A PROOF OF STIEFEL'S PARALLELIZABILITY THEOREM

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Abstract. This paper gives a proof of Stiefel's parallelizability theorem, which states that all compact, orientable 3-manifolds are parallelizable. In doing so, Stiefel-Whitney classes, Steenrod squaring operations, Wu classes, and some basics of obstruction theory are introduced. This proof is as outlined in Problem 12-B of [1], and most of the work leading up to it follows the same text. Knowledge of basic algebraic topology and manifolds is expected.

Contents

1. Stiefel-Whitney classes and Steenrod operations

We will begin by stating the definition of a vector bundle, and some related concepts as necessary to state the axioms for Stiefel-Whitney classes.

Definition 1.1. A **vector bundle**, ξ, is defined over a given **base space** B, and also includes the **total space** $E(\xi)$ and a continuous **projection map**, $\pi : E \to B$. This must also meet the conditions, for any $b \in B$, that $\pi^{-1}(b)$ (known as the **fiber over** b) has the structure of a vector space, and that there is some neighborhood U around b such that $f : U \times \mathbb{R}^n \to \pi^{-1}(U)$ (for some nonnegative integer n) produces an isomorphism of vector spaces (using the structure defined) $x \mapsto f(u, x)$ for any $u \in U$.

Definition 1.2. A **trivial bundle** is one for which the entire base space B can be chosen as the neighborhood U in order to meet the second condition described above.

Definition 1.3. Given a bundle ξ, the **induced bundle** $g^*\xi$ can be defined over any space B_1, given a map $g : B_1 \to B$, to have total space contained within $B_1 \times E$ and include all points (b, e) such that $g(b) = \pi(e)$ (and a projection map that outputs b).

Definition 1.4. Given two bundles, ξ_1 and ξ_2, their **Cartesian product**, $\xi_1 \times \xi_2$, can be defined to have total space $E(\xi_1) \times E(\xi_2)$ and projection map $\pi_1 \times \pi_2(e_1, e_2) =$

$(\pi_1(e_1), \pi_2(e_2))$. Then, taking $d : B \to B \times B$ to be the diagonal embedding, the **Whitney sum** of two bundles over the base space B can be defined, $\xi_1 \oplus \xi_2 = d^*(\xi_1 \times \xi_2)$.

Definition 1.5. Over a given projective space, P^n, we can define the **canonical line bundle**, γ_n^1, with total space the subset of $P^n \times \mathbb{R}^{n+1}$ including all points $(\{\pm x\}, v)$ such that v is a scalar multiple of x (and projecting onto $\{\pm x\}$).

We now have enough to state the axioms defining the Stiefel-Whitney classes of a vector bundle ξ, which give a sequence of classes of the singular cohomology groups, with coefficients in $\mathbb{Z}/2$, of the base space.

Axiom 1.6. We define a sequence of **Stiefel-Whitney classes**, $w_i(\xi) \in H^i(B(\xi); \mathbb{Z}/2)$, such that $w_0(\xi) = 1$ and $w_i(\xi) = 0$ for $i > n$, n being the vector dimension of the bundle fibers.

Axiom 1.7. If $f : B(\xi) \to B(\eta)$ is the restriction to the base spaces of a continuous function on the total spaces that brings fibers isomorphically into fibers, then $w_i(\xi) = f^* w_i(\eta)$ (pullback being in the cohomology sense).

Axiom 1.8. If ξ and η share a base space, then $w_k(\xi \oplus \eta) = \sum (w_i(\xi) \smile w_{k-i}(\eta))$. (Going forward, such products will be assumed to be the cup product, $\smile$.)

Axiom 1.9. $w_1(\gamma_1^1)$ is nonzero. (In practice, this prevents the theory from being trivial.)

Going forward, coefficients in $\mathbb{Z}/2$ will be assumed unless otherwise specified.

Before continuing, we will prove the existence of Stiefel-Whitney classes defined according to these axioms. This requires first to define the Steenrod operations, which will remain important later on.

Definition 1.10. Acting on a cohomology group $H^n(X, Y)$, for two spaces $X \supset Y$, the **Steenrod squaring operations** give a sequence of additive homomorphisms $Sq^i : H^n(X, Y) \to H^{n+i}(X, Y)$ that meet the following three properties:

Given $f : (X, Y) \to (X', Y')$, $Sq^i \circ f^* = f^* \circ Sq^i$.

Given $a \in H^n(X, Y)$, $Sq^0(a) = a$, $Sq^n(a) = a \smile a$, and $Sq^i(a) = 0$ for $i > n$.

Given a, b such that $a \smile b$ is defined, $Sq^k(a \smile b) = \sum Sq^i(a) \smile Sq^{k-i}(b)$.

We must now also work to define the Thom isomorphism. Note that E_0 and F_0 refer to the nonzero subsets of total space E and any given fiber $F = \pi^{-1}(b)$.

Definition 1.11. The **fundamental cohomology class** is the unique class $u \in H^n(E, E_0)$ such that for each fiber F, it restricts as $u|(F, F_0) \in H^n(F, F_0)$ to the unique nonzero class.

It can be proved as a theorem (for example in chapter 10 of [1]) that u exists and $x \mapsto x \smile u$ defines an isomorphism.

Definition 1.12. The **Thom isomorphism** $\phi : H^k(B) \to H^{k+n}(E, E_0)$ is defined such that $\phi(a) = \pi^*(a) \smile u$.

We can now define the Stifel-Whitney classes by $w_i(\xi) = \phi^{-1} Sq^i \phi(1)$, with $1 \in H^0(B)$ and by showing that this definition meets the given axioms, will have proven their existence by construction.

Lemma 1.13. *This definition of w_i satisfies Axiom 1.6.*

Proof. We see by following which cohomology group the output of each function is in that $w_i \in H^i(B)$. The other points of the axiom follow from the second property required of Sq^i. □

Lemma 1.14. *This definition of w_i satisfies Axiom 1.7.*

Proof. If $f : E \to E'$ brings fibers isomorphically into fibers, it must induce a map $g : (E, E_0) \to (E', E'_0)$. Then, if u' is the fundamental cohomology class of (E, E'), $g^*(u') = u$ by uniqueness. We then see that $g^* \circ \phi' = \phi \circ f^*$ (when considering the restriction of f to the base space), which applying the first property required of Sq^i, gives $f^* w_i(\xi') = w_i(\xi)$. □

Lemma 1.15. *This definition of w_i satisfies Axiom 1.8.*

Proof. Consider a Cartesian product of bundles, $\xi'' = \xi \times \xi'$. Given their fundamental cohomology classes, we can take $u \times u' \in H^{n+m}(E \times E', E \times E'_0 \cup E_0 \times E')$. It follows from the definition then that $u \times u' = u''$, and then we can relate the Thom isomorphisms, $\phi''(a \times b) = \phi(a) \times \phi(b)$. This gives

$$\phi''(w_i(\xi'')) = Sq^i(u'') = \sum_j Sq^j(u) \times Sq^{i-j}(u') = \sum_j \phi(w_j(\xi)) \times \phi'(w_{i-j}(\xi')) =$$

$$\sum_j \phi''(w_j(\xi) \times w_{i-j}).$$

Then, applying $(\phi'')^{-1}$ gives $w_i(\xi'') = \sum_j w_j(\xi) \times w_{i-j}(\xi)$, which becomes the formula required by the axiom after the pullback of the diagonal embedding. □

Lemma 1.16. *This definition of w_i satisfies Axiom 1.9.*

Proof. Considering the vectors of length ≤ 1 in the total space of γ_1^1, these form a Mobius strip M, with boundary ∂M a circle. Since these are deformation retracts of E and E_0 respectively, $H^*(M, \partial M) \cong H^*(E, E_0)$. Since $\mathbb{R}P^2 - D^2$ is homeomorphic to M, by the excision theorem,

$$H^*(M, \partial M) \cong H^*(\mathbb{R}P^2, D^2) \cong H^*(\mathbb{R}).$$

Then, $u \in H^1(E, E_0)$, which is obviously nonzero, must correspond under the composition of isomorphisms to the generator $a \in H^1(\mathbb{R}P^2)$. Then $Sq^1(u)$ corresponds to $Sq^1(a) = a \smile a$, which is nonzero because it is the generator of $H^2(\mathbb{R}P^2)$. Applying the inverse of the Thom isomorphism gives that $w_1(\gamma_1^1)$ is nonzero. □

2. Wu classes

We will now begin considering characteristic classes in relation to manifolds. A simple definition begins this.

Definition 2.1. Over a smooth manifold M, we can define the **tangent bundle**, τ_M, with total space including all points (x, v) where $x \in M$ and v is tangent to M at x and projection map $\pi(x, v) = x$. If τ_M is trivial, then M is called **parallelizable**.

Now we must build the concept of an orientation.

Definition 2.2. Given a point $x \in M$ for some manifold M, a **local orientation**, μ_x, is a choice of generator of $H_n(M, M - x; \mathbb{Z})$.

Definition 2.3. An **orientation** on M assigns to each $x \in M$ a local orientation μ_x in a "continuous" fashion, that is, such that for each compact neighborhood N, there is a class $\mu_N \in H_n(M, M - N; \mathbb{Z})$ whose image under the isomorphism into $H_n(M, M - x; \mathbb{Z})$ is μ_x for every $x \in N$.

If M is compact, it follows that we can take μ_M defined as such. Then we define $\mu = \mu_M$ to be the **fundamental homology class**.

Then, using $\langle\rangle$ to represent evaluation, i.e. the Kronecker index, we can consider a sequence of cohomology classes $v_k \in H^k(M)$ such that $\langle v_k \smile x, \mu\rangle = \langle Sq^k(x), \mu\rangle$ for every $x \in H^{n-k}(M)$. These v_k represent the **Wu classes** of a manifold M. We can observe that v_k must be zero whenever $k > n - k$.

Both the existence and uniqueness of such classes and the theorem below derive from the following formulation of the Poincare Duality Theorem.

Theorem 2.4. *Given a basis $b_1, \ldots, d_r$ of $H^*(M)$, there is a corresponding dual basis $b_1^*, \ldots, b_r^*$ such that $\langle b_i \smile b_j^*, \mu\rangle = 1$ for $i = j$ and is zero otherwise.*

Theorem 2.5. *The Stiefel-Whitney class of a manifold's tangent bundle can be derived from its Wu classes:*

$$w_k = \sum Sq^i(v_{k-i})$$

Some more work will be needed to build up to the proof of this theorem.

First, see that a tangent bundle τ_M can be associated with the normal bundle of the diagonal embedding of M in $M \times M$ by $(x, v) \mapsto ((x, x), (v, -v))$. From this we can define a fundamental cohomology class

$$u' \in H^n(M \times M, M \times M - \Delta M).$$

Definition 2.6. The **diagonal cohomology class**, $u'' \in H^n(M \times M)$, is the restriction of u', defined above, under $H^n(M \times M, M \times M - \Delta M) \to H^n(M \times M)$.

Given we are working with coefficients in $\mathbb{Z}/2$, which is a field, we have that $H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$, which lets us easily define a new operation.

Definition 2.7. The **slant product** is a map $/ : H^{p+q}(X \times Y) \otimes H_q(Y) \to H^p(X)$ such that $(a \times b)/\beta = a\langle b, \beta\rangle$.

Lemma 2.8. *Given a compact manifold, its (well-defined) fundamental homology class μ can be related to the diagonal cohomology class u'' by $u''/\mu = 1 \in H^0(M)$.*

Proof. For a given $x \in M$, consider a homomorphism $H^n(M \times M) \to H^n(x \times M)$ and note that it will take u'' to $1 \times f_x^*(u'')$, where $f_x(y) = (x, y)$ gives an embedding of M into $x \times M$. This is then mapped into $H^0(x)$ by $/\mu$, and the output of this map will be the restriction to x of u''/μ.

Specifically, $(1 \times f_x^*(u''))/\mu = \langle f_x^*(u''), \mu\rangle$, and the Kronecker index must evaluate to 1 based on how u'' is defined from u'. Then, since the restriction of u''/μ to $H^0(x)$ is 1 for every $x \in M$, u''/μ must itself be 1. □

Lemma 2.9. *For M compact and smooth, the Stiefel-Whitney classes of τ_M can be calculated $w_i = Sq^i(u'')/\mu$.*

Proof. Recalling the formula used to prove the existence of SW classes, see that $\pi^*(w_i) \smile u = Sq^i(u)$, for $u \in H^n(E, E_0)$ being the fundamental cohomology class. Considering the isomorphism $H^*(E, E_0) \cong H^*(M \times M, M \times M - \Delta M)$, and subsequently the restriction to $H^n(M \times M)$, it then follows $Sq^i(u'') = (w_i \times 1) \smile u''$.

Applying $/\mu$ to both sides and using the linearity of the slant product gives $Sq^i(u'')/\mu = w_i \smile (u''/\mu)$. Applying the above lemma to the second part of the cup product, it follows that $Sq^i(u'') = w_i$. □

We have now done enough to offer a proof of Theorem 2.5.

Proof. By choosing a basis and corresponding dual basis of $H^*(M)$, as in Theorem 2.4, we can derive, for any cohomology class x,

$$x = \sum b_j \langle x \smile b_j^*, \mu \rangle.$$

Then,

$$v_m = \sum b_j \langle Sq^m(b_j^*), \mu \rangle.$$

Then, we look at the formula given in the theorem to see

$$\sum Sq^i(v_{k-i}) = \sum (\sum Sq^i(b_j) \langle Sq^{k-i}(b_j^*), \mu \rangle) = \sum (\sum (Sq^i(b_j) \times Sq^{k-i}(b_j^*))/\mu).$$

We can eliminate i from the sum by using the third property required in Definition 1.10, and j by considering the construction of the diagonal cohomology class to simplify this to

$$Sq^k(u'')/\mu,$$

which equals w_k by Lemma 2.9. □

3. Obstruction theory

In order to continue, we will address some of the basics of obstruction theory. Loosely speaking, this involves the procedure of extending continuous maps defined on CW complexes to higher dimensions.

(This theory will thus be useful here because smooth compact manifolds can be given the structure of CW complexes.)

Definition 3.1. A **Stiefel manifold**, $V_k(\mathbb{R}^n)$, is defined as all linearly independent k-frames in $\mathbb{R}^n$, that is, all k-tuples of linearly independent vectors, considered as a subset of the Cartesian product of k copies of $\mathbb{R}^n$.

Definition 3.2. Given a bundle, ξ, we can define the associated **Stiefel manifold bundle**, $V_k(\xi)$, with total space including all points $(x, (v_1, \dots, v_k))$, that is, all points in the base space with k-frames in their respective fibers. Fibers in this new bundle are thus Stiefel manifolds.

Definition 3.3. A **cross-section** of a bundle is a continuous map from its base space into its total space that takes each point into a point in its respective fiber. It is **nowhere zero** if in every fiber, it maps to a nonzero vector.

Note that a cross-section of a Stiefel manifold bundle is, in fact, a k-tuple of linearly independent cross-sections of the original bundle. Note also that it is simple to construct a cross-section of $V_k(\xi)$ over the $(n-k)$-skeleton of B. It is the question of extending this cross-section to higher dimensions that allows the definition of obstruction classes, as in [2].

Definition 3.4. First defining $j = n - k + 1$, we define a **primary obstruction class**, $o_j(\xi) \in H^j(B; \pi_{n-k}(V_k(\mathbb{R}^n)))$ (n still representing the dimension of a fiber), such that $V_k(\xi)$ has a cross-section of the $(n-k+1)$-skeleton of B if and only if o_j is zero.

Before stating a theorem describing these classes, I will provide a couple of definitions necessary to its proof.

Definition 3.5. An **infinite Grassmann manifold**, G_n, is the manifold constructed from the set of n-dimensional subspaces of $\mathbb{R}^\infty$, which is the set of infinite sequences of real numbers with all but finitely many elements zero.

Definition 3.6. A **universal bundle**, γ^n, is constructed over G_n to have total space including all pairs of elements in G_n (which are subspaces), and points in that subspace.

It is an important fact that for any bundle ξ over a paracompact base space (such as a CW complex), there is a continuous map from ξ into γ^n that brings fibers isomorphically into fibers (see chapter 5 in [1]).

Theorem 3.7. *If we reduce the coefficients of the obstruction class into $\mathbb{Z}/2$, using some homomorphism h (which may be an isomorphism if j is even and less than n), then the induced class, $h_* o_j(\xi) \in H^j(B;\mathbb{Z}/2)$, equals the Stiefel-Whitney class, $w_j(\xi)$.*

Proof. As shown in chapter 7 of [1], $H^*(G_n)$ is a polynomial algebra generated by the Stiefel-Whitney classes of γ^n, and thus we can write

$$h_* o_j(\gamma^n) = f_j(w_1(\gamma^n), \dots, w_n(\gamma^n)),$$

for a polynomial f_j.

Then, using the relation of γ^n to any bundle ξ over a CW complex, Axiom 1.7, and Theorem 35.7 in [2], we state

$$h_* o_j(\xi) = f_j(w_1(\xi), \dots, w_n(\xi)).$$

From this, we observe that the polynomial expression must have dimension $j \leq n$, and thus based on the dimensions of the w_i, we can rewrite

$$f_j(w_1, \dots, w_j) = f'(w_1, \dots, w_{j-1}) + \lambda w_j,$$

with $\lambda \in \mathbb{Z}/2$. We first will show that the polynomial $f' = 0$.

Define a bundle $\eta = \gamma^{j-1} \oplus \epsilon^{n-j+1}$, and see it is obvious that V_{n-j+1} has a cross-section over the j-skeleton, so $o_j(\eta)$ must be zero. It also follows from construction that $w_j(\eta) = 0$ and $w_i(\eta) = w_i(\gamma^{j-1})$ for $i < j$. Thus, it must hold that

$$f'(w_1(\gamma^{j-1}), \dots, w_{j-1}(\gamma^{j-1})) = 0,$$

and since these classes are independent, $f' = 0$. It remains only to prove that $\lambda = 1$.

Now, restrict γ^n to denote a new bundle $\chi = \gamma_1^j \oplus \epsilon^{n-j}$ over $G_j(\mathbb{R}^{j+1}) \cong \mathbb{R}P^j$. The isomorphism follows from the correspondence between j-planes and normal lines, and then γ_1^j can be described by antipodal points on S^j paired with vectors in the plane(s) tangent to the sphere at those points.

This bundle has a cross section that is nonzero over all but one point by taking the part of any other vector normal to a given vector. If we choose this point in the interior of a j-cell of $\mathbb{R}P^j$, then we have a cross-section of the $(j-1)$-skeleton and a way to assign to the j-cell a generator of $\pi_{n-k}(V_k(\mathbb{R}^n))$. By theorem 25.6 in [2], this group is either $\mathbb{Z}$ or $\mathbb{Z}/2$, and thus nonzero. Thus, $h_* o_j(\gamma_1^j)$ is nonzero, and so λ must also be nonzero, and the proof follows. □

4. Orientable bundles

The main result of this section will require an important construction known as a **Gysin sequence**.

Lemma 4.1. *Given a bundle ξ, where $\pi_0 : E_0 \to B$ is the restriction of π to the nonzero subset of E, there must exist an exact sequence*

$$\cdots \to H^{j-n}(B) \xrightarrow{\smile w_n} H^j(B) \xrightarrow{\pi_0^*} H^j(E_0) \to H^{j-n+1}(B) \to \cdots .$$

Proof. The exactness axiom of cohomology gives the sequence

$$\cdots \to H^j(E, E_0) \to H^j(E) \to H^j(E_0) \xrightarrow{\delta} H^{j+1}(E, E_0) \to \cdots .$$

Then, using the isomorphism $\smile u$, we can replace $H^j(E, E_0)$ with $H^{j-n}(E)$. This gives the exact sequence

$$\cdots \to H^{j-n}(E) \xrightarrow{g} H^j(E) \to H^j(E_0) \to H^{j-n+1}(E) \to \cdots ,$$

where g represents a cup product with the restriction of u to E.

Now, note that there is an isomorphism between $H^*(E)$ and $H^*(B)$. This takes u to $w_n(\xi)$ by the Thom isomorphism, as in Definition 1.12, which yields the required exact sequence. □

A 2-fold cover $\tilde{B} \to B$ is one such that each point of B has a neighborhood with an inverse image that is two disjoint copies of itself. Given such, we can generate a line bundle, η, over B by considering $\tilde{B} \times \mathbb{R}$ and associating (b, t) and $(b', -t)$ given b, b' distinct points that map to the same point in B.

Since $\tilde{B}$ is a deformation retract of E_0, it follows from the above lemma that there must be an exact sequence

$$\cdots \to H^{j-1}(B) \xrightarrow{\smile w_1} H^j(B) \to H^j(\tilde{B}) \to H^j(B) \to \cdots .$$

Now, we can begin to discuss those vectors bundles to which we can give an orientation.

Definition 4.2. An **orientation** of a vector bundle, ξ, requires giving an orientation to every fiber such that for any $b \in B$, there is a neighborhood U such that the the function constructed by the condition of local triviality is orientation-preserving when restricted to each $b' \in U$.

Now, denote by $\tilde{G}_n$ a Grassmann manifold of oriented n-planes, which provides a 2-fold cover of G_n, since a plane has two possible orientations. Then, we can denote an oriented universal bundle, $\tilde{\gamma}^n$. For any oriented bundle ξ, the map $\xi \to \gamma^n$ produces a unique map into $\tilde{\gamma}^n$ that preserves orientation.

Theorem 4.3. *If a vector bundle ξ over a paracompact base space is orientable, then $w_1(\xi) = 0$.*

Proof. It will be sufficient here to show that $w_1(\tilde{\gamma}^n) = 0$, based on the map that is known to exist from ξ.

As previously described, there must exist an exact sequence

$$\cdots \to H^{j-1}(G_n) \xrightarrow{\smile w_1(\eta)} H^j(G_n) \to H^j(\tilde{G}_n) \to H^j(G_n) \to \cdots ,$$

where η is the line bundle associated with the 2-fold covering. Suppose $w_1(\eta) = 0$.

Then, we consider the exact sequence

$$0 \to H^0(G_n) \to H^0(\tilde{G}_n) \to H^0(G_n) \xrightarrow{\smile w_1(\eta)} \cdots ,$$

and since the rightmost arrow here is a zero map, $H^0(\tilde{G}_n) \cong H^0(G_n) \oplus H^0(G_n)$. This would imply that $\tilde{G}_n$ is not continuous, which is not true, so $w_1(\eta)$ is nonzero.

Since η is a bundle over G_n, and $H^*(G_n)$ is generated by the Stiefel-Whitney classes of γ^n, it follows by dimension that $w_1(\eta) = w_1(\gamma^n)$. Then, since the image of $\smile w_1(\gamma^n)$ is in the kernel of all maps into $H^*(\tilde{G}_n)$, we can conclude that $w_1(\tilde{\gamma}^n) = 0$. □

5. Proof of theorem

At this point we will state the main result of this paper.

Theorem 5.1. *Let M be any orientable, compact 3-manifold. Then, M is parallelizable.*

First, note that if M is orientable, then τ_M is also orientable. Specifically, the definition of orientation for τ_M is equivalent to assigning a preferred generator to $H_n(DM_x, DM_x - 0)$ in the same "continuous" fashion seen in Definition 2.3. Since $H_n(DM_x, DM_x - 0)$ is isomorphic to $H_n(M, M - x)$, the equivalence of the existence of the two orientations follows smoothly.

We will continue by synthesizing much of what has been already proven to prove a lemma.

Lemma 5.2. *Given such an orientable, compact 3-manifold M, $w_2(\tau_M) = 0$.*

Proof. We know from Theorem 2.5 that

$$w_2 = \sum Sq^i(v_{2-i}).$$

Applying Definition 1.10, note that

$$Sq^0(v_2) = v_2,\ Sq^1(v_1) = v_1 \smile v_1,\ Sq^2(v_0) = 0.$$

Moreover, v_2 must be zero because it derives from Sq^2 acting on elements of $H^1(M)$, which means that $w_2 = v_1 \smile v_1$. We can also reason that $w_1 = Sq^0(v_1) = v_1$ (since $Sq^1(v_0) = 0$).

Now, since τ_M is orientable, it follows from Theorem 4.3 that $w_1 = 0$, so $v_1 = 0$, and thus $w_2 = v_1 \smile v_1 = 0$. □

I will now give a proof of Theorem 5.1, and thus conclude the paper.

Proof. By Theorem 3.7, since w_2 is zero and h is a homomorphism, the obstruction class $o_2(\tau_M)$ is zero. This means, by Definition 3.4, that the obvious cross section of $V_2(\tau_M)$ over the 1-skeleton of M can be extended over the 2-skeleton.

Then, we have another obstruction class, $o_3(\tau_M) \in H^3(M; \pi_2(V_2(\mathbb{R}^3)))$, that relates to the extension of this section over the 3-skeleton of M (i.e. all of M). However, $\pi_2(V_2(\mathbb{R}^3)) \cong \pi_2(\mathrm{SO}(3)) = 0$ (following from that SO(3) is diffeomorphic to $\mathbb{R}P^3$). Therefore, this new obstruction class must also be zero, so there exists a cross section of $V_2(\tau_M)$ over all of M.

There then exist two linearly independent cross-sections of τ_M, and so $\tau_M = \epsilon^2 \oplus \lambda$ for a trivial 2-dimensional sub-bundle ϵ^2, since the cross-section maps can be used for the triviality condition. Note that for any trivial bundle, $w_i = 0$ for

$i > 0$, since a map can be constructed into a bundle over a single point. Thus, $w_1(\tau_M) = w_1(\epsilon^2) + w_1(\lambda) = w_1(\lambda) = 0$, since τ_M is orientable. Then, repeating the obstruction theoretic approach above shows that λ must also be a trivial bundle. Since τ_M is the Whitney sum of two trivial bundles, it is trivial itself.

Then, since M has a trivial tangent bundle, it is parallelizable, and the proof is complete. □

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