A_1 CALCULATION

JULIAN KAUFMANN

ABSTRACT. In this paper I examine the A_1 sub-algebra of the Steenrod Algebra. In particular, the cohomology groups, $H^{*,*}(A_1)$ over odd prime fields are calculated. This gives a modern treatment of the computation and elaborates on other computations which are concentrated at the odd prime 3. The main computation tool used for the paper will be [6].

CONTENTS

1. INTRODUCTION

Much of algebraic topology is concentrated at the prime 2. This also extends to computations involving the Steenrod Algebra and its subalgebra. Let A be the mod p Steenrod algebra for a prime p. For odd p, let A_1 be the subalgebra of the Steenrod algebra generated by $\{\beta, P^1\}$.

Much computation has been done for the equivalent notion at the prime 2. Recently, there has been some interest in computations of A_1 at the prime 3, as well as higher odd primes. In particular, Culver gives a computation using Massey products for the prime 3 in [1], while Hill gives a computation for the same object in [2]. However, both references do not continue to higher odd primes, but restrict themselves to the prime 3.

To compute the necessary algebras and coalgebras in order to completely determine the cohomology of A_1 at all odd primes, several preliminary results must be recalled.

2. BACKGROUND ON LIE ALGEBRAS

All of the computational tools used in this paper are found in [6] and some of the key results are summarized below.

I will focus on describing the algebra and coalgebra structure used in constructing spectral sequences which are used to compute the homology of A_1 over the field of

Date: Sep. 22, 2024.

characteristic p. The coalgebra structure is then integral in the calculations that follow.

First, before creating a resolution of restricted Lie algebras, I will review the resolution of non-restricted Lie algebras. Let L be any bigraded Lie algebra over a field K, of prime characteristic p, with universal enveloping algebra $U(L)$. Then, a $U(L)$-free resolution of K as a trivial right $U(L)$-module will be constructed. As a K module, the resolution will be given as $Y(L) = \bar{Y}(L) \otimes U(L)$ where $\bar{Y}(L) = \Gamma(sL^-) \otimes E(sL^+)$. Here, L^- denotes the odd degree elements in the Lie algbera, L^+ the even degree elements, Γ the standard divided polynomial algebra, E the standard exterior algebra, and sL indicates a copy of L where all elements are given a new, homological, degree of 1.

If one gives $\bar{Y}(L)$ a structure of a an algebra over the Hopf algebra $U(L)$, then $Y(L)$, as a $U(L)$-module, can be defined as the semi-tensor product, $\bar{Y}(L) \odot U(L)$, which is originally defined by Massey and Peterson in [3]. Most important for our calculation however, is the algebra, coalgebra, and differential structure that can be created on $Y(L)$.

Lemma 2.1. [6] *$Y(L)$ may be given an algebra structure by requiring the product to agree with the natural one on $\bar{Y}(L)$ and on $U(L)$ and to satisfy the relations:*

$$u\langle y\rangle = (-1)^{deg\ u}\langle y\rangle u + (-1)^{deg\ u} s[u,y],$$
$$u \in L, \quad \langle y\rangle = sy \in sL^+.$$
$$u\gamma_r(x) = \gamma_r(x)u + (-1)^{deg\ u}\gamma_{r-1}(x)s[u,x],$$
$$u \in L, \quad \gamma_1(x) = sx \in sL^-.$$

Y(L) may be given a Hopf algebra structure with coproduct D by requiring D to be a morphism of algebras and to agree with the natural coproduct on $\bar{Y}(L)$ and on $U(L)$. In particular, the coalgebra structure on genrators is given as:

$$D\langle y\rangle = \langle y\rangle \otimes 1 + 1 \otimes \langle y\rangle$$
$$D\gamma_r(x) = \sum_{i+j=r} \gamma_i(x) \otimes \gamma_j(x)$$

Additionally, $Y(L)$ can also be given a differential:

Theorem 2.2. [6] *Define a differential d on $Y(L)$ by*

$$d(ab) = d(a)b + (-1)^{deg\ a} a d(b), \quad a, b \in Y(L)$$

and

$$d(u) = 0, \quad d\langle y\rangle = y, \quad d\gamma_r(x) = \gamma_{r-1}(x)x - \frac{1}{2}\gamma_{r-2}(x)\langle[x,x]\rangle,$$

where $u \in U(L)$, $\langle y\rangle \in sL^+$, $\gamma_1(x) \in sL^-$, and $\gamma_{-1}(x) = 0$. Then, $Y(L)$ is a $U(L)$-free resolution of K, and is also a differential coalgebra over $U(L)$ with differential defined dually, i.e. for $d(x)$ nonzero,

$$\delta(d(x)) = x$$

3. Background on Restricted Lie Algebras

Now, to continue to restricted Lie algebras, some additional constructions are necessary. Again, K is a field of characteristic p, this time over Z_p for some prime p, L is again the (restricted) Lie algebra, and $V(L)$ is the universal enveloping algebra of L. Let $W(L) = \bar{Y}(L) \odot V(L)$ and $X(L) = \Gamma(s^2\pi L^+) \otimes W(L)$ as a K-module. Here, $s^2\pi L$ denotes a copy of L with all degree multiplied by p and with all elements having a homological degree of 2. After $X(L)$ is endowed with additional structure, it will become the desired resolution.

The additional structure twists both the coproduct and the differential within $X(L)$. Most important for us is the differential twisting. First we need some preliminary definitions.

Definition 3.1. [6] We give $R = Hom(\Gamma(s^2\pi L^+), Y(L^+))$ a structure of a differential algebra with differential δ and product $\cup$. The differential is defined by

$$\delta(r)(b) = d(r(b)) + (-1)^{\deg r+1} r(d(b))$$

for $b \in \Gamma(s^2\pi L^+)$ and $r \in R$. Here, d is the differential on the approriate algebra. Additionally, the product is defined by

$$(r \cup r')(b) = \pi(r \otimes r')D(b),$$

where π is the product on $Y(L^+)$, $r, r' \in R$, and D is the coproduct on $\Gamma(s^2\pi L^+)$.

Then, we introduce some additional structure.

Definition 3.2. [6] We give $\Gamma(s^2\pi L^+) \otimes W(L)$ the structure of a left differential $R = Hom(\Gamma(s^2\pi L^+), Y(L^+))$-module with module product $\cap$. This is defined by

$$r \cap (g \otimes w) = (1 \otimes \sigma)(1 \otimes r \otimes 1)(D \otimes 1)(g \otimes w).$$

In the equation above, $r \in R$, $g \in \Gamma(s^2\pi L^+)$, $w \in W(L)$, D is the coproduct on $\Gamma(s^2\pi L^+)$, and σ is determined by the epimorphism of differential algebras, $Y(L^+) \to W(L^+)$, and by the algebra structure of W(L).

Then, the differential above can be twisted by a homomorphism t.

Definition 3.3. [6] Then, for $t \in R^1$, so that $t_n : \Gamma(s^2\pi L^+)_n \to Y(L^+)_{n-1}, n \geq 1$, we may define $d_t : \Gamma(s^2\pi L^+) \otimes W(L) \to \Gamma(s^2\pi L^+) \otimes W(L)$ by

$$d_t(g \otimes w) = d(g \otimes w) + t \cap (g \otimes w)$$

From this definition, a simple calculation yields $d_t^2(b \otimes f) = (\delta(t) + t \cup t) \cap (b \otimes f)$.

Definition 3.4. [6] t is said to be a twisting cochain if $\delta(t) + t \cup t = 0$ and $\epsilon t_1 = 0$, where ϵ is the counit $Y(L^+) \to K$. In particular, t is a twisting cochain if,

$$dt_n + t_{n-1}d + \sum_{i=1}^{n-1} t_i \cup t_{n-i} = 0, \quad n > 1.$$

If this map t is a twisting cochain, then d_t is a differential for $\Gamma(s^2\pi L^+) \otimes W(L)$. For the next calculations, we would like to chosse t to have a specific property. In particular, we can (and will) choose the homomorphism t such that $t_2 : \Gamma(s^2\pi L^+)_2 \to Y(L^+)_1$ satisfies

$$t_2(\gamma_1(\tilde{y})) = \langle y \rangle y^{p-1} - \langle \xi(y) \rangle.$$

In the equation above, $\tilde{y}$ denotes the copy of y in $\Gamma(s^2\pi L^+)$, and ξ is the restriction.

Finally, let us prove a short Lemma on the general structure of A_1.

Lemma 3.5. *(Structure of A_1) A_1 is a primitively generated Hopf algebra with its restriction being identically zero. Hence, it is isomorphic to the universal restricted enveloping algebra of the restricted Lie algebra of its primitive elements, i.e. $A_1 = V(PA_1)$. Additionally, when considered as an unrestricted Lie algebra, A_1 is isomorphic to the universal enveloping algebra of its primitive elements, i.e. $A_1 = U(PA_1)$.*

Proof. A_1 is the subalgebra of the Steenrod algebra generated by the first Steenrod operation, $P^1 = P$, and the Bockstein homomorphism, $\beta = Q_0$. According to Milnor [4], the coproduct on generating elements is given by

$$D(Q_0) = Q_0 \otimes 1 + 1 \otimes Q_0$$
$$D(P) = \sum_{i+j=1} P^i \otimes P^j = P \otimes 1 + 1 \otimes P.$$

indicating Q_0 and P are both primitive. From the structure of the coproduct we also see that $Q_1 = [P, Q_0]$ is also primitive. Since both P and Q_0 are primitive, A_1 is primitively generated and thus, $A_1 = V(PA_1)$ by a theorem of Milnor and Moore [5], where PA_1 is the algebra of primitive elements (or $A_1 = U(PA_1)$ when considered without restriction). The structure of PA_1 is also easy to compute. Namely, since all other elements are products of P, Q_0, and Q_1, PA_1 has basis $\{Q_0, Q_1, P\}$.

Next, the restriction on PA_1 is also considered. Note that $Q_0^p = 0$ as $Q_0^2 = 0$ and $p > 2$. Similarly, $Q_1^p = (PQ_0 - Q_0P)^p = (PQ_0)^p - (Q_0P)^p = 0$ as we are working in characteristic p. Finally, by the Adem relations, $P^p = \binom{p-2}{1}\binom{p-2}{2}\dots\binom{p-2}{p-1}P^p = 0$. So, the restriction is identically zero and the Lemma above holds. □

4. Calculation for Unrestricted Lie Algebra

With this background, we can now proceed to a preliminary calculation of the cohomology of A_1 without considering its restriction. In this case the differential graded algebra (DGA) we are considering to compute the cohomology of A_1 is $Y(L)^* = (\bar{Y}(L) \otimes U(L))^* = ((\Gamma(sL^-) \otimes E(sL^+)) \otimes U(L))^*$

Theorem 4.1. *If $p \geq 3$, then, when A_1 is considered as an unrestricted Lie algebra,*

$$H^{*,*}(U(A_1)) = P[a_0, a_1] \otimes E[\gamma_j | 0 \leq j \leq p-1]$$

modulo the relations

$$a_0\gamma_j = 0 \text{ for } j \leq p-2 \text{ and } \gamma_i\gamma_j = 0 \text{ for all } i \text{ and } j.$$

In terms of basis elements, these generators are given by

$$a_0 = q_0, \quad a_1 = q_1^p, \text{ and } \gamma_i = q_1^i p,$$

where the elements $a_0, a_1, and \gamma_i$ have respective (s, t) bidegrees

$$(1,1), \quad (p, p(q+1)), \text{ and } \quad (j+1, (j+1)q + j),$$

for $q = 2(p-1)$.

Proof. First, the DGA mentioned above must be computed. Since P has degree $2(p-1)$, Q_0 has degree 1, and Q_1 has degree $2(p-1)+1$, L^+ has basis $\{P\}$, and L^- has basis $\{Q_0, Q_1\}$. To compute $H^*(U(L)) = H^*(PA_1)$, we must first examine

$$\bar{Y}(L)^* = P((sL^+)^*) \otimes E(sL^+)$$

Letting q_0, q_1, and p denote the basis elements corresponding to Q_0, Q_1, and P, the DGA has the form

$$\bar{Y}(L)^* = P(q_0, q_1) \otimes E(p)$$

with all basis elements having homological degree 1. The differential in this DGA takes elements of bidegree (a,b) to elements of bidegree (a+1,b). Examining degrees yields

$$d(q_0) = 0, \quad d(q_1) = q_0 p \quad d(p) = 0$$

Recalling that

$$d(ab) = d(a)b + (-1)^{deg(a)deg(b)} a d(b)$$

from above, this yields that the differential on all possible elements is given by

$$d(q_0^i q_1^j) = j q_0^{i+1} q_1^{j-1} p, \quad d(q_0^i q_1^j p) = 0.$$

So, a basis for the cohomology of $\bar{Y}(L)^*$ is given by elements of the form

$$q_0^i q_1^p \quad q_1^j p \quad q_0^{i+1} q_1^{pj-1} p$$

where $i \geq 0, j \geq 0$. Letting a_0, a_1, and γ_i denote the cohomology classes represented by q_0, q_1^p, and $q_1^i p$, respectively, allows the theorem above to follow readily. The bidegrees of the generators are also easily computed to coincide with the theorem above. □

5. Calculation for Restricted Lie Algebra

Before continuing to the calculation of A_1 as a restricted Lie algebra, some preliminary results regarding the structure of the spectral sequence that will be employed should be covered.

Theorem 5.1. [6] *There exists a spectral sequence $\{E^r L\}$ of differential coalgebras which converges to $H_*(V(L))$ and satisfies*

$$E^2 L = \Gamma(s^2 \pi L^+) \otimes H_*(U(L)).$$

The dual spectral sequence $\{E_r L\}$ of differential algebras converges to $H^*(V(L))$ and satisfies

$$E_2 L = P((s^2 \pi l^+)^*) \otimes H^*(U(L))$$

Remark 5.2. Note that in the spectral sequence above, one way to filter the complex $X(L) \otimes_{V(L)} K = \bar{X}$ is by

$$F_i \bar{X}_n(L) = \oplus_{m \leq i} \Gamma_m(s^2 \pi L^+) \otimes \bar{Y}_{n-m}(L).$$

This yields the E^0 term to be

$$E^0 L = \Gamma(s^2 \pi L^+) \otimes \bar{Y}(L).$$

It should be noted that the first differential, d_0 has no effect on $\Gamma(s^2 \pi L^+)$ since its elements are all of degree two or higher, and, by the unrestricted calculations above, the homology of $\bar{Y}(L)$ is $H_*(U(L)$. This gives the desired $E^2 L$ term as above. The next non-zero differential, d_2, is mainly focused on the twisting cochain. Since the differential on $\Gamma(s^2 \pi L^+)$ is assumed to be zero, the differential d_t simplifies to

$$d_t(g \otimes y) = g \otimes d(w) + t \cap (g \otimes w).$$

Of course, when considering only elements $w \in H_*(U(L))$, this further simplifies to

$$d_t(g \otimes y) = t \cap (g \otimes w).$$

Finally, it should also be remarked that when the restriction is zero, such as in the case of A_1, the map t_2 is given by

$$t_2(\gamma_1(\tilde{y})) = \langle y \rangle y^{p-1}.$$

Note that this does not mean t_2 is identically zero, as the module action of w on the image of t_2 is not necessarily zero.

This lets us now compute the desired cohomology.

Theorem 5.3. *If $p \geq 3$, then*

$$H^{*,*}(A_1) = P[a_0, a_1, \delta] \otimes E[\gamma_j | 0 \leq j \leq p-2].$$

In terms of basis elements, these generators are given by

$$a_0 = q_0, \quad a_1 = q_1^p, \text{ and } \gamma_i = q_1^i p,$$

modulo the relations

$$a_0\gamma_j = 0 \text{ for } j \leq p-2 \text{ and } \gamma_i\gamma_j = 0 \text{ for all } i \text{ and } j.$$

Here the elements $a_0, a_1, \gamma_0, \gamma_1$ and δ have respective (s, t) bidegrees

$$(1,1), \quad (p, p(q+1)), \quad (j+1, (j+1)q+j), \text{ and } (2, pq)$$

for $q = 2(p-1)$.

Proof. From the unrestricted calculation we have that a basis for the cohomology of $\bar{Y}(L)^*$ is given by elements of the form

$$q_0^i q_1^p \quad q_1^j p \quad q_0^{i+1} q_1^{pj-1} p.$$

This basis is then generated by elements q_0, q_1^p, and $q_1^i p$. Now, from the above remark, the effect that t_2 has on these elements must be examined. In particular, we will see that the dual differential has $d_t(q_o^{p-1}\delta) = pq_1^{p-1}$ and no other non-zero values, indicating that $d^t(pq_1^{p-1}) = q_0^{p-1}\delta$ and that all other cohomology classes remain.

Examining the bidegrees of the generators above shows that none of them are affected by tensoring with $P(s^2\pi L^{+^*})$ except $q_1^2 p$. This has bidegree $(p, p(q+1)-1)$, which indicates its cohomological differential has bidegree $(p, p(q+1)$. There is a non-zero element with this bidegree, namely $q_0^{p-1}\delta$. Indeed, direct computation also yields the same result,

$$\begin{aligned} d_t(q_0^{p-1}\delta) &= t \cap (\delta \otimes \gamma_1(q_0)^{p-1}) \\ &= (1 \otimes \sigma)(1 \otimes t \otimes 1)(D \otimes 1)(\delta \otimes \gamma_1(q_0)^{p-1}) \\ &= 1 \otimes \langle p \rangle \otimes p^{p-1}(\gamma_1(q_0)^{p-1}) = pq_1^{p-1} \end{aligned}$$

where the last step is completed by induction. Indeed, recalling that multiplication in $\overline{Y}(L) \odot V(L)$ is given by

$$\begin{aligned} vz_1z_2 &= (vz_1)z_2 + (-1)^{deg(v)deg(z_1)} z_1vz_2 \\ v\gamma_r(x) &= \gamma_r(x)v + (-1)^{deg(v)}\gamma_{r-1}(x)s[u,x] \end{aligned}$$

For $n = 1$ we have

$$p\gamma_1(q_0) = s[p, q_0] = q_1$$

and assuming it is true for $n = k - 1$, we get, for $n = k$,

$$\begin{aligned} &p^n \gamma_1(q_0)^n \\ = p(\gamma_1(q_1)^{n-1}\gamma_1(q_0) + \gamma_1(q_1)^{n-2}\gamma_1(q_0)\gamma_1(q_1) + &\cdots + \gamma_1(q_0)\gamma_1(q_1)^{n-1}) \\ &= nq_1^n \end{aligned}$$

No other new homology elements can occur since examining the formula for t shows the only product containing δ that has non-zero differential is δq_1^2. So, the spectral sequence collapses for larger degrees, and the conclusion follows. □

Acknowledgments

It is a pleasure to thank my mentor, Prof. Peter May, for the interesting problem and much guidance throughout the program as well as organizing the entire program.

References

[1] D. Culver. The Adams spectral sequence for 3-local tmf. arXiv:1902.04230v3. 2020.
[2] M. A. Hill. The 3-local tmf homology of BΣ3. Proceedings of the American Mathematical Society. 135:12, 4076-4086. 2007.
[3] W.S. Massey and F. P. Peterson. The cohomology structure of certain fiber spaces I. Topology 4, 47-65. 1965.
[4] J. W. Milnor. The Steenrod Algebra and Its Dual. Annals of Mathematics 2:67:1, 150-171, 1958.
[5] J. W. Milnor and J. C. Moore. On the Structure of Hopf Algebras. Annals of Mathematics, 81:2, 211-264, 1965.
[6] J. P. May. The Cohomology of Restricted Lie Algebras and of Hopf Algebras. Journal of Algebra 3, 123-146. 1966.