INTRODUCTION TO FRACTAL DIMENSIONS USING ALGORITHMIC INFORMATION

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Abstract. This expository paper describes Hausdorff and packing dimensions, commenting on theorems which demonstrate duality between the two fractal dimensions. It mainly presents algorithmic fractal dimension and the Point to Set Principle, which relates the dimension of a set to the Kolmogorov complexity of its points. The Point to Set Principle reveals additional dualities between Hausdorff and packing dimensions and significantly simplifies some classical proofs of fractal dimension theorems. Furthermore, a concept resulting from the Point to Set Principle, 'optimal oracles,' provides another category of sets that replaces the analytic requirement in Marstrand's Projection Theorem, and may be applied in the future to broaden other notable theorems.

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1. Introduction

Lebesgue measure, informally, gives a notion of the volume of a set, but can classify drastically different sets as identically having measure zero. For example, in $\mathbb{R}^3$, a plane, a line, and a point all have measure zero. Similarly, in $\mathbb{R}$, a one-thirds Cantor set and a Cantor set constructed by removing the middle 9/10ths have Lebesgue measure zero. The amount of information this measure can provide about a set depends on the space that the set is considered in.

An alternative measure, Hausdorff s-measure, returns values proportional to Lebesgue measure (when taken in $\mathbb{R}^n$ with $s = n$) while also possessing a method for determining the 'appropriate' dimension in which to evaluate a set. This dimension

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is the Hausdorff dimension of a set. With a slightly different construction, packing measure and packing dimensions can also be defined. Both Hausdorff dimension and packing dimension are examples of fractal dimensions.

Using algorithmic information theory, it is possible to find effective dimensions for individual points in $\mathbb{R}^n$. The Point to Set Principle associates the Hausdorff and packing dimension of a set with the effective Hausdorff and packing dimensions of its points, allowing algorthmic information and computability results to be used to prove theorems about fractal dimensions. Additionally, since the Point to Set Principle does not assume that sets are Borel or analytic, this alternative method may be able to remove these requirements from some classical theorems.

This expository paper will summarize both classical analysis results on duality between Hausdorff and packing dimension, as well as the Point to Set Principle and results using effective dimension. It will also present examples of sets in which the inequality relations between Hausdorff and packing dimensions of sets and their Cartesian product are equalities and strict inequalities. Finally, it describes a quality of a set (possessing "optimal oracles") which indicates the set is, in a sense, well behaved. The quality of having optimal oracles has replaced analytic requirements on sets in some theorems, significantly broadening the theorem's applicability, and this quality has the potential to be applied to other theorems in fractal geometry.

2. Hausdorff and Packing Dimensions

This section will define Hausdorff dimension and packing dimension, which are both types of fractal dimensions. Although there are other notable fractal dimensions, Hausdorff and packing dimension are the most relevant to this paper. First, it is necessary to present the Hausdorff and packing measures, and define dimension relative to how these measures behave for various values of s. The definition for Hausdorff dimension follows the construction in [5], and the packing dimension definition follows [8] in N. Lutz's section introducing classical fractal dimensions.

Definition 2.1. [5] Let $E \subseteq \mathbb{R}^n$. If $E \subseteq \bigcup_i U_i$ and for all i, it is true that $0 < |U_i| \leq \delta$ (where $|U_i|$ is the diameter of U_i), then $\{U_i\}$ is called a *δ-cover* of E. For non-negative s and $\delta > 0$, define $\mathscr{H}^s_\delta(E)$ as

$$\mathscr{H}^s_\delta(E) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s\right\}$$

where the infimum is over all countable δ-covers $\{U_i\}$ of E.

The *s-dimensional Hausdorff outer measure* of E is then

$$\mathscr{H}^s(E) = \lim_{\delta\to 0} \mathscr{H}^s_\delta(E).$$

Finally, the *Hausdorff dimension* of E is

$$\dim_H(E) = \inf\{s > 0 : \mathscr{H}^s(E) = 0\}.$$

Notice that, because of how the Hausdorff measure is defined, for any set E and any s, t with

$$0 \leq s < \dim_H(E) < t,$$

then $\mathscr{H}^s(E) = \infty$ and $\mathscr{H}^t(E) = 0$. Therefore, the Hausdorff dimension of E indicates, informally, the appropriate dimension for measuring E with the Hausdorff measure. For example, if F is the one thirds Cantor set, then even though the

Hausdorff 1-measure of F is zero, the s-dimensional Hausdorff measure of F is 1 for $s = \dim_H(F) = \frac{\log(2)}{\log(3)}$ [5]. A 9/10ths Cantor set, despite also having Hausdorff 1-measure zero, would have a smaller Hausdorff dimension of $\frac{\log(2)}{\log(20)}$, effectively distinguishing the two 'differently sized' sets.

Furthermore, note that, if $\dim_H(E) = 0$, then $\mathscr{H}^0(E)$ 'counts' the number of points in $E \subseteq \mathbb{R}^n$. If E has one point, it can be covered by one set of arbitrary diameter, so $\inf\left\{\sum_{i=1}^{\infty} |U_i|^0\right\} = 1 = \mathscr{H}^0(E)$, for example.

Defining packing dimension similarly requires first defining packing s-measure.

Definition 2.2. [8] Let $B_\delta(x)$ indicate the open ball with diameter $\delta > 0$ and center at x. Then define

$$P^s_\delta(E) = \sup\left\{\sum_{i=1}^{\infty} |B_{\delta_i}(x_i)|^s\right\}$$

where the supremum is taken over all countable collections $\{B_{\delta_i}(x_i)\}$ of open, pairwise disjoint balls with diameters at most δ and centers $x_i \in E$. Now define $P^s_0(E)$ as

$$P^s_0(E) = \lim_{\delta\to\infty} P^s_\delta(E).$$

The *s-dimensional packing outer measure* of E is then

$$P^s(E) = \inf\left\{\sum_{i=1}^{\infty} P^s_0(E_i) : E \subseteq \bigcup_{i=1}^{\infty} E_i\right\}$$

where the infimum is taken over possible countable covers of E.
The *packing dimension* is then

$$\dim_p(E) = \inf\left\{s > 0 : P^s(E) = 0\right\}.$$

Notice that, for any set $E \subseteq \mathbb{R}^n$, then $0 \le \dim_H(E) \le \dim_p(E) \le n$.

3. Some Classical Results

This section will present some basic results about these two fractal dimensions, as well as an interesting theorem on duality between Hausdorff and packing dimensions for additive complements. (Although this theorem is presented to discuss the duality between Hausdorff and packing dimension, providing the detail necessary to fully prove the theorem would distract from the focus of this paper. The complete proof and discussion can be found in [9].)

Note that the term "classical" contrasts against results which use algorithmic information and effective dimension. The results obtained with algorithmic information and the Point-to-Set Principle are discussed in the remaining sections of this paper. The proofs of the following results also present introductory examples of techniques used in obtaining classical fractal dimension theorems.

Proposition 3.1. *Let $E \subseteq \mathbb{R}^n$ and fix $z \in \mathbb{R}^n$. If $E+z$ is defined as the translation of E by z, or equivalently*

$$E + z = \{x + z : x \in E\},$$

then

$$\mathscr{H}^s(E + z) = \mathscr{H}^s(E).$$

Proof. Let $E \subseteq \mathbb{R}^n$ and $z \in \mathbb{R}^n$, and let $\{U_i\}$ be any δ-cover of E. Construct $\{V_i\}$ to be a δ-cover of $E+z$ by choosing $V_i = U_i + z$ for all $i \in \mathbb{N}$, noticing that $|V_i| = |U_i|$, since translation does not change the diameter of a set. Therefore,

$$\mathscr{H}^s_\delta(E) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s\right\} = \inf\left\{\sum_{i=1}^{\infty} |V_i|^s\right\} = \mathscr{H}^s_\delta(E+z)$$

so that $\mathscr{H}^s(E) = \mathscr{H}^s(E+z)$.

□

The following lemmas, presented partially in [5], prove that the Hausdorff dimension of any projection of a set is never larger than the dimension of the original set.

Lemma 3.2. *Let $E \subseteq \mathbb{R}^n$ and $F \subseteq \mathbb{R}^m$, and let $f : E \to F$ be a surjective mapping such that*

$$\|f(x) - f(y)\| \leq c\|x - y\|$$

for some constant $c \in [0, \infty)$, for all $x, y \in E$. Then for any $s \in [0, \infty)$,

$$\mathscr{H}^s(F) \leq c^s \mathscr{H}^s(E).$$

Proof. First note that, if $c = 0$, then $f(x) = f(y)$ for all $x, y \in E$. If E is the empty set, then so is F, and the result is trivial. If E has at least one element, then since f is surjective, $F = \{f(x)\}$ and has $\dim_H(F) = 0$ so that $\mathscr{H}^0(F) = 1 \leq \mathscr{H}^0(E)$. For any $s > 0$, then by definition $\mathscr{H}^s(F) = 0$ so the result holds for all s.

Now assume that $c > 0$. Let $\{U_i\}$ be some countable δ-cover of E, and construct $\{V_i\}$ such that for all i, the set has $V_i = f(U_i \cap E)$. Notice that, since f is surjective,

$$F \subseteq \bigcup_{i=1}^{\infty} V_i.$$

Furthermore, notice that $\{V_i\}$ is a $c\delta$-cover of F, since $|f(U_i \cap E)| \leq c|U_i| \leq c\delta$. Therefore, for any s,

$$\mathscr{H}^s_{c\delta}(F) = \inf\left\{\sum_i |V_i|^s\right\} \leq \inf\left\{\sum_i c^s|U_i|^s\right\} = c^s \mathscr{H}^s_\delta(E).$$

Since $\lim_{\delta\to 0} \mathscr{H}^s_\delta(F) = \lim_{c\delta\to 0} \mathscr{H}^s_{c\delta}(F)$, then taking the limit gives

$$\mathscr{H}^s(F) \leq c^s \mathscr{H}^s(E)$$

completing the proof.

□

Lemma 3.3. *Let $E \subseteq \mathbb{R}^n$, let Π be any subspace of $\mathbb{R}^n$, and let $\mathrm{proj}_\Pi(E)$ denote the orthogonal projection of E onto the subspace Π. Then for any $s \geq 0$,*

$$\mathscr{H}^s(\mathrm{proj}_\Pi(E)) \leq \mathscr{H}^s(E)$$

and

$$\dim_H(\mathrm{proj}_\Pi(E)) \leq \dim_H(E).$$

Proof. Note that orthogonal projection onto Π does not increase distances, so for any $x, y \in E$,

$$\|\text{proj}_\Pi(x) - \text{proj}_\Pi(y)\| \leq \|x - y\|$$

and therefore Lemma 3.2 can be applied directly to give the first result. The second result follows from the above and the definition of $\dim_H(E)$. If $\mathscr{H}^s(F) \leq \mathscr{H}^s(E)$ for any s, then

$$\inf\{s > 0 : \mathscr{H}^s(F) = 0\} \leq \inf\left\{d > 0 : \mathscr{H}^d(E) = 0\right\}$$

necessarily. Therefore, $\dim_H(\text{proj}_\Pi(E)) \leq \dim_H(E)$.

□

Straightforward classical inequalities, like the ones above and the first portion of the following lemma, are often proven in [5] by demonstrating that the inequality holds for any Hausdorff s-measure and later extending the result to Hausdorff dimension. Due to the Point-to-Set Principle, algorithmic proofs rarely use the Hausdorff or packing measures directly.

The following lemma is from [5] and [9], and is briefly used for the later theorem on duality between additive complements. The first portion of the lemma, from [5], does not require the sets to be Borel. However, the statements are combined here for simplicity, and the most complete version of the lemma is proven in Section 6 using algorithmic dimension.

Lemma 3.4. *For Borel sets E and F,*

$$\dim_H(E) + \dim_H(F) \leq \dim_H(E \times F) \leq \dim_H(E) + \dim_p(F).$$

The following results on Hausdorff and packing dimension duality with respect to additive complements are from [9], and are briefly commented on at the end of Section 6.

Theorem 3.5. [9] *Let $A \subseteq \mathbb{R}^n$ be a nonempty Borel set. Then*

$$\begin{aligned} \dim_H(A) &= n - \inf\left\{\overline{\dim_M}(B) : B \subseteq \mathbb{R}^n \text{ is compact, } int(A+B) \neq \varnothing\right\} \\ &= n - \inf\left\{\dim_p(B) : B \subseteq \mathbb{R}^n \text{ is compact, } int(A+B) \neq \varnothing\right\} \\ &= n - \inf\left\{\dim_p(B) : B \subseteq \mathbb{R}^n \text{ is Borel, } A+B = \mathbb{R}^n\right\}, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \dim_p(A) &= n - \inf\left\{\dim_H(B) : B \subseteq \mathbb{R}^n \text{ is compact, } int(A+B) \neq \varnothing\right\} \\ &= n - \inf\left\{\dim_H(B) : B \subseteq \mathbb{R}^n \text{ is Borel, } A+B = \mathbb{R}^n\right\}. \end{aligned} \tag{3.7}$$

The following two lemmas are necessary to prove Theorem 3.5, and their proofs can be found in detail in [9] (in which they are referred to as Lemma 1.6 and Theorem 1.9, respectively). Note that $A + B = \{x + y : x \in A, y \in B\}$. It should also be noted that $\overline{\dim_M}(B)$ is the upper Minkowski dimension of B, and $\dim_p(E) \leq \overline{\dim_M}(B)$ for any set B. The precise definition of upper Minkowski dimension and related results will not be discussed in depth here.

Lemma 3.8. [9] *Let $n \in \mathbb{N}$ and let $A \subseteq \mathbb{R}^n$ be a nonempty Borel set with $\dim_H(A) > s$. Then there exists a compact E with $\overline{\dim_M}(E) = n - s$ such that $A + E$ has nonempty interior.*

Lemma 3.9. [9] *For any analytic $A \subseteq \mathbb{R}^n$ with $\dim_p(A) > s$, there exists a compact $B \subseteq \mathbb{R}^n$ such that*

$$\mathscr{H}^{n-s}(B) = 0$$

and int$(A + B) \neq \varnothing$.

Proof of Theorem 3.5. Since Lemma 3.4 holds for Borel sets E and F and $E + F$ is a projection of $E \times F$, then for Borel sets A and B such that $A + B = \mathbb{R}^n$, it is the case that

$$\dim_H(A) + \dim_p(B) \geq \dim_H(A \times B) \geq n.$$

This gives

$$\dim_H(A) \geq n - \inf \left\{\dim_p(B) : B \subseteq \mathbb{R}^n \text{ is Borel, } A + B = \mathbb{R}^n\right\} \text{ and}$$
$$\dim_p(A) \geq n - \inf \left\{\dim_H(B) : B \subseteq \mathbb{R}^n \text{ is Borel, } A + B = \mathbb{R}^n\right\},$$

so it is now only necessary to prove that the opposite inequality also holds for both statements. Note that, in the case that either $\dim_H(A) = 0$ or $\dim_p(A) = 0$, and since $\dim_H(B)$ and $\dim_p(B)$ are both at most n, then the desired result follows immediately.

Consider equation (3.6) from the theorem. Say that $\dim_H(A) = d > 0$. Then, by Lemma 3.8, for any s such that $d > s$, there exists a compact $B_s \in \mathbb{R}^n$ such that

$$\overline{\dim_M}(B_s) = n - s$$

and $\text{int}(A + B_s) \neq \varnothing$. Then,

$$\dim_H(A) = d \leq n - \inf \left\{\overline{\dim_M}(B_s)\right\}$$

which gives the first line of the theorem.

Applying the fact that packing dimension is always less than or equal to upper Minkowski dimension gives the second line of the first statement, and the fact that countable translations of sets with nonempty interior can cover $\mathbb{R}^n$ gives the final line.

Now consider (3.7), and say that $\dim_p(A) = c$. By Lemma 3.9, for $s < c$ there exists a compact B_s such that $\mathscr{H}^{n-s}(B_s) = 0$ and $\text{int}(A + B_s) \neq \varnothing$. Therefore, since $\mathscr{H}^{n-s}(B_s) = 0$, then

$$\dim_H(B_s) \leq n - s$$

by definition. Therefore,

$$\inf\{\dim_H(B_s) : B_s \text{ compact, } \text{int}(A + B_s) \neq \varnothing\} \leq n - c$$

so that

$$\dim_p(A) = c \leq n - \inf\{\dim_H(B) : B \text{ compact, } \text{int}(A + B) \neq \varnothing\}$$

and so, combining both inequalities for (3.6) and (3.7) completes the proof. □

4. Kolmogorov Complexity and Algorithmic Dimension

This section introduces relevant subjects from information theory, including Kolmogorov complexity, conditional Kolmogorov complexity, oracles, and effective Hausdorff and packing dimensions. From this point, note that points in $\mathbb{R}^n$ are assumed to be expressed in binary, as Kolmogorov complexity takes in binary strings.

Definition 4.1. Let $\sigma \in \{0,1\}*$ be a finite binary string and let U be a universal prefix-free Turing machine. Then the *Kolmogorov complexity* of σ is

$$K(\sigma) = \min_{\beta \in \{0,1\}*} \{\ell(\beta) : U(\beta) = \sigma\},$$

where $\ell(\beta)$ returns the length of the string β.

Informally, the Kolmogorov complexity of σ gives the length of the shortest string which, when given to U, causes it to output σ. See [3] for clarification on universal Turing machines and their role in Kolmogorov complexity.

Example 4.2. If σ is the string $101010\ldots10$ where $'10'$ is repeated 32 times, then one possible string β such that $U(\beta) = \sigma$ would be $\beta = \sigma$. However, this would not be the shortest possible string that gives σ due to the repetition in σ. As 100000 in binary is equal to 32, then it would only be necessary to give the computer 8 bits of information on amount of repetition and content being repeated. Therefore, the Kolmogorov complexity of σ is much shorter than its length. Similarly, for a string of zeros of length r, the Kolmogorov complexity is on the order of $\log(r)$, not r.

Kolmogorov complexity effectively measures, in a sense, the randomness of a string, and this measurement can be extended to arbitrary points in $\mathbb{R}^n$ by comparing them with sufficiently close points in $\mathbb{Q}^n$.

Definition 4.3. Let $x \in \mathbb{R}^n$, and let $r \in \mathbb{N}$. Then define

$$K_r(x) = \inf\{K(\sigma) : \sigma \in B_{2^{-r}}(x) \cap \mathbb{Q}^n\}.$$

Then $K_r(x)$ is called the *Kolmogorov complexity* of x at precision r. If $r \in (0, \infty)$, then $K_r(x)$ is defined as $K_{\lceil r \rceil}(x)$.

In this definition, the Kolmogorov complexity of σ, since σ is rational, is identical to the complexity of the binary encoding string of σ, up to an additive constant [3].

Note that $x[r]$ indicates the value in the rth place of the binary expansion of x. Informally, it can be thought that $K_r(x)$ truncates x to the rth place and calculates the complexity of the resulting rational number.

We will now define *conditional complexity* and *oracles*, which both quantify how providing additional information to compute a point may reduce the Kolmogorov complexity of that point.

Definition 4.4. [8] First note that, for two finite strings $\sigma, \pi \in \{0,1\}*$, the conditional Kolmogorov complexity of σ given π is defined as

$$K(\sigma \mid \pi) = \min_{\beta \in \{0,1\}*} \{\ell(\beta) : U(\beta, \pi) = \sigma\}$$

where the universal Turing machine is also given the information in π for outputting σ.

Notice that, using this format, $K(\sigma)$ as defined previously is equal to $K(\sigma \mid \pi)$ where π is the empty string.

The types of Kolmogorov complexity which take strings as inputs are only used in establishing the related definitions which take points in $\mathbb{R}^n$ as inputs. Whenever Kolmogorov complexity or conditional Kolmogorov complexity is referenced outside of these definitions, we assume that the complexity is taken with respect to points, unless explicitly stated otherwise.

Definition 4.5. Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $r, s \in \mathbb{N}$. Then the *conditional Kolmogorov complexity* of x to precision r given y to precision s is defined as

$$K_{r,s}(x \mid y) = \max\{\min\{K(p \mid q) : p \in B_{2^{-r}} \cap \mathbb{Q}^n\} : q \in B_{2^{-s}} \cap \mathbb{Q}^m\}$$

Note that $K_{r,r}(x \mid y)$ is written as $K_r(x \mid y)$.

Notice that, informally, if y is somehow random with respect to x or otherwise not useful for calculating x, then $K_r(x \mid y)$ will be similar to $K_r(x)$. If y is equivalent to x up to the rth place, then $K_r(x \mid y)$ will be much lower than $K_r(x)$, since more information about x was provided. (The idea of randomness and "random with respect to an oracle" is meant in the Martin-Löf sense, as discussed in [4], and will be briefly elaborated on later.)

We will now consider a broader way to provide the universal Turing machine with information. An *oracle* or *oracle set* $A \subseteq \mathbb{N}$ is countable information that may be provided while calculating Kolmogorov complexity, replacing the universal Turing machine with a universal oracle machine with access to the established oracle.

Definition 4.6. Let σ be a string and let $A \subseteq \mathbb{N}$ be an oracle. The *relativized Kolmogorov complexity* $K^A(\sigma)$ is defined as

$$K^A(\sigma) = \inf\{\ell(\beta) : U^A(\beta) = \sigma\}$$

where the universal prefix-free Turing machine from the definition of Kolmogorov complexity is replaced with U^A, a universal oracle machine with access to A. Note that U^A may use a computational step to query membership in A.

The definitions of $K_r^A(x)$ and $K_{r,s}^A(x \mid y)$ for points x, y are identical to their definitions excluding the oracle, except the universal Turing machine they reference is replaced with a universal oracle machine with access to A.

A very useful property that Kolmogorov complexity possesses is the *Symmetry of Information.* The following forms of this symmetry are presented in [3] and [7], where it is also referred to as a chain rule.

Theorem 4.7. *Let $n, m, r, s \in \mathbb{N}$ with $r \geq s$ and let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Then*

(1) $|K_r(x \mid y) + K_r(y) - K_r(x, y)| \leq O(\log r) + O(\log \log \|y\|)$
(2) $|K_{r,s}(x \mid x) + K_s(x) - K_r(x)| \leq O(\log r) + O(\log \log \|x\|)$
(3) $K_r(x, y) = K_r(x \mid y) + K_r(y) + o(r)$

Now I will define effective Hausdorff and packing dimension, which describe the 'dimension' of points in $\mathbb{R}^n$ by using the ratio of the points' Kolmogorov complexity up to precision r to the value of r.

Definition 4.8. Let $x \in \mathbb{R}^n$. The *effective Hausdorff dimension* of x is

$$\dim(x) = \liminf_{r\to\infty} \frac{K_r(x)}{r}$$

and the *effective packing dimension* of x is

$$\mathrm{Dim}(x) = \limsup_{r\to\infty} \frac{K_r(x)}{r}.$$

The definition presented above was originally a theorem, for $\dim(x)$ and $\mathrm{Dim}(x)$ defined with respect to *gales*, as described in [8]. In this paper, these equivalences will be taken as definitions instead of theorems.

From this definition, notice that for any $x \in \mathbb{R}^n$,

$$0 \leq \dim(x) \leq \mathrm{Dim}(x) \leq n.$$

The last inequality holds because, if a point has n coordinates, then $K_r(x)$ may be up to nr, such as in the case that all of the coordinates are random.

Lemma 4.9. [3] *Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Then*

$$\begin{aligned}\dim(x \mid y) + \dim(y) &\leq \dim(x, y)\\ &\leq \mathrm{Dim}(x \mid y) + \dim(y)\\ &\leq \mathrm{Dim}(x, y)\\ &\leq \mathrm{Dim}(x \mid y) + \mathrm{Dim}(y).\end{aligned}$$

Notice that this lemma on effective dimensions follows from the Symmetry of Information and properties of addition of sequences with lim inf and lim sup.

It should be noted that *conditional effective dimension* and *relativized effective dimensions* $\dim(x \mid y), \dim^A(x), \mathrm{Dim}(x \mid y)$ and $\mathrm{Dim}^A(x)$ are defined by replacing the Kolmogorov complexity in the original definition with the corresponding conditional or relativized Kolmogorov complexities. The above lemma also holds for relativized effective dimensions, since its proof does not depend on the presence or absence of relativized Kolmogorov complexity.

Note also that, for $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, it is possible to define an oracle A_y which provides the expansion of y [8]. In this case, Kolmogorov complexity or dimension calculations with access to oracle A_y are conventionally written as $K_r^y(x)$, $\dim^y(x)$, etc. instead of $K_r^{A_y}(x)$. The value $K_r^y(x)$ is distinct from $K_r(x \mid y)$ in the fact that the former has access to the entire countable expansion of y, while the latter only has access to y up to precision r (or finite precision s, in the case of $K_{r,s}(x \mid y)$).

Lemma 4.10. [8] *Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and $r \in \mathbb{N}$. Then*

$$K_r^y(x) \leq K_r(x \mid y) + O(\log n),$$

so that $\dim^y(x) \leq \dim(x \mid y)$ *and* $\mathrm{Dim}^y(x) \leq \mathrm{Dim}(x \mid y)$.

Remark 4.11. [1] Note that any finite or countable combination of oracles is also an oracle. The *join* of oracles A and B is denoted by (A, B). Providing more information may not increase dimension; for any oracles A, B and $x \in \mathbb{R}^n$,

$$\dim^{A,B}(x) \leq \dim^A(x)$$

and similarly,

$$\mathrm{Dim}^{A,B}(x) \leq \mathrm{Dim}^A(x).$$

5. The Point to Set Principle

The Point to Set Principle provides methods to manipulate Hausdorff and packing dimensions by using effective dimension and algorithmic information. This principle allows some statements about fractal dimension to be proven much more concisely or without unnecessarily restrictive requirements on sets.

Theorem 5.1 (Point to Set Principle)**.** [3] *Let* $E \subseteq \mathbb{R}^n$*. Then*

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x)$$

and

$$\dim_p(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \operatorname{Dim}^A(x).$$

Note that the minimums, instead of infimums, in both equivalences indicate that there are oracles such that the minimum value is achieved.

Definition 5.2. Let $E \subseteq \mathbb{R}^n$ and let $A \subseteq \mathbb{N}$ be an oracle such that

$$\sup_{x \in E} \dim^A(x) = \dim_H(E)$$

then A is called a *Hausdorff oracle* for E. Let $B \subseteq \mathbb{N}$ be an oracle such that

$$\sup_{x \in E} \operatorname{Dim}^B(x) = \dim_p(E)$$

then B is called a *packing oracle* for E.

Remark 5.3. Note the following results from Theorem 5.1 and properties of effective dimension and oracles:

1. [8] Let $E \subseteq \mathbb{R}^n$ and $\epsilon > 0$. For any oracle $A \subseteq \mathbb{N}$, there is an $x \in E$ such that $\dim^A(x) \geq \dim_H(E) - \epsilon$ and there is a $y \in E$ such that $\operatorname{Dim}^A(y) \geq \dim_p(E) - \epsilon$.
2. Since the addition of more oracles cannot increase dimension, then notice that, if A is a Hausdorff (or packing) oracle for a set E, and B is an arbitrary oracle, then (A, B) is also a Hausdorff (or packing) oracle for E. Therefore, since every set has a Hausdorff oracle and a packing oracle by the Point to Set Principle, then every set has infinitely many Hausdorff and packing oracles.
3. [6] For any oracle $A \subseteq \mathbb{N}$, almost every $x \in \mathbb{R}^n$ is *Martin-Löf random* relative to A. This formally means that there exists a constant c such that $\forall r \in \mathbb{N}$, then $K_r^A(x) \geq nr - c$, or $\dim^A(x) = \operatorname{Dim}^A(x) = n$. Informally, the oracle A is entirely unhelpful for calculating x.

The form of the Point to Set Principle can provide intuition about what information an oracle may encode. For example, since the dimension of a set is unchanged by translation, rotation, or scaling, it is clear that an oracle must be able to encode these types of transformations. Additionally, since countable sets have dimension zero, and oracles may encode countable information, it is clear that there is an oracle that, informally, contains all or most of the information of a countable set.

6. Initial Results Using Algorithmic Dimension

Before demonstrating the value of the Point to Set Principle for expanding or proving major theorems in fractal geometry, it is useful to introduce how this principle may be used to prove initial results about Hausdorff and packing dimensions. This exercise also provides insight on the different techniques required to use algorithmic information for completing proofs, as compared to classical methods.

This first result is a useful description of the relationship between the Hausdorff and packing dimension of any sets E, F and the fractal dimensions of their product $E \times F$.

Theorem 6.1. [8] *Let $E \subseteq \mathbb{R}^n$ and $F \subseteq \mathbb{R}^m$. Then*

$$\begin{aligned}\dim_H(E) + \dim_H(F) &\leq \dim_H(E \times F)\\ &\leq \dim_H(E) + \dim_p(F)\\ &\leq \dim_p(E \times F)\\ &\leq \dim_p(E) + \dim_p(F)\end{aligned}$$

Proof. Let $E \subseteq \mathbb{R}^n$ and $F \subseteq \mathbb{R}^m$. This proof generally follows each inequality in Lemma 4.9 by establishing specific oracles and points in E, F, and $E \times F$ using parts 1 and 2 of Remark 5.3.

Inequality one:

Consider $E \times F$, and let A be a Hausdorff oracle of $E \times F$. Let $\epsilon > 0$. Then the following are true, by Remark 5.3:

(1) Since A is a Hausdorff oracle, then $\dim_H(E \times F) = \sup_{z \in E \times F} \dim^A(z)$
(2) There exists an $x_1 \in E$ such that $\dim^A(x_1) \geq \dim_H(E) - \frac{\epsilon}{2}$. Fix this x_1.
(3) There exists a $y_1 \in F$ such that $\dim^{A,x_1}(y_1) \geq \dim_H(F) - \frac{\epsilon}{2}$. Fix this y_1.

Therefore,

$$\dim_H(E \times F) = \sup_{z \in E \times F} \dim^A(z) \geq \dim^A(x_1, y_1)$$

since (x_1, y_1) is in $E \times F$ by definition. By Lemma 4.9,

$$\dim^A(x_1, y_1) \geq \dim^A(x_1) + \dim^A(y_1 \mid x_1)$$

and by Lemma 4.10, $\dim^A(y_1 \mid x_1) \geq \dim^{A,x_1}(y_1)$, so

$$\dim_H(E \times F) \geq \dim^A(x_1) + \dim^{A,x_1}(y_1) \geq \dim_H(E) + \dim_H(F) - \epsilon.$$

Furthermore, since ϵ may be chosen to be arbitrarily small, it must be the case that

$$\dim_H(E) + \dim_H(F) \leq \dim_H(E \times F),$$

proving the first line of the theorem.

Inequality two:

By Remark 5.3, there exist oracles B_p and B_H and a point $(x_2, y_2) \in E \times F$ such that

(1) $\dim_p(E) = \sup_{x \in E} \mathrm{Dim}^{B_p}(x)$,
(2) $\dim_H(F) = \sup_{y \in F} \dim^{B_H}(y)$, and
(3) $\dim^{B_p, B_H}(x_2, y_2) \geq \dim_H(E \times F) - \epsilon$

using the same ϵ as was established initially. Therefore,

$$\dim_p(E) + \dim_H(F) \geq \mathrm{Dim}^{B_p}(x_2) + \dim^{B_H}(y_2) \geq \mathrm{Dim}^{B_p,B_H}(x_2) + \dim^{B_p,B_H}(y_2)$$

since the addition of oracles cannot increase dimension. Since adding information does not increase dimension, or, formally, $\mathrm{Dim}^{B_p,B_H}(x_2) \geq \mathrm{Dim}^{B_p,B_H}(x_2 \mid y_2)$, then

$$\dim_p(E) + \dim_H(F) \geq \mathrm{Dim}^{B_p,B_H}(x_2 \mid y_2) + \dim^{B_p,B_H}(y_2) \geq \dim^{B_p,B_H}(x_2, y_2)$$

by Lemma 4.9. Therefore, it holds that

$$\dim_p(E) + \dim_H(F) \geq \dim_H(E \times F) - \epsilon$$

for any $\epsilon > 0$, so the second inequality is proven.

Inequality three:
Let ϵ, B_p, B_H be as previously defined. Let B' be a packing oracle of $E \times F$, and let B be the join of oracles B_p, B_H, and B'. Let x_3 and y_3 be elements in E and F such that

(1) $\dim^B(y_3) \geq \dim_H(F) - \frac{\epsilon}{2}$ and
(2) $\mathrm{Dim}^{B,y_3}(x_3) \geq \dim_p(E) - \frac{\epsilon}{2}$.

Then, by Remark 5.3, Lemma 4.10, Lemma 4.9, and the fact that relativization can only decrease dimension, the following series of inequalities hold:

$$\begin{aligned}
\dim_p(E \times F) &\geq \mathrm{Dim}^{B'}(x_3, y_3) \\
&\geq \mathrm{Dim}^{B}(x_3, y_3) \\
&\geq \mathrm{Dim}^{B}(x_3 \mid y_3) + \dim^B(y_3) \\
&\geq \mathrm{Dim}^{B,y_3}(x_3) + \dim^B(y_3) \\
&\geq \dim_p(E) + \dim_H(F) - \epsilon
\end{aligned}$$

so the third inequality is proven.

Inequality four:
Let ϵ, B_p be as defined previously, let C be a packing oracle for F, and let $D = (B_p, C)$. Then find $(x_4, y_4) \in E \times F$ such that

$$\mathrm{Dim}^D(x_4, y_4) \geq \dim_p(E \times F) - \epsilon.$$

Then

$$\begin{aligned}
\dim_p(E) + \dim_p(F) &\geq \mathrm{Dim}^{B_p}(x_4) + \mathrm{Dim}^C(y_4) \\
&\geq \mathrm{Dim}^{B_p}(x_4 \mid y_4) + \mathrm{Dim}^C(y_4) \\
&\geq \mathrm{Dim}^D(x_4 \mid y_4) + \mathrm{Dim}^D(y_4) \\
&\geq \mathrm{Dim}^D(x_4, y_4) \\
&\geq \dim_p(E \times F) - \epsilon
\end{aligned}$$

so inequality four is proven, and the proof of the theorem is complete.

□

The straightforward nature of the above proof makes clear the benefit of applying effective dimension and algorithmic information to theorems on fractal dimensions. The following is an immediate corollary from Theorem 6.1.

Corollary 6.2. *Let $E \in \mathbb{R}^n$ such that $\dim_H(E) = \dim_p(E) = s$ for some s. Then $\forall F \in \mathbb{R}^m$,*

$$s + \dim_H(F) = \dim_H(E \times F)$$

and

$$s + \dim_p(F) = \dim_p(E \times F).$$

In light of the generality provided by Theorem 6.1, we will briefly return to comment on Theorem 3.5 and the possibility of removing Borel requirements in portions of its proof.

Remark 6.3. Note that for any $E, F \in \mathbb{R}^n$, it is the case that $E+F$ is a projection of $E \times F$, so by Lemma 3.3,

$$\dim_H(E + F) \leq \dim_H(E \times F).$$

Since Theorem 6.1 holds for any sets E and F, then the fact that

$$\dim_H(E + F) \leq \dim_H(E \times F) \leq \dim_H(E) + \dim_p(F)$$

gives

$$\dim_H(E) \geq n - \dim_p(F)$$

and

$$\dim_p(E) \geq n - \dim_H(F)$$

for any F such that $\dim_H(E + F) = n$.

However, the Borel requirement for the opposite direction of Theorem 3.5 is not easily removed with the application of algorithmic fractal dimension.

It should also be noted that, in [8], N. Lutz presents a statement equating packing dimension of a set to a difference between Hausdorff dimension of two related sets. The statement was previously proven for analytic sets; Lutz's proof extends the result to arbitrary sets with the Point to Set Principle. Although this result has key differences from Theorem 3.5, including lacking a requirement on additive complements, it demonstrates a similar duality between packing dimension and Hausdorff dimension.

Theorem 6.4. [8] *Let $E \subseteq \mathbb{R}^n$. Then*

$$\dim_p(E) = \sup_{F \subseteq \mathbb{R}^n} \{\dim_H(E \times F) - \dim_H(F)\}.$$

There are a variety of other notable proofs that make use of the Point to Set Principle. This principle has been successfully applied to finding the Hausdorff dimension of plane Kakeya sets [3], finding the dimension of a pinned distance set [2], and describing a broader condition for sets which ensures that Marstrand's projection theorem holds [1]. Most of these examples are cited as evidence for the value of the Point to Set Principle, and will not be presented in full in this paper. However, Section 8 will describe the result on Marstrand's projection theorem and the condition of having optimal oracles in detail.

7. Examples of Equality and Strict Inequality in Theorem 6.1

It is useful to provide insight on Theorem 6.1, and demonstrate that the inequalities present cannot be replaced with equalities, by constructing examples of sets for which equality and strict inequality hold.

Because of Corollary 6.2, it is straightforward to generate examples where equality holds in part or in all of the theorem. For example, if $E = F = \varnothing$, or both E and F are sets with equivalent packing and Hausdorff dimension, then clearly $\dim_H(E) + \dim_H(F) = \dim_p(E) + \dim_p(F)$, and all of the inequalities in Theorem 6.1 are equal. If $\dim_H(E) = \dim_p(E) = s$ but $\dim_H(F) \neq \dim_p(F)$, then

$$\dim_H(F) + s = \dim_H(E \times F) < \dim_p(E \times F) = \dim_p(F) + s,$$

so some portions of the theorem are equal and others are strict inequalities.

The proposition below provides an example of sets E, F for which

$$\dim_H(E)+\dim_H(F) < \dim_H(E\times F) = \dim_H(E)+\dim_p(F) < \dim_p(E)+\dim_p(F).$$

The construction presented in this proposition is similar to one described in [5], but Falconer's proof is classical. The use of effective dimension for this proof, and specifically the method for proving that $\dim_H(E) = 0$ as well as the intuition behind the construction of the sets, was explained to me by my mentor, Iqra Altaf.

Proposition 7.1. *There exist sets E and F such that $\dim_H(E) = \dim_H(F) = 0$ and $\dim_H(E \times F) = 1$.*

Proof. Let $\{m_n\}$ be a sequence of integers such that $m_1 = 1$ and for all $n \in \mathbb{N}$, $m_{n+1} \geq n m_n$. Recall that $x[r]$ indicates the value in the rth place of the binary expansion of x. Let E be the set of $x \in [0, 1]$ such that for all $r \in \mathbb{N}$ with $m_{2j-1}+1 \leq r \leq m_{2j}$ for some j, then $x[r] = 0$.

Note that, since E includes all x of this form, then for any $z \in [0, 1]$, it is possible to find a point x_z in E which has $x_z[r] = z[r]$ for $r \notin [m_{2j-1} + 1, m_{2j}]$ for some j. Therefore, since almost every point in $[0, 1]$ is Martin Löf random, it is possible to find points in E that have random strings in their expansion between the strings that are set to be zero.

Similarly, construct F as the set of all $y \in [0, 1]$ such that for all r with $m_{2j} + 1 \leq r \leq m_{2j+1}$ for some $j \in \mathbb{N} \cup \{0\}$ (and setting $m_0 = 0$ for clarity), then $y[r] = 0$.

The following is an informal picture, intended to provide intuition about the construction. Say, for example, that $m_n = n!$, and say that having $x[r]$ represented as z for some value r within a binary expansion indicates that the value is random or may vary. Then all $x \in E$ and $y \in F$ are of the forms

$$x = 0.\ z\ 0\ z\ z\ z\ z\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ z \ldots$$

$$y = 0.\ 0\ z\ 0\ 0\ 0\ 0\ z\ z\ z\ z\ z\ z\ z\ z\ z\ z\ z\ z\ z\ z\ z\ z\ z\ z\ 0 \ldots$$

where spacing is exaggerated to demonstrate alignment between random sections and zero sections of x and y, and the length of each section is determined by the values of $\{m_n\}$. With this visualization and the construction of E and F complete, we will return to the proof.

Consider the dimensions of E and F. I claim that, for any $x \in E$,

$$\liminf_{r\to\infty} \frac{K_r(x)}{r}$$

occurs for the subsequence of $r = m_{2j}$ for $j \in \mathbb{N}$. Informally, this is true because, for r such that $m_{2j}+1 \leq r \leq m_{2j+1}$, then $x[r]$ may be random as discussed above, so $\frac{K_r(x)}{r}$ may be up to one. For r such that $m_{2j-1}+1 \leq r < m_{2j}$, then since all places after $x[m_{2j-1}]$ are zeroes, they only contribute to $K_r(x)$ proportionally to $\log(r)$. Since, in this case, $\frac{\log(r)}{r} \geq \frac{\log(m_{2j})}{m_{2j}}$, then the claim holds.

Returning to the proof of the Hausdorff dimension of E, fix $x \in E$. By the symmetry of information,

$$\begin{aligned} K_{m_{2j}}(x) &\leq K_{m_{2j},m_{2j-1}}(x \mid x) + K_{m_{2j-1}}(x) + o(m_{2j}) \\ &\leq \log(m_{2j}) + m_{2j-1} + o(m_{2j}). \end{aligned}$$

Notice that $K_{m_{2j},m_{2j-1}}(x)$ is approximately $\log(m_{2j})$ because, by construction, this finds the complexity of the $m_{2j} - m_{2j-1}$ long string of zeros in this section of the expansion. Therefore,

$$\dim(x) = \liminf_{r\to\infty} \frac{K_r(x)}{r} = \lim_{j\to\infty} \frac{K_{m_{2j}}(x)}{m_{2j}}$$

where

$$\frac{K_{m_{2j}}(x)}{m_{2j}} \leq \frac{m_{2j-1}}{m_{2j}} + \frac{\log(m_{2j})}{m_{2j}} + \frac{o(m_{2j})}{m_{2j}}$$

and since $\frac{m_{2j-1}}{m_{2j}} \leq \frac{1}{2j-1}$, then, taking the limit,

$$\dim(x) \leq 0$$

for all $x \in E$, since this was computed with respect to an arbitrary x. Furthermore, since the addition of oracles cannot increase effective dimension, then

$$\min_{A\in\mathbb{N}} \sup_{x\in E} \dim^A(x) = \dim_H(E) = 0.$$

Since $y \in F$ is constructed in an almost identical manner as $x \in E$, then by the same argument,

$$\dim_H(F) = 0.$$

Now consider the Hausdorff dimension of $E \times F$. First note that $E + F$ is a projection of $E \times F$ onto the line $y = x$. Take any $z \in [0,1]$. By construction, it is possible to find an $x \in E$ and a $y \in F$ such that, $\forall r \in \mathbb{N}$ and $j \in \mathbb{N} \cup \{0\}$, if $m_{2j}+1 \leq r \leq m_{2j+1}$, then $x[r] = z[r]$, and if r is such that $m_{2j-1}+1 \leq r \leq m_{2j}$, then $y[r] = z[r]$. In a sense, it is possible to find x, y such that z is split between them, with x and y alternating between a section of the expansion of z and a string of zeroes. Then, since x, y are zero everywhere except for the sections assigned to equal z, it is true that $x + y = z$.

Therefore, for any $z \in [0,1]$, it is the case that $z \in E + F$, so $\dim_H(E + F) = 1$. By Lemma 3.3,

$$\dim_H(E \times F) \geq \dim_H(E + F) = 1.$$

To determine that $\dim_H(E \times F) = 1$, consider $\dim_p(E)$. By Theorem 6.1,

$$1 \leq \dim_H(E \times F) \leq \dim_p(E) + \dim_H(F) = \dim_p(E)$$

and since $E \subseteq \mathbb{R}$, then $\dim_p(E) \leq 1$. Notice that the same argument holds for $\dim_p(F)$. Therefore, $\dim_p(E) = 1 = \dim_p(F)$, and $\dim_H(E \times F) = 1$. This completes the proof, and also demonstrates that

$$\begin{aligned}\dim_H(E) + \dim_H(F) &< \dim_H(E \times F) \\ &= \dim_H(E) + \dim_p(F) = \dim_p(E) + \dim_H(F) \\ &< \dim_p(E) + \dim_p(F).\end{aligned}$$

□

8. Optimal Oracles

Another concept motivated by the Point to Set Principle and Hausdorff and packing oracles is the idea of optimal oracles and sets which possess optimal oracles. Broadly, optimal Hausdorff or packing oracles are Hausdorff or packing oracles which minimize the complexity of most points. This section follows [1], which defines optimal oracles, constructs sets without optimal oracles, and replaces the requirement of sets being analytic in Marstrand's projection theorem with a requirement for sets to have optimal oracles. This section aims to briefly introduce the main ideas of [1].

Say that E is a set without optimal Hausdorff oracles, and let A be a Hausdorff oracle for E. Then, even though A achieves the minimum for $\sup_{x\in E} \dim^A(x)$, it would be the case that, for some oracle B and $x \in E$, the value $\dim^{A,B}(x)$ is significantly less than $\dim^A(x)$ (but B leaves $\sup_{x\in E} \dim^{A,B}(x)$ unchanged). The existence of optimal oracles for a set establishes that there is an oracle for which the addition of any further oracles only minimally reduces the Kolmogorov complexity of most points in the set, preventing most points from dramatically changing complexity with the addition of more oracles. The optimal oracle is a 'best' oracle for reducing the complexity at most points.

Definition 8.1. [1] Let $E \subseteq \mathbb{R}^n$ and let $A \subseteq \mathbb{N}$ be a Hausdorff oracle for E. Then A is called a *Hausdorff optimal oracle* for E if, for every oracle B and $\epsilon > 0$, there exists an $x \in E$ such that

$$\dim^{A,B}(x) \geq \dim_H(E) - \epsilon$$

and, for almost every $r \in \mathbb{N}$,

$$K_r^{A,B}(x) \geq K_r^A(x) - \epsilon r.$$

To demonstrate the value of this definition of an optimal oracle, it is also necessary to define the following set. If B is an oracle and $\epsilon > 0$, then

$$N(A, B, \epsilon) = \{x \in E \mid (\forall^\infty r) K_r^{A,B}(x) \geq K_r^A(x) - \epsilon r\},$$

where $\forall^\infty r$ indicates 'for almost every r.'

Proposition 8.2. [1] *Let $E \subseteq \mathbb{R}^n$ have positive Hausdorff dimension, and let A be an oracle. Then A is a Hausdorff optimal oracle for E if and only if A is a Hausdorff oracle and, for every $B \subseteq \mathbb{N}$ and $\epsilon > 0$,*

$$\dim_H(N(A, B, \epsilon)) = \dim_H(E).$$

Proof. [1] To prove the forward direction, assume that A is a Hausdorff optimal oracle for E. The result that A is a Hausdorff oracle for E follows trivially. Let $B \subseteq \mathbb{N}$ and fix $\epsilon > 0$. Let C be a Hausdorff oracle for $N(A, B, \epsilon)$. Notice that, by construction, $\dim_H(N(A, B, \epsilon)) \leq \dim_H(E)$. Suppose for contradiction that there is some $\lambda > 0$ (assuming, without loss of generality, that $\epsilon > \lambda$) such that

$$\dim_H(N(A, B, \epsilon)) < \dim_H(E) - \lambda.$$

Then, for every $x \in N(A, B, \epsilon)$, it is true that

$$\dim^{A,(B,C)}(x) \leq \dim^C(x) \leq \dim_H(N(A, B, \epsilon)) < \dim_H(E) - \lambda$$

and, since A is a Hausdorff optimal oracle, there exists a point $x \in E$ such that $\dim^{A,(B,C)}(x) \geq \dim_H(E) - \lambda$ and, for almost every r, it is the case that $K_r^{A,(B,C)}(x) \geq K_r^A(x) - \lambda r$. Any such x that fulfills these properties cannot be in the set $N(A, B, \epsilon)$, by the above. Since $x \notin N(A, B, \epsilon)$, then, for infinitely many r,

$$K_r^{A,(B,C)}(x) < K_r^A(x) - \epsilon r$$

which produces a contradiction. Therefore, $\dim_H(N(A, B, \epsilon)) = \dim_H(E)$.

To prove the backward direction, let A be a Hausdorff oracle for E, and assume that, for any B and $\epsilon > 0$, then $\dim_H(N(A, B, \epsilon)) = \dim_H(E)$. Fix $B \subseteq \mathbb{N}$ and $\epsilon > 0$. Then

$$\dim_H(E) = \dim_H(N(A, B, \epsilon)) \leq \sup_{x \in N(A,B,\epsilon)} \dim^{A,B}(x).$$

Therefore, there exists an $x \in E$ such that $\dim^{A,B}(x) \geq \dim_H(E) - \epsilon$ and, for almost every r, it holds that $K_r^{A,B} \geq K_r^A(x) - \epsilon r$. This proves that A is a Hausdorff optimal oracle.

□

Even though the definition of having optimal oracles only requires that Kolmogorov complexity is minimized at one point in the set, the above proposition demonstrates that this requirement ensures that the Kolmogorov complexity of most points are minimized by the optimal oracle.

Just as any oracle joined with a Hausdorff or packing oracle is also a Hausdorff or packing oracle, the join of any oracle and a Hausdorff optimal oracle is an optimal oracle.

Lemma 8.3. [1] *Let $E \subseteq \mathbb{R}^n$. Let A be a Hausdorff optimal oracle for E. Then for any oracle $B \subseteq \mathbb{N}$, the join (A, B) is a Hausdorff optimal oracle for E.*

Proof. [1] Let A be a Hausdorff optimal oracle for E, and let B be an oracle. By Remark 5.3, the join (A, B) is a Hausdorff oracle for E. Let $\epsilon > 0$, and let C be an oracle. Let $x \in E$ such that

$$\dim^{A,(B,C)}(x) \geq \dim_H(E) - \epsilon$$

and also, for almost every r,

$$K_r^{A,(B,C)}(x) \geq K_r^A(x) - \epsilon r$$

which are both possible by Remark 5.3 and the fact that A is a Hausdorff optimal oracle.

For large enough r, it is true that

$$K_r^{(A,B),C}(x) = K_r^{A,(B,C)}(x).$$

Since

$$\begin{aligned} K_r^{A,(B,C)}(x) &\geq K_r^A(x) - \epsilon r \\ &\geq K_r^{A,B} - \epsilon r \end{aligned}$$

then

$$K_r^{(A,B),C}(x) \geq K_r^{A,B} - \epsilon r,$$

which holds for any arbitrary oracle C. Therefore, (A, B) is also a Hausdorff optimal oracle for E.

□

It is also possible to define optimal packing oracles, which have broadly similar properties as optimal Hausdorff oracles.

Definition 8.4. [1] Let $E \subseteq \mathbb{R}^n$ and let $A \subseteq \mathbb{N}$ be a packing oracle for E. Then A is a *optimal packing oracle* for E if, for every oracle B and $\epsilon > 0$, there exists a point $x \in E$ with $\text{Dim}^{A,B}(x) \geq \dim_p(E) - \epsilon$ and, for almost every $r \in \mathbb{N}$,

$$K_r^{A,B}(x) \geq K_r^A(x) - \epsilon r.$$

The packing optimal equivalents of Proposition 8.2 and Lemma 8.3 also both hold.

Proposition 8.5. [1] *Let $E \subseteq \mathbb{R}^n$ have positive packing dimension, and let A be an oracle. Then A is a packing optimal oracle for E if and only if A is a packing oracle and, for every $B \subseteq \mathbb{N}$ and $\epsilon > 0$,*

$$\dim_p(N(A, B, \epsilon)) = \dim_p(E).$$

Lemma 8.6. [1] *Let $E \subseteq \mathbb{R}^n$. Let A be a packing optimal oracle for E. Then for any oracle $B \subseteq \mathbb{N}$, the join (A, B) is a packing optimal oracle for E.*

Remark 8.7. These additional qualities of optimal oracles and sets with optimal oracles are presented and proven in [1].

(1) Let $F \subseteq E$. If $\dim_H(F) = \dim_H(E)$ and E has optimal Hausdorff oracles, then F has optimal Hausdorff oracles. If $\dim_p(F) = \dim_p(E)$ and E has optimal packing oracles, then F has optimal packing oracles.
(2) Let $\{E_i\}_{i\in\mathbb{N}}$ be a sequence of sets such that $E = \bigcup_i E_i$. If each E_i has optimal Hausdorff oracles, then E has optimal Hausdorff oracles. If each E_i has optimal packing oracles, then E has optimal packing oracles.
(3) Let F such that $\dim_H(F) = \dim_p(F)$. Then, for a set $E \subseteq \mathbb{R}^n$, the set $E \times F$ has optimal Hausdorff oracles if and only if E has optimal Hausdorff oracles.

The following are some sufficient conditions for a set having optimal oracles.

(1) If $E \subseteq \mathbb{R}^n$ is analytic, it has optimal Hausdorff oracles and optimal packing oracles.
(2) If E is such that $\dim_H(E) = \dim_p(E)$, then it has optimal Hausdorff oracles and optimal packing oracles.
(3) Assume the Axiom of Determinacy. Then every set has optimal Hausdorff oracles.

Not every set has optimal Hausdorff oracles or optimal packing oracles [1]. Assuming the axiom of choice (AC) and the continuum hypothesis (CH), for $s \in (0, 1)$, it is possible to construct a set $E \subseteq \mathbb{R}$ with Hausdorff dimension s such that E does not have optimal Hausdorff oracles. It is also possible to extend this construction to make sets $E \subseteq \mathbb{R}^n$ with Hausdorff dimension $s \in (0, n)$ that do not have optimal Hausdorff oracles. Similarly, it is possible to construct a set E, under the assumptions of AC and CH, such that, for $0 < s_1 < s_2 \leq 1$, the set has $\dim_H(E) = s_1$ and $\dim_p(E) = s_2$ and E does not have optimal Hausdorff oracles or optimal packing oracles.

The above propositions and lemmas present notable features of sets with optimal oracles. We will now consider a major motivation for using the property of possessing optimal oracles: the extension of Marstrand's projection theorem.

Theorem 8.8 (Marstrand's projection theorem for sets with optimal oracles)**.** [1] *Let $E \subseteq \mathbb{R}^2$ be a set with optimal Hausdorff oracles. For $\theta \in [0, \pi]$, let $\text{proj}_\theta(E)$ indicate the projection of E onto the line that makes angle θ with the origin. Then for almost every $\theta \in [0, \pi]$,*

$$\dim_H(\text{proj}_\theta(E)) = \min\{\dim_H(E), 1\}.$$

Beyond Marstrand's projection theorem and the results that have already been achieved using these methods, optimal oracles and the Point to Set Principle have significant potential for augmenting classical results on fractal dimensions.

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