_i$ have respective (s, t) bidegrees

$$(1,1), \quad (p, p(q+1)), \text{ and } \quad (j+1, (j+1)q + j),$$

for $q = 2(p-1)$.

Proof. First, the DGA mentioned above must be computed. Since P has degree $2(p-1)$, Q_0 has degree 1, and Q_1 has degree $2(p-1)+1$, L^+ has basis $\{P\}$, and L^- has basis $\{Q_0, Q_1\}$. To compute $H^*(U(L))$, we must first examine

$$\bar{Y}(L)^* = P((sL^+)^*) \otimes E(sL^+).$$

Letting q_0, q_1, and p^1 denote the basis elements corresponding to Q_0, Q_1, and P, the DGA has the form

$$\bar{Y}(L)^* = P(q_0, q_1) \otimes E(p^1)$$

with all basis elements having homological degree 1. The differential in this DGA takes elements of bidegree (a,b) to elements of bidegree (a+1,b). Examining degrees yields

$$d(q_0) = 0, \quad d(q_1) = q_0p^1, \quad d(p^1) = 0.$$

Recalling that

$$d(ab) = d(a)b + (-1)^{deg(a)deg(b)} ad(b)$$

from above, this yields that the differential on all possible elements is given by

$$d(q_0^i q_1^j) = j q_0^{i+1} q_1^{j-1} p^1, \quad d(q_0^i q_1^j p^1) = 0.$$

So, a basis for the cohomology of $\bar{Y}(L)^*$ is given by elements of the form

$$q_0^i q_1^p, \quad q_1^j p^1, \quad q_0^{i+1} q_1^{pj-1} p^1$$

where $i \geq 0, j \geq 0$. Letting a_0, a_1, and γ_i denote the cohomology classes represented by q_0, q_1^p, and $q_1^i p^1$, respectively, allows the above theorem to follow readily. The bidegrees of the generators are also easily computed to coincide with the theorem above. ☐

5. Calculation for Restricted Lie Algebra

Before continuing to the calculation of the cohomology of A_1 as a restricted Lie algebra, some preliminary results regarding the structure of the spectral sequence that will be employed should be covered.

Theorem 5.1. [6] *There exists a spectral sequence $\{E^r L\}$ of differential coalgebras which converges to $H_*(V(L))$ and satisfies*

$$E^2 L = \Gamma(s^2 \pi L^+) \otimes H_*(U(L)).$$

The dual spectral sequence $\{E_r L\}$ of differential algebras converges to $H^*(V(L))$ and satisfies

$$E_2 L = P((s^2 \pi l^+)^*) \otimes H^*(U(L)).$$

Remark 5.2. Note that in the spectral sequence above, one way to filter the complex $X(L) \otimes_{V(L)} K = \bar{X}$ is by

$$F_i \bar{X}_n(L) = \oplus_{m \leq i} \Gamma_m(s^2 \pi L^+) \otimes \bar{Y}_{n-m}(L).$$

This yields the E^0 term to be

$$E^0 L = \Gamma(s^2 \pi L^+) \otimes \bar{Y}(L).$$

Note, the second differential in this spectral sequence, which will be denoted ∂_2, is inherited from the differential on $E^0 L = \Gamma(s^2 \pi L^+) \otimes \bar{Y}(L)$, d_t, as defined in Definition 3.4. The first differential of this spectral sequence, ∂_0, has no effect on $\Gamma(s^2 \pi L^+)$ since its elements are all of degree two or higher, and, by the unrestricted calculations above, the homology of $\bar{Y}(L)$ is $H_*(U(L)$. This gives the desired $E^2 L$ term.

The next nonzero differential, ∂_2, is zero on $\Gamma(s^2 \pi L^+)$ by assumption. So, the differential simplifies to

$$\partial_2(g \otimes y) = d_t(g \otimes y) = g \otimes d(y) + t \cap (g \otimes y).$$

A few additional remarks about the differential will be helpful later. When considering elements $y \in H_*(U(L))$, the second differential further simplifies to

$$\partial_2(g \otimes y) = d_t(g \otimes y) = t \cap (g \otimes y).$$

Furthermore, when the restriction is zero (as in A_1), the chosen map t_2, from which the total map t and the differential d_t can be computed via Definition 3.4, is given by

$$t_2(\gamma_1(\tilde{y})) = \langle y \rangle y^{p-1}. \tag{5.3}$$

Note that this does not mean t_2 is identically zero, as the module action of y on the image of t_2 is not necessarily zero.

This now allows us to compute the desired cohomology.

Theorem 5.4. *If $p \geq 3$, then*

$$H^{*,*}(A_1) = P[a_0, a_1, \delta] \otimes E[\gamma_j | 0 \leq j \leq p-2]$$

modulo the relations

$$a_0^{p-1}\delta = 0, \quad a_0\gamma_j = 0 \text{ for } j \leq p-2, \text{ and } \gamma_i\gamma_j = 0 \text{ for all } i \text{ and } j.$$

Here the elements $a_0, a_1, \gamma_0, \gamma_1$ and δ have respective (s, t) bidegrees

$$(1,1), \quad (p, p(q+1)), \quad (j+1, (j+1)q+j), \text{ and } (2, pq)$$

for $q = 2(p-1)$. In terms of Massey products, the generators are given by

$$\gamma_j = \langle a_0, \gamma_{j-1}, \gamma_1 \rangle \text{ and } a_1 = \langle a_0, \gamma_0, a_0, \gamma_{p-2} \rangle.$$

Proof. From the unrestricted calculation we have that a basis for the cohomology of $\bar{Y}(L)^*$ is given by elements of the form

$$q_0^i q_1^p, \quad q_1^j p^1, \quad q_0^{i+1} q_1^{pj-1} p^1.$$

This basis is then generated by elements q_0, q_1^p, and $q_1^i p^1$. Now, the image these elements have under the new differential, $\partial^2 = d^t$, must be examined. Recall that d^t is the differential twisted by an element $t \in Hom(\Gamma(s^2\pi L^+), Y(L^+))$ (given in 3.4). In particular, we will see that the dual differential has $d_t(q_0^{p-1}\delta) = q_1^{p-1}p^1$ and no other nonzero values, indicating that all cohomology classes carry over from the unrestricted case, except for $q_1^{p-1}p^1$ since $d^t(q_1^{p-1}p^1) = q_0^{p-1}\delta$.

Examining the bidgrees of the above generators shows that no element has a new nonzero differential after tensoring with $P(s^2\pi L^{+*})$ except for $q_1^{p-1}p^1$. This element has bidegree $(p, p(q+1)-1)$, which indicates its cohomological differential has bidegree $(p, p(q+1)$. There is a nonzero element with this bidegree, namely $q_0^{p-1}\delta$. Direct computation confirms the result:

$$\begin{aligned} d_t(q_0^{p-1}\delta) &= t \cap (\delta \otimes \gamma_1(q_0)^{p-1}) \\ &= (1 \otimes \sigma)(1 \otimes t \otimes 1)(D \otimes 1)(\delta \otimes \gamma_1(q_0)^{p-1}) \\ &= (1 \otimes \sigma)(1 \otimes t(\delta) \otimes \gamma_1(q_0)^{p-1}) \\ &= 1 \otimes \langle p^1 \rangle (p^1)^{p-1} \otimes \gamma_1(q_0)^{p-1}) \\ &= 1 \otimes \langle p^1 \rangle \otimes (p^1)^{p-1}(\gamma_1(q_0)^{p-1}) = q_1^{p-1}p^1 \end{aligned}$$

where the last step, $(p^1)^{p-1}(\gamma_1(q_0)^{p-1}) = q_1^{p-1}$, is completed by induction.

The induction argument is presented below. Recall that multiplication in $\overline{Y}(L) \odot V(L)$ is given by

$$vz_1z_2 = (vz_1)z_2 + (-1)^{deg(v)deg(z_1)} z_1vz_2$$
$$v\gamma_r(x) = \gamma_r(x)v + (-1)^{deg(v)}\gamma_{r-1}(x)s[u, x].$$

Then, for $n = 1$ we have

$$p^1\gamma_1(q_0) = s[p^1, q_0] = q_1$$

and assuming the claim holds for $n = k-1$, we get, for $n = k$,

$$\begin{gathered}(p^1)^n \gamma_1(q_0)^n \\ = p^1(\gamma_1(q_1)^{n-1}\gamma_1(q_0) + \gamma_1(q_1)^{n-2}\gamma_1(q_0)\gamma_1(q_1) + \cdots + \gamma_1(q_0)\gamma_1(q_1)^{n-1}) \\ = nq_1^n.\end{gathered}$$

This direct calculation further confirms no other elements can have new nonzero differentials since $t \in Hom(\Gamma(s^2\pi L^+), Y(L^+))$ is completely determined by $t(\delta) = t_2(\delta)$, and the formula for the map t_2 indicates that $t(\delta) = \langle p^1 \rangle (p^1)^{p-1}$, so products containing δ must be generated by $q_1^{p-1}p^1$. So, the spectral sequence collapses for larger degrees, and the conclusion follows. Then, in terms of basis elements the generators are given by

$$a_0 = q_0, \quad a_1 = q_1^p, \text{ and } \gamma_i = q_1^i p^1.$$

□

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