THE EULER CHARACTERISTIC AND SMOOTH VECTOR FIELDS

MICHAEL ANDRE KAIBNI

Abstract. The goal of this paper is to show how the Euler characteristic of a space determines the structure of its smooth vector fields. To that end, a proof of the Poincaré-Hopf theorem for orientable manifolds and a corollary of the Lefschetz fixed point theorem will be presented. Multiple proofs of weaker statements are discussed to develop the language necessary for these theorems.

Contents

1. Introduction

This paper describes how the Euler characteristic of a topological space M dictates the structure of its smooth vector fields. If M is a closed and smooth manifold with a non-zero Euler characteristic, then all smooth vector fields on M must vanish at some point. This is a corollary of a classical result in topology called the Lefschetz fixed point theorem (Section 4). Additionally, one can establish an invariant of smooth vector fields on M: an integer sum characterizing the field's behavior near such vanishing points. The Poincaré-Hopf theorem states this invariant will always equal the Euler characteristic of the manifold; we will show this for orientable manifolds (Section 5). We prove the Brouwer fixed point theorem (Section 2) and the hairy ball theorem (Section 3) to review some basic concepts in algebraic and differential topology before approaching these theorems.

2. The Brouwer Fixed Point Theorem

We wish to show all continuous functions $f : D^n \to D^n$ have a fixed point. We will prove this statement twice: once using a differential argument and again using a more algebraic approach. For a quick review, we define the following.

Definition 2.1. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. A map $f : X \to Y$ is a **diffeomorphism** if both f^{-1} and f are bijective and smooth.

Definition 2.2. A **manifold** M of dimension m is a topological space such that for any $x \in M$ there exists a neighborhood $U_x \subset M$ which can be mapped homeomorphically to an open ball $B \subset \mathbb{R}^m$. An **atlas** $\mathscr{A}$ for M is a collection of such mappings whose domain is M. If for any $\varphi, \psi \in \mathscr{A}$ with intersecting domains the map $\psi \circ \varphi^{-1}$ is diffeomorphic, then $\mathscr{A}$ is a **smooth atlas**. A smooth atlas $\mathscr{A}$ is called **maximal** if it is not properly contained within a larger smooth atlas.

Definition 2.3. A **smooth manifold** M is a topological manifold with a maximal smooth atlas $\mathscr{A}$ on M.

Let M and N be smooth manifolds of dimension m and n with atlases $\mathscr{A}$ and $\mathscr{B}$. A map $f : M \to N$ is considered smooth if for any $x \in M$ there exists $\varphi \in \mathscr{A}$, $\psi \in \mathscr{B}$, $U_x \subset M$, and $V_{f(x)} \subset N$ such that $f(U_x) \subseteq V_{f(x)}$ and the composite map $\psi \circ f \circ \varphi^{-1}$ is smooth. Such maps will be our main focus. One can associate to any $x \in M$ an m-dimensional vector space TM_x; the map f induces the linear transformation $df_x : TM_x \to TN_{f(x)}$ for any $x \in M$. By the Whitney embedding theorem (see Guillemin and Pollack [9], page 48) any smooth compact manifold M of dimension m can be embedded into some $\mathbb{R}^k$ where $m < k$. In this embedding, df_x is a Jacobian matrix.

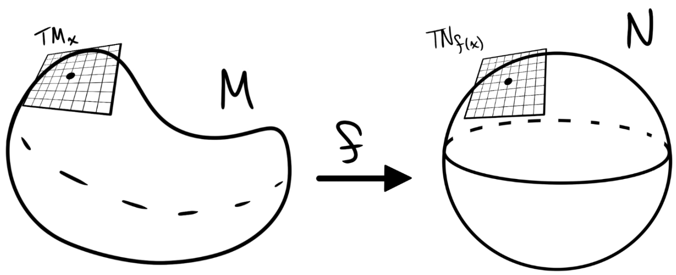


FIGURE 1. A visualization of $f : M \to N$ and tangent spaces TM_x and $TN_{f(x)}$ where M and N are both two dimensional manifolds.

Definition 2.4. A point $x \in M$ is considered a **regular point** if the matrix df_x has *rank* equal to n. If df_x has *rank* less than n, then x is a **critical point**.

Definition 2.5. A point $y \in$ N is called a **regular value** if the set $f^{-1}(y)$ contains only regular points. If $f^{-1}(y)$ contains a critical point, then y is a **critical value**.

Any point $y \in N$ is either a regular or critical value. This fact is useful when considered with the following classical result:

Theorem 2.6. ***(Sard's)*** *Let $f : U \to \mathbb{R}^n$ be a smooth function where $U \subset \mathbb{R}^m$ is an open subset and let $C = \{x \in U \mid rank(df_x) < n\}$ be the set of critical points in U. Then $f(C) \subset \mathbb{R}^n$ has Lebesgue measure zero.*

One can use a countable cover of M named $\{U_i\}$ to generate countably many measure zero sets $f(C_i)$. Their union is the set of all critical values in N and must have measure zero. Therefore, the set of regular values of f must be dense in N. This fact is incredibly useful once one realizes the structure and properties regular points impose on smooth maps. Consider the following.

Lemma 2.7. *Let $f : M \to N$ be a smooth map between smooth manifolds M and N such that $m \geq n$ and y is a regular value. Then $f^{-1}(y)$ is a smooth manifold of dimension $m - n$.*

Proof: Let $x \in f^{-1}(y)$. Since y is a regular value, df_x maps TM_x onto TN_y. Therefore the kernel of df_x is a vector space of dimension $m - n$. Choose a linear map $L : \mathbb{R}^m \to \mathbb{R}^{m-n}$ such that $\text{Im}(L|_{ker(df_x)})$ has dimension $m - n$. We can then define the function $F : M \to N \times \mathbb{R}^{m-n}$ such that $F(x) = (f(x), L(x))$ and $dF_x(h) = (df_x(h), L(h))$. The map dF_x is a square $m \times m$ matrix with $rank = m$. This implies dF_x is invertible. By the inverse function theorem, there exists an open neighborhood $U_x \subset M$ on which F is diffeomorphic to an open neighborhood $V_y \subset N \times \mathbb{R}^{m-n}$ of $(y, L(x))$. The set $\{y\} \times \mathbb{R}^{m-n} \cap V$ is a manifold of dimension $m - n$. Notice F^{-1} maps $\{y\} \times \mathbb{R}^{m-n} \cap V_y$ to $f^{-1}(y) \cap U_x$. One can then create the necessary diffeomorphic mappings by considering the composition of F with the projective map $\varphi : V \times \mathbb{R}^{m-n} \to \mathbb{R}^{m-n}$. □

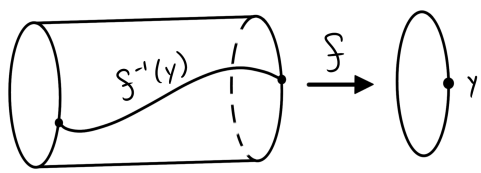


FIGURE 2. "Example" of Lemma 2.7

Notice the figure above does not neatly fit into our proof: the endpoints of $f^{-1}(y)$ are not consistent with our definition of a smooth manifold. The space $M = S^1 \times [0, 1]$ depicted above is called a smooth manifold with boundary. Such objects are defined using the closed half-space $\mathbb{H}^m = \{(x_1, ..., x_m) \in \mathbb{R}^m \mid x_m \geq 0\}$ under the half-disk topology. The definition of a **manifold with boundary** is a space where neighborhoods of points can be mapped homeomorphically to open balls or "half-open" balls centered on the hyperplane $\mathbb{R}^{m-1} \times \{0\}$. The definition of **smooth manifold with boundary** is simply the definition of a smooth manifold where $\mathbb{R}^m$ is replaced with $\mathbb{H}^m$. The subset $\partial M \subset M$, called the **boundary** of M, is the set of points mapped to the hyperplane $\mathbb{R}^{m-1} \times \{0\}$. Note this implies ∂M is a smooth manifold of dimension $m - 1$. Lemma 2.7 can be extended to include such objects (see Milnor [3], page 13) and to find the following.

Lemma 2.8. *Let M be a smooth manifold with boundary ∂M. Let N be a smooth manifold. Let $f : M \to N$ be a smooth function. Let $y \in N$ be a regular value for both f and $f|_{\partial M}$. Then $\partial f^{-1}(y) = f^{-1}(y) \cap \partial M$.*

Lemma 2.9. *Let M be a compact manifold with boundary ∂M. Then there exists no smooth map $f : M \to \partial M$ which leaves ∂M pointwise fixed.*

Proof: The dimension of M is m and the dimension of ∂M is $m - 1$. Suppose such a map f exists, then $f|_{\partial M} = id$. Notice all values of ∂M are regular under the identity transformation. Choose any regular value of f (possible since regular values are dense in ∂M). By Lemma 2.7 and Lemma 2.8, $f^{-1}(y)$ is a 1-manifold with boundary $f^{-1}(y) \cap \partial M$. Since $f|_{\partial M}$ is the identity, the only point within this intersection is y. Since f is continuous, $f^{-1}(y)$ is closed. Closed subsets of compact spaces are compact. Therefore $f^{-1}(y)$ is also compact. All compact one manifolds are disjoint unions of line segments with boundary and loops without boundary. Since line segments have an even number of boundary points, the number of boundary points in $f^{-1}(y)$ must be even. But the only boundary point is $\{y\}$. Therefore no such mapping f may exist. $\square$

This statement is trivial for cases like $S^1 \times [0,1]$ but is useful for examining manifolds with connected boundaries. Take the closed unit disk D^n where $\partial D^n = S^{n-1}$. Lemma 2.9 shows there exists no smooth map which collapses D^n to S^{n-1} in a "fixed" manner. This fact leads directly to the Brouwer fixed point theorem.

Lemma 2.10. *There does not exist a smooth map $f : D^n \to D^n$ without a fixed point.*

Proof : We will show such an f implies the existence of a smooth map r which contradicts Lemma 2.9. Given any value $x \in D^n$ draw a line connecting x to $f(x)$ and extending to ∂D^n. Define $r(x)$ to be the intersection closer to x than $f(x)$.

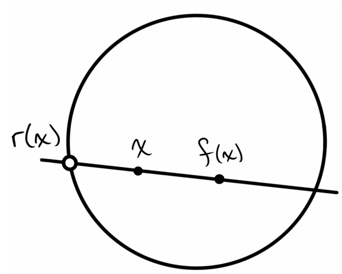


FIGURE 3. A visualization of the function $r(x)$ on D^2

Since D^n is centered at zero, this function can be formalized as $r(x) = x + tu$, where $u = \frac{x-f(x)}{||x-f(x)||}$ and $t = -x \cdot u + \sqrt{1 - x \cdot x + (x \cdot u)^2}$. Since f is smooth and $f(x) \neq x$ for any $x \in D^n$, r is smooth and defined for all x. Note that for $x \in \partial D^n$, $r(x) = x$. Thus we have a smooth map $r : D^n \to \partial D^n$ which is fixed along the boundary: a contradiction. $\square$

Theorem 2.11. ***(Brouwer fixed point)*** *There exists no continuous function $G : D^n \to D^n$ without a fixed point.*

Proof: By the Stone-Weirstrauss Theorem, any continuous map $f : D^n \to D^n$ can be approximated by a polynomial P such that $||f - P|| < \epsilon$ for any $x \in D^n$. Suppose $G : D^n \to D^n$ is continuous and without a fixed point. Then for any $\epsilon > 0$ there exists a polynomial $P_1(x)$ such that $||G(x) - P_1(x)|| < \epsilon$ for any $x \in D^n$. We can then form the function $P(x) = \frac{P_1(x)}{1+\epsilon}$ whose image is contained within D^n. Note $||G(x)|| \leq 1$ for any $x \in D^n$. Therefore

$$||G(x) - P(x)|| = ||G(x) - \frac{P_1(x)}{1+\epsilon}|| = \frac{1}{1+\epsilon}||\epsilon G(x) + G(x) - P_1(x)||$$
$$\leq \frac{1}{1+\epsilon}(\epsilon||G(x)|| + ||G(x) - P_1(x)||) < \frac{2\epsilon}{1+\epsilon} < 2\epsilon$$

Note $||G(x) - x||$ maps the compact set D^n into $\mathbb{R}$. Therefore the image is closed and bounded. A lower bound of 0 implies a fixed point. Therefore there must exist some $\mu > 0$ such that $||G(x) - x|| > \mu$. Set $\epsilon = \frac{\mu}{2}$. Then $||G(x) - P(x)|| < \mu$. If $P(x) = x$ at any point, then we obtain the contradiction $\mu < ||G(x) - x|| < \mu$.

Therefore $P(x)$ has no fixed points. This contradicts lemma 2.10. □

The algebraic argument involves examining how continuous maps between spaces induce maps between their fundamental groups. For those familiar with category theory, we are defining a functor between the category of topological spaces and the category of groups. As review, we define the following.

Definition 2.12. A **path** is a continuous map f from the interval $[0,1] = I$ to a space X. A **loop** is a path where $f(0) = f(1) = x_0 \in X$. Such an x_0 is called the loop's **basepoint**.

Definition 2.13. Let f and g be paths such that $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$. They are **homotopic** (denoted $f \simeq g$) if there exists a continuous function $F : I^2 \to X$ such that $F(x,0) = f(x)$, $F(x,1) = g(x)$, $F(0,t) = x_0$, and $F(1,t) = x_1$ for any $t, x \in [0,1]$. Note $\simeq$ is an equivalence relation.

Definition 2.14. Elements of the **fundamental group** $\pi_1(X, x_0)$ are equivalence classes ($\simeq$) of loops in X with basepoint x_0.

(i) For any $[f], [g] \in \pi_1(X, x_0)$ the group operation $[f] * [g] = [f * g]$ where

$$f * g = \begin{cases} f(2x) & x \in [0, \frac{1}{2}] \\ g(2x-1) & x \in (\frac{1}{2}, 1] \end{cases}$$

(ii) The identity element $[e]$ is the equivalence class of the constant loop.

(iii) The inverse of $[f]$ is the equivalence class with the reverse parameterization. If $f_1(x) \in [f]$, then $f_1(1-x) \in [f]^{-1}$.

Note $f * g$ defined above simply concatenates loops; we can use a similar action to examine path connected spaces. Consider such a space X and groups $\pi_1(X, x_0)$ and $\pi_1(X, y_0)$ formed with different basepoints. Let h be the path from y_0 to x_0 and h^{-1} the reverse parameterization. Any loop f_x based at x_0 corresponds to a loop $h * f_x * h^{-1} = f_y$ based at y_0. Under $\simeq$ this operation is homomorphic and has a well-defined inverse. Therefore $\pi_1(X, x_0) \cong \pi_1(X, y_0)$, allowing us to write $\pi_1(X)$ for a path connected space.

Proposition 2.15. *The fundamental group $\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$*

The proof can be found in Hatcher [1], page 29. Each element of $\pi_1(S^1)$ is determined by the net number of rotations. When loops with opposite parameterization are stitched together, they can be unwound into a net sum. Figure 4

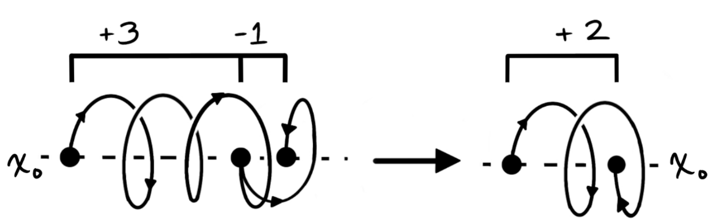


FIGURE 4. Group operations in $\pi_1(S^1)$ visualized in $S^1 \times I$

demonstrates this process. Negative integers describe clockwise rotations; positive integers counter-clockwise. This naturally leads to a fundamental group with the structure of $\mathbb{Z}$.

Proposition 2.16. *Let $\varphi : X \to Y$ be a continuous function where $\varphi(x_0) = y_0$. Then the map φ induces a homomorphism $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.*

Proof: Given loops $f_1, f_2 \in [f] \in \pi_1(X)$, one receives two new loops in Y through the function $\varphi \circ f_1$ and $\varphi \circ f_2$. If there exists a homotopy $F(x,t)$ between f_1 and f_2, then $\varphi \circ F$ is a homotopy connecting $\varphi \circ f_1$ and $\varphi \circ f_2$. This implies φ_* is well-defined. One can further show $\varphi(f * g) \simeq \varphi(f) * \varphi(g)$. Therefore the induced map $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a homomorphism. $\square$

Furthermore $f = \varphi_2 \circ \varphi_1$ implies $f_* = \varphi_{2*} \circ \varphi_{1*}$ and $f = id$ implies $f_* = id_*$.

Proposition 2.17. *The fundamental group of the unit disk $\pi_1(D^n)$ contains only the identity element.*

Proof: The unit disk is a convex space. Therefore, the equation $f(t) = x_1(1 - t) + x_2 t$ defines a path in D^n between any two points $x_1, x_2 \in D^n$. For any loop $f(x)$, one can then choose the basepoint $x_0 \in D^n$ and define the homotopy $F(x,t) = x_0(1-t) + f(x)t$. This map contracts f to the constant loop. Therefore $\pi_1(D^n) = [e]$. $\square$

This argument works for all convex spaces and naturally leads to the following proof of the Brouwer fixed point theorem.

Theorem 2.18. ***(Brouwer fixed point for two dimensions)*** *Any continuous mapping $f : D^2 \to D^2$ must have a fixed point.*

Proof: Given a continuous function $f : D^2 \to D^2$ without a fixed point, one can define a map $r : D^2 \to S^1$ fixed along the boundary S^1 as in Lemma 2.10. Note that this map is now continuous, not smooth. Let $i : S^1 \to D^2$ be the inclusion map of S^1. The composition $(r \circ i) : S^1 \to S^1$ is the identity map. Therefore $(r \circ i)_* : \pi_1(S^1) \to \pi_1(S^1)$ is the identity homomorphism. As stated, $(r \circ i)_* = r_* \circ i_*$ where $i_* : \pi_1(S^1) \to \pi_1(D^2)$ and $r_* : \pi_1(D^2) \to \pi(S^1)$. But $\pi_1(D^2) = [e]$, so the image of r_* is only $[e] \in \pi_1(S^1)$. Since $\pi_1(S^1) \cong \mathbb{Z}$, then $r_* \circ i_* \neq id_*$. This is a contradiction. $\square$

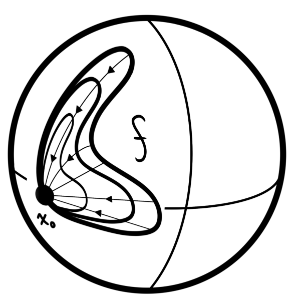


FIGURE 5. A homotopy bringing a trivial loop f on S^2 to the constant loop at x_0

Unlike the differential proof, this argument fails at higher dimensions. The structure of $\pi_1(S^n)$ becomes trivial for $n \geq 2$ (as visualized in figure 5). In order to generalize to any n, we need an algebraic tool with the same properties but which retains its structure for higher n-spheres. If one interprets the fundamental group as mappings of S^1 into a space, a higher dimensional analog using S^n (called π_n) seems natural. Unfortunately, these groups are unwieldy to calculate, limiting their wider applications. We will explore an alternative in the next section.

3. The Hairy Ball Theorem

Take a sphere and smoothly assign to each point $x \in S^2$ a vector on the tangent space TM_x. The hairy ball theorem states such a smooth vector field must include a vanishing point. Both a differential and algebraic argument extend this property to all S^n where n is even.

Definition 3.1. An **orientation** of a finite dimensional real vector space $\mathbb{R}^n$ is an equivalence class of ordered bases. Two bases $\beta_0 = \{\vec{v}_1, ..., \vec{v}_n\}$ and $\beta_1 = \{\vec{w}_1, ..., \vec{w}_n\}$ spanning $\mathbb{R}^n$ are the **same orientation** if the matrix $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $A(\vec{v}_i) = \vec{w}_i$ for any $i \in \{1, ..., n\}$ has a positive determinant. If $det(A) < 0$, we say β_0 and β_1 are **opposite orientations**.

There are two orientations for all ordered bases of $\mathbb{R}^n$. The orientation of the trivial basis $\{e_1, ..., e_n\}$ is called $+1$ while the opposite is -1.

Example 3.2. Let β_0 be an ordered basis of $\mathbb{R}^n$. Let β_1 be the basis formed by reflecting β_0 across a trivial basis vector. The matrix describing this transformation is the identity matrix with a diagonal entry replaced by -1. Therefore β_0 and β_1 are opposite orientations.

Example 3.3. Let β_0 and β_1 be ordered bases of $\mathbb{R}^m$. Let α_0 and α_1 be ordered bases of $\mathbb{R}^n$. Let 0_m and 0_n denote the zero vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$. One can create an ordered basis for the vector space $\mathbb{R}^m \times \mathbb{R}^n$ by considering $(\beta_0 \times 0_n, 0_m \times \alpha_0) = \{v_1 \times 0_n, ..., v_m \times 0_n, 0_m \times w_1, ..., 0_m \times w_n\}$ where $v_i \in \beta_0$ and $w_i \in \alpha_0$. Suppose $d\beta$ transforms β_0 into β_1 and $d\alpha$ transforms α_0 into α_1. Then the matrix

$$A = \begin{pmatrix} d\beta & 0 \\ 0 & d\alpha \end{pmatrix}$$

transforms $(\beta_0 \times 0_n, 0_m \times \alpha_0)$ into $(\beta_1 \times 0_n, 0_m \times \alpha_1)$ and $det(A) = det(d\beta)\ det(d\alpha)$.

Definition 3.4. An **oriented smooth manifold** M is an m-manifold with a choice of orientation for each tangent space TM_x such that

(i) For any $x \in M$, there exists a neighborhood U_x and a diffeomorphic map $h : U_x \to V$ where V is an open subset of $\mathbb{R}^m$ or $\mathbb{H}^m$.
(ii) For any $z \in U_x$ the map $dh_z : TM_z \to \mathbb{R}^m$ must transform the assigned basis of TM_z into the standard basis on $\mathbb{R}^m$.

The **orientation of a smooth manifold M** is a choice of orientation for each TM_x fulfilling the above requirement. Note dh_z is an isomorphism for any $z \in U_x$. The continuous change of dh guarantees a neighborhood U'_x where $\det(dh)$ does not change sign. Orientation is therefore locally constant on M. If M is connected, there are only two orientations of M: one where every TM_x is $+1$ and another where every TM_x is -1.

We will add an additional requirement on orientation for a manifold with boundary M. For $x \in \partial M$, there are three types of vectors in TM_x: those tangent to ∂M, those pointing into the manifold, and those pointing outward. Let $\{v_2,\ ...\ , v_m\}$ be an ordered basis on $T(\partial M)_x$ and let n_1 be the outward vector. We declare that the orientation of $\{n_1, v_2, \ ... \ , v_m\}$ is the same orientation as $\{v_2, \ ... \ , v_m\}$, also called the **boundary orientation**.

Notice a 1-manifold with boundary N only has inward and outward vectors on $x \in \partial N$. In this case, we simply say a boundary with an inward vector has an orientation of -1 and a boundary with an outward vector has an orientation of $+1$ (one can see this choice is consistent with example 3.3). As such, a smooth choice of orientation forces every oriented line segment to have both a positively oriented boundary and a negatively oriented boundary.

Definition 3.5. Let $f : M \to N$ be a smooth map. Let M and N be smooth orientable manifolds of dimension n such that M is compact and N is connected. Let $y \in N$ be a regular value of f. The **degree** of f at y is defined as

$$\deg(f, y) = \sum_{x \in f^{-1}(y)} sign \ (df_x)$$

Since M and N are of the same dimension, $f^{-1}(y)$ is a collection of points. For any $x_i \in f^{-1}(y)$, there exists a neighborhood U_{x_i} where f is invertible. Therefore all such x_i are locally isolated. Since M is compact, $f^{-1}(y)$ is finite. Therefore $\deg(f, y)$ is well-defined for any regular value $y \in N$.

For each $x_i \in f^{-1}(y)$ there is a neighborhood U'_{x_i} where orientation is constant. Let $A_i = U_{x_i} \cap U'_{x_i}$. Then $\bigcap_{i=1}^{k} f(A_i) - f(M - A_1 - ... - A_k)$ is an open neighborhood of $y \in N$ where degree is constant. Since regular values are dense and N is connected, $\deg(f, y)$ is constant for all regular values of N. We can replace $\deg(f, y)$ with $\deg(f)$.

Example 3.6. Let $r : S^n \to S^n$ be the antipodal map defined by $r(x) = -x$ for any $x \in S^n$. The map is bijective and the Jacobian of any x is

$$df_x = \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{bmatrix}$$

Therefore $\deg(r) = sign(df_x) = (-1)^{n+1}$ since df_x is an $n+1 \times n+1$ matrix.

Degree is also additive over compact sets. Let $f : M \to N$ be smooth. Let $M = M_1 \cup M_2$ where M_1 and M_2 are disconnected compact orientable manifolds. Let $A_1 = M_1 \cap f^{-1}(y)$. Let $A_2 = M_2 \cap f^{-1}(y)$. Then

$$\begin{aligned} \deg(f) = \sum_{x \in A_1 \cup A_2} sign \ (df_x) &= \sum_{x \in A_1} sign \ (df_x) + \sum_{x \in A_2} sign \ (df_x) \\ &= \deg(f|_{M_1}) + \deg(f|_{M_2}) \end{aligned}$$

Lemma 3.7. *If $f : M \to N$ extends to a smooth map $F : W \to N$ where W is a compact orientable manifold of dimension $k+1$ and $\partial W = M$, then* $\deg(f) = 0$.

Proof: Let f extend to $F : W \to N$ as above. Therefore $f = F|_{\partial W}$. Let y be a regular value of F and f. Consider the 1 manifold $F^{-1}(y)$. By Lemma 2.8, $\partial F^{-1}(y) = \partial W \cap F^{-1}(y) = M \cap F^{-1}(y)$. The manifold $F^{-1}(y)$ is a closed subset of a compact space W. Therefore $F^{-1}(y)$ must be compact: a finite union of arcs and circles which inherit the orientation of M. Let $A_i \subset F^{-1}(y)$ be one of these arcs. Then $\{a\} \cup \{b\} = \partial A_i$. Let $\{a\}$ have an inward vector and $\{b\}$ have an outward vector. Notice $a, b \in M = \partial W$. The orientations of TW_b and TW_a must agree since W is connected. The basis of TW_a is $\{-n_1, v_2, \ldots, v_m\}$ where $\{v_2, \ldots, v_m\}$ is the oriented basis of TM_a. Similarly, the basis of TW_a is $\{n_1, w_2, \ldots, w_m\}$ where $\{w_2, \ldots, w_m\}$ is the oriented basis of TM_b. Therefore TM_b and TM_a must be oppositely oriented. This implies $\deg(f|_{A_i}) = 0$. Since circles have no intersection on M, $\deg(f) = \sum \deg(f|_{A_i})$. Therefore $\deg(f) = 0$. □

Definition 3.8. Let $f, g : M \to N$ be smooth maps. Let $F : M \times I \to N$ be a homotopy such that F is smooth, $F(x, 0) = f(x)$, and $F(x, 1) = g(x)$. Then F is called a **smooth homotopy** and the two mappings f, g are considered smoothly homotopic (written $f \sim g$).

Lemma 3.9. *Let $f, g : M \to N$ be smooth maps between manifolds of the same dimension. If $f \sim g$, then $deg(f) = deg(g)$.*

Proof: Let $F : M \times I \to N$ be the smooth homotopy between f and g. Note that the domain $M \times I$ is a smooth orientable manifold with boundary (simply attach to every $TM_x \times \{t\}$ the trivial vector along I). By example 3.3, $M \times I$ inherits the orientation of M. Note $\partial(M \times I)$ is the union of disconnected components $M \times \{0\} = A$ and $M \times \{1\} = B$ where TA_α has an inward vector for $\alpha \in A$ and TB_β has an outward vector for $\beta \in B$. As in the proof of Lemma 3.7, the orientations of TM_α and TM_β must be opposite. Define $H : A \cup B \to N$ such that $H|_A = f$ and $H|_B = g$. Then F is a smooth extension which satisfies the properties in Lemma 3.7. Therefore $\deg(H) = 0$. Since the TA_α has the inward vector, orientations in the preimage reverse. Therefore $\deg(H) = \deg(f) - \deg(g)$. This implies $\deg(f) = \deg(g)$. □

Definition 3.10. Let M be a smooth manifold. The **tangent bundle** TM is the union of all $\{x\} \times TM_x$ for $x \in M$. A **smooth vector field** v on M is a smooth map $v : M \to TM$ where $v(x)$ is a vector in TM_x.

Theorem 3.11. ***(hairy ball)*** *There can exist no non-vanishing smooth vector field v on an even n-sphere S^n.*

Proof: For any $x \in S^n$ embedded in $\mathbb{R}^{n+1}$, the tangent space $TS^n_x = \{\vec{v} \in \mathbb{R}^{n+1} | \ \vec{x} \cdot \vec{v} = 0\}$. Suppose a smooth non-vanishing vector field v exists on S^n. Define a new smooth vector field $w(x) = \frac{v(x)}{||v(x)||}$. Since $v(x) \neq 0$ for any $x \in S^n$, $w(x)$ is well defined. For $x \in S^n$ and $t \in [0, \pi]$, we can define the smooth homotopy $F(x, t) = \vec{x}\cos(t) + w(x)\sin(t)$. Note that

$$||F(x,t)||^2 = (\vec{x} \cdot \vec{x})\cos^2(t) + 2(\vec{x} \cdot w(x))\sin(t)\cos(t) + (w(x) \cdot w(x))\sin^2(t) = 1$$

since both $\vec{x}$ and $w(x)$ are unit vectors. Therefore $F(x, t) \in S^n$ for any $x \in S^n$ and $t \in [0, \pi]$. But $F(x, 0) = \vec{x}$ and $F(x, \pi) = -\vec{x}$. Therefore $F(x, t)$ is a smooth homotopy between the identity map and the antipodal mapping $r(x) = -x$. Notice $\deg(id) = 1$. By lemma 3.9, $\deg(r) = 1$. By example 3.6, this is only possible when n is odd. □

We will now present an algebraic approach which involves introducing a more limited definition of degree. Its construction will be rooted in homology, a subject too rich to capture within this paper. For a rigorous introduction see chapter 2 of Hatcher [1]; what follows is a quick outline for review.

An **n-cell**, written e^n, is a space homeomorphic to the n-dimensional unit disk D^n. Let X_0 be a collection of 0-cells. Map a collection of 1-cells onto elements of X_0 such that the 0-cells form the boundaries of the 1-cells. This collection is called X_1. In general, X_n is created by mapping n-cells onto X_{n-1} such that the previous elements form the boundaries of the n-cells. A **cell complex** of X is the collection of sets X_n and the connective maps defining their construction. Let $C_n(X)$ be the abelian group generated by the n-cells in X_n. A **chain complex** is the structure

$$0 \xrightarrow{\partial_{m+1}} C_m(X) \xrightarrow{\partial_m} ... \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

where ∂_n are homomorphic maps from $C_n(X)$ to $C_{n-1}(X)$ formed by mapping n-cells to linear combinations of their boundaries. These boundary maps are such that $\partial_n \circ \partial_{n+1} = 0$ for any n. In other words, $Im(\partial_{n+1}) \subset ker(\partial_n)$. We define the **homology group** $H_i(X) = Ker(\partial_i)/Im(\partial_{i+1})$.

A proper introduction to the subject replaces n-cells e^n with rigid manifolds called n-simplexes Δ_n (discussed in the next section). The triangular structure of such objects leads to a natural definition of ∂_n such that $\partial_{n+1} \circ \partial_n = 0$. Homology groups follow as before. One can experiment a bit with the procedure, generating the abelian groups $C_n(X)$ with the uncountable set of continuous functions $f : \Delta_n \to X$. One can use ∂_n on the simplex to define a new boundary map on these functions; despite being a radically different approach, both methods produce equivalent homology groups. The former, called **simplicial homology**, allows for quick calculations, while the latter, called **singular homology**, allows for natural generalizations to any topological space. The procedure with n-cells outlines **cellular homology**, another equivalent method, which we will use below.

Example 3.12. The unit sphere S^n can be constructed with a n-cell and a 0-cell. Therefore $C_n(S^n) \cong C_0(S^n) \cong \mathbb{Z}$ and $C_k(S^n) \cong 0$ for any other k. Notice

$$0 \xrightarrow{\delta_{n+1}} \mathbb{Z} \xrightarrow{\delta_n} 0 \xrightarrow{\delta_{n-1}} ... \xrightarrow{\delta_2} 0 \xrightarrow{\delta_1} \mathbb{Z} \xrightarrow{\delta_0} 0$$

Since ∂_i are homomorphisms, the homology group $H_k(S^n) \cong \mathbb{Z}$ if $k = n$ or 0. The group is trivial for any other value of k.

Definition 3.13. Given a continuous function $f : S^n \to S^n$, we have an induced homomorphism $f_* : H_n(S^n) \to H_n(S^n)$. Since $f_* : \mathbb{Z} \to \mathbb{Z}$, $f_*(x) = \alpha x$ where $\alpha \in \mathbb{Z}$,. The **homology degree** of f, denoted $\deg_H(f)$, is the integer α.

To learn the specifics of such induced mappings see Hatcher [1], page 111. One can understand f_* in this context as a mapping of "dimensional holes". To be well-defined, a continuous function $f : S^n \to S^n$ must wrap the n-sphere around itself. The degree α tracks the number of "rotations" around the image per "rotation" in the domain. A negative α accounts for a flip in orientation.

Proposition 3.14. *If f and g are continuous maps $f, g : S^n \to S^n$, then* $\deg_H(f \circ g) = \deg_H(f)\deg_H(g)$.

Proof: Note $(f \circ g)_* = f_* \circ g_*$. Suppose $g_*(x) = \alpha x$ and $f_*(x) = \beta x$, then $f_* \circ g_*(x) = f_*(\alpha x) = \alpha\beta x$. Therefore $\deg(f \circ g) = \deg(g)\ \deg(f)$. □

One can further show homotopic maps induce the same function on a space's homology groups (Hatcher [1], page 112). Therefore if $f, g : S^n \to S^n$ are homotopic, then $\deg_H(f) = \deg_H(g)$.

Proposition 3.15. *Let $r : S^n \to S^n$ be the antipodal mapping of the n sphere, then $\deg_H(r) = (-1)^{n+1}$.*

Proof: Note r can be written as $r = \varphi_{n+1} \circ ... \circ \varphi_1$ where $\varphi_i(x_1, ..., x_i, ..., x_{n+1}) = (x_1, ..., -x_i, ..., x_{n+1})$. As previously discussed, $\deg_H(\varphi_i) = -1$. By proposition 3.14, the homology degree of r is $(-1)^{n+1}$. □

Note that the new definition of degree has the same properties as the differential version (though homotopy invariance is a bit stronger). As such, we can borrow the argument from the differential proof.

Theorem 3.16. ***(hairy ball)*** *There can exist no non-vanishing smooth vector field v on an even n-sphere S^n.*

Proof: As before, assume a smooth non-vanishing vector field v on S^n exists. Create the new vector field $w(x) = \frac{v(x)}{||v(x)||}$. Once again, $F(x,t) = \vec{x}\cos(t) + w(x)\sin(t)$ proves $r \simeq id$. Therefore $\deg_H(r) = (-1)^{n+1} = \deg_H(id) = 1$. This is impossible when n is even. □

4. The Lefschetz Fixed Point Theorem

The Lefschetz fixed point theorem allows us to deduce the behavior of f from f_*. If $f_* : H_n(X) \to H_n(X)$ has a specific property, quantified by the Lefschetz number, the function $f : X \to X$ must have a fixed point. This fact shows if M admits a smooth non-vanishing vector field, then its Euler Characteristic is zero.

To begin, consider a CW complex built from rigid triangular n-cells described by $\{\alpha_1\vec{v}_1 + ... + \alpha_n\vec{v}_n | \sum \alpha_i \leq 1\}$, where vectors $\vec{v}_i$ form a basis in $\mathbb{R}^n$. A 1-cell becomes a straight line, a 2-cell a triangle, a 3-cell a tetrahedron, and so on. Such an n-cell is called a **simplex**, and such a cellular complex a **simplicial complex**. A map $f : K \to L$ between simplicial complexes K and L is considered **simplicial** if each simplex in K is mapped to a simplex in L by a linear function taking vector to vector. Futhermore, one can define a standard way of dividing a simplicial complex to create a finer structure, called a **barycentric subdivision**, where each n-simplex divides into finitely many n-simplices. Iterative barycentric subdivisions allow one to pick an $\epsilon > 0$ and divide a simplicial complex K into a finite collection of simplexes $\{\sigma_i\}^k$ such that $\sup\{\delta(x,y)|\ x, y \in \sigma_i\} < \epsilon$ for any $i \in \{1, ..., k\}$. The quantity $\sup\{\delta(x,y)|\ x, y \in \sigma_i\}$ is known as the diameter of σ_i.

Lemma 4.1. ***(simplicial approximation)*** *If K is a finite simplicial complex and L is an arbitrary simplicial complex, then any map $f : K \to L$ is homotopic to a map $g : K \to L$ that is simplicial with respect to some iterated barycentric subdivision of K.*

The proof relies on the diameter restriction for each σ_i. Picking a small enough ϵ, one can force x to be mapped into the same simplex by both $f(x)$ and a new simplicial function $g(x)$. The full argument can be found in Hatcher [1], page 178.

Definition 4.2. Let homology groups have coefficients in $\mathbb{Z}$. The **Lefschetz number** $\tau(f)$ of a map $f : X \to X$ is defined as

$$\tau(f) = \sum_{n=0} (-1)^n tr(f_* : H_n(X)/Tor(H_n(X)) \to H_n(X)/Tor(H_n(X))$$

The set $Tor(X)$ refers to the cyclic elements of a group X. The Lefschetz number is only concerned with the free part of the homology groups.

Theorem 4.3. ***(Lefschetz fixed point)*** *Let X be a finite simplicial complex and let $f : X \to X$ be such that $\tau(f) \neq 0$. Then f has a fixed point.*

Proof: Suppose $f : X \to X$ has no fixed points. Since X is compact, there must exist an $\epsilon > 0$ such that $\delta(x, f(x)) > \epsilon$. Let L be the simplicial subdivision of X such that each simplex $\sigma \in L$ has a diameter less than $\frac{\epsilon}{2}$. Consider $f : L \to L$. According to lemma 4.1, there exists a subdivision of L named K such that $f : L \to L$ is homotopic to the simplicial map $g : K \to L$. For any $x \in X$, $f(x)$ and $g(x)$ are contained within the same simplex of L. Therefore $\delta(f(x), g(x)) < \frac{\epsilon}{2}$. Since $\delta(x, f(x)) > \epsilon$, we know $\delta(g(x), x) > \frac{\epsilon}{2}$ by the triangle inequality. Therefore any $x \in K$ must be mapped to a different simplex in L by g. So $g(\sigma) \cap \sigma = \emptyset$ for any simplex $\sigma \subset K$. Consider $g_* : H_n(K)/Tor(H_n(K)) \to H_n(K)/Tor(H_n(K))$. Each basis element $\sigma_i \in H_n(K)$ will be sent to $\sigma_j \in H_n(K)$ where $j \neq i$. Therefore, the value of any diagonal entry along the matrix representing g_* must be 0. Therefore, $\tau(g) = 0$. Since $g \simeq f$, we know $f_* = g_*$. Therefore $\tau(g) = \tau(f) = 0$. □

Classical results in topology have shown that a compact smooth manifold M is homeomorphic to some finite simplicial complex X (see Whitehead [5]). One can use this fact to extend the Lefschetz fixed point theorem to any compact smooth manifold (see Bredon [2] page 256). We will use this extension to examine the implications for compact boundaryless manifolds, also called **closed** manifolds.

Definition 4.4. Let M be a smooth manifold. Let v be a smooth vector field on M. The **flow** generated by v on the manifold is the map $F : M \times I \to M$ where $F(x, 0) = x$ for any $x \in M$, $F(t, F(s, x)) = F(t + s, x)$ for any $s, t \in I$, and $\frac{\partial F}{\partial t}|_x = v(x)$ for any $x \in M$.

One can picture flow as smooth lines embedded on a manifold tracking how position changes with time t. Any smooth vector field generates such a function.

Lemma 4.5. *Let M be a closed and smooth manifold. Let v be a non-vanishing smooth vector field on M. Let F be the flow generated by v. Then there exists T such that $F(x, t) \neq x$ for any $t \in (0, T]$ and $x \in M$.*

Proof: Choose a neighborhood U_x whose diffeomorphic mapping to $V \subset \mathbb{R}^m$ turns v into a locally constant vector field. Such a value of T_x must exist for U_x. One can create a covering $\{U_{x_i}\}$ of the manifold M. Since M is compact, there is a finite covering. The minimum T_{x_i} satisfies the property above. □

Definition 4.6. The **Euler characteristic** of a space X, written $\chi(M)$, is the alternating sum of the rank of X's homology groups:

$$\chi(M) = \sum (-1)^i \operatorname{rank}(H_i(X))$$

Corollary 4.7. *Let M be a smooth and closed manifold. If M admits a smooth non-vanishing vector field v, then the Euler Characteristic $\chi(M) = 0$.*

Proof: Let v be a smooth non-vanishing vector field on M. Let F be the flow generated by v. By Lemma 4.5, there exists T such that $F(x, T) \neq x$. Let $f(x) = F(x, T)$. Therefore f is a function without fixed points. By the extension of Theorem 4.3, $\tau(f) = 0$. Since $F(x, 0) = id$, the flow gives a smooth homotopy between f and id. Therefore $\tau(f) = \tau(id)$, which implies

$$\tau(f) = \tau(id) = \sum(-1)^n tr(id_* : H_n(M)/Tor(H_n(M)) \to H_n(M)/Tor(H_n(M)))$$
$$= \sum(-1)^n rank(H_n(M)) = \chi(M)$$

since rank does not consider the torsion group. Therefore $\chi(M) = 0$. □

Note that this corollary gives us a much more general version of the hairy ball theorem. Using example 3.12, the Euler characteristic $\chi(S^n) = 1 + (-1)^n$. Therefore, there may exist no non-vanishing smooth vector field on S^n when n is even. The corollary also serves to define a vacuous case of the following theorem.

5. The Poincaré-Hopf Theorem

The Poincaré-Hopf theorem describes an explicit relation between the Euler Characteristic of a closed smooth manifold M and its smooth vector fields. This section presents a proof for closed smooth orientable manifolds by establishing the index sum as an invariant on such objects, then relating it to the Euler characteristic using basic Morse theory. To that end, we define the following.

Definition 5.1. Let M be a smooth manifold of dimension m. Let v be a smooth vector field on M. A **zero point** of v is a value $z \in M$ where $v(z) = 0$. Let the zero points of v be isolated from each other. Take a local neighborhood U_z such that $\bar{U}_z$ is homeomorphic to D^m and consider the function $\bar{v} : \partial\bar{U}_z \to S^{m-1}$ such that $\bar{v}(x) = \frac{v(x)}{||v(x)||}$. The **index** of v at z is defined to be $i = \deg(\bar{v})$.

Note we use the differential degree, not the homology equivalent.

Definition 5.2. A zero point z of a vector field v is called **nondegenerate** if the Jacobian of the smooth vector field dv_z is invertible.

By the inverse function theorem, nondegenerate zeros are isolated from each other. Furthermore, one can find that $\det(dv_z) > 0$ implies the index is 1 and $\det(dv_z) < 0$ implies the index is -1 (see Milnor [3] page 37).

Definition 5.3. Let X be a compact smooth manifold of dimension m with boundary ∂X. The **Gauss map** $G : \partial X \to S^{m-1}$ maps each x to its outward unit vector.

Lemma 5.4. *Let X be a compact smooth orientable manifold with boundary ∂X. Let v be a smooth vector field on X with only nondegenerate zeros such that $x \in \partial X$ implies $v(x)$ points outwards. Then the sum of all indexes of v, written $\Sigma(i)$, is equal to the degree of the Gauss Mapping $G : \partial X \to S^{n-1}$.*

Proof: Let $\{z_l\}$ be the set of all zero points. Since X is compact and each z_l is isolated, this set is a finite collection $\{z_l\}^k$. Each z_l has a local open ball B_l centered on itself. Consider the new manifold $X' = X - \{B_l\}^k$. Notice $\partial X' = \partial X \cup \{\partial D_l\}^k$ where D_l is the closure of B_l. Consider $\bar{v} : \partial X' \to S^{m-1}$ where $\bar{v}(x) = \frac{v(x)}{||v(x)||}$ is smooth and well defined. The map can be smoothly extended to all X' since zero points of v were removed. By Lemma 3.7, $\deg(\bar{v}) = 0$. The components ∂X and ∂D_l are disjoint and compact. Therefore $\deg(\bar{v}) = \deg(\bar{v}|_{\partial X}) + \Sigma_l^k \deg(\bar{v}|_{\partial D_l \subset X'})$. Note $G \simeq \bar{v}|_{\partial X}$ since $\bar{v}$ points outwards on ∂X. Therefore $\deg(G) = \deg(\bar{v}|_{\partial X})$. Examine the following image. Removing an open ball from an embedded oriented

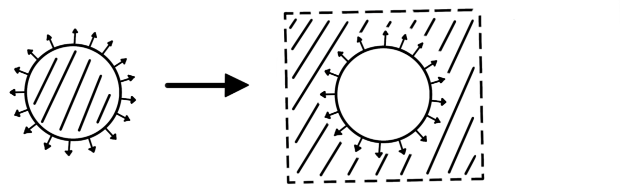


FIGURE 6. Flip in the direction of the inward and outward vector.

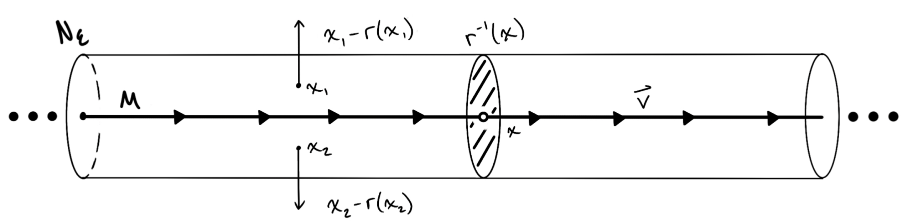
manifold flips the direction of the inward and outward vectors at the boundary. As such, $\deg(\bar{v}|_{\partial D_l \subset X}) = -\deg(\bar{v}|_{\partial D_l \subset X'})$. The index of z_l is $\deg(\bar{v}|_{\partial D_l \subset X})$. Therefore $\deg(\bar{v}) = \deg(G) - \Sigma(i) = 0$. $\square$

This lemma defines a specific invariance which can be used to find a more general case. One can use M embedded in $\mathbb{R}^k$ to create a k dimensional manifold. This space, named N_ϵ, is the set of all $x \in \mathbb{R}^k$ where there exists $y \in M$ such that $\delta(x, y) \leq \epsilon$. One can think of N_ϵ as an expansion of M in $\mathbb{R}^k$.

Lemma 5.5. *: Let M be a closed smooth orientable manifold of dimension m embedded in $\mathbb{R}^k$. Let v be a smooth nondegenerate vector field on M. Then the index sum $\Sigma(i)$ is equal to the degree of the Gauss Mapping $G : \partial N_\epsilon \to S^m$.*

Proof: Let $r : N_\epsilon \to M$ be the function $r(x) = \{y \in M |\ \delta(x, y) = \inf(\delta(x, y))\}$. For a small enough ϵ, the map r is well defined and smooth. Note r brings each x to its closest point on M. The vector $x - r(x)$ is then perpendicular to the tangent space TM_x for any $x \in N_\epsilon$ and smoothly defined. Consider the function $\varphi = ||x - r(x)||^2$ and its gradient $\nabla\varphi = 2(x - r(x))$. Note that the boundary $\partial N_\epsilon = \varphi^{-1}(\epsilon^2)$. We can form the outward unit vector along ∂N_ϵ using $\nabla\varphi$ such that $G(x) = \frac{\nabla\varphi}{||\nabla\varphi||} = \frac{x - r(x)}{\epsilon}$. Now use the vector field v on M to define a new smooth vector field $w : N_\epsilon \to TN_\epsilon$ where $w(x) = (v \circ r)(x) + x - r(x)$. Note $(v \circ r)(x) \in TM_x$, so $(v \circ r)(x) \perp (x - r(x))$. The TM_x component of w vanishes at the zeros of v and the remaining component vanishes when $r(x) = x$. Therefore the zeros of v are also the zeros of w. Note $r^{-1}(x)$ is a closed ball of dimension $k - m$ and radius ϵ. The vectors $(x - r(x))$ lie on these balls. Since $r(x)$ is constant on this space, $x - r(x)$ only changes with x. Therefore $d(x - r(x))$ is $id : \mathbb{R}^{k-m} \to \mathbb{R}^{k-m}$ on the ball. Given a vanishing point z,

$$dw_z = \begin{pmatrix} dv_z & 0 \\ 0 & id \end{pmatrix}$$



Since dv_z is invertible, the matrix dw_z is invertible. Therefore all zero points of w are nondegenerate. Since $\det(dw_z) = \det(dv_z)$, the index sum of v equals the index sum of w. Note $w \cdot G = \frac{1}{\epsilon}||(x - r(x))||^2 = \epsilon$. Therefore the vector field w is outward for $x \in \partial N_\epsilon$. By Lemma 5.4, the index sum of w equals $\deg(G)$. Therefore the index sum of v equals $\deg(G)$. □

The index sum of a nondegenerate vector field v on a closed smooth manifold M is thereby constant. Before generalizing this invariance to vector fields with degenerate zeros, we will address the relation between $\chi(M)$ and the index sum.

Examine the torus pictured below with the height function $f : \mathbb{T}^2 \to \mathbb{R}$. One can see 4 critical points: a_1, a_2, a_3, & a_4. Define the set $M_z = \{x | f(x) \leq z\}$. Notice M_z changes structure when $z = f(a_i)$. For $z \in (a_1, a_2)$, M_z is homeomorphic to D^2. Yet for $z \in (a_2, a_3)$, M_z is homeomorphic to $S^1 \times I$. Morse theory gives us a way of formalizing these changes with n-cells.

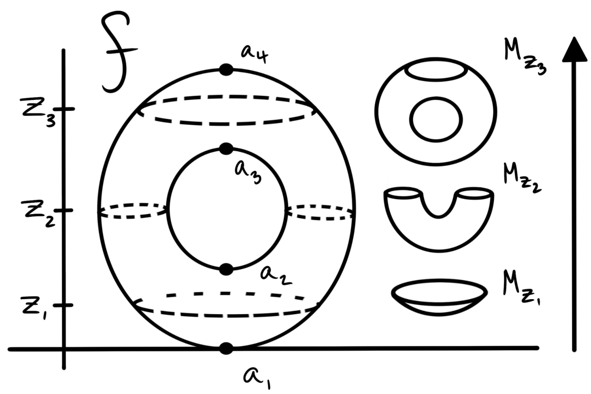


FIGURE 7. Motivation for Morse Theory

Definition 5.6. Let $f : M \to \mathbb{R}$ be smooth. Let M be a manifold of dimension m. The **Hessian** H_f of f is defined to be the following matrix:

$$H_f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

Definition 5.7. A critical point p of f is called **Morse nondegenerate** if the Hessian at p is invertible.

Definition 5.8. The **index** I_H of the Hessian H_f at a point is the number of negative eigenvalues.

Lemma 5.9. ***(Morse's)*** *Let p be a Morse nondegenerate critical point of f on M. Then there exists a local coordinate system $(y_1(x), ..., y_n(x))$ of a neighborhood $U_p \subset M$ such that $y_i(p) = 0$ and*

$$f = f(p) - y_1^2 - y_2^2 - ... - y_\lambda^2 + y_{\lambda+1}^2 + ... + y_n^2$$

holds for all $x \in U_p$.

We now have a quick tool for calculating I_H at critical points. The index I_H is invariant with changes in coordinate systems. The Hessian with respect to this coordinate system is a diagonal matrix with λ entries equal to -2 and $n-\lambda$ entries equal to 2. Therefore $I_H = \lambda$.

Notice negative values on the diagonal of H_f indicate which directions on the manifold M form a "peak" of f. They determine what connects the critical point p to the set $M_{(f(p)-\epsilon)}$ for some small $\epsilon > 0$. The more negative eigen values, the higher the dimension of this "downward path". We can formalize this with the following.

Lemma 5.10. *Let $f : M \to \mathbb{R}$ be smooth. Let $p \in M$ be a non-degenerate critical point with index λ. Let $f(p) = c$. Suppose $f^{-1}[c-\epsilon, c+\epsilon]$ is compact and contains no other critical points. Then the set $M_{c+\epsilon}$ deformation retracts to $M_{c-\epsilon} \cup e^\lambda$ where e^λ is a cell complex of dimension λ attached to $M_{c-\epsilon}$.*

Lemma 5.11. *Let $f : M \to \mathbb{R}$ be smooth. Let $a, b \in \mathbb{R}$ such that $f^{-1}[a, b]$ is compact and contains no critical points of f. Then M_b deformation retracts to M_a.*

A **deformation retract** is a retraction map defined by a homotopy $F : X \times I \to Y$ where $Y \subset X$ and $F(x, t) = x$ for any $x \in Y$ and $t \in I$. If X deformation retracts to a subspace Y, then $H_n(Y) \cong H_n(X)$. As such, lemma 5.10 and 5.11 can be used to calculate homology groups. For the remainder of the paper, assume coefficients in $\mathbb{Z}$.

Definition 5.12. Let X and $A \subset X$ be topological spaces. Let $C_n(X, A)$ be the quotient group $C_n(X)/C_n(A)$. The homomorphisms $\partial_i : C_i(X) \to C_{i-1}(X)$ induce maps φ_i on these quotient groups, forming the chain complex

$$0 \xrightarrow{\varphi_{n+1}} C_n(X,A) \xrightarrow{\varphi_n} ... \xrightarrow{\varphi_2} C_1(X,A) \xrightarrow{\varphi_1} C_0(X,A) \xrightarrow{\varphi_0} 0$$

A **relative homology group** $H_i(X, A) = ker(\varphi_i)/Im(\varphi_{i+1})$.

Relative homology can be thought of as homology "mod" some space. We will take advantage of this framework to give a generalization of the Euler characteristic.

Definition 5.13. Given two spaces $Y \subset X$, the **Euler characteristic** $\chi(X, Y) = \sum(-1)^i$ rank $H_i(X, Y)$.

Notice that when $Y = \emptyset$, we recover our original definition $\chi(X, \emptyset) = \chi(X)$. The switch to relative homology also allows us to make use of the following properties.

Definition 5.14. A **good pair** is a pair of topological spaces (X, A) where $A \subset X$ and A is a deformation retract of a neighborhood of X.

Proposition 5.15. *If (X, A) is a good pair, then $H_n(X, A) \cong H_n(X/A)$ for $n \geq 1$ and $H_0(X, A) \times \mathbb{Z} = H_0(X/A)$.*

Proposition 5.16. *Given $Z \subset Y \subset X$ then $\chi(X, Z) = \chi(X, Y) + \chi(Y, Z)$.*

Proof: We can construct the following long exact sequence of relative homology groups

$$0 \xrightarrow{t_z} H_v(Y,Z) \xrightarrow{t_z - 1} ...$$

$$... \longrightarrow H_i(Y,Z) \xrightarrow{t_{i+1}} H_i(X,Z) \xrightarrow{t_i} H_i(X,Y) \xrightarrow{t_{i-1}} H_{i-1}(Y,Z) \longrightarrow ...$$

$$... \xrightarrow{t_1} H_0(X,Y) \xrightarrow{t_0} 0$$

Notice $ker(t_0) = H_0(X,Y)$, so $Im(t_1) \cong H_0(X,Y)$. Therefore, $\text{rank}(H_0(X,Y)) = \text{rank}(Im(t_1)) = \text{rank}(H_0(X,Z)) - \text{rank}(Im(t_2)) = \text{rank}(H_0(X,Z)) - \text{rank}(H_0(Y,Z)) + \text{rank}(Im(t_3))$. The iterative process terminates at t_z with the equation

$$\sum(-1)^n(\text{rank}(H_n(X,Y)) + \text{rank}(H_n(Y,Z))) = \sum(-1)^n \text{rank}(H_n(X,Z))$$

Therefore $\chi(X,Z) = \chi(X,Y) + \chi(Y,Z)$. □

Proposition 5.17. ***(excision)*** *Let $Z \subset A \subset X$ be topological spaces. Let the closure of Z be contained in A. Then $H_n(X,A) \cong H_n(X - Z, A - Z)$ for any n.*

Lemma 5.18. *Let M be a compact manifold of dimension m. Let $f : M \to \mathbb{R}$ be a smooth function with isolated and non-degenerate critical points. Then*

$$\chi(M) = \chi(M, \emptyset) = \sum_{\lambda=0}^{m}(-1)^\lambda C_\lambda$$

where C_λ is the number of critical points with the index λ.

Proof: Let $M_n = M$ and $M_0 = \emptyset$. Choose a smooth function $f : M \to \mathbb{R}$ as above such that the relation $f(a_1) < f(a_2) < ... < f(a_n)$ holds for the set of all critical points $\{a_1, ..., a_n\}$. Consider subsets $M_0 \subset M_1 \subset ... \subset M_n$ where $M_i = M_{c_i}$ and $c_i \in (f(a_i), f(a_{i+1}))$. By proposition 5.16

$$\chi(M, \emptyset) = \sum_{i=0}^{n-1} \chi(M_{i+1}, M_i)$$

$$\chi(M_{i+1}, M_i) = \sum_{k=0}(-1)^k \text{ rank } H_k(M_{i+1}, M_i)$$

By Lemma 5.10 and 5.11, M_{i+1} deformation retracts to $M_i \cup e^\lambda$ where λ is the degree of the critical point a_{i+1}. One can show $H_k(M_{i+1}, M_i) = H_k(M_i \cup e^\lambda, M_i)$. By excision, $H_k(M_i \cup e^\lambda, M_i) = H_k(e^\lambda, \partial e^\lambda)$. Note $(e^\lambda, \partial e^\lambda)$ is a good pair and the space $e^\lambda/\partial e^\lambda$ is the λ-dimensional unit sphere. By proposition 5.15, $H_k(e^\lambda, \partial e^\lambda) \cong \mathbb{Z}$ for $k = \lambda$ and $H_k(e^\lambda, \partial e^\lambda) \cong 0$ for $k \neq \lambda$. Therefore, $\chi(M_{i+1}, M_i) = (-1)^{\lambda_{i+1}}$ where λ_{i+1} is the degree of a_{i+1}. Furthermore

$$\chi(M) = \chi(M, \emptyset) = \sum \chi(M_{i+1}, M_i) = \sum_{\lambda=0}^{m}(-1)^\lambda C_\lambda \quad \square$$

Such an $f : M \to \mathbb{R}$ can be chosen since the set of smooth functions with non-degenerate zeros is dense in $C^\infty(M, \mathbb{R})$ when M is compact (see Audin and Damian [8], page 12). With this, we return to our examination of the index sum.

Proposition 5.19. *Let M be a compact smooth manifold of dimension M. Let v be a smooth vector field with isolated degenerate zeros on M. Then there exists a vector field v' with the same index sum containing only nondegenerate zeros.*

Proof: Let z be an isolated degenerate zero of v. Let U_z be a local neighborhood homeomorphic to D^m. Let U'_z be a similar neighborhood where $\bar{U}'_z \subset U_z$. Let $\lambda : U_z \to [0,1]$ be the smooth bump function where $\lambda(x)|_{U'_z} = 1$ and $\lambda(x)|_{\partial U_z} = 0$. Define the smooth vector field $v'(x) = v(x) - y\lambda(x)$ where y is a regular value of $v(x)$. The zeros z'_i of v' are points where $v(z'_i) = y$ in $\bar{U}'_z$. Since y is a regular value, such zeros are nondegenerate. One can show the index sum of v' on U_z equals the index of v using an argument similar to the proof of lemma 5.4. One can replace all degenerate zeros in the same manner, preserving the index sum on M. □

Theorem 5.20. ***(Poincaré-Hopf)*** *Let M be a closed smooth orientable manifold of dimension m. Let v be a smooth vector field on M containing only isolated zeros. Then the index sum $\sum(i)$ of v equals the Euler Characteristic $\chi(M)$.*

Proof: By Lemma 5.5, the index sum $\sum(i)$ is the degree of the Gaussian mapping of N_ϵ for some small $\epsilon > 0$. Let $f : M \to R$ be a smooth function with exclusively isolated and morse non-degenerate critical points. Define a vector field w such that $w(x) = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_m})$. The zeros of w are the critical points of f. Furthermore

$$dw_x = \begin{pmatrix} \frac{\partial^2 f}{\partial^2 x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_m} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial^2 x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_m \partial x_1} & \frac{\partial^2 f}{\partial x_m \partial x_2} & \cdots & \frac{\partial^2 f}{\partial^2 x_m} \end{pmatrix} = H_{f_x} = A^T H_{f_y} A$$

where $A^T H_{f_y} A$ is the local change in coordinates guaranteed by Morse's Lemma. Since all critical points are Morse non-degenerate, the zeros of the vector field are nondegenerate. Note $\det(dw_x) = \det(A)^2 \det(H_{f_y})$. Therefore $\text{sign}(\det(dw_z)) = \text{sign}(\det(H_{f_y})) = (-1)^\lambda$ where λ is the degree of the critical point. Since the sign of the determinant determines the index, we can find $\sum(i) = \sum(-1)^\lambda C_\lambda$. Therefore $\chi(M) = \sum(i)$ by lemma 5.18. □

Besides showing how the Euler characteristic of a space influences its smooth vector fields, this theorem defines a new method for calculating the Euler characteristic which avoids homology. Take S^2 and define a smooth vector field pointing from the north to the south pole at every $x \in S^2$. Let the poles be the only zeros. Both have an index of 1, so by Poincaré-Hopf $\chi(S^2) = 2$. We can check this is correct by computing the Euler characteristic directly with example 3.12. The extension to non-orientable manifolds generalizes this method to all closed smooth manifolds.

6. Acknowledgements

I would like to thank Zhong Zhang for her time and mentorship during the University of Chicago REU. This paper would be incoherent without her guidance and edits. I would also like to thank Tobias Shin for suggesting this topic and discussing it with me. A final thanks to Professor Peter May for organizing the program.

References

[1] Allen Hatcher. Algebraic Topology. Cambridge University Press. 2001
[2] Glen E. Bredon. Topology and Geometry. Springer-Verlag. 1993
[3] John W. Milnor. Topology From a Differentiable Viewpoint. Princeton University Press. 1965.
[4] John W. Milnor. Morse Theory. Princeton University Press. 1963.
[5] J. H. C. Whitehead. On C^1-complexes. Annals of Mathematics. 1940
[6] John M. Lee. Introduction to Smooth Manifolds. Springer. 2012
[7] J. P. May. A Concise Coures in Algebraic Topology.
[8] Michéle Audin and Mihai Damian. Morse Theory and Floer Homology. Springer. 2013.
[9] Victor Guillemin and Alan Pollack. Differential Topology. AMS Chelsea Publishing. 1974