ALGORITHMIC METHODS FOR KAKEYA AND FURSTENBERG SETS IN THE PLANE

JULIAN HOVANEC

Abstract. This paper gives an introductory overview of the applications of algorithmic complexity theory to classical geometric measure theory problems using a recent point-to-set principle, which gives an algorithmic characterization of Hausdorff dimension. We introduce the notions of Kolmogorov complexity in Euclidean space and effective dimension, prove the point-to-set principle, as well as a lower bound on the effective dimension of points on a line, and use these results to provide alternate proofs of the Kakeya problem in the plane and a special case of the Furstenberg set conjecture.

Contents

1. Introduction

Hausdorff dimension is one of the most important tools in geometric measure theory, especially for the study of fractal sets. However, it is often very difficult to compute. This paper provides an introduction from a geometric perspective to the concepts, proof, and some applications of a recent *point-to-set principle* due to J. Lutz and N. Lutz in [1], which gives an alternate method for proving lower bounds on Hausdorff dimension using the *effective dimension* of points in $\mathbb{R}^n$. Throughout the paper, we assume basic knowledge of Hausdorff dimension, but no background in complexity theory.

In Section 2, we give an intuitive introduction to the Kolmogorov complexity of finite strings and how this idea can extend into Euclidean space, as well as defining the effective dimension of a point $x \in \mathbb{R}^n$. In Section 3, we introduce *oracles* and prove the point-to-set principle, which characterizes the Hausdorff dimension of a set $E \subset \mathbb{R}^n$ as follows:

$$\dim_H(E) = \min_{A \subset \mathbb{N}} \sup_{x \in E} \dim^A(x).$$

Date: August 7, 2024.

In Section 4, we provide a geometric interpretation of a result due to N. Lutz and D. M. Stull in [2], which proves a lower bound on the effective dimension of points $(x, ax+b) \in \mathbb{R}^2$ in terms of the dimensions of a, b, and x. Finally, we use this result in Section 5, in conjunction with the point-to-set principle, to provide straightforward algorithmic proofs that Kakeya sets in $\mathbb{R}^2$ are two-dimensional and that (α, β) -Furstenberg sets in the plane have dimension greater than $\alpha + \min\{\alpha, \beta\}$, a result first shown in [2] which contributed to the recent resolution of the Furstenberg set conjecture in $\mathbb{R}^2$ in [4]. Through these applications, we seek to demonstrate the power of algorithmic methods in geometric measure theory.

2. Kolmogorov Complexity and Effective Dimension

Let $\{0,1\}^*$ denote the set of finite binary strings. Given a string $\sigma \in \{0,1\}^*$, how "hard" is it to compute σ? Kolmogorov complexity quantifies the difficulty of this computation using Turing machines.

Definition 2.1. Fix U to be a universal Turing machine (to avoid tedious details, think of a computer equipped with any common coding language). Define the *Kolmogorov complexity* of σ as follows:

$$K(\sigma) = \inf\{\ell(\beta) \text{ such that } \beta \in \{0,1\}^* \text{ and } U(\beta) = \sigma\},$$

where $\ell(\beta)$ is the length of the string β.

Intuitively, the Kolmogorov complexity of σ is the shortest input one can give a computer program such that it will return σ. For a formal treatment of complexity theory, see [8].

Example 2.1. Let $\sigma = 1010...10$, where 10 is repeated sixty-four times. For a computer to output σ, we only need to tell it to repeat the string "10" sixty-four times, so a sufficient input would be the number 64 in binary, followed by the string "10". This input is just 8 bits long, so that $K(\sigma) = 8$, which is much smaller than the length of σ.

However, if σ is more or less "random", then $K(\sigma)$ is going to be close to the length of σ, meaning that there is no shortcut to computing σ, and the best one can do is input σ itself into the computer and tell it to simply output the input.

This brings us to the following definitions, which extend the ideas of Kolmogorov complexity into Euclidean space.

Definition 2.2. Given $x \in \mathbb{R}^n$ and $r \in \mathbb{N}$, the *Kolmogorov complexity of x at precision r* is

$$K_r(x) = \inf\{K(\sigma) : \sigma \in B_{2^{-r}}(x) \cap \mathbb{Q}^n\}.$$

If x is a real number, one can think of $K_r(x)$ as the complexity of the finite string obtained by cutting off the infinite binary expansion of x at r digits.

Definition 2.3. The *effective dimension* (also called the *algorithmic dimension,* or simply *dimension*) of a point $x \in \mathbb{R}^n$ is

$$\dim(x) = \liminf_{r\to\infty} \frac{K_r(x)}{r}.$$

Effective dimension measures the difficulty of computing an element of $\mathbb{R}^n$ in an asymptotic way. It follows directly from the definition that, for any $x \in \mathbb{R}^n$, $0 \leq \dim(x) \leq n$, but more is true:

Lemma 2.4. *For any real number $\alpha \in [0, n]$, there are uncountably many $x \in \mathbb{R}^n$ with* $\dim(x) = \alpha$.

This statement was first proven in [7], and is very important for any proof involving effective dimension because it allows us to choose points with any dimension we want, particularly "totally random" points, or those with effective dimension n.

Example 2.2. $\dim(0) = 0$, since to compute r digits of 0 we only need to tell the computer to repeat the digit 0 r times, which takes around $\log r$ bits of information because we only need to tell it the number r in binary (note that all logarithms in this paper are base 2). Therefore, $\dim(0) = \liminf_{r\to\infty} \frac{\log r}{r} = 0$. Similarly, $\dim(q) = 0$ if q is rational, since the binary expansion of q eventually repeats.

There exists a larger class of elements of $\mathbb{R}^n$ with effective dimension 0 than just the rationals, which are called *computable* [9]. However, we are not as concerned with the dimension of individual points in $\mathbb{R}^n$, only that we can always find points of any dimension.

3. The Point to Set Principle

In this section, we will introduce and prove the central theorem of [1], which relates the Hausdorff dimension of a set $E \subset \mathbb{R}^n$ (a global property) to the effective dimensions of points in E (a pointwise property).

The key to the point to set principle involves slightly modified versions of the definitions in the previous section, where our Turing machine now has access to an *oracle*, which can be thought of as some arbitrary countable information that "costs nothing" to compute. Without irrelevant details, an *oracle* is a subset $A \subset \mathbb{N}$, where at each computational step, our Turing machine can ask whether a number it has computed is in the set A and receive a yes or no answer in only one computational step. For us, an oracle can drastically shorten the necessary input to compute a certain string by providing extra information that we then don't need to include in the input. Importantly, oracles can sometimes lower the dimension of points in $\mathbb{R}^n$. We will use the following relativized definitions:

Definition 3.1. Let $\sigma \in \{0,1\}^*$ and $A \subset \mathbb{N}$ be any oracle. The *Kolmogorov complexity of σ relative to A* is

$$K^A(\sigma) = \inf\{\ell(\beta) \text{ such that } \beta \in \{0,1\}^* \text{ and } U^A(\beta) = \sigma\},$$

where the superscript U^A means that the Turing machine U has access to the oracle A.

Definition 3.2. Similarly, given $x \in \mathbb{R}^n$ and $r \in \mathbb{N}$, the *Kolmogorov complexity of x at precision r relative to the oracle A* is

$$K_r^A(x) = \inf\{K^A(\sigma) : \sigma \in B_{2^{-r}}(x) \cap \mathbb{Q}^n\}.$$

Definition 3.3. Finally, the *dimension relative to the oracle A* of a point $x \in \mathbb{R}^n$ is

$$\dim^A(x) = \liminf_{r\to\infty} \frac{K_r^A(x)}{r}.$$

If the oracle set A encodes the countable binary expansion of a real number $b \in \mathbb{R}$, we may also write $\dim^b(x)$ to denote the same thing as $\dim^A(x)$.

Example 3.1. To see how oracles can drastically lower the dimension of points, let $x \in \mathbb{R}$ be totally random, and let A encode the binary expansion of x. Then $\dim^A(x) = 0$, while $\dim(x) = 1$, since at each precision r, we only need to tell the computer to read off r digits from A, requiring only $\log r$ bits.

Even though an oracle can lower the dimension of certain points, Lemma 2.4 still holds relative to any oracle, so that no matter what the oracle A is, we can always find uncountably many points that are totally random relative to A [1]. These random points depend on A, but they always exist.

The point-to-set principle concerns the Hausdorff dimension of sets in $\mathbb{R}^n$, so let us briefly state the relevant definitions, following Falconer's text [3].

Definition 3.4. Let $E \subset \mathbb{R}^n$ and let $\{U_i\}$ be a countable (or finite) collection of sets. We say $\{U_i\}$ is a *δ-cover* of E if $E \subset \bigcup_{i=1}^{\infty} U_i$ and for each $i \in \mathbb{N}$, $|U_i| < \delta$, where $|U_i| := \sup\{|x-y| : x, y \in U_i\}$ is the *diameter* of U_i.

Definition 3.5. Let $E \subset \mathbb{R}^n$, $s \geq 0$, and $\delta > 0$. Define

$$\mathscr{H}^s_\delta(E) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s \right\}$$

where the infimum is taken over all countable δ-covers of E. Letting δ tend to 0, define the *s-dimensional Hausdorff measure of E* to be

$$\mathscr{H}^s(E) = \lim_{\delta \to 0} \mathscr{H}^s_\delta(E)$$

In fact, $\mathscr{H}^s(E)$ is a measure on $\mathbb{R}^n$ for any s, and $\mathscr{H}^n(E)$ is equal to n-dimensional Lebesgue measure up to a constant multiple depending only on n.

For a given $E \subset \mathbb{R}^n$, if for some s we have $\mathscr{H}^s(E) < \infty$, then for every $t > s$, $\mathscr{H}^t(E) = 0$, so that as soon as E has finite s-dimensional measure, it will immediately have 0 t-dimensional measure for any $t > s$. Thus, for each E there exists a unique s where the Hausdorff measure of E "jumps" from ∞ to 0. This value of s is called the *Hausdorff dimension of E*.

Definition 3.6. Formally, the *Hausdorff dimension* of a set $E \subset \mathbb{R}^n$ is

$$\dim_H(E) = \inf\{s : \mathscr{H}^s(E) = 0\} = \sup\{s : \mathscr{H}^s(E) = \infty\}.$$

Now we can state the following theorem, due to J. Lutz and N. Lutz in [1].

Theorem 3.7 (Point-to-set Principle)**.** *For any $E \subset \mathbb{R}^n$,*

$$\dim_H(E) = \min_{A \subset \mathbb{N}} \sup_{x \in E} \dim^A(x) \tag{3.8}$$

Importantly, the right hand side of this equation is not an infimum, but a minimum, which means that there exists some oracle $A \subset \mathbb{N}$ such that

$$\dim_H(E) = \sup_{x \in E} \dim^A(x).$$

If A is such a minimizing oracle, we call A the *Hausdorff oracle* for E. In fact, there are infinitely many Hausdorff oracles for each set E, since any oracle B containing a Hausdorff oracle A also contains information of A, so it cannot increase the complexity of points in E.

Example 3.2. Suppose $E = \{x\}$ is a singleton set. Then, letting A encode the binary expansion of x, we find that $\dim^A(x) = 0$, so that $\dim_H(E) = 0$, as expected. Similarly, we can see that the dimension of a countable set is 0, since we can encode the information of every point in a single oracle.

Beyond the simple cases of singletons and countable sets, one might wonder how the point-to-set principle can hold if $\dim_H(E)$ is invariant under rotation, translation, and rescaling, while the right hand side of (3.8) seems intimately connected with the particular points that make up E. For example, what happens if we translate E by a "random" point, that is, $x \in \mathbb{R}^n$ with $\dim(x) = n$? Since one point is just countable information, we can simply add the information of this point to the minimizing oracle, so that our computer can calculate points in the translated version of E just as well as in the original, since it has access to arbitrarily good approximations of the translation for free. Part of the power of using oracles is that we can always add more countable information to them. Rotation and rescaling can be dealt with in a similar way, so that the minimum in (3.8) is in fact invariant under all these transformations.

Proof of Point-to-Set Principle. The proof proceeds in two parts. First, we will show that for all oracles $A \subset \mathbb{N}$, $\sup_{x\in E} \dim^A(x) \geq \dim_H(E)$. Then, we will construct the Hausdorff oracle for E, showing that there exists some oracle A for which this inequality is, in fact, an equality.

For the first part, let $E \subset \mathbb{R}^n$, let $d = \dim_H(E)$, and let $A \subset \mathbb{N}$ be any oracle. Then, let $d' = \sup_{x\in E} \dim^A(x)$. We would like to show that $d' \geq d$.

Our definition of d' tells us that for all $x \in E$, $\dim^A(x) \leq d'$, or in other words

$$\liminf_{r\to\infty} \frac{K_r^A(x)}{r} \leq d',$$

that is, for all $x \in E$ there are infinitely many r such that $K_r^A(x) \leq d'r$. By the definition of Kolmogorov complexity, this means that for each x and r there is a finite binary string $\sigma_{x,r}$ such that $\ell(\sigma_{x,r}) \leq d'r$ and

$$x \in B_{2^{-r}}(U^A(\sigma_{x,r})).$$

Now, let

$$\mathcal{B}_r = \{B_{2^{-r}}(q) \,:\, q \in \mathbb{Q} \text{ and } K(q) \leq d'r\}.$$

How many balls are in the collection $\mathcal{B}_r$? Well, $\#\mathcal{B}_r$ should be the same as the number of rationals q such that $K(q) \leq d'r$, which cannot exceed the number of binary strings of length less than or equal to $d'r$. This is because each input to a universal Turing machine uniquely determines an output, so no two q can have the same minimal input. Therefore, $\#\mathcal{B}_r \leq 2^{d'r+1}$, which is the number of strings of length less than $d'r$.

Now, since $U^A(\sigma_{x,r})$ is a rational point with complexity less than $d'r$,

$$B_{2^{-r}}(U^A(\sigma_{x,r}) \in \mathcal{B}_r,$$

so that each x is in a ball contained in the collection $\mathcal{B}_r$ for infinitely many r. Letting

$$\mathcal{W}_i = \bigcup_{r=i}^{\infty} \mathcal{B}_r,$$

we obtain a sequence of countable covers of E, and for any $\delta > 0$, if i is large enough such that $2^{-i} < \delta$, then $\mathcal{W}_i$ will be a δ-cover of E.

Now, suppose for contradiction that $d' < d$, and let $s \in (d', d)$. For a sufficiently large i,

$$\begin{aligned}\mathscr{H}^s_\delta(E) &\leq \sum_{B\in\mathcal{W}_i} |B|^s \\ &= \sum_{r=i}^{\infty} \sum_{B\in\mathcal{B}_r} |B|^s \\ &\leq \sum_{r=i}^{\infty} \sum_{1}^{2^{d'r+1}} (2^{-r})^s \\ &= \sum_{r=i}^{\infty} 2^{r(d'-s)+1},\end{aligned}$$

which converges because $d' - s < 0$. Because this sum converges, as $\delta \to 0$, that is, as $i \to \infty$, the sum goes to 0, so that $\mathscr{H}^s(E) = 0$. But this contradicts the fact that $d = \dim_H(E)$, since $s < d$. Therefore, $d' \geq d$, so for all $A \subset \mathbb{N}$,

$$\sup_{x\in E} \dim^A(x) \geq \dim_H(E).$$

For the second part of the proof, we want to construct the Hausdorff oracle, or the oracle that realizes the above inequality. Again, let $d = \dim_H(E)$, and let $s \in \mathbb{Q}$ and $s > d$. This means $\mathscr{H}^s(E) = 0$, so for a small enough δ, we have $\mathscr{H}^s_\delta(E) < 1$.

Now, pick $t \in \mathbb{N}$ such that $2^{-t} < \delta$. Then there exists a cover $\{U_i\}^{t,s}$ of E with

$$\sum_{i=1}^{\infty} |U_i^{t,s}|^s \leq 1, \tag{3.9}$$

and $|U_i^{t,s}| < 2^{-t}$ for all $i \in \mathbb{N}$. This gives a countable sequence of covers which we will encode in the minimizing oracle, A.

Specifically, A will encode, for each $U_i^{t,s}$, the information of i, t, s, one rational point $q \in U_i^{t,s}$, and a rational approximation r of $|U_i^{t,s}|$. Now, suppose $x \in E$, meaning $x \in U_i^{t,s}$ for some i. To compute x up to precision r relative to A, we only need to specify which $U_i^{t,s}$ it is in, and ensure that $|U_i^{t,s}|$ is around 2^{-r}.

Supposing $2^{-(r+1)} \leq |U_i^{t,s}| \leq 2^{-r}$, (3.9) tells us that there are at most $2^{(r+1)s}$ such $|U_i^{t,s}|$, so that to specify one of them requires at most $(r+1)s$ bits. Therefore,

$$K_r^A(x) \leq (r+1)s + O(\log r),$$

where the error term comes from technical details about Turing machines which we have omitted. Therefore, for every $x \in E$,

$$\dim^A(x) = \liminf_{r\to\infty} \frac{K_r^A(x)}{r} \leq \liminf_{r\to\infty} \frac{(r+1)s}{r} + \frac{O\log r}{r} \leq s,$$

so that for each rational $s > d$,

$$\sup_{x\in E} \dim^A(x) \leq s,$$

implying that $\sup_{x\in E} \dim^A(x) = d$. ◻

The following corollary demonstrates an important application of the point-to-set principle, which allows us to prove lower bounds for $\dim_H(E)$, typically a much more difficult task than proving upper bounds.

Corollary 3.10. *Let $E \subset \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. If for every oracle $A \subset \mathbb{N}$ and every $\varepsilon > 0$, there is a point $x \in E$ such that $\dim^A(x) \geq \alpha - \varepsilon$, then $\dim_H(E) \geq \alpha$. In particular, we only need to find such an x when A is a Hausdorff oracle for E.*

Proof. Our assumption that there is always an $x \in E$ with $\dim^A(x) \geq \alpha - \varepsilon$ implies that $\sup_{x \in E} \dim^A(x) \geq \alpha - \varepsilon$ for all oracles $A \subset \mathbb{N}$, so that

$$\dim_H(E) = \min_{A \subset \mathbb{N}} \sup_{x \in E} \dim^A(x) \geq \alpha - \varepsilon.$$

Since ε was arbitrary, $\dim_H(E) \geq \alpha$. □

This corollary will allow us to easily derive meaningful statements about Hausdorff dimension from the examination of the effective dimension of points relative to an arbitrary oracle.

4. The Effective Dimension of Points on a Line

The rest of this paper will be dedicated to applying the point-to-set principle through an exposition of [2], which proves a bound for the effective dimension of points on lines in $\mathbb{R}^2$. Using the point-to-set principle, this bound immediately leads to bounds on the Hausdorff dimension of Kakeya sets in $\mathbb{R}^2$ and generalized Furstenberg sets in $\mathbb{R}^2$, demonstrating how algorithmic methods can be useful for classical geometric measure theory problems. However, this line result is just one among many, and the point-to-set principle has also extended Marstrand's projection theorem to a much wider class of sets and led to other results involving packing dimension. For a more thorough summary of the applications of algorithmic methods in geometric measure theory, see [5].

The bound in [2] uses a concept developed in [1] called *conditional Kolmogorov complexity*, which will be summarized in the following definitions.

Definition 4.1. Let $\sigma, \tau \in \{0,1\}^*$. The *conditional Kolmogorov complexity of σ given τ* is

$$K(\sigma|\tau) = \min\{\ell(\beta) : U(\beta, \tau) = \sigma\}.$$

This definition is identical to the definition of Kolmogorov complexity, except that τ is now allowed as input, so that $K(\sigma) = K(\sigma|\tau)$, where τ is the empty string. Basically, we are asking how easy it is to calculate σ once we already know τ.

We can now extend this definition into Euclidean space the same way we did for Kolmogorov complexity.

Definition 4.2. For $x \in \mathbb{R}^n$ and $q \in \mathbb{Q}^n$, let

$$K_r(x|q) = \min\{K(p|q) : p \in B_{2^{-r}}(x) \cap \mathbb{Q}^n\}$$

Now, if $x \in \mathbb{R}^n, y \in \mathbb{R}^m$, *the conditional Kolmogorov complexity of x at precision r given y at precision s* is

$$K_{r,s}(x|y) = \max\{K_r(x|q) : q \in B_{2^{-s}}(y) \cap \mathbb{Q}^m\},$$

and for shorthand, we write $K_r(x|y)$ to mean $K_{r,r}(x|y)$.

These definitions may all be extended to include an oracle in the same way as in Section 2, where anytime the superscript A appears, it means we are working relative to the oracle set A.

Definition 4.3. Finally, the *conditional dimension of x given y* is

$$\dim(x|y) = \liminf_{r\to\infty} \frac{K_r(x|y)}{r},$$

and the *conditional dimension of x given y relative to an oracle $A \subset \mathbb{N}$* is

$$\dim^A(x|y) = \liminf_{r\to\infty} \frac{K_r^A(x|y)}{r}.$$

Note that $K_{r,s}^A(x|y) \le K_r^A(x) + O(1)$, and therefore $\dim^A(x|y) \le \dim^A(x)$, since the additional information of y can only decrease the length of input needed to calculate x.

One important tool in the study of complexity and effective dimension is called *symmetry of information*, and quantifies the intuitive result that computing two things at the same time is more or less the same as computing the first one, then the second one *given* the first one. We present the version of symmetry of information in Euclidean space from [2].

Lemma 4.4 (Symmetry of Information in Euclidean Space)**.** *For all $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and $r, s \in \mathbb{N}$ with $r \ge s$,*

(i) $|K_r(x|y) + K_r(y) - K_r(x, y)| \le O(\log r) + O(\log\log \|y\|)$

(ii) $|K_{r,s}(x|x) + K_s(x) - K_r(x)| \le O(\log r) + O(\log\log \|x\|)$,

and in particular, if $A \subset \mathbb{N}$ is any oracle set,

$$\dim^A(x, y) = \dim^A(x|y) + \dim^A(y).$$

Symmetry of information, as well as the following lemma from [1], which specifies the relationship between conditional complexity and complexity relative to an oracle, will be useful to us later when working with conditional dimension.

Lemma 4.5. *Given $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, there is a constant c, depending only on m and n, such that*

$$K_r^y(x) \le K_{r,t}(x|y) + K(t) + c,$$

and in particular, $\dim^y(x) \le \dim(x|y)$.

With these definitions, we can present the main result of [2], which makes progress on the following question: Given a line $L_{a,b} := \{(x, ax+b) : x \in \mathbb{R}\} \subset \mathbb{R}^2$, what can we say about the dimension of points on this line? In particular, is there a line whose points all have the same dimension? The following theorem, due to N. Lutz and D. M. Stull, gives a bound for the dimension of points on a line in terms of the complexity of its slope, y-intercept, and the first coordinate x.

Theorem 4.6. *For any $a, b, x \in \mathbb{R}$, and any oracle $A \subset \mathbb{N}$,*

$$\dim^A(x, ax+b) \ge \dim^A(x|a, b) + \min\{\dim^A(a, b), \dim^{a,b}(x)\}.$$

To clarify notation, $\dim^A(x, ax+b)$ is the same thing as $\dim^A(z)$ where $z \in \mathbb{R}^2$ is the point with coordinates $(x, ax+b)$, whereas $\dim^A(x|(a, b))$ means for each r, the first r digits of *both* a and b are given as input. Finally, when the real numbers

a and b appear in the superscript, it means the oracle set encodes the countable binary expansions of a and b.

We will attempt to provide a geometric interpretation of the proof of Theorem 4.6, due to Lutz and Stull. Much of the literature regarding effective dimension is written from an information-theoretic perspective, but geometric interpretations of many of the proofs are possible, and are often more digestible for those coming from an analysis perspective, rather than a computer science one.

This proof requires several lemmas, the first of which, roughly speaking, provides sufficient conditions for a point $(x, ax+b)$ to have complexity close to $K_r(x, a, b)$, which by symmetry of information is roughly equivalent to $K_r(x|a, b) + K_r(a, b)$.

Lemma 4.7 (Lemma 6 in [2]). *Let $a, b, x \in \mathbb{R}$, $\delta > 0$, $\varepsilon, \eta \in \mathbb{Q}_+$, and r sufficiently large. Suppose that the following hold:*

(i) $K_r(a, b) \leq (\eta + \varepsilon)r$
(ii) For all pairs (u, v) such that $ux + v = ax + b$,

$$K_r(u, v) \geq (\eta - \varepsilon)r + \delta(r - t),$$

where t is such that $\|(a, b) - (u, v)\| = 2^{-t}$ (and $r \geq t$).

If conditions (i) and (ii) hold, then for all oracles $A \subset \mathbb{N}$,

$$K_r^A(x, ax + b) \geq K_r^A(a, b, x) - \frac{4\varepsilon}{\delta} r - O(\log r).$$

As an informal explanation, the conclusion of this lemma essentially guarantees that $K_r^A(x, ax+b)$ is close to the maximum possible, since $K_r^A(a, b, x) \geq K_r^A(x, ax+b) + O(\log r)$ because the information of a, b, and x is enough to calculate $(x, ax+b)$ to a similar precision. Condition (i) says that $K_r(a, b)$ is small, while condition (ii) says that if $ux + v = ax + b$, the only way for $K_r(u, v)$ to be small is for (u, v) to be close to (a, b), meaning they determine very similar lines.

Proof. To show

$$K_r^A(x, ax + b) \geq K_r^A(a, b, x) - \frac{4\varepsilon}{\delta} r - O(\log r), \tag{4.8}$$

we need to show that the information of $(x, ax+b)$ up to precision r, along with $\frac{4\varepsilon}{\delta} r$ more bits of information is enough to know (a, b, x) up to precision r. However, we really only need to calculate (a, b), since we already know x from the pair $(x, ax+b)$.

Finding (a, b) up to precision r is the same thing as finding out, among all possible (u, v) such that $ux + v = ax + b$ (of which there are only finitely many that are unique up to precision r), which one is (a, b). Geometrically, this means we need $(u, v) \in B_r(a, b)$. Conditions (i) and (ii) provide the following restrictions on the possible pairs (u, v):

Because $(u, v) \in B_r(a, b)$, their first r digits are identical, so

$$K_r(u, v) = K_r(a, b) \leq (\eta + \varepsilon)r,$$

by condition (i). Since $ux + v = ax + b$, by condition (ii),

$$K_r(u, v) \geq (\eta - \varepsilon)r + \delta(r - t).$$

Therefore,

$$(\eta - \varepsilon)r + \delta(r-t) \le (\eta+\varepsilon)r$$
$$\implies r - \frac{2\varepsilon}{\delta} r \le t$$
$$\implies 2^{-t} \le 2^{-r} \cdot 2^{\frac{2\varepsilon}{\delta} r}.$$

This means that at most $2^{\frac{4\varepsilon}{\delta} r}$ balls of size 2^{-r} can fit inside a 2^{-t} ball (up to a constant multiple). These balls of radius 2^{-r} are precisely the candidates for (u,v) we are seeking, since by definition of t, all our candidates are within 2^{-t} of (a,b).

Now, to find out (a,b) up to precision r, we just need to specify which of these $2^{\frac{4\varepsilon}{\delta} r}$ balls is the one containing (a,b). This takes at most $\frac{4\varepsilon}{\delta} r$ bits, because we simply enumerate the balls, and then tell the computer the number that corresponds to the ball we want. Therefore, (4.8) holds. □

With this next lemma, we seek to bound the complexity of any (u,v) such that $ux+v = ax+b$ in terms of the complexity of a, b, and x. We will later use this bound to show that condition (ii) from the previous lemma is satisfied for any a, b, and x.

Lemma 4.9 (Lemma 7 in [2]). *Let $a, b, x \in \mathbb{R}$. Suppose $ux + v = ax + b$, and let t be such that $\|(a,b) - (u,v)\| = 2^{-t}$. Then for all $r \ge t$,*

$$K_r(u,v) \ge K_t(a,b) + K_{r-t,r}(x|a,b) - O(\log r)$$

This lemma is presented in a rearranged form which is useful for the final proof, but it can be summarized geometrically. The equation $ux + v = ax + b$ tells us that the points (u,v) and (a,b) are on a line with slope x. This means if (u,v) and (a,b) are very far apart, so that t is small relative to r, knowing (u,v) and (a,b) up to precision r means we know x to a much higher precision, while if they are very close, we know much less about x, but knowing (u,v) gives much more information of (a,b), since t is closer to r.

Proof. Suppose we know (u,v) *and* (a,b) up to precision r. Since $ux + v = ax + v$,

$$x = \frac{b-v}{u-a}.$$

What error do we get in x from the error of 2^{-r} in u, v, a, and b? Well, $b - v$ has error around 2^{-r}, so that

$$\Delta x \le 2^{-r}\frac{1}{|u-a|} \approx \frac{2^{-r}}{2^{-t}} = 2^{-(r-t)}$$

We claim $|u-a| \approx 2^{-t}$, because if u and a were much closer together than 2^{-t}, then by the equation $ux + v = ax + b$, the parameters v and b would also have to be much closer together than 2^{-t}, but this contradicts our assumption that $2^{-t} = \|(a,b) - (u,v)\|$. Therefore, $\Delta x \approx 2^{-(r-t)}$, which tells us that knowing (a,b) and (u,v) up to precision r is enough to know x up to precision $r - t$. Since we know (a,b) up to precision r already,

$$K_{r-t,r}(x,(a,b)) \le K_r(a,b,u,v) \le K_r(u,v) + K_{r,t}((a,b)|(a,b)). \tag{4.10}$$

The second inequality follows because knowing (u,v) up to precision r automatically tells us (a,b) up to precision t, since $(a,b) \in B_{2^{-t}}((u,v))$, so we only need to calculate the rest of (a, b) given the first t digits. Finally, by a direct application of symmetry of information (Lemma 4.4),

$$K_{r,t}((a,b)|(a,b)) = K_r(a,b) - K_t(a,b) + O(\log r) \tag{4.11}$$

and

$$K_{r-t,r}(x,(a,b)) = K_{r-t,r}(x|(a,b)) + K_r(a,b) + O(\log r). \tag{4.12}$$

Rearranging and combining (4.10), (4.11), and (4.12), we get

$$\begin{aligned} K_r(u,v) &\geq K_{r-t,r}(x,(a,b)) - K_{r,t}((a,b)|(a,b)) \\ &\geq K_{r-t,r}(x|(a,b)) + K_r(a,b) - K_r(a,b) + K_t(a,b) - O(\log r) \\ &\geq K_t(a,b) + K_{r-t,r}(x|(a,b)) - O(\log r), \end{aligned}$$

as desired. □

The final lemma constructs a useful oracle set D_r, which guarantees that conditions (i) and (ii) of Lemma 4.7 are met relative to D_r, while $K_r^{D_r}(a,b,x)$ is not lowered too much. We will eventually use this oracle to prove Theorem 4.6 relative to *any* oracle set, by the fact that it is constructed to be more useful than any other oracle in certain key areas, while being less useful than any other oracle otherwise.

Lemma 4.13 (Lemma 8 in [2]). *Let $z \in \mathbb{R}^n$ and $\eta \in \mathbb{Q}$ with $0 \leq \eta \leq \dim(z)$. For all $r \in \mathbb{N}$ there is an oracle D_r, (also depending on n, η, and z) such that*

(1) For all $t \leq r$, $K_t^{D_r}(z) = \min\{\eta r, K_t(z)\} + O(\log r)$
(2) For all $t \in \mathbb{N}$ and $y \in \mathbb{R}^m$, $K_{t,r}^{D_r}(y|z) = K_{t,r}(y|z) + O(\log r)$ and $K_t^{z,D_r}(y) = K_t^z(y) + O(\log r)$.

Basically, D_r helps as much as possible in computing z up to any precision less than r, but doesn't help compute any other $y \neq z$ any more than z does.

Proof. Since $\eta \leq \dim(z) = \liminf_{s\to\infty} \frac{K_s(z)}{s}$, there exists at least one $t \leq r$ such that $K_{t-1}(z) < \eta r$. Let t_0 be the largest such t, and let D_r encode the digits of z from t_0 to r. Note that this makes D_r a finite oracle.

To show that conclusion (1) holds, if $t \geq t_0$, then $K_t(z) \geq K_{t_0}(z) = \eta r$ (this equality obscures some technical details which we skip for brevity; see the appendix of [2]). In fact,

$$K_t^{D_r}(z) = K_{t_0}(z) + O(\log r) = \eta r + O(\log r)$$

because we only need to compute the first t_0 digits, since the remaining $t - t_0$ digits are encoded in D_r, information we can access for free.

On the other hand, if $t \leq t_0$, then $K_t(z) \leq \eta r$, and so

$$K_t^{D_r}(z) \geq K_t(z) + O(\log r), \tag{4.14}$$

meaning D_r can't help too much with the first t digits of z. See the appendix of [2] for the proof of (4.14).

Skipping over technical details, (2) is true because the information encoded in D_r is redundant. If we are computing y conditioned on the first r digits of z, we are already given all of the information of D_r through these digits of z, so

$$K_{t,r}^{D_r}(y|z) \geq K_{t,r}(y|z) + O(\log r),$$

and equality follows because a priori oracles can only lower complexity. Finally,

$$K_t^{z,D_r}(y) = K_t^z(y) + O(\log r)$$

for the same reasons. All of z is already in the oracle set, so we don't need D_r. □

In the main proof of Theorem 4.6, we will use this oracle, along with Lemma 4.9, to show that the hypotheses of Lemma 4.7 hold for any $a, b, x \in \mathbb{R}$, relative to D_r. This requires that the complexity of (a, b) be low, while any other pair (u, v) such that $ux + v = ax + b$ has high complexity unless it is very close to (a, b). Once we have the conclusion of Lemma 4.7, straightforward manipulations yield the desired bound on $\dim^A(x, ax + b)$.

Proof of Theorem 4.6. Let $a, b, x \in \mathbb{R}$ and $\varepsilon \in \mathbb{Q}_+$. Let $A \subset \mathbb{N}$, and let

$$\eta \in \mathbb{Q} \cap [0, \dim^A(a, b)] \cap [0, \dim^{a,b}(x)),$$

with $\delta := \dim^{a,b}(x) - \eta > 0$ (we choose η and δ this way to artificially achieve the minimum term in the final inequality). For each $r \in \mathbb{N}$, let D_r be the oracle described in Lemma 4.13, with $z = (a, b)$. By conclusion (1) of the lemma,

$$\begin{aligned} K_r^{D_r}(a, b) &= \min\{\eta r, K_r(a, b)\} + O(\log r) \\ &\leq (\eta + \varepsilon) r \end{aligned}$$

for every large enough r, so that condition (i) of Lemma 4.7 is satisfied relative to D_r. To show that condition (ii) is also satisfied, let $u, v \in \mathbb{R}$ such that $ux+v = ax+b$, and let t be such that $2^{-t} = \|(a, b) - (u, v)\|$. We want to show

$$K_r^{D_r}(u, v) \geq (\eta - \varepsilon) r + \delta(r - t). \tag{4.15}$$

Again, this statement geometrically means that if (u, v) is not close to (a, b), then it has high complexity. Now, by Lemma 4.9, relativized to D_r,

$$K_r^{D_r}(u, v) \geq K_t^{D_r}(a, b) + K_{r-t,r}^{D_r}(x|(a, b)) - O(\log r),$$

and by Lemma 4.5 and the properties of D_r from Lemma 4.13,

$$\begin{aligned} K_r^{D_r}(u, v) &\geq \min\{\eta r, K_t(a, b)\} + K_{r-t,r}(x|(a, b)) - O(\log r) \\ &\geq \min\{\eta r, K_t(a, b)\} + K_{r-t}^{a,b}(x) - O(\log r) \\ &= \min\{\eta r, \dim(a, b) \cdot t - o(t)\} + \dim^{a,b}(x)(r - t) - o(r). \end{aligned}$$

Finally, by our choice of η and δ,

$$\begin{aligned} K_r^{D_r}(u, v) &\geq \min\{\eta r, \eta t - o(t)\} + (\eta + \delta)(r - t) - o(r) \\ &= \eta t - o(t) + \eta r - \eta t + \delta(r - t) - o(r) \\ &= \eta r - o(r) + \delta(r - t) \\ &\geq (\eta - \varepsilon) r + \delta(r - t) \end{aligned}$$

for every sufficiently large r. Since condition (ii) is now satisfied, the conclusion of Lemma 4.7 holds for any oracle, so that

$$K_r^A(x, ax + b) \geq K_r^{A,D_r}(x, ax + b) \geq K_r^{A,D_r}(a, b, x) - \frac{4\varepsilon}{\delta} r - O(\log r).$$

By symmetry of information and Lemma 4.13,

$$\begin{aligned}K_r^{A,D_r}(a,b,x) &= K_r^{A,D_r}(x|(a,b)) + K_r^{A,D_r}((a,b)) - O(\log r)\\ &= K_r^{A}(x|(a,b)) + \eta r - O(\log r),\end{aligned}$$

so that

$$\frac{K_r^A(x, ax+b)}{r} \geq \frac{K_r^A(x|(a,b))}{r} + \frac{\eta r}{r} - \frac{4\varepsilon r}{\delta r} - \frac{O(\log r)}{r}.$$

Now, taking liminfs yields

$$\dim^A(x, ax+b) \geq \dim^A(x|(a,b)) + \eta - \frac{4\varepsilon}{\delta}, \tag{4.16}$$

and since ε was arbitrary and (4.16) is true for any $\eta \in \mathbb{Q} \cap [0, \dim^A(a,b)] \cap [0, \dim^{a,b}(x))$, we have

$$\dim^A(x, ax+b) \geq \dim^A(x|(a,b)) + \min\{\dim^A(a,b), \dim^{a,b}(x)\}.$$

□

5. Applications in Geometric Measure Theory

Theorem 4.6 provides a short proof of the Kakeya problem in $\mathbb{R}^2$ via the point-to-set principle. The Kakeya problem concerns the dimension of a Kakeya set, which is a set in $\mathbb{R}^n$ containing a unit line segment in every direction. It has been known since 1928 that Kakeya sets exist of arbitrarily small measure, and in fact, there are multiple constructions of null sets that contain a line segment in every direction [10]. In the plane, Kakeya sets were shown to have Hausdorff dimension 2 by Davies in 1971, but for $n \geq 3$, the dimension of Kakeya sets in $\mathbb{R}^n$ is unknown [6]. The following conjecture, however, is widely believed to be true, and has implications across analysis.

Conjecture 5.1 (Kakeya Conjecture)**.** *If $K \subset \mathbb{R}^n$ contains a line segment in every direction, then $\dim_H(K) = n$.*

For more details on the full conjecture and its implications, see [10]. In [1], J. Lutz and N. Lutz provide a new information-theoretic proof of the Kakeya problem in $\mathbb{R}^2$ using conditional dimension. However, we will use Theorem 4.6 to show the same result. The following lemma will be necessary.

Lemma 5.2. *For any oracle $A \subset \mathbb{N}$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$,*

$$\dim^A(x,y) \geq \dim^A(x).$$

Proof. By a relativized version of symmetry of information,

$$\dim^A(x,y) = \dim^A(x|y) + \dim^A(y).$$

Now, we claim that for any $r \in \mathbb{N}$,

$$K_r^A(x|y) + K_r^A(y) \geq K_r^A(x).$$

This is true because the input length on the left hand side is sufficient to compute x up to precision r. We simply first use $K_r^A(y)$ bits of input to compute y, then $K^A(x|y)$ more bits to compute x given y. Now, taking liminfs yields

$$\dim^A(x|y) + \dim^A(y) \geq \dim^A(x),$$

so that $\dim^A(x,y) \geq \dim^A(x)$. □

As a corollary to Theorem 4.6, we now present the following proof that Kakeya sets in $\mathbb{R}^2$ have dimension 2.

Corollary 5.3 (Kakeya conjecture with $n = 2$)**.** *If $K \subset \mathbb{R}^2$ contains a line segment in every direction, then* $\dim_H(K) = 2$.

Proof. K containing a line segment in every direction means that for every $a \in \mathbb{R}$, K contains a line segment with slope a. Given an oracle $A \subset \mathbb{N}$, we want to find $a, b, x \in \mathbb{R}$ such that the point $(x, ax+b) \in K$ and $\dim^A(x, ax+b) = 2$. If we found such a point for every oracle, then by Corollary 3.10 to the point-to-set principle, we would have $\dim_H(K) \geq 2$, and therefore $\dim_H(K) = 2$

Given an oracle $A \subset \mathbb{N}$, let $a \in \mathbb{R}$ be such that $\dim^A(a) = 1$. K contains a line segment with slope a, so let b be the y-intercept of this line segment, and let (x_L, y_L) and (x_R, y_R) be the left and right endpoints, respectively. We want to choose x so that the following hold:

(i) $\dim^A(x|(a, b)) = 1$

(ii) $\dim^{a,b}(x) = 1$

(iii) $\dim^A(a, b) \geq 1$

By the fact that relative to any given oracle, almost every $x \in \mathbb{R}$ is random, we can choose $x \in [x_L, x_R]$ such that $\dim^{A,a,b}(x) = 1$. With this choice of x,

$$\dim^{a,b}(x) \geq \dim^{A,a,b}(x) = 1,$$

and by Lemma 4.5,

$$\dim^A(x|a, b) \geq \dim^{A,a,b}(x) = 1,$$

so that (i) and (ii) hold. Condition (iii) follows from Lemma 5.2, since

$$\dim^A(a, b) \geq \dim^A(a) = 1.$$

Finally, by Theorem 4.6 and our choices of a, b, and x,

$$\begin{aligned}\dim^A(x, ax+b) &\geq \dim^A(x|a, b) + \min\{\dim^A(a, b), \dim^{a,b}(x)\} \\ &\geq 1 + 1 = 2,\end{aligned}$$

so by the point-to-set principle, $\dim_H(K) = 2$. $\square$

This proof illustrates the power of the point-to-set principle and Theorem 4.6. All we had to do was choose a random slope a and a random coordinate x, and the fact that random points are so abundant and that K contains lines of every slope guaranteed that we could choose these random points such that $(x, ax + b) \in K$.

Theorem 4.6 gives insight into another closely related classical geometric measure theory problem: the Furstenberg set conjecture, which has recently been fully resolved in the plane by K. Ren and H. Wang in [4]. The Furstenberg set conjecture concerns the Hausdorff dimension of a generalization of a Kakeya set, where instead of a line in every direction, our set contains an α-dimensional subset of a line in a β-dimensional subset of directions.

Definition 5.4. $E \subset \mathbb{R}^2$ is an (α, β)-Furstenberg set (and we write $E \in F_{\alpha,\beta}$) if there exists $J \subset S^1$ such that $\dim_H(J) \geq \beta$ and for each $e \in J$ there is a line ℓ_e in the direction e such that $\dim_H(E \cap \ell_e) \geq \alpha$.

Note that Kakeya sets are in $F_{\alpha,\beta}$ for any $\alpha, \beta \leq 1$, and in particular are a special type of $(1,1)$-Furstenberg sets. Like the Kakeya conjecture, the Furstenberg set conjecture asks about a lower bound on the Hausdorff dimension of sets in $F_{\alpha,\beta}$.

Theorem 5.5 (Furstenberg set conjecture)**.** *If $E \in F_{\alpha,\beta}$, then*

$$\dim_H(E) \geq \min\left\{\alpha + \beta, \frac{3\alpha + \beta}{2}, \beta + 1\right\}.$$

This theorem was proven in [4] using classical methods, but following [2], we can very straightforwardly prove the $\alpha+\beta$ term in the bound for the case $0 \leq \beta \leq \alpha \leq 1$ using a similar strategy as in our proof of the Kakeya problem in the plane.

Theorem 5.6 (Theorem 12 in [2])**.** *Let $\alpha, \beta \in (0,1]$. If $E \in F_{\alpha,\beta}$, then*

$$\dim_H(E) \geq \alpha + \min\{\alpha, \beta\}.$$

In particular, if $0 \leq \beta \leq \alpha \leq 1$, then $\dim_H(E) \geq \alpha + \beta$.

Proof. By the definition of $F_{\alpha,\beta}$, there exists some $J \subset S^1$ with $\dim_H(J) \geq \beta$ and $\dim_H(E \cap \ell_e) \geq \alpha$ for every $e \in J$. Let $M = \{\tan(e) : e \in J\}$ be the set of slopes of the lines in the directions in J. We claim that $\dim_H(J) = \dim_H(M)$, since the tangent map is locally bi-Lipschitz. Let A be the Hausdorff oracle for E, so that

$$\dim_H(E) = \sup_{x \in E} \dim^A(x).$$

By the point-to-set principle for M,

$$\dim_H(M) \leq \sup_{a \in M} \dim^A(a),$$

so that given $\varepsilon > 0$, there exists some $a \in M$ such that $\dim^A(a) \geq \beta - \varepsilon$. Now, let b be such that $\dim_H(E \cap L_{a,b}) \geq \alpha$. By Lemma 5.2,

$$\dim^A(a,b) \geq \dim^A(a) \geq \beta - \varepsilon.$$

Now we will use the point-to-set principle for $E \cap L_{a,b}$. This tells us that there is some $x \in \mathbb{R}$ such that $(x, ax+b) \in E \cap L_{a,b}$ and $\dim^{A,a,b}(x) \geq \alpha - \varepsilon$. By Lemma 4.5,

$$\dim^A(x|a,b) \geq \dim^{A,a,b}(x) \geq \alpha - \varepsilon,$$

and by basic properties of oracles,

$$\dim^{a,b}(x) \geq \dim^{A,a,b}(x) \geq \alpha - \varepsilon,$$

so that by Theorem 4.6,

$$\begin{aligned}\dim^A(x, ax+b) &\geq \alpha - \varepsilon + \min\{\beta - \varepsilon, \alpha - \varepsilon\} \\ &= \alpha + \min\{\beta, \alpha\} - 2\varepsilon.\end{aligned}$$

Since ε was arbitrary, we are done by Corollary 3.10. $\square$

In the above proof, we straightforwardly applied the point-to-set principle, illustrating the fact that whenever you have a bound on Hausdorff dimension, the point-to-set principle gives significant information about the effective dimension of points, which, after some effort understanding the structure of effective dimension, as in the work of Theorem 4.6, can immediately be translated into bounds on Hausdorff dimension via the point-to-set principle.

Continued study of the effective dimension of points on lines or other geometric structures, especially extended into higher dimensions, could give insight into important problems in geometric measure theory, such as the Kakeya problem and Furstenberg conjecture, which remain unsolved in higher dimensions.

Acknowledgments

This paper would not be possible without my mentor, Iqra Altaf, who has helped me greatly in understanding these algorithmic methods and has been there for me in navigating the crazy world of mathematics. Enormous thanks to Iqra. I would also like to thank Daniel Chen for his help proofreading, and J. Peter May for organizing the University of Chicago REU year after year. Finally, a huge thanks to Novak Chernesky for helping me actually complete this paper and learn to express my ideas with clarity.

References

[1] Jack H. Lutz and Neil Lutz. Algorithmic information, plane Kakeya sets, and conditional dimension. https://arxiv.org/abs/1511.00442. 2016.
[2] Neil Lutz and D. M. Stull. Bounding the Dimension of Points on a Line. https://arxiv.org/abs/1612.00143. 2017.
[3] Kenneth J. Falconer. The Geometry of Fractal Sets. Cambridge University Press. 1985.
[4] Kevin Ren and Hong Wang. Furstenberg sets estimate in the plane. https://arxiv.org/abs/2308.08819. 2023.
[5] Jack H. Lutz and Elvira Mayordomo. Algorithmic Fractal Dimensions in Geometric Measure Theory. https://arxiv.org/abs/2007.14346. 2020.
[6] Roy O. Davies. Some Remarks on the Kakeya problem. *Proc. Cambridge Phil. Soc.,* 69:417-421, 1971.
[7] Krishna B. Athreya, John M. Hitchcock, Jack H. Lutz, and Elvira Mayordomo. Effective strong dimension in algorithmic information and computational complexity. *SIAM J. Comput.*, 37(3):671-705, 2007.
[8] Ming Li and Paul M.B. Vitányi. *An Introduction to Kolmogorov Complexity and Its Applications*. Springer, third edition, 2008.
[9] Jack H. Lutz. The Dimensions of Individual Strings and Sequences. https://arxiv.org/abs/cs/0203017. 2002.
[10] Izabella Laba. The Kakeya Problem, and connections to harmonic analysis. https://secure.math.ubc.ca/~ilaba/kakeya.html. 2001.