# RUSSO-SEYMOUR-WELSH THEORY FOR PERCOLATION MODELS

AARON HO

ABSTRACT. Russo-Seymour-Welsh (RSW) Theory studies the behavior of crossing probabilities in percolation models, focusing on crossings of rectangular "boxes". It asserts that a lower bound on the probability of crossing a rectangle of aspect ratio $\alpha$ implies a lower bound on the probability of crossing a rectangles of larger aspect ratio $\beta$. This paper examines RSW theory in both the discrete and continuum percolation settings, highlighting the distinctions and challenges when switching between different structural settings.

## CONTENTS



## 1. INTRODUCTION

In short, Percolation Theory is the study of properties of randomly generated subgraphs. These subgraphs are formed by selecting each edge or vertex with a certain probability $p$, which determines whether they are included in the random subgraph.

We consider the following standard model in Percolation Theory, which is the $d$-dimensional integer lattice with vertex set $\mathbb{Z}^d$ and edge set consisting of all vertices that are Euclidean distance 1 apart. In this section, let $\Lambda$ denote this graph. In general, $\Lambda$ will denote infinite unoriented graphs, while $\overrightarrow{\Lambda}$ will denote infinite oriented graphs. Standard terminology in Percolation Theory refers to vertices and edges as sites and bonds, respectively.

We consider the following percolation on the infinite graph $\Lambda$ by fixing $p \in [0, 1]$ and selecting each edge independently with probability $p$ to be *open*, and otherwise with probability $1-p$ to be *closed*. The process of selecting the edges to be open is called *bond percolation* of strength $p$ on $\Lambda$. In contrast, selecting the vertices to be open is called *site percolation* of strength $p$ on $\Lambda$. For site percolation, we consider the induced subgraph by the open sites. Let $\Lambda_p^b$ and $\Lambda_p^s$ denote the open subgraphs in bond and site percolation, respectively. Note that given a graph $\Lambda$, the collection of all induced random subgraphs by site percolation on $\Lambda$ is a subset of the random subgraphs by bond percolation on $\Lambda$.

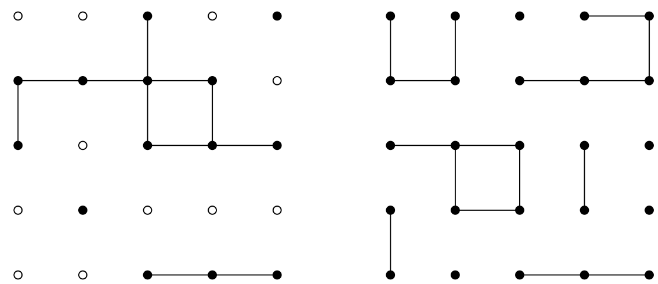

FIGURE 1. Site Percolation (left) and Bond Percolation (right) on $5 \times 5$ lattice grid subset of $\mathbb{Z}^2$. On the left, the filled circles represent the selected vertices. Credit to [1].

Once we have our percolation on $\Lambda$, it is natural to ask whether an infinite cluster of connected vertices exists. More specifically, we have the following motivational question for our random subgraph.

**Question 1.1.** When do infinite cluster(s) exist i.e. how does the formulation of infinite clusters depend on $p$?

To answer this question, we must first start by formally defining the measure space and operations on the random subgraph.

Let $\Omega = \{0,1\}^E$, where $E = |E(\Lambda)|$. $\Omega$ is the space of all possible outcomes of subgraphs called *configurations* of $\Lambda$. Let $\mathbb{P}_{\Lambda,p} = \prod \mu_p$ denote the product probability measure on $\Lambda$ with strength parameter $p$, where $\mu_p(1) = p$ and $\mu_p(0) = 1-p$. Here, $\mathbb{P}_{\Lambda,p}$ is the *Bernoulli percolation* probability measure. We shall ignore events of probability 0. Let $C(x)$ denote the connected component containing the vertex $x$, and $|C(x)| := |V(C(x))|$, the number of open vertices in the cluster, sometimes called the *size* of the cluster. For convenience, let $C$ denote the component containing the origin.

Next, we introduce a fundamental concept in Percolation Theory: the percolation function, which represents how the probability of an infinite open cluster containing the origin depends on the fixed probability parameter $p \in [0,1]$. Formally, let $\theta_x(p)$ be a probability that an infinite cluster contains $x$ in the random subgraph of $\Lambda$. Since our cluster is infinite, it makes sense to "center" it at a specific vertex, which we take to be the origin.

Here, we make a few observations.

**Lemma 1.1.** *$\theta_x(p) = 0$ for every site $x$ or $\theta_x(p) > 0$ for every site $x$.*

*Proof.* Let $x$ and $y$ be sites such that they are graph distance $n$ units apart. Then clearly, $\theta_x(p) \geq p^n \theta_x(p)$. □

Through Lemma 1.1, it suffices to just observe the value $\theta_0(p)$, which we abbreviate as $\theta(p) = \mathbb{P}_{\Lambda,p}[|C| = \infty]$ for the sake of notation convenience. For site percolation, we select the origin to be open with probability 1. In most cases, our infinite graph will be $\mathbb{Z}^d$, so we omit the graph reference in the subscript of our probability measure.

**Remark 1.2.** $|C| = \infty$ is equivalent to the existence of an infinite self-avoiding path (SAP) starting at 0 and connected with open edges.

**Lemma 1.3.** *$\theta(p)$ is a nondecreasing function of $p$.*

*Proof.* We use a traditional coupling argument. Let $p \leq p'$. Observe that $\mathbb{1}\{|C| = \infty\}$ is an increasing function of $\omega = \{0,1\}^{|E|}$ where E is the edge set of our graph, and Bernoulli distributed. Let $\{U_e\}_{e\in E}$ be i.i.d random variables uniformly distributed over $[0,1]$. Fix $p \in [0,1]$, and for each $e \in E$, define $\eta_e(p) = \mathbb{1}\{U_e < p\}$. Then the collection $\{\eta_e(p)\}_{e\in E}$ is i.i.d random variables and Bernoulli distributed with parameter $p$. Thus, $\eta_e(p) \leq \eta_e(p')$. Using monotonicity of expectation, we have $\mathbb{E}[\eta_e(p)] \leq \mathbb{E}[\eta_e(p')]$ and the following:

$$\theta(p) = \mathbb{E}_p[\mathbb{1}\{|C| = \infty\}] = \mathbb{E}[\eta_e(p)] \leq \mathbb{E}[\eta_e(p')] = \mathbb{E}_{p'}[\mathbb{1}\{|C| = \infty\}] = \theta(p')$$

Since both variables have the same probability distribution, these random variables can be viewed as the same, and thus obtain equivalent expectations. □

**Lemma 1.4.** *$\theta(p) > 0$ if and only if $\mathbb{P}_p[\exists x$ such that $|C(x)| = \infty] = 1$ i.e. there exists an infinite cluster with probability 1.*

*Proof.* Here $|C(x)| = \infty$ is an event that does not depend on a finite number of edges being open or closed. Thus, it resides in a tail field, allowing for the application of Kolmogorov's Zero-One Law i.e $\mathbb{P}_p[\text{some } C(x) \text{ is infinite}] = \{0,1\}$. The proof follows directly from Lemma 1.1. □

**Theorem 1.5.** *$\theta(p)$ is a right continuous function of $p$ on $[0,1]$; And $\theta(p)$ is continuous on $(p_c, 1]$.*

Now, that we have established a few basic properties of $\theta(p)$, we motivate the reason for the critical probability value. Since $\theta(0) = 0$ and $\theta(1) = 1$ and $\theta(p)$ is nondecreasing in $p$, we would expect a value $p_c \in (0,1]$ such that $\theta(p)$ becomes strictly positive for the first time. Formally, let

$$p_c \coloneqq \sup\{p \mid \theta(p) = 0\} = \inf\{p \mid \theta(p) > 0\}$$

**Remark 1.6.** However, not all percolation functions are continuous on $[0,1]$. Consider percolation on $\mathbb{Z}$. We have $p_c = 1$, and thus a single jump discontinuity at $p = 1$. Its percolation function is still upper semicontinuous. For dimensions $d \geq 2$, it has been proven that $p_c \in (0,1)$.

We can define another critical probability for the expected size of the cluster around the site $x$.

$$\chi_x(p) = \mathbb{E}_p[|C(x))|]$$

where $\mathbb{E}_p$ is the expectation with respect to $\mathbb{P}_p$. As before, we let $\chi(p) \coloneqq \chi_0(p)$. The critical probability for this model is defined as

$$p_T \coloneqq \sup\{p \mid \chi(p) < \infty\} = \inf\{p \mid \chi(p) = \infty\}$$

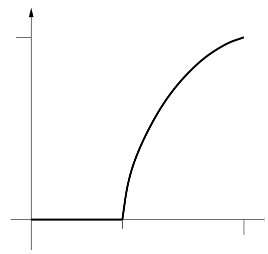

FIGURE 2. Graph of a continuous Percolation function. Credit to [2].

The subscript $T$ refers to the mathematician Temperley. Here, $p_T \leq p_c$ for any infinite graph $\Lambda$. In Section 3, we will establish the conditions that guarantee equivalence between the critical percolation probability $p_c$ and the threshold related to the cluster's expected size $p_T$.

**Theorem 1.7** (**Uniqueness of Infinite Cluster).** *If $\theta(p) > 0$, then $\mathbb{P}_p$[the infinite cluster $C$ of the origin is unique] $= 1$.*

*Proof.* Refer to [2] pages 10 to 13 for full proof by Burton and Keane. □

We conclude the section by having answered the motivational questions about percolation on $\Lambda = \mathbb{Z}^d$.

## 2. Probabilistic Tools and Facts

Here, we outline several important theorems and inequalities that are extensively used in Percolation and Probability Theory. While some statements may not be directly used in this paper, they are essential in the proof(s) of many related results.

**Theorem 2.1** (**Kolmogorov's Zero-One Law).** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $A_1, \ldots, A_n, \ldots \in \mathcal{F}$ be a sequence of independent events with tail field $\tau$. If $E \in \tau$, then $\mathbb{P}[E] = \{0, 1\}$. Alternately, if $A_1, \ldots, A_n, \ldots$ are independent events then for event*

$$E \in \bigcap_{i=1} \sigma(A_{i+1}, A_{i+2}, \ldots)$$

*We have that $\mathbb{P}[E] = \{0, 1\}$.*

Kolmogorov's Zero-One Law states that certain events either almost surely occur or almost surely do not occur.

**Definition 2.2.** Consider a configuration space $\Omega := \{0, 1\}^{|E|}$, where $E$ is the edge set of our subgraph. Consider the partial order on $\Omega$ given by $\omega \preceq \omega'$ if and only if $\omega_e \leq \omega'_e$ for all $e \in E$. $\omega_e = 1$ and $\omega_e = 0$ implies the edge is open and closed, respectively. A function $f : \{0, 1\}^{|E|} \mapsto \mathbb{R}$ is increasing if $\omega \preceq \omega'$ implies $f(\omega) \leq f(\omega')$. An event is increasing if its indicator function is increasing.

**Example 2.3.** Consider the event $E = \{\Lambda_p^b$ contains at least 50 open edges$\}$. Then $E$ is an increasing event.

**Lemma 2.4** (**Harris-FKG inequality**). *Let $\mathcal{A}$ and $\mathcal{B}$ be both increasing or decreasing events. Then:*

$$\mathbb{P}_p[\mathcal{A} \cap \mathcal{B}] \geq \mathbb{P}_p[\mathcal{A}]\mathbb{P}_p[\mathcal{B}], \tag{2.5}$$

*where $\mathbb{P}_p$ is the product probability measure associated with Bernoulli percolation. We also have the following equivalent expression:*

$$\mathbb{E}_p[\mathbb{1}_{\mathcal{A}}\mathbb{1}_{\mathcal{B}}] \geq \mathbb{E}_p[\mathbb{1}_{\mathcal{A}}]\mathbb{E}[\mathbb{1}_{\mathcal{B}}]$$

**Remark 2.6.** Lemma 2.4 also holds for the probability measure $\mathbb{P}_p$ associated with random Voronoi percolation in $\mathbb{R}^2$, when $\mathcal{A}$ and $\mathcal{B}$ be both increasing or decreasing *black* events. This will be expanded on in much more detail in Section 5.

Intuitively, Harris-FKG inequality suggests that the probability of associated (both increasing or decreasing) events occurring together is higher than the probability of them occurring independently. When one event occurs, it provides information about the underlying configuration, making the occurrence of the other event more probable.

**Corollary 2.7** (**Nth Root Trick**). *Let $A_1, \ldots, A_n$ be increasing events with equal probabilities such that $\mathbb{P}[A_1 \cup \ldots \cup A_n] \geq p$. Then for some index $j \in \{1, \ldots, n\}$ we have:*

$$\mathbb{P}[A_j] \geq 1 - (1-p)^{\frac{1}{n}}$$

*Proof.* Let $A = A_1 \cup \ldots \cup A_n$. Then by the Harris-FKG inequality, we have:

$$\prod_i \mathbb{P}[A_i^c] \leq \mathbb{P}[A^c] = 1 - \mathbb{P}[A]$$

which immediately implies:

$$\mathbb{P}[A_j^c] \leq (\mathbb{P}[A^c])^{\frac{1}{n}}$$

for some index $j \in \{1, \ldots, n\}$. Thus, we have:

$$\mathbb{P}[A_j] \geq 1 - (\mathbb{P}[A^c])^{\frac{1}{n}} \geq 1 - (1-p)^{\frac{1}{n}}$$

□

Nth Root Trick implies that given a large collection of similar events whose union has a high probability, we expect at least one of the events to occur with high probability.

**Proposition 2.8** (**First Moment Formula**). *If $X$ is a non-negative, integer-valued, random variable, then*

$$\mathbb{P}[X > 0] \leq \mathbb{E}[x > 0]$$

*Proof.* Let $a = 1$. Then by Markov's Inequality:

$$\mathbb{P}[X > 0] \leq \mathbb{P}[X > 1] \leq \frac{\mathbb{E}[X > 0]}{1} = \mathbb{E}[X > 0]$$

□

**Definition 2.9.** For $\mathcal{A}, \mathcal{B} \in \mathcal{P}(S)$, we define the box operation as:

$$\mathcal{A} \,\square\, \mathcal{B} = \left\{ C \subset S \;\middle|\; \begin{array}{l} \text{There exist sets } Y, Z \in S \text{ such that} \\ D \cap Y = C \cap Y \text{ implies } D \in \mathcal{A}, \text{ and} \\ D \cap Z = C \cap Z \text{ implies } D \in \mathcal{B} \end{array} \right\}$$

**Theorem 2.10** (**Van Den Berg–Kesten Inequality**).

$$\mathbb{P}_p[\mathcal{A} \,\square\, \mathcal{B}] \leq \mathbb{P}_p[\mathcal{A}]\mathbb{P}_p[\mathcal{B}] \tag{2.11}$$

The Van Den Berg-Kesten inequality can be interpreted as the partial "converse" of the Harris-FKG inequality. It is used to prove the exponential decay of zero-cluster in the subcritical regime. However, the proof of several theorems related to this exponential decay of zero-cluster in the subcritical regime are omitted but can be found in [1].

**Corollary 2.12.** *By definition of the box operation,*

$$\mathcal{A} \,\square\, \mathcal{B} \subset \mathcal{A} \cap \mathcal{B}$$

## 3. RSW Theory

Discrete RSW Theory focuses on box crossings of lattice structures often equipped with Bernoulli percolation. However, before we discuss basic RSW Theory on $\mathbb{Z}^2$, we first introduce percolation on its dual space, $(\mathbb{Z}^2)^*$. The dual space of $\mathbb{Z}^2$ is $(\mathbb{Z}^2)^* = \mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$. Notice that every edge of $(\mathbb{Z}^2)^*$ crosses exactly one edge of $\mathbb{Z}^2$. We can couple the graph and its dual together by considering each edge $e$ to be open if and only if the corresponding dual edge $e^*$ is closed. The following is a useful lemma that relates strong results of percolation on $\mathbb{Z}^2$ with its dual.

**Lemma 3.1.** $|C| < \infty$ *if and only if there exists a simple cycle in* $(\mathbb{Z}^2)^*$ *surrounding* $0$ *consisting of all open dual edges.*

*Proof.* Consider overlapping $\mathbb{Z}^2$ and $(\mathbb{Z}^2)^*$. Then we have this "ring" or "wall" of open dual edges surrounding 0. By our coupling method above, open dual edges correspond to closed edges. Thus, any cluster of open edges containing the origin will have a finite size. □

### 3.1. **Discrete Box Crossings.**

We can now begin RSW Theory on $\mathbb{Z}^2$. RSW Theory is the study of generalizing crossings of a $n \times n$ square to crossings of a $\rho n \times n$ rectangle for $\rho \geq 1$, where $n \in \mathbb{N}$. Here, $\rho$ is the *aspect ratio*, and crossings are a connected self-avoiding path of open edges from the left to the right side of the rectangle. This information allows us to determine certain properties of connected components in our random subgraph, such as exponential decay of the cluster at a sub-critical probability i.e. $p < p_c$. By generalizing these crossings, we can deduce information of much larger sections of $\mathbb{Z}^2$ by a, significantly smaller, $n \times n$ square.

The rectangle $R$ is the subgraph of the induced vertices in $[a, b] \times [c, d]$, where $a \leq b$ and $c \leq d$. Let $k = b - a + 1$ and $l = d - c + 1$. Then $R$ is a $k \times l$ rectangle. Note that there are $kl$ sites and $(k-1)l + (l-1)k = 2kl - k - l$ bonds.

The *horizontal dual* of $R$ is $R^h = [a + \frac{1}{2}, b - \frac{1}{2}] \times [c - \frac{1}{2}, d + \frac{1}{2}]$. Analogously, the *vertical dual* of $R$ is $R^v = [a - \frac{1}{2}, b + \frac{1}{2}] \times [c + \frac{1}{2}, d - \frac{1}{2}]$. Let $H(R)$ denote the event that a horizontal (left-to-right) crossing of $R$ occurs and similarly let $V(R)$ denote the event that a vertical (top-to-bottom) crossing of $R$ occurs. From now on, let $R$ denote a $[0, n] \times [0, n]$ rectangle and $R(\rho)$ denote a $[0, \rho n] \times [0, n]$ rectangle.

**Lemma 3.2.** *Consider bond percolation on a rectangle* $R$ *in* $\mathbb{Z}^2$. *Then exactly one of the events* $H(R)$ *or* $V(R^h)$ *holds.*

*Proof.* We consider the following partial tiling of the overlap of $R$ and $R^h$. Replace each vertex in R with degree $d$ with a black-colored $2d-$gon. Similarly, for each vertex in $R^h$ with degree d, replace it with a white-colored $2d-$gon. For each edge $e$ in $R$, replace it with a black-colored$4-$gon if $e$ is open, else, replace it with a white-colored $4-$gon. Since $d = 4$ for each vertex, we have tiling called $R'$ of octagons and squares of $R$ and $R^h$. The states of vertical edges of the left and right side of $R$ do not affect any horizontal crossings so we color these edges black.

Observe that $H(R)$ occurs if and only if we have a path of black shapes inside $R'$ spanning from the left to right of $R'$. Analogously, $V(R^h)$ holds if and only if we have a path of white shapes inside $R'$ spanning from the top to the bottom of $R'$.

More formally, let $L$ be the *interface graph* formed by taking the boundary (edges) of the octagons and squares that separate the black regions from the white ones. Let $L$ have the black regions on its right side and the white regions on its left side. From our construction of the partial tiling, there are only 2 entrance points and 2 exit points for $L$ at the diagonal spots of the overlap of $R$ and $R^h$. Since $L$ does not terminate in the overlap of $R$ and $R^h$, we have that either $H(R)$ or $V(R^h)$ occurs. It's easy to see that given $H(R)$ or $V(R^h)$ the interface graph traces out the path $P$ such that $P$ is the top-most horizontal crossing or the left-most vertical crossing, respectively. Figure 3 shows the partial tiling with the interface graph. Note that the path $P$ that traverses from the left to right of $R$ is the top-most horizontal crossing. □

This argument can be generalized to the following graphs embedded in $\mathbb{Z}^2$. More can be found in [1] section 3.

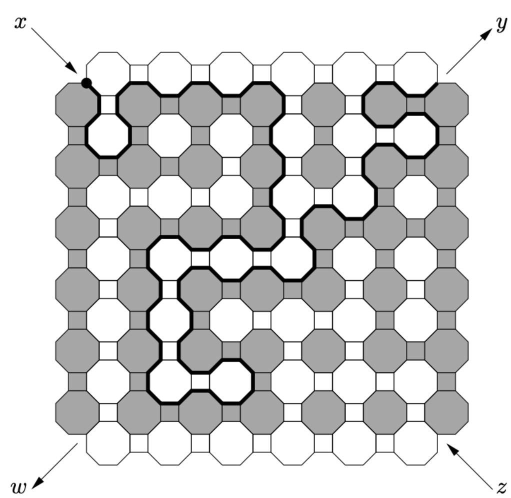


FIGURE 3. Interface graph of percolation in $\mathbb{Z}^2$. Credit to [1].

We emphasize the importance of the left-most path for vertical crossings and the top-most path for horizontal crossings. Let S be a square. If $P$ is a vertical crossing of $S$, let $\{\mathrm{LV(S)} = \mathrm{P}\}$ be the event that $P$ is the left-most vertical crossing. Similarly, let $\{TH(S) = P\}$ be the event that $P$ is the top-most horizontal crossing.

Observe that $V(S)$ is the disjoint union of events {LV(S) = P}. {LV(S) = P} is independent of any configuration of bonds to the right of $P$. Analogously, {TH(S) = P} is independent of any condition of bonds below $P$. Knowing the left-most vertical path of a square allows us to deduce information about potential paths intersecting it. The following proposition demonstrates this.

**Proposition 3.3.** *Let $R$ be a $m \times 2n$ be a rectangle, where $m \geq n$. Let $X(R)$ be the event that there are open paths $P_1$ and $P_2$, such that $P_1$ vertically crosses the $n \times n$ square $S$, which we take to be nested in the bottom left corner of $R$, and $P_2$ connects some site on $P_1$ to some site on the right-hand side of $R$. Then,*

$$\mathbb{P}_p[X(R)] \geq \mathbb{P}_p[H(R)]\mathbb{P}_p[V(R)]/2$$

Now, we present some useful inequalities and deductions about crossings of rectangles with different aspect ratios.

**Corollary 3.4.**

*(i) If $\rho_1 \leq \rho_2$, we have*

$$\mathbb{P}_p[H(R(\rho_2))] \leq \mathbb{P}_p[H(R(\rho_1))]$$

.

*(ii) If $R$ and $R'$ are $k \times l - 1$ and $k - 1 \times l$ rectangles in $\mathbb{Z}^2$, respectively, then*

$$\mathbb{P}_p[H(R)] + \mathbb{P}_{1-p}[V(R')] = 1$$

*(iii) If $R$ is an $n + 1 \times n$ rectangle, then*

$$\mathbb{P}_{\frac{1}{2}}[(H(R))] = \frac{1}{2}$$

*(iv) If $S$ is an $n \times n$ square, then*

$$\mathbb{P}_{\frac{1}{2}}[V(S)] = \mathbb{P}_{\frac{1}{2}}[H(S)] \geq \frac{1}{2}$$

.

*Proof.*

(i) Every horizontal crossing of $R(\rho_2)$ is a crossing of $R(\rho_1)$

(ii) By Lemma 3.2, we have that $\mathbb{P}_p[H(R)] + \mathbb{P}_p[V(R^h)] = 1$. $V(R^h)$ can be defined by the dual graph, so each edge is open with probability $1-p$. Since $V(R^h)$ and $V(R')$ are both vertical crossings of $k - 1 \times l$ rectangles, using the translational invariance of $\mathbb{P}_p$, we deduce $\mathbb{P}_p[V(R^h)] = \mathbb{P}_{1-p}[V(R')]$.

(iii) Follows directly from (ii) with $p = 1/2$.

(iv) Follows from rotational symmetry and (i).

□

We can now begin to understand the relationship between the crossings of squares and the crossings of rectangles. RSW Theory states that a nonzero probability of a horizontal square crossing guarantees a nonzero probability of rectangle crossings. We state the main statement and the strongest result of RSW Theory. Note that, $\mathbb{P}_p$ refers to the Bernoulli percolation measure.

**Theorem 3.5 (RSW).** *Let $R$ and $R(\rho)$ for some $\rho \geq 1$ be rectangles. Then for any $n \geq 1$,*

$$\mathbb{P}_p[H(R)] > 0 \implies \mathbb{P}_p[H(R(\rho))] > 0$$

*For the generalized version, let* $0 < \rho_1 < \rho_2 < \infty$ *and* $\alpha > 0$ *There exists a constant* $c_1 = c_1(\rho_1, \rho_2, c_0) > 0$ *such that*

$$\mathbb{P}_p[H(R(\rho_1))] > c_0 \implies \mathbb{P}_p[H(R(\rho_2))] > c_1$$

*Note that* $c_1$ *does not depend on the square height* $n$ *and probability parameter* $p$.

**Remark 3.6.** For RSW, we mainly consider the case when $0 < \rho_1 \leq 1 < \rho_2$ or $0 < \rho_1 < \rho_2 \leq 1$. The other case, namely when $1 < \rho_1 < \rho_2 < \infty$, follows directly from repeated applications of Harris-FKG inequality and Lemma 3.7.

**Lemma 3.7.** *Let* $\rho_1, \rho_2 \geq 1$ *Then:*

$$\mathbb{P}_p[H(R(\rho_1 + \rho_2 - 1))] \geq \mathbb{P}_p[H(R(\rho_1))]\mathbb{P}_p[H(R(\rho_2))]\mathbb{P}_p[V(R(1))]$$

*Proof.* Let $R$ be a $(\rho_1 + \rho_2 - 1)n \times n$ rectangle. We can divide $R$ into 3 smaller rectangles: $R_1$, a $\rho_1 n \times n$ rectangle; $R_2$, a $\rho_2 n \times n$ rectangle; and $S$, a $n \times n$ square (rectangle). Overlap $R_1$ and $R_2$ such that $S$ is their intersection. Every horizontal crossing of $R_1$ and $R_2$ and vertical crossing of $S$ is a crossing of $R$. By Harris-FKG inequality, we have

$$\begin{aligned}\mathbb{P}_p[H(\rho_1 + \rho_2 - 1)] &\geq \mathbb{P}_p[H(\rho_1) \cap H(\rho_2) \cap V(R(1))] \\ &\geq \mathbb{P}_p[H(\rho_1)]\mathbb{P}_p[H(\rho_2)]\mathbb{P}_p[V(R(1))]\end{aligned}$$

Figure 4 displays this observation. □

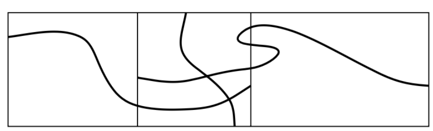

FIGURE 4. The concatenation of the horizontal crossings of 2 rectangles and a vertical crossing of a square is a crossing of a much larger rectangle. This technique can be generalized to construct crossing paths over a large number of similar rectangles. Credit to [1].

3.2. **Discrete Exponential Decay.** In this section, we explore the size, measured through the number of vertices, of the random connected zero-cluster $C$. Although Percolation Theory is mainly concerned with the existence of an infinite component of the origin, in this section, we will focus on studying subcritical probability component(s) of the origin.

First, we briefly touch on the growth of $C$, and characterizing components of finite size.

**Definition 3.8.** An *animal* is a finite connected subgraph of $\mathbb{Z}^d$ containing the origin. Let $A$ be an animal. Let $n$ and $m$ denote the number of vertices and edges contained in $A$, respectively. Let $b$ denote the number of edges on the boundary of $A$ i.e. one vertex is in $A$ and the other isn't. Let $\mathcal{A}_{n,m,b}$ be the set of all animals with $n$ vertices, $m$ edges, and $b$ boundary edges; and let $a_{n,m,b} = |\mathcal{A}_{n,m,b}|$.

If $C$ is a finite-sized connected subgraph and contains the origin, then $C$ is an *random animal*. Since the number of vertices is finite, we can choose each edge independently with probability $p$. Thus, we have

$$\mathbb{P}_p[C = A] = p^m(1-p)^b$$

for all $A \in \mathcal{A}$, the set of all animals on $\mathbb{Z}^d$. Altogether, the probability that $C$ is of size $n$ is

$$\mathbb{P}_p[|C| = n] = \sum_{m,b} a_{n,m,b} p^m (1-p)^b$$

This probability helps us deduce an upper bound on the number of animals of $n$ vertices, $m$ edges, and $b$ boundary edges.

**Lemma 3.9.** *For fixed $n \geq 1$:*

$$\sum_{m,b} a_{n,m,b} \leq \left(\frac{27}{4}\right)^{dn}$$

*Proof.* For fixed $n \geq 1$, we have $a_{n,m,b} \neq 0$ if $1 \leq b \leq 2dn$ and $n-1 \leq m \leq dn$. It suffices to only consider these conditions in finding an upper bound on the number of animals.

$$\sum_{m,b} a_{n,m,b} p^{dn} (1-p)^{2dn} \leq \sum_{m,b} a_{n,m,b} p^m (1-p)^b \leq 1$$

Since our summation is independent of $p \in [0,1]$,

$$\sum_{m,b} a_{n,m,b} \leq \left((p(1-p)^2\right)^{-dn}$$

we have the minimizer $p^* = 1/3$ for $\left((p(1-p)^2\right)^{-dn}$, and thus

$$\sum_{m,b} a_{n,m,b} \leq \left(\frac{4}{27}\right)^{-dn} = \left(\frac{27}{4}\right)^{dn}$$

□

Now, we resume the discussion about the size of the subcritical probability component(s) of the origin. For the rest of the following section, we will consider the site percolation on an infinite-oriented graph $\overrightarrow{\Lambda}$. This model extends to oriented bond and unoriented site bond percolation through the transformation of graphs and *equivalence* of certain percolation measures.

For convenience, let $\mathbb{P}^s_{\overrightarrow{\Lambda},p}$ be abbreviated as $\mathbb{P}_p$. The *out-subgraph* of $\overrightarrow{\Lambda}$ centered at site $x$ denoted $\overrightarrow{\Lambda}^+_x$ contains all sites and bonds reachable by (oriented) paths from $x$. Two sites $x$ and $y$ are called *out-like* if $\overrightarrow{\Lambda}^+_x$ and $\overrightarrow{\Lambda}^+_y$ are isomorphic as rooted oriented graphs. The *out-class* $[x]$ is the equivalence class under the out-like relation containing $x$. Let $C_{\overrightarrow{\Lambda}}$ denote the *out-class graph* of $\overrightarrow{\Lambda}$, whose vertices are these equivalence classes $[x]$. $[x]$ and $[y]$ are connected by an oriented edge if and only if there are sites $x' \in [x]$ and $y' \in [y]$ such that $\overrightarrow{x'y'} \in E(\overrightarrow{\Lambda})$. Suppose there is $y' \in [y]$ such that $\overrightarrow{xy'}$ is an edge of $\Lambda$. Then there exists an (oriented) edge from $[x]$ directed to $[y]$ in $C_{\overrightarrow{\Lambda}}$. Note that connectivity in $C_{\overrightarrow{\Lambda}}$ allows for loops. An oriented graph is *strongly connected* if, for every site $x$ and $y$, there is an oriented path $P$ from $x$ to $y$. Similarly, as in its unoriented case, we can define $C(x),\ \theta_x(p),\ p^s_c(\overrightarrow{\Lambda}; x))$.

**Example 3.10.** Consider $\overrightarrow{\mathbb{Z}^2}$ with each vertical edge directed North and each horizontal edge directed East. The out-class graph, $C_{\overrightarrow{\mathbb{Z}^2}}$, consists of only one vertex with a loop.

Fix a site $x$ and $n \geq 1$. Let $S_n^+(x)$ be the *sphere* of radius $n$ centered at $x$, that is, $S_n^+(x) = \{y \in V(\overrightarrow{\Lambda}) \mid d(x,y) = n\}$, where $d(\cdot,\cdot)$ is the graph distance between sites. Let $B_n(x) = \bigcup_{i=0}^n S_i^+(x)$, be the *ball* of radius $n$ centered at $x$. Let $\{x \xrightarrow{n}\}$ be the event there is an open self-avoiding path from $x$ to some site $y \in S_n^+(x)$. And $R_n(x)$ be the event there is an open path from an out-neighbor of $x$ to some site in $S_n^+(x)$ i.e. $R_n(x) = \{x^+ \xrightarrow{n}\}$. Observe that $R_n(x)$ is independent of the state of $x$ and $\{x \xrightarrow{n}\} = \{x \text{ is open}\} \cap R_n(x)$. Lastly, let $r(C(x))$ denote the *radius* of the open out-cluster of $x$ i.e. $r(C(x)) = \sup\{n \mid C(x) \cap S_n^+(x) \neq \phi\}$. It is important to remember that these definitions, although not explicitly notated, all rely on the percolation probability $p$.

**Lemma 3.11.** *Let $\overrightarrow{\Lambda}$ be a locally finite multi-graph with $C_{\overrightarrow{\Lambda}}$ strongly connected. Then there exist $p_c^s(\overrightarrow{\Lambda}), p_T^s(\overrightarrow{\Lambda}) > 0$ such that*

$$p_c^s(\overrightarrow{\Lambda}; x) = p_c^s(\overrightarrow{\Lambda})$$
$$p_T^s(\overrightarrow{\Lambda}; x) = p_T^s(\overrightarrow{\Lambda})$$

*for all sites $x$.*

*Proof.* Let $x$ and $y$ be sites of $\overrightarrow{\Lambda}$. Since, $C_{\overrightarrow{\Lambda}}$ is strongly connected, there is an oriented path from $[x]$ to $[y]$. Consequently, there is also a path $P$ from $x$ to some $y' \in [y]$. It follows that $\theta_x(p) \geq p^{|P|}\theta_{y'}(p) = p^{|P|}\theta_y(p)$ and $\chi_x(p) \geq p^{|P|}\chi_{y'}(p) = p^{|P|}\chi_y(p)$. Thus, $\theta(x) > 0$ or $\theta(x) = 0$ for all sites $x$, and analogously, $\chi_x(p) < \infty$ or $\chi_x(p) = \infty$ for all sites $x$. $p_c^s(\overrightarrow{\Lambda}; x)$ and $p_T^s(\overrightarrow{\Lambda}; x)$ are independent on their center site $x$. □

Lemma 3.11 serves as a generalization of Lemma 1.1 for oriented graphs by extending the equality of percolation functions across all sites to their respective critical probabilities. From this point forward, we will omit the site $x$ when expressing the critical probability of the percolation function. As one might intuitively expect, isomorphic subgraphs behave identically under Bernoulli percolation, sharing the same critical probabilities.

With the prerequisite material established, we can now begin examining the exponential decay of various characteristics of $C$. The following series of statements will build up to the desired exponential decay of the size, the number of (open) vertices, of $C$. First, we start by observing the occurrence of an out-path of length $n$ for a site $x$ as $n \to \infty$.

**Lemma 3.12.** *Let $x$ be a site of $\overrightarrow{\Lambda}$, a locally finite oriented multi-graph with $C_{\overrightarrow{\Lambda}}$ finite and strongly connected, and let $p \in (0,1)$. Then $p < p_c^s(\overrightarrow{\Lambda})$ implies $\sup_{x \in V(\overrightarrow{\Lambda})} \mathbb{P}_p^s[R_n(x)] \to 0$ as $n \to \infty$.*

*Proof.* It's clear that for every site $x$, $\mathbb{P}_p^s[R_n(x)] \to 0$ as $n \to \infty$. However, $\mathbb{P}_p^s[R_n(x)]$ depends only on the out-class $[x]$ of $x$. Finitely many out-classes guarantees $\sup_{x \in V(\overrightarrow{\Lambda})} \mathbb{P}_p^s[R_n(x)] = \max_{[x] \in C_{\overrightarrow{\Lambda}}} \mathbb{P}_p^s[R_n(x)] \to 0$ as $n \to \infty$. □

Note that the following statements will stress the necessity and implications of $C_{\overrightarrow{\Lambda}}$ being finite. If our graph is too "large" (contains infinitely many non-isomorphic subgraphs), then the characterization of any sort of decay becomes inconclusive.

**Theorem 3.13** (**Hammersley's Result**)**.** *Let $\overrightarrow{\Lambda}$ be a locally finite multi-graph with $C_{\overrightarrow{\Lambda}}$ finite and strongly connected. Let $p < p_T^s(\overrightarrow{\Lambda})$. Then there exists $\alpha > 0$ such that*

$$\mathbb{P}_p^s[\{x \xrightarrow{n}\}] \leq \exp(-\alpha n)$$

Since $\{x \xrightarrow{n}\} \subset \{|C(x)| \geq n\}$, the previous theorem can be strengthened to tackle the size of $C(x)$ instead of the (oriented) path of length $n$.

**Theorem 3.14** (**Discrete Exponential Decay**)**.** *Let $\overrightarrow{\Lambda}$ be a locally finite multi-graph with $C_{\overrightarrow{\Lambda}}$ finite and strongly connected. Let $p < p_T^s(\overrightarrow{\Lambda})$. Then there exists an $\alpha > 0$ such that*

$$\mathbb{P}_p^s[|C(x)| \geq n] \leq \exp(-\alpha n)$$

*for all sites $x$ and $n \geq 1$.*

The property in Theorem 3.14 is referred to as the *exponential decay* of zero-cluster in a subcritical probability model when $p < p_T^s$. This decay reflects the rapid change of the cluster's probability of occurrence and the magnitude of its size near critical probabilities. It also describes the localization of $C$ and exemplifies the absence of connections (open paths) to "$\infty$", the vertices at an infinite distance away from the origin.

Note, this result holds for all $p < p_T^s \leq p_c^s$. However, conditioning on the magnitude of growth of the sphere of radius $n$, we can achieve equality between $p_T^s$ and $p_c^s$, as proven by Menshikov. As mentioned previously, $p_T \leq p_c$ holds for any infinite graph $\overrightarrow{\Lambda}$.

**Theorem 3.15** (**Menshikov's Theorem**)**.** *Let $\overrightarrow{\Lambda}$ be a locally finite multi-graph with $C_{\overrightarrow{\Lambda}}$ finite and strongly connected. If there exists $C > 0$ such that $|S_n^+(x)| \leq \exp\left(\frac{Cn}{\log(n^3)}\right)$ for every site $x$ and $n \geq 1$, we have*

$$p_c^s(\overrightarrow{\Lambda}) = p_T^s(\overrightarrow{\Lambda})$$

Altogether, we achieve the desired result of *exponential decay* of the zero-cluster in a subcritical probability model when $p < p_c^s$.

## 4. Continuum Percolation Models

In this section, we transition from discrete percolation to continuum percolation models by examining both the Gilbert disc model and the Voronoi model. While these models extend many results, arguments, and conceptual ideas from their discrete settings, they also introduce new challenges. Such challenges arise from the complexity of continuous spaces.

### 4.1. **Gilbert Disc Model.**

Fix $\lambda > 0$. Let $\{X_{i,j} | (i,j) \in \mathbb{Z}^2\}$ be independent random variables, each with mean $\lambda$, that is for $k \in \mathbb{N}$,

$$\mathbb{P}[X_{i,j} = k] = e^{-\lambda} \frac{\lambda^k}{k!}$$

Let $C_{i,j}$ be the unit square (side length 1) with its bottom left vertex at $(i,j) \in \mathbb{Z}^2$. Formally, $C_{i,j} = \{(x,y) \in \mathbb{R}^2 \mid i \leq x \leq i+1, j \leq y \leq j+1\}$.

For every square $C_{i,j}$, select $X_{i,j}$ points independently and uniformly from the square, and let $\mathcal{P}_\lambda = \bigcup_{i,j} X_{i,j}$. Let us write $\mu_\lambda(U)$ for the number of points of $\mathcal{P}_\lambda$ in a bounded Borel set $U$. $\mathcal{P}_\lambda$ is called *homogeneous Poisson process of intensity* $\lambda$ and satisfies the following properties:

(i) If $U_1, \ldots, U_n$ are pairwise disjoint bounded Borel sets, then the random variables $\mu_\lambda(U_1), \ldots, \mu_\lambda(U_n)$ are independent.
(ii) For every bounded Borel set $U$, the random variable $\mu_\lambda(U)$ is a Poisson random variable with mean $\lambda m(U)$, where $m(U)$ is the Lebesgue measure of $U$.

The purpose of defining $\mathcal{P}_\lambda$ is to create a "uniform" collection of points in $\mathbb{R}^2$. Points in $\mathcal{P}_\lambda$ are "well-distributed", not appearing in noticeable or distinct patterns. More specifically, with probability 1, $\mathcal{P}_\lambda$ has no limit points and any Borel set of measure 0 contains no points of $\mathcal{P}_\lambda$. We shall strengthen this by assuming that $\mathcal{P}_\lambda \cap B = \phi$ for all measure zero sets $B$. With this distribution of points, we can define the *Gilbert disc model.*

Fix $r, \lambda > 0$. Let $G_{r,\lambda}$ be the *Gilbert disc model of radius* $r$ *and density* $\lambda$ with the vertex set $\mathcal{P}_\lambda$. Sites $x$ and $y$ are connected by an unoriented edge if their Euclidean distance is at most $r$. The standard Gilbert disc model is abbreviated as $G_r = G_{r,\lambda}$, where $\lambda = 1$. If $x \in \mathcal{P}_\lambda$, then the (graph) degree of $x$ is Poisson-distributed with mean $\pi r^2 \lambda$. We also write $G(a) = G_{r,\lambda}$ to emphasize the dependence on the *connection area* $a = \pi r^2 \lambda$, in which we can freely adjust $\lambda$ and $r$, provided $a = \pi r^2 \lambda$ stays constant.

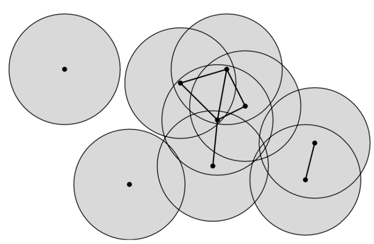

FIGURE 5. The random graph $G(a)$ can represent an infinite communication network where two transceivers can communicate if they are at most $r$ apart. Credit to [1].

Bernoulli percolation on the Gilbert disc model is analogous to the original lattice case. We can define site and bond percolation analogously on $G_r$. For clarity, fix $p \in [0,1]$. Selecting edges to be *open* with probability $p$ is called *bond percolation*, while selecting vertices to *open* is called *site percolation.* As before, for site percolation, we consider the induced subgraph by the open sites. For more terminology and definitions, $G(a)$ *percolates* if there exists an infinite open component. Let $\theta(a) = \theta(r,\lambda)$ be the probability that the component containing the origin is infinite. Since Poisson point processes are translational invariant, we shall condition that the origin lies inside $\mathcal{P}_\lambda$.

It's easy to see that $\theta(0) = 0$ and $\theta(a) = 1$ as $a \to \infty$. Since $\theta(a)$ is nondecreasing in $a$, there exists a *critical area* $a_c \in (0,\infty)$ such that $\theta(a) = 0$ for all $a < a_c$ and

$\theta(a) > 0$ for all $a > a_c$. Again, as in the discrete case, the value of $\theta(a_c)$ is inconclusive i.e. we cannot determine with certainty that $\theta(a_c) = 0$ or $\theta(a_c) > 0$.

The results from discrete percolation are easily applied to the Gilbert disc model.

**Theorem 4.1.** *For the standard Let $a_c$ be the critical area for the Gilbert disc model $G(a)$ with Then,*

$$2.184 \leq a_c \leq 10.588$$

**Theorem 4.2.** *In the Gilbert model $G(a) = G_{r,\lambda}$, there is almost surely at most one infinite component.*

**Theorem 4.3.** *Let $a = \pi r^2 \lambda < a_c$, where $a_c$ is the critical area of $G(a)$ and $|C(G(a))|$ denote the number of points in the component of the origin in $G(a)$. Then there exist an $\alpha > 0$ such that*

$$\mathbb{P}[|C(G(a)| \geq n] \leq \exp(-\alpha n)$$

*and $a_T = a_c$, where $a_T = \inf\{a \mid \mathbb{E}[G(a)] = \infty\}$.*

The Gilbert disc model serves as the our first example for defining percolation in a continuous space, serving as a set-up and introduction for the Voronoi model in the next section.

4.2. **Voronoi Model.** Let $\mathcal{P} = \mathcal{P}_\lambda \subset \mathbb{R}^d$ be the Poisson point process of intensity 1. For every $x \in \mathcal{P}$, let $V_x$ be the *closed Voronoi cell* of $x$, defined as

$$V_x = V_x(\mathcal{P}) = \{y \in \mathbb{R}^d \mid d(x,y) \leq d(z,y) \text{ for all } z \in \mathcal{P}\}$$

That is, the Voronoi (pronounced Vo-ro-noi) cell of $x$ is the set of all points closer to $x$ than to any other point in $\mathcal{P}$. Two Voronoi cells are *adjacent* if they share a $(d-1)$-dimensional face.

Let $V(\mathcal{P}) = \{V_x \mid x \in \mathcal{P}\}$ be the *Voronoi tessellation* of $\mathbb{R}^d$ with respect to $\mathcal{P}$. $V = V(\mathcal{P})$ is the *random Voronoi tessellation* of $\mathbb{R}^d$. $V(\mathcal{P})$ defines a graph $G_{\mathcal{P}}$ with vertex set $\mathcal{P}$ and $x, y \in \mathcal{P}$ are joined by an unoriented edge if their Voronoi cells $V_x$ and $V_y$ are adjacent. $V(\mathcal{P}_\lambda)$ can be analyzed and interpreted through both a graph-theoretical and geometric approach.

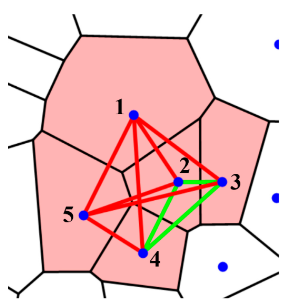


FIGURE 6. Translating between graph-theoretical and geometric interpretations of Voronoi diagram. Credit to [11].

Given the Poisson point process, $V(\mathcal{P})$ can be constructed using half-planes. Fix $x, y \in \mathcal{P}$. Let $h(x,y)$ denote the perpendicular bisector of the line segment $\overline{xy}$, including all the points on the part that contains $x$. Then the closed Voronoi cell of $x$ can be expressed as the following:

$$V_x = \bigcap_{y \in \mathcal{P} \setminus \{x\}} h(x, y)$$

By this alternate definition, it's easy to see that each Voronoi cell is convex because it is the intersection of convex sets. And for any $k$-dimensional face of $V_x$, there are exactly $d + 1 - k$ Voronoi cells $V_z$ containing it. This can be shown using reverse induction.

Since edges of $V(\mathcal{P})$ are $(d-1)$-dimensional and vertices are $(d-2)$-dimensional, each vertex has degree 3, and each edge bisects 2 Voronoi cells.

We begin by describing some basic properties of Voronoi cells and tessellations.

**Proposition 4.4.**

*(i)* *$V(\mathcal{P})$ is connected and its edges are all line segments or half lines.*
*(ii)* *Suppose $|\mathcal{P}_\lambda| = n$ then the number of vertices, faces and edges in $V(\mathcal{P})$ are all $O(n)$.*
*(iii)* *Each Voronoi cell is a convex $d$-dimensional polytope with finitely many $(d-1)$ faces.*

*Proof.*

(i) By the construction of each Voronoi cell, the edges are all straight lines. Suppose there exists an edge $e$ in $V(\mathcal{P})$ that is a full line (infinite in both directions). This edge is a perpendicular bisector of two Voronoi cells $V_x$ and $V_y$. Let $z \in \mathcal{P}$. By our Poisson point process, $z$ is not co-linear to $x$ and $y$ in $\mathbb{R}^d$, and thus $h(z, y)$ intersects $e$ at a point $z_0 \in \mathbb{R}^d$. However, the part of $e$ contained inside $h(z, y)$ cannot lie on the boundary of $V_y$ i.e. $e \cap int(h(z, y)) \neq \phi$.

If $V(\mathcal{P})$ is not connected, there would be some infinite Voronoi cell $V_x$, dividing the plane. By the convexity of cells, $V_x$ would be a vertical segment bounded by parallel lines. However, all edges in $V\mathcal{P}$ are line segments or half lines.

(ii) Adding a vertex $q_\infty$ at "infinity" makes $V(\mathcal{P}_\lambda)$ a planar graph. By construction of the Voronoi tessellation, $|F| = n$. Since each vertex has degree 3, $2E \geq 3(V + 1)$. By Euler's Formula, we have $(V + 1) + E - F = 2$. Thus, $V + 1 \leq 2(n - 2)$ and $E \leq 3n - 6$.

(iii) It was established earlier that each Voronoi cell $V_x$ is a $d$-dimensional convex structure. It suffices to show that $V_x$ is bounded. Suppose, for the sake of contradiction, that $V_x$ is unbounded i.e. there exists $\{y_n\}_{n \in \mathbb{N}}$ such that $B_n(x)$ only contains one point of $\mathcal{P}$, namely $x$ itself. However, $\mathbb{P}[\mu(B_n(x)) = 1] = e^{(-\pi n^2)}(\pi n^2) \to 0$ as $n \to \infty$. Thus, $V_x$ is a convex $d$-dimensional polytope with finitely many $(d - 1)$ faces.

□

As established in the previous section, points of $\mathcal{P}_\lambda$ don't reside in measure 0 sets. Therefore, no more than 2 points in $\mathcal{P}_\lambda$ are co-linear and no more than 3 points are co-circular.

**Lemma 4.5.**

*(i) $q$ is a vertex of $V(\mathcal{P})$ if and only if the largest circle with its interior containing no points of $\mathcal{P}$, denoted $C_{\mathcal{P}}(q)$, contains exactly three points in $\mathcal{P}$ on its boundary.*

*(ii) A point $p$ is on an edge of $V(\mathcal{P})$ if and only if the largest circle with its interior containing no points of $\mathcal{P}$, denoted $C_{\mathcal{P}}(q)$, contains at least two points in $\mathcal{P}$ on its boundary.*

*Proof.* The proof follows directly from the fact that each vertex has degree 3 and that every edge in $V(\mathcal{P})$ is a perpendicular bisector of two Voronoi cells. □

The *Delaunay graph* is the dual of the $V(\mathcal{P})$. Since each vertex of $V(\mathcal{P})$ has degree 3, each face of the Delaunay graph is a triangle, and consequently, this graph is referred to as the *Delaunay Triangulation*. The graph $G_{\mathcal{P}}$ is exactly the Delaunay Triangulation of $\mathcal{V}(\mathcal{P})$.

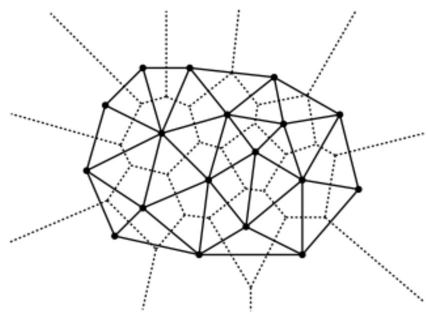

FIGURE 7. Voronoi Tessellation of $\mathbb{R}^2$ (dashed lines) and its Delaunay Triangulation (solid lines). Credit to [5].

The fundamental concepts from percolation in discrete models naturally extend to the continuum case. To start, consider *cell or face* percolation of the random Voronoi tessellation $V$ with respect to the Poisson point process $\mathcal{P}$ of intensity 1. Fix $p \in [0, 1]$. For each $x \in \mathcal{P}$, $V_x$ is *open* with probability $p$, independent of other cells. This formulation is equivalent to site percolation on $G_{\mathcal{P}}$.

Let $\mathcal{P}^+$ and $\mathcal{P}^-$ be points in $\mathcal{P}$ that have been selected to be open and closed, respectively. Then, $\mathcal{P}^+$ and $\mathcal{P}^-$ are disjoint and their union is $\mathcal{P}$. And, for $x \in \mathcal{P}$, $x \in \mathcal{P}^+$ with probability $p$, independent of any other points of $\mathcal{P}$. Let $\mathbb{P}_{\mathcal{V}(\mathcal{P}),\mathbb{R}^d,p} = \mathbb{P}_p$ be the probability measure associated to $(\mathcal{P}^+, \mathcal{P}^-)$ with respect to the Voronoi tessellation on $\mathbb{R}^d$. $\mathbb{P}_p$ is defined on $\Omega$ consisting of *configurations* $\omega = (X^+, X^-)$, where $X^+$ and $X^-$ are disjoint discrete subsets of $\mathbb{R}^d$. As in the discrete case, we shall ignore events of probability 0. The Voronoi percolation measure is invariant under translation, $\pi/2$-rotations, and horizontal reflections.

We approach percolation on $V$ through a geometric perspective. For each $x \in \mathcal{P}^+$, color the associated Voronoi cell $V_x$ *black*, and for each $x \in \mathcal{P}^-$, color the associated Voronoi cell $V_x$ *white*. We say that a point $z \in \mathbb{R}^d$ is *black* if it lies in a black cell, *white* otherwise. Note that this definition allows for some points in $\mathbb{R}^d$ to be colored both black and white, namely points on the boundary of Voronoi cells or the edges of $V$.

The *open cluster* of $x \in \mathcal{P}$ is the maximal connected subgraph of open sites in $G_{\mathcal{P}}$. Equivalently, the *black cluster* is the maximal connected set of black points in $\mathbb{R}^d$. Let $z_0 \in \mathcal{P}$ be the point such that the origin is inside $V_{z_0}$. Let $C_0^G$ denote the *open cluster* of the origin in $G_{\mathcal{P}}$, and $C_0$ denote the *black cluster* of the origin. We assume $z_0$ is open, otherwise $C_0^G = C_0 = \phi$. Given the definition of the black cluster, it is not meaningful to consider $|C_0| = \infty$, since each open Voronoi cell already contains an uncountable number of black points. Instead, we focus on the unboundedness of $C_0$. Let $\theta(p) = \mathbb{P}_p[|C_0| = \infty] = \mathbb{P}_p[C_0^G \text{ is unbounded}]$ be the Voronoi percolation function. As in the discrete version, we observe the occurrence of the associated critical probabilities $p_c$ and $p_T$.

Kolmogorov's Zero-One Law guarantees that $\mathbb{P}_p[\exists x \in \mathcal{P} \text{ such that } |C_0^G(x)| = \infty] = \{0, 1\}$ for any $p$. An event $E$ defined with respect to the Poisson point process $(\mathcal{P}^+, \mathcal{P}^-)$ is called *black increasing* if $E$ is preserved under the addition of black points or the removal of white points. More formally, for every configuration $\omega_1 = (X_1^+, X_1^-) \in E$ and for every configuration $\omega_2 = (X_2^+, X_2^-)$ such that $X_1^+ \subset X_2^+$ and $X_1^+ \supset X_2^+$, we have $\omega_2 = (X_2^+, X_2^-) \in E$. Again, as in the discrete case, we have the following analogous continuum theorems.

**Theorem 4.6.** *For any $\lambda > 0$ and $p \in (0, 1)$. There is almost surely at most one infinite open cluster in $G_{\mathcal{P}}$.*

**Theorem 4.7.** *For random Voronoi percolation in the plane, the critical probability $p_c = \frac{1}{2}$.*

*Proof.* The full proof is the main result of [12]. □

Analyzing the properties of random Voronoi percolation proves to be more challenging than percolation on a lattice or the Gilbert disc model. For example, if $x, y \in \mathbb{R}^d$, the event $\{x \text{ is black}\}$ and $\{y \text{ is black}\}$ are not independent, as $x$ and $y$ might lie in the same open Voronoi cell. Let $LV(R)$ denote the left-most vertical crossing of rectangle $R$, which is obtained through the interface graph through $R$. Then the event $\{\mathrm{LV(R) = P}\}$ is not independent of points in $(\mathcal{P}^+, \mathcal{P}^-)$ located to the right of $P$. Given $LV(R)$, we can infer information about the distribution of $(\mathcal{P}^+, \mathcal{P}^-)$ in small $\epsilon$-neighborhoods around points $p \in P$. Independence in Voronoi tessellations holds when points are arbitrarily far apart.

To address this problem, we confine events to a localized region $R$ ensuring that $(\mathcal{P}^+, \mathcal{P}^-)$ when restricted to $R$ remains unaffected by $(\mathcal{P}^+, \mathcal{P}^-)$ outside of $R$. The following lemma, while intuitive, provides an example of such a restriction.

**Proposition 4.8.** *Fix $\rho, s \geq 1$. Let $R_s(\rho) \subset \mathbb{R}^2$ be a $\rho s \times s$ rectangle. Let $r = 2\sqrt{\log(s)}$ and let $F_r(R_s(\rho))$ be the event that every $B_r(x), x \in R_s(\rho)$ contains at least one point of $\mathcal{P}$. Then $F_r(R_s(\rho))$ occurs with probability 1 as $s \to \infty$. Moreover, if $E(R_s(\rho))$ is an arbitrary event defined by the color of points in $R_s(\rho)$, then $E(R_s(\rho)) \cap F_r(R_s(\rho))$ depends only on the points $\mathcal{P}$ restricted to $B_r(R_s(\rho))$.*

*Proof.* Cover $R_s(\rho)$ with finitely (dependent on $s$) many squares, $S_i$, of length $r/2$. The probability that $S_i$ contains no points of $\mathcal{P}$ is $1/s^2$ i.e. $\mathbb{P}[\mu(S_i) = 0] = 1/s^2 \to 0$ as $s \to \infty$. Thus, every $S_i$ contains a point of $\mathcal{P}$ with probability 1 as $s \to \infty$, which is exactly the event $F_r(R_s(\rho))$.

The second statement follows directly, as $F_r(R_s(\rho))$ depends only on $\mathcal{P}$ restricted to $B_r(R_s(\rho))$. □

# 5. Continuum RSW Theory

With all the relevant material now introduced, we can proceed with the discussion of Continuum RSW Theory. In Section 3, our focus was on box crossings within a fixed lattice structure. We now shift to analyzing these crossings in continuous structures and spaces.

5.1. **RSW Theory on Voronoi Tessellations.** To start, we introduce familiar concepts from classical RSW theory on lattice structures. Throughout this section, we will consider Voronoi percolation on *the plane*, $\mathbb{R}^2$. RSW Theory breaks down in dimensions $d \geq 3$ for several reasons. There is no concrete correspondence between "left-to-right" and "top-to-bottom" crossings. Additionally, many of the discrete RSW results rely heavily on the self-dual nature of $\mathbb{Z}^2$.

Let $R = [a, b] \times [c, d]$ be the a rectangle in $\mathbb{R}^2$. A *horizontal black crossing* of $R$ is a continuous path $P$ from $\{a\} \times [c, d]$ to $\{b\} \times [c, d]$ such that every point $p \in P$ is black. Let $H_b(R)$ denote this event. Let $H_w(R)$ denote the event of a *horizontal white crossing*. Similarly, we can define $V_b(R)$ and $V_w(r)$ for *vertical black crossings* and *vertical white crossings*, respectively. Using an equivalent graph-theoretical, a *horizontal open crossing* is a path $P = q_1, \dots, q_n$ of open vertices such that $V_{q_1}$ intersects the left-hand side of $R$ and $V_{q_n}$ intersects the right-hand side of $R$, with neighboring vertices $q_i$ and $q_{i+1}$ (partially or fully) meeting inside of $R$ for $i \in \{1, \dots, n-1\}$. The next few statements will be the continuous analog of basic facts from discrete RSW Theory.

**Lemma 5.1.** *Consider random Voronoi percolation $V$ on a rectangle $R$ in $\mathbb{R}^2$. Then exactly one of the events $H_b(R)$ or $V_w(R)$ holds.*

*Proof.* The proof is essentially the same as Lemma 3.2. For clarity, let $I$ be the *interface graph* i.e. the set of all points that are colored both black and white, that is, residing on the boundary of Voronoi cells. Referring to Figure 8, the tessellation $V$ with the interface graph $I$ admits to 2 entrance points and 2 exit points. It is clear that both $H_b(R)$ and $V_w(R)$ cannot occur. □

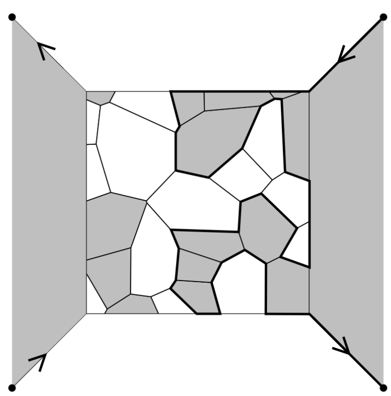

Figure 8. Interface graph for Voronoi Tessellation of the plane. For this configuration, the square admits a vertical white crossing. Credit to [1].

**Corollary 5.2.** *Fix $h > 0$. Let $S = [x, x+h] \times [y+y+h]$ be a square in $\mathbb{R}^2$. Then*

$$\mathbb{P}_{\frac{1}{2}}[H_b(S)] = \frac{1}{2}$$

*Proof.* Let $p = 1/2$. From Lemma 5.1, we have $\mathbb{P}_p[H_b(R)] + \mathbb{P}_p[V_w(R)] = 1$. The result follows from the rotational symmetry of the Poisson process $\mathbb{P}_{1-p}[V_b(R)] = \mathbb{P}_{1-p}[H_b(R)]$, and $\mathbb{P}_p[V_w(R)] = \mathbb{P}_{1-p}[V_b(R)]$. □

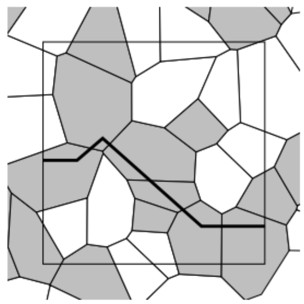

FIGURE 9. Example of a horizontal black crossing of the rectangle $R$ in a random Voronoi tessellation of the plane. It is interesting to observe that there are either 0 or $\infty$ horizontal or vertical crossings of any rectangle. Credit to [1].

Fix $s \in \mathbb{R}_{>0}$. Let $f_p(\rho, s) = \mathbb{P}_p[H_b([0, \rho s] \times [0, s])]$ be the probability (associated with the Voronoi percolation measure) that a $\rho s \times s$ rectangle has a black horizontal crossing.

Let $B_s = [-s, s]^2 \subset \mathbb{R}^2$, and let $A_{s,2s} = B_{2s} \setminus B_s$, be a square annulus of inner radius $s$ and outer radius $2s$. Lastly, let $\mathcal{A}_{s,2s}$ denote the event of a black circuit contained in the square annulus $A_{s,2s}$. For simplicity, $\mathcal{A}_{s,2s}$ will be abbreviated as $\mathcal{A}_s$. The following proposition is a generalization of Lemma 3.7 using the repeated intersection of squares technique applied to the Voronoi percolation measure. Both results are an immediate consequence of Harris-FKG inequality.

**Proposition 5.3.**

*(i)* $f_p(1 + ik, s) \geq f_p(1 + k, s)^i f_p(1, s)^{i-1}$ *for any* $k > 0$ *and* $i \geq 1$.
*(ii)* $f_p(2, s) \geq \mathbb{P}[\mathcal{A}_s]$.

*Proof.*

(i) Consider partitioning $(1+ik)s\times s$ rectangle into $i$ many $(1+k)s\times s$ rectangles with $i-1$ nested $s \times s$ squares. Any event consisting of a horizontal crossing of every $(1+ik)s \times s$ rectangle and a vertical crossing of every $s \times s$ square induces a horizontal crossing of $(1 + ik)s \times s$ rectangle.
(ii) Any black circuit around $A_{s,2s}$ will have to horizontally pass through a $2s \times s$ rectangle or vertically pass through a $s \times 2s$ rectangle.

□

We are now prepared to state (and prove some of) the main results of RSW Theory for random Voronoi percolation.

**Theorem 5.4** (**Continuum RSW**)**.** *Fix $\rho > 0$ and let $p \in (0,1)$. If $\liminf_{s\to\infty} f_p(1,s) > 0$, then $\limsup_{s\to\infty} f_p(\rho,s) > 0$.*

The following theorem is a weaker version of the discrete version RSW as it fails due to the lack of independence between Voronoi cells. Furthermore, Continuum RSW requires a lower bound across infinitely many scales $s$. This only ensures crossing probabilities for larger $\rho s \times s$ rectangles at arbitrary large scales, but not necessarily all scales of $s$.

**Lemma 5.5.** *There exist the following constants and sequence of scales: $c_0 > 0, C > 4$, and $\{s_i\}_{i\geq 1}$ such that for all $i \geq 1$,*

*(i)* $4s_i \leq s_{i+1} \leq Cs_i$

*(ii)* $\mathbb{P}_p[\mathcal{A}_{s_i}] \geq c_0$.

The previous lemma plays a key role in the proof of the following Theorem, an upgraded, stronger, version of Continuum RSW. The full proof of the lemma can be found in [3], Lemma 3.3.

**Theorem 5.6** (**Stronger Continuum RSW**)**.** *Let $p \in (0,1)$. If $\inf_{s\geq 1} f_p(1,s) > 0$, then for every $\rho > 1$, we have $\inf_{s\geq 1} f_p(\rho,s) > 0$.*

*Proof.* Fix $\rho > 1$. Since $\inf_{s\geq 1} f_p(1,s) > 0$, we have a $c_1 > 0$ such that $f_p(1,s) \geq c_1$ for all scales $s$. Let $c_1 > 0$ and $\{s_i\}_{i\geq 1}$ as there were in Lemma 5.5. Thus,

$$\inf_{s\geq 1} \mathbb{P}_p[\mathcal{A}_s] \geq \inf_{i\geq 1} \mathbb{P}_p[\mathcal{A}_{s_i}] \geq c_0$$

An application of both items in Proposition 5.3 yields the following inequality. Note that, $\rho > 1$, guarantees that $\lceil \rho \rceil \geq 2$.

$$\begin{aligned} f_p(\rho,s) &\geq f_p(\lceil \rho \rceil, s) \\ &= f_p(1 + (\lceil \rho \rceil - 1), s) \\ &\geq f_p(2,s)^{\lceil \rho \rceil - 1} f_p(1,s)^{\lceil \rho \rceil - 2} \\ &\geq \left(\mathbb{P}_p[\mathcal{A}_s]\right)^{\lceil \rho \rceil - 1} c_1^{\lceil \rho \rceil - 2} \\ &\geq c_0^{\lceil \rho \rceil - 1} c_1^{\lceil \rho \rceil - 2} \end{aligned}$$

Since this lower bound of $f_p(\rho,s)$ is independent of the scale $s$, $\inf_{s\geq 1} f_p(\rho,s) > 0$. □

Stronger Continuum RSW asserts that for every $\epsilon > 0$, if the probability of a horizontal square crossing is non-zero, then the probability of horizontal $(1+\epsilon)s \times s$ rectangle crossing is also nonzero. Intuitively, this states that information from square crossings can be easily extended to slightly (or much) larger rectangles, and relates to earlier discussions about how a square can "encode" crossing information for much larger rectangles.

**Theorem 5.7** (**Box Crossing Probability**)**.** *Fix $p \in (0,1)$. For any $\rho > 0$, there exists $c = c(\rho) > 0$ such that for any $s \geq 1$, we have*

$$c \leq f_p(\rho,s) \leq 1 - c$$

Box Crossing Probability gives a nonzero probability of a horizontal black crossing of a $\rho s \times s$ rectangle. The upper bound, provides a uniform bound strictly less than 1, which can be used to demonstrates convergence to zero for certain associated crossing events together with the Harris-FKG inequality.

Finishing the discussion of nonzero events, we now shift our focus toward examining high-probability events. More specifically, if a horizontal crossing of an $s \times s$ square occurs almost surely, one might expect that a horizontal crossing of a $\rho s \times s$ rectangle either occurs almost surely or at least have a nonzero probability. This expectation draws from concepts in RSW Theory, which suggest that events on a square can provide information about much larger aspect ratio rectangles in the plane.

**Theorem 5.8** (**High Probability Continuum RSW**). *Let* $p \in (0,1)$. *If* $\lim_{s\to\infty} f_p(1,s) = 1$, *then for all* $\rho > 1$, *we have* $\lim_{s\to\infty} f_p(\rho,s) = 1$.

*Proof.* Fix $\epsilon > 0$, $p \in (0,1)$, and $\rho > 1$. Let $H_s(a,b)$ denote the event that there is a black horizontal crossing from the left side of $B_{s/2}$ to $\{s/2\} \times [a,b]$.

By Stronger Continuum RSW, there exists a constant $c > 0$ such that $\mathbb{P}_p[\mathcal{A}_s] \geq c$ for all scales $s$. A consequence of Proposition 4.8 is the existence of some $\eta > 0$ such that

$$\mathbb{P}_p[\mathcal{A}_{\eta s, s/4}] \geq 1 - \epsilon$$

for large $s$.

Consider $B_{s/2}$. We subdivide the vertical segment of length $s$ into $\lceil \frac{1}{2\eta} \rceil$ segments of length $2\eta s$. Given the crossing probability $f_p(1,s)$, a black path will intersect one of the $2\eta s$-length segments on the right side. Thus, by the Nth Root Trick, there exists a $y_s \in [-s/2, s/2]$ such that

$$\mathbb{P}_p[H_s(y_s - \eta s, y_s + \eta s)] \geq 1 - (1 - f_p(1,s))^{\lceil \frac{1}{2\eta} \rceil}$$

Now, we define the following three black increasing events.

(i) Let $E_1$ be the event that there is a black path from left to $\{s/2\} \times [y_s - \eta s, y_s + \eta s]$ in $B_{s/2}$.
(ii) Let $E_2$ be the event that there is a black path from $\{s/2\} \times [y_s - \eta s, y_s + \eta s]$ to right in $(s,0) + B_{s/2}$.
(iii) Let $E_3$ be event of a black circuit in $(s/2, y_s) + A_{\eta s, s/4}$.

By overlapping the events $E_1, E_2$, and $E_3$, we observe that the intersection of these events induces a left-to-right crossing of $[-s/2, 3s/2] \times [-s/2, s]$ or $[-s/2, 3s/2] \times [-s, s/2]$. The choice is dependent on where the value $y_s$ lies on the vertical segment $\{s/2\} \times [-s/2, s/2]$. Applying the Harris-FKG inequality, we obtain the following relationship for all large $s$.

$$f_p(4/3, 3s/2) \geq \mathbb{P}_p[E_1 \cap E_2 \cap E_3] \geq \mathbb{P}_p[E_1]\mathbb{P}_p[E_2]\mathbb{P}_p[E_3]$$

Combining inequalities we have that,

$$f_p(4/3, 3s/2) \geq \left(1 - (1 - f_p(1,s))^{\lceil \frac{1}{2\eta} \rceil}\right)^2 (1 - \epsilon)$$

And taking the infimum over $s \geq 1$, we derive

$$\liminf_{s\to\infty} f_p(4/3, s) \geq (1 - \epsilon)$$

Thus, $f_p(4/3, s) \to 1$ as $s \to \infty$. The rest of the proof is similar to the computations in Theorem 5.6. As before, note the fixed value of $\rho > 1$ guarantees that $\lceil \rho \rceil \geq 2$.

$$\begin{aligned}
f_p(\rho, s) &\geq f_p(\lceil \rho \rceil, s) \\
&= f_p(1 + 3(\lceil \rho \rceil - 1)(3/3), s) \\
&\geq f_p(4/3, s)^{3(\lceil \rho \rceil - 1)} f_p(1, s)^{3(\lceil \rho \rceil - 1) - 1}
\end{aligned}$$

The desired result is obtained by taking the limit inferior of the inequality as $s \to \infty$. □

5.2. **Continuum Exponential Decay.** We now shift our focus towards the "size" of the infinite cluster $C$ in random Voronoi percolation in the plane. Since we cannot define the size by counting the number of black points in $C$, we instead characterize it geometrically through its radius or area. Alternatively, we can count the number of open sites in its graph structure $G_{\mathcal{P}}$. Thus, we have the following adapted continuum statement of Discrete Exponential Decay. $\mathbb{P}_p$ denotes the Voronoi percolation measure.

**Theorem 5.9** (**Continuum Exponential Decay).** *Let $|C|$ denote the radius of $C$, the area of $C$, or the number of open Voronoi cells in $C$. Then for any $p < p_c = 1/2$, there exists a constant $c = c(p)$ such that*

$$\mathbb{P}_p[|C| \geq n] \leq e^{-c(p)n}$$

*for every $n \geq 1$.*

It's important to realize that the continuum version has fewer restrictions compared to Discrete Exponential Decay. We also have the continuum analog of Menshikov's Theorem, which asserts that the critical probability for the percolation function coincides with the threshold probability for the expectation that the size of the zero-cluster is infinite. For random Voronoi percolation Theorem 4.7 establishes that the critical probability $p_c$ and the threshold probability $p_T$ are both equal to $1/2$.

More interesting connections between the exponential decay of the zero cluster, box-crossing probabilities, and RSW (both discrete and continuum) can be found in [7].

## Acknowledgements

First and foremost, I would like to thank my mentor, Professor Minjae Park His invaluable guidance has made this paper possible, and I am truly grateful for his advice and encouraging support. I would also like to thank Professor Peter May for his extraordinary efforts in running the REU. Lastly, thank you to my roommates and friends, Andrew Razborov and Yash Rastogi, for their motivation and the countless late-night conversations throughout the program.

## References

[1] Béla Bollobás and Oliver Riordan. *Percolation*. 2009.
[2] Jeffrey Steif. *A mini course on percolation theory*. 2009. https://www.math.chalmers.se/~steif/perc.pdf.
[3] Geoffrey Grimmet. *Percolation*. 1999.
[4] Hugo Duminil-Copin. *Introduction to Bernoulli percolation*. October 7, 2018. https://www.ihes.fr/~duminil/publi/2017percolation.pdf.
[5] Jared Saia. *My notes on Voronoi Diagrams, Delaunay Triangulations and More Dual Transformations*. 2020. https://www.cs.unm.edu/~saia/classes/506-s20/lec/VoronoiDelaunayandHulls.pdf.
[6] Vincent Tassion. *Crossing probabilities for Voronoi percolation*. 2015. https://arxiv.org/pdf/1410.6773.
[7] Hugo Duminil-Copin and Vincent Tassion. *RSW and Box-Crossing Property for Planar Percolation*. 2016. https://www.unige.ch/~duminil/publi/2015IAMPproceedings.pdf.

[8] Kim Christensen. *Percolation Theory*. 2002. https://web.mit.edu/ceder/publications/Percolation.pdf.

[9] Leandro P. R. Pimentel. *On some fundamental aspects of polyominoes on random Voronoi tilings*. 2013.

[10] Burton and Keane. *Density and uniqueness in percolation.* 1989. Pages 501–505.

[11] A. Gilevich, S. Shoval, M. Nosonovsky, M. Frenkel, E. Bormashenko. *Converting Tessellations into Graphs: From Voronoi Tessellations to Complete Graphs* 2024. https://doi.org/10.3390/math12152426.

[12] Béla Bollobás and Oliver Riordan. *The critical probability for random Voronoi percolation in the plane is 1/2*. 2004. https://arxiv.org/abs/math/0410336