RICCI FLOW WITH SURGERY

HE YIKE

Abstract. Ricci flows typically develop singularities within finite time. Perelman's groundbreaking work allowed for the classification of these singularities and introduced the concept of Ricci flow with surgery. This innovation was crucial in establishing the existence and long-term behavior control of Ricci flow with surgery, ultimately contributing to the proof of Thurston's geometrization conjecture. In this paper, we outline the construction of Ricci flow with surgery.

Contents

1. Introduction

One of the central themes of modern geometry is the construction of canonical objects on specific geometric spaces. A classical result of this kind is the uniformization theorem for simply connected Riemann surfaces [26], which ultimately leads to the topological classification of compact Riemann surfaces. The canonical objects of interest here are metrics with constant Gaussian curvature. In more recent developments, during the 1980s, Simon Donaldson investigated the moduli space of anti-self-dual connections [11], which can be considered as "canonical" because they minimize the Yang-Mills functional. His work resulted in a striking theorem

Date: August, 2024.

about 4-dimensional differential topology. Another example is the Kähler-Einstein metric and the related notion called K-stability, both of which play a crucial role in constructing good moduli spaces for Fano varieties.

In the context of three-dimensional topology, in 1982, Richard Hamilton [12] first put out the idea of studying an evolution equation of metric: the Ricci flow

$$\frac{\partial g_{ij}}{\partial t} = -2\,\mathrm{Ric}_{ij}(g).$$

Starting with a metric of positive Ricci curvature and evolving it under Ricci flow, Hamilton showed that the metric will finally tend to a metric with constant curvature in an appropriate sense. If one can show that, given some condition on the fundamental group, we can construct a metric of positive constant curvature on such manifold, then the Poincaré conjecture is proved because the only simply connected closed manifolds admitting such metrics are spheres.

However, when starting with an arbitrary metric and evolving it under the Ricci flow, finite time singularity will appear. Therefore, one must have a better understanding of the singularity and try to extend the Ricci flow for a longer time. Before Perelman's seminal work, people usually applied the maximum principle to Ricci flow, which only gave us relatively weaker control of the Ricci flow near finite time singularity.

In three remarkable papers [23],[24],[25], Grisha Perelman significantly advanced the theory of Ricci flow. The central idea behind many of Perelman's arguments is the **compactness-contradiction argument**. If we want to prove some claim is true for general Ricci flow, we may consider a sequence of Ricci flows which contradicts our claim. By applying some compactness results, we may pass to a limit Ricci flow. Such limit flows turn out to be more understandable and combine our knowledge of them with the starting assumption we may draw a contradiction and prove our claim.

On the one hand, Perelman introduced new quantities associated with the flow, enabling a qualitative classification of limit flows. On the other hand, he also drew some stronger control over curvature, allowing him to obtain compactness in less favorable settings. With these preparations, Perelman finally classified the finite time singularities of the Ricci flow and defined the notion of Ricci flow with surgery. Analyzing the long-time behavior of Ricci flow with surgery leads to the proof of Thurston's geometrization conjecture.

In this paper, we primarily focus on Perelman's advances, assuming prior results as given. The reader can find results about variation formulas of Ricci flow, short-time existence of Ricci flow and generalized maximum principle of Ricci flow in [21],[10],[12] and [30]. For results from comparison geometry, see [7] and [19]. For detailed explanation of Perelman's argument, see [21],[22],[17] and [6]. Relevant facts about Ricci flow are also listed in the Appendix.

Finally, we mention some recent developments in the three-dimensional Ricci flow. One unsatisfying aspect of Perelman's Ricci flow with surgery is that its construction involves many artificial choices of parameters. Thus, the Ricci flow with surgery is by no means canonical and cannot be uniquely determined by initial data. Recently, in [16] and [2], Bruce Kleiner, John Lott, and Richard Bamler defined the notion of singular Ricci flow, which is some kind of "weak" solution of Ricci flow, and proved that it is uniquely determined by initial data. Further, one may regard Ricci flow with surgery as a regularization of singular Ricci flow. This new notion is used to construct homotopy in the space of Riemannian metrics, which finally gives a proof of generalized Smale conjecture [3]. Besides, in [4], Bamler proved that one can choose parameters appropriately such that there are only finitely many surgeries in the Ricci flow with surgery. These two advances were originally conjectured by Perelman. Ricci flow has also been useful in studying the relationship between curvature and various dynamical and geometric quantities of interest. See [1] [20] [18].

We now fix some notations. Let (M, g) be a Riemannian manifold. Locally, we write the metric as (g_{ij}). Denote by (g^{ij}) the inverse of the matrix (g_{ij}). The Christoffel symbols are given by $\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij})$ and the Levi-Civita connection is denoted by ∇. Here ∇ naturally extends to any tensor fields. Define the Riemann curvature tensor as $\mathrm{Rm}_{ijkl} = g_{ks}(\partial_i \Gamma^s_{jl} - \partial_j \Gamma^s_{il} + \Gamma^t_{jl}\Gamma^s_{it} - \Gamma^t_{il}\Gamma^s_{jt})$, the Ricci curvature as $\mathrm{Ric}_{ij} = \partial_l \Gamma^l_{ij} - \partial_j \Gamma^l_{il} + \Gamma^l_{lt}\Gamma^t_{ij} - \Gamma^l_{ip}\Gamma^p_{lj}$, and the scalar curvature as $R = g^{ij}\,\mathrm{Ric}_{ij}$. We denote the volume form by dV or dV_g. Norms $|\cdot|^2$ are always defined with respect to g (for example $|\,\mathrm{Rm}\,|^2 := g^{ij} g^{ab} g^{kl} g^{cd}\,\mathrm{Rm}_{iakc}\,\mathrm{Rm}_{jbld}$). The Laplacian is defined by $\Delta A := g^{ij}(\nabla_{\partial i}\nabla_{\partial j} A - \nabla_{\nabla_{\partial i}(\partial j)} A)$, where A is a tensor field. The Hessian is defined by $\mathrm{Hess}(f) := \nabla df$, where f is a smooth function. We denote the diffeomorphism group of M by $\mathbf{Diff}(M)$.

2. Blow-up limit of finite time singularity

This section describes how to obtain a κ-solution from Ricci flow with finite-time singularity. In Subsection 2.1, We will introduce Perelman's W-functional and show its monotonicity. In Subsection 2.2, the monotonicity is used to prove the non-collapsing theorem. In Subsection 2.3, with the non-collapsing theorem, we can apply Hamilton's convergence theorem in [13] to extract a limit Ricci flow. We will then define the notion of κ-solutions and prove that the blow-up limit is indeed a κ-solution using results in section 2 and [12] and [30].

κ-solutions are important because they appear as limit Ricci flow in the **compactness-contradiction argument**. As indicated in the introduction, the properties of the limit Ricci flow help us draw conclusions about general Ricci flows. On the other hand, the non-collapsing theorem will be used to provide lower bounds for injectivity radius. The non-collapsing theorem is an inequality of volume, which is easier to understand compared to the injectivity radius. We shall obtain control over the volume by estimating integrands in the W-functional.

2.1. **Perelman's W-functional.**

Consider the functional

$$W(g, f, \tau) := \int_M [\tau(R + |\nabla f|^2) + f - n](4\pi\tau)^{-\frac{n}{2}} e^{-f} dV, \tag{2.1}$$

where M is a smooth manifold, g is a Riemannian metric, R is the scalar curvature of g, $f; M \to \mathbb{R}$ is smooth, τ is positive. We also require that

$$\int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV = 1. \tag{2.2}$$

Denote $(4\pi\tau)^{-\frac{n}{2}} e^{-f}$ by function u. We say g, f, and τ are compatible if (2.2) is satisfied. This functional was introduced by Perelman in [23]. The W-functional is invariant under diffeomorphism $W(g, f, \tau) = W(p^* g, p^* f, \tau)$ for $p \in \mathbf{Diff}(M)$ and rescaling $W(g, f, \tau) = W(\lambda g, f, \lambda\tau)$.

The following proposition shows that the W-functional is non-decreasing under the Ricci flow if f and τ are made to evolve appropriately. This monotonicity property will play a crucial role in establishing the non-collapsing theorem.

Proposition 2.3. *Let M be a closed smooth manifold. If g, f, and τ evolve according to*

$$\frac{\partial g_{ij}}{\partial t} = -2\operatorname{Ric}_{ij}, \frac{d\tau}{dt} = -1, \frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau},$$

then the W-functional evolves according to

$$\frac{d}{dt}W(g,f,\tau) = 2\tau\int_M |\operatorname{Ric} + \operatorname{Hess}(f) - \frac{g}{2\tau}|^2 (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV, \tag{2.4}$$

which implies that W is non-decreasing.

Proof. We define the smooth function v by

$$v := [\tau(2\Delta f - |\nabla f|^2 + R) + f - n]u. \tag{2.5}$$

Then it is sufficient to prove that

$$\Box^* v = -2\tau|\operatorname{Ric} + \operatorname{Hess}(f) - \frac{g}{2\tau}|^2 u. \tag{2.6}$$

Here, $\Box^* = -\frac{\partial}{\partial t} - \Delta + R$ is the conjugate heat operator. Combining this with the fact that $\int_M \Delta u dV = \int_M (\Delta f - |\nabla f|^2) u dV = 0$ (by the divergence theorem) and remark 6.3.2 in [30], we have $W = \int_M v dV$ and $\frac{d}{dt}W = \frac{d}{dt}\int_M v = -\int_M \Box^* v$, which is exactly the conclusion of the proposition.

We now prove that

$$\Box^* v = -2\tau|\operatorname{Ric} + \operatorname{Hess}(f) - \frac{g}{2\tau}|^2 u.$$

Splitting v as a product of u and $\frac{v}{u}$, we see that

$$\Box^* v = \Box^*(u\frac{v}{u}) = -u(\frac{\partial}{\partial t} + \Delta)(\frac{v}{u}) - 2\langle\nabla(\frac{v}{u}), \nabla u\rangle,$$

For the first term, we have

$$-(\frac{\partial}{\partial t} + \Delta)v = (2\Delta f - 2|\nabla f|^2 + 2R) - \tau(\frac{\partial}{\partial t} + \Delta)(2\Delta f - 2|\nabla f|^2 + 2R) - \frac{n}{2\tau}.$$

Now we need some standard evolution formulas of Ricci flow listed as follows

$$\frac{\partial}{\partial t}R = \Delta R + 2|\operatorname{Ric}|^2,$$

$$\frac{\partial}{\partial t}\Delta f = \Delta\frac{\partial}{\partial t}f + 2\langle\operatorname{Ric}, \operatorname{Hess}(f)\rangle,$$

$$\frac{\partial}{\partial t}|\nabla f|^2 = 2\operatorname{Ric}(\nabla f, \nabla f) + 2\langle\nabla f, \nabla\frac{\partial f}{\partial t}\rangle.$$

These formulas follow from standard calculation. See [10] for complete calculations.

Combine these with Bochner formula, we have

$$\begin{aligned} u^{-1}\Box^* v &= -\frac{n}{2\tau} + 2\Delta f + 2R + \tau(-4\langle\operatorname{Ric}, \operatorname{Hess}(f)\rangle - 2|\operatorname{Ric}|^2 - 2|\operatorname{Hess}(f)|^2) \\ &= -2\tau|\operatorname{Ric} + \operatorname{Hess}(f) - \frac{g}{2\tau}|^2. \end{aligned}$$

This finishes the proof. □

Remark 2.7. If the W-funcional is constant along some Ricci flow $(M, g(t))$, then the proposition implies that

$$\text{Ric}(g_{ij}(t)) + \text{Hess}(f) = -\frac{g(t)}{2\tau}. \tag{2.8}$$

This flow is a gradient shrinking soliton in light of Theomrem A.1 and Example A.3. Therefore, in nonlinear PDE's terms, we can say that the W-functional is critical (scaling-invariant) and coercive (geometric-controlling). See [29] for more discussions from PDE perspectives. Such critical, coercive quantities play an important role in studying nonlinear evolution systems. Before Perelman's work, researchers had only weaker control using the generalized maximum principle. See[12][10][30].

The following purely analytic lemma will be used in the proof of the non-collapsing theorem. The lemma essentially states that the infimum of the W-functional is attained by a smooth function f and this infimum is strictly positive. Recall that we required g, f, and τ to be compatible.

Lemma 2.9. *For a closed Riemannian manifold (M, g_{ij}), and $\tau > 0$, the infimum of $W(g, f, \tau)$ is attained by a smooth compatible function. Moreover, defining μ as*

$$\mu(g, \tau) := \inf_{f \in C^\infty} W(g, f, \tau),$$

the function $\mu(g, \tau)$ is bounded below as τ varies within any fixed finite interval $(0, \hat{\tau}]$

To prove Lemma 2.9, we change the variables of W-functional by defining $\phi := e^{-\frac{f}{2}}$. We then obtain

$$\hat{W}(g, \phi, \tau) := W(g, f, \tau) = (4\pi\tau)^{-\frac{n}{2}} \int_M [\tau(4|\nabla\phi|^2 + R\phi^2) - 2\phi^2 ln\phi - n\phi^2]dV.$$

The condition (2.2) becomes $\int_M u = (4\pi\tau)^{-\frac{n}{2}} \int_M \phi^2 = 1$. The first statement of Lemma 2.9 is proved in [27], and the second statement is a claim of [23] on page 9.

Remark 2.10. We may define the quantity ν,

$$\nu(g, \tau_0) := \inf_{\tau \in (0, \tau_0]} \mu(g, \tau) > -\infty, (\tau_0 > 0).$$

This quantity will give us a lower bound for volume in Subsection 2.2.

2.2. **Non-collapsing theorem.** Given a complete n-dimensional Riemannian manifold (M, g), a point $p \in M$, and a positive number s, we denote $V_g(p, s) := \text{Vol}(B_g(p, s))$ as the volume of the geodesic ball $B_g(p, s)$ centered at p with radius s, with respect to the metric g.

Definition 2.11. We say that a metric g is κ-non-collapsed on the scale ρ if every geodesic ball B of radius $s < \rho$, which satisfies $|\text{Rm}(p)| \le s^{-2}$ for any $p \in B$, has $\text{Vol}(B) \ge \kappa r^n$.

Theorem 2.12. *(Non-collapsing). Let $(M, g(t))$ be a Ricci flow on an n-dimensional closed manifold for $t \in [0, T]$. Suppose $T \in (0, T_0]$, where $T_0 > 0$ is some fixed number. Then the metric $g(T)$ is κ-non-collapsed on the scale ρ, where ρ is some positive number and κ depends on $n, g(0), \rho$ and T_0.*

Remark 2.13. Actually, we will prove a stronger result. For every geodesic ball B of radius $s < \rho$, which satisfies $|R(p)| \le s^{-2}$ (replacing Riemannian curvature tensor with scalar curvature) for any $p \in B$, we have $\text{Vol}(B) \ge \kappa r^n$.

We first prove the theorem assuming the following proposition.

Proposition 2.14. *Let $(M, g(t))$ be a Ricci flow on an n-dimensional closed manifold, for $t \in [0, T]$, r be a positive number and $p \in M$. Then we have*

$$\gamma \leq \frac{V_{g(T)}(p,r)}{V_{g(T)}(p,r/2)} + ln\left[\frac{V_{g(T)}(p,r)}{r^n}\right] + \frac{r^2}{V_{g(T)}(p,r/2)}\int_{B(p,r)}|R|dV, \tag{2.15}$$

for some $\gamma \in \mathbb{R}$ depending on $n, g(0)$, and upper bounds for r and T.

Proof. (Non-collapsing theorem) let T_0 be the upper bound of T. By Proposition 2.14, γ depends on $n, g(0), \rho$ and T_0. When $s \in (0, r]$, we have that $|R| \leq r^{-2} \leq s^{-2}$ on $B(p, s)$. Therefore, we have

$$\gamma \leq 2\frac{V_{g(T)}(p,r)}{V_{g(T)}(p,r/2)} + ln[V_{g(T)}(p,r)r^{-n}].$$

Define ω_n to be the volume of the unit ball in the n-dimensional Euclidean space. We have $V_{g(T)}(p,r)r^{-n} \to \omega_n$ as s tends to zero. We claim that we can take $\kappa := \min\{\frac{\omega_n}{2}, e^{\gamma - 2^{n+1}}\}$.

Assuming the contrary, if $V_{g(T)}(p,r)r^{-n} \leq \kappa \leq e^{\gamma-2^{n+1}}$, we have

$$\gamma \leq 2V_{g(T)}(p,r)/V_{g(T)}(p,r/2) + \gamma - 2^{n+1}.$$

We see that $2^n \leq V_{g(T)}(p,r)/V_{g(T)}(p,r/2)$ and hence that $\kappa \geq V_{g(T)}(p,r)r^{-n} \geq V_{g(T)}(p,r/2)(r/2)^{-n}$. Using this inequality iteratively, we have $V_{g(T)}(p, \frac{r}{2^m})/(\frac{r}{2^m})^n \leq \kappa < \omega_n$ for any $m \geq 1$. This contradicts the fact that $V_{g(T)}(p, \frac{r}{2^m})/(\frac{r}{2^m})^n \to \omega_n$ as m tends to infinity. □

We now return to the proof of Proposition 2.14. We will obtain the lower bound by the monotonicity of W-functional and the following lemma, in which we relate the W-functional with geometric information.

Lemma 2.16. *For any n-dimensional closed Riemannian manifold (M, g), and $r > 0, p \in M$ and $\lambda > 0$,*

$$\mu(g, \lambda r^2) \leq 36\lambda\frac{V(p,r) - V(p,\frac{r}{2})}{V(p,\frac{r}{2})} + ln\left[\frac{V(p,r)}{(4\pi\lambda r^2)^{\frac{n}{2}}}\right] + \frac{\lambda r^2}{V(p,\frac{r}{2})}\int_{B(p,r)}|R|. \tag{2.17}$$

Proof. Recall we have

$$\hat{W}(g,\phi,\lambda r^2) = (4\pi\lambda r^2)^{-\frac{n}{2}}\int_M[\lambda r^2(4|\nabla\phi|^2 + R\phi^2) - 2\phi^2 ln\phi - n\phi^2]dV,$$

and $(4\pi\lambda r^2)^{-n/2}\int_M \phi^2 dV = 1$.

We now choose a special test function ϕ and estimate the integral. Let $\varphi(x) : [0, \infty) \to [0, 1]$ be a smooth function, such that $\text{supp}(\varphi(x)) \subseteq [0, 1], \varphi(x) = 1$ for $x \in [0, 1/2]$ and $|d\varphi/dx| \leq 3$. Then we let $\phi = e^{-c/2}\varphi(d_g(x,p)/r)$, where c is determined by the compatibility condition. Therefore, we have $V(p, r/2) \leq e^c(4\pi\lambda r^2)^{-n/2} \leq V(p, r)$. Now we estimate $\hat{W}$ term by term. For the first term, we use $|\nabla\phi| \leq e^{-c/2}r^{-1}\sup|d\varphi/dx| \leq 3r^{-1}e^{-c/2}$. For the second term, we apply $\phi^2 \leq e^{-c} \leq (4\pi\lambda r^2)^{n/2}/V(p, r/2)$. For the third term, we apply Jensen's inequality. The fourth term is given by compatibility condition. □

Proof. (Proposition 2.14) Take $\lambda = 1/100$ and $g = g(T)$ in (2.15). We take a smooth function f_T such that $W(g(T), f_T, \frac{1}{100}r^2) = \mu(g(T), \frac{1}{100}r^2)$. We then set $\tau = T + \frac{1}{100}r^2 - t$ and f evolve as in Proposition 2.3 ($f(T) = f_T$). Then by monotonicity, we have

$$\mu(g(0), \frac{1}{100}r^2 + T) \leq W(g(0), f(0), \frac{1}{100}r^2 + T)$$
$$\leq W(g(T), f(t), \frac{1}{100}r^2) = \mu(g(T), \frac{1}{100}r^2).$$

This, coupled with inequality (2.17), gives us the lower bound γ. □

We have finished the proof of the non-collapsing theorem.

2.3. **Blow-up limits and κ-solutions.** In this section, we prove that the blow-up limit of a Ricci flow at a finite time singularity is a κ-solution.

Definition 2.18. (κ-solution) An ancient solution is a Ricci flow $(\hat{M}, \hat{g}(t))$ defined on $t \in (-\infty, 0]$ such that each time slice $(\hat{M}, \hat{g}(t))$ is a connected, complete, non-flat Riemannian manifold whose curvature operator Rm is bounded and non-negative. A κ-solution is an ancient solution that is κ-non-collapsed on all scales for each time slice.

Remark 2.19. The scalar curvature $R_{g(t)}$ of a κ-solution $(M, g(t))$ is strictly positive because $(M, g(t))$ is non-flat and Rm is non-negative.

First, we introduce the sequence of parabolically re-scaled Ricci flows. By Theorem A.1, we suppose that $(M, g(t))$ is a Ricci flow defined on its maximal interval. From theorem 14.1 of [12], we know that $\sup |\operatorname{Rm}| \to +\infty$ as $t \to T$. To control the curvature, we rescale the metric near T. Specifically, pick (p_i, t_i) from $M \times [0, T - \frac{1}{i}]$ such that for any $(p, t) \in M \times [0, T - \frac{1}{i}]$ we have $|\operatorname{Rm}(p_i, t_i)| \geq |\operatorname{Rm}(p, t)|$. We define a sequence of Ricci flow $(M, g_i(t))$ by $g_i(t) = |\operatorname{Rm}(p_i, t_i)| g(t_i + \frac{t}{|\operatorname{Rm}(p_i, t_i)|})$, which is defined on the time interval $[-t_i|\operatorname{Rm}(p_i, t_i)|, (T - t_i)|\operatorname{Rm}(p_i, t_i)|]$. From dimensional analysis, one can see that the rescaled metric also satisfies the Ricci flow equation and that $|\operatorname{Rm}|$ is now bounded by 1. From Chapter 8, section 4 of [10], there is a positive number b such that $b \leq (T - t_i)|\operatorname{Rm}(p_i, t_i)|$ holds for any i. We call $(M_i, g_i(t), p_i)$, where $t \in (-t_i|\operatorname{Rm}(p_i, t_i)|, b)$, a sequence of parabolically re-scaled Ricci flows

Definition 2.20. Let $(M_i, g_i(t), p_i)$ and $(\hat{M}, \hat{g}(t), \hat{p})$ be smooth families of complete Riemannian manifolds for $t \in (a, b)$, where $-\infty \leq a < b \leq +\infty$. We say that

$$(M_i, g_i(t), p_i) \to (\hat{M}, \hat{g}(t), \hat{p})$$

as $i \to \infty$ if there exist a sequence (Ω_i, ϕ_i) such that

(i) $\{\Omega_i\}$ is a compact exhausting sequence of $\hat{M}$. Namely, we have $\Omega_i \subseteq \Omega_{i+1}$, $\hat{p} \in \operatorname{int}(\Omega_i)$ for each i and that any compact subset K of $\hat{M}$ is contained in some Ω_i for sufficiently large i.

(ii) ϕ_i is a sequence of smooth maps satisfying $\phi_i(p) = p_i$ and $\phi_i(\Omega_i)$ diffeomorphic to Ω_i.

(iii) $\phi_i^* g_i(t)$ converge to $\hat{g}(t)$ as $i \to \infty$ in C^∞ topology.

In [13], Hamilton proved the following compactness theorem of Ricci flows. This theorem will be used in Theorem 2.22.

Theorem 2.21. *(Compactness of Ricci flows) Let $(M_i, g_i(t), p_i)$ be a sequence of complete Ricci flow defined on $t \in (a, b)$, where $-\infty \le a < 0 < b \le +\infty$, for all i. Suppose that (i) $|\operatorname{Rm}|$ is uniformly bounded with respect to both t and i; (ii) injectivity radii of complete pointed Rimanian manifolds $(M_i, g_i(0), p_i)$ have a positive lower bound.*

Then there exist a pointed complete Ricci flow $(\hat{M}, \hat{g}(t), \hat{p})$, where $t \in (a, b)$ and $\hat{p} \in \hat{M}$, such that, after passing to a sub-sequence,

$$(M_i, g_i(t), p_i) \to (\hat{M}, \hat{g}(t), \hat{p}),$$

as $i \to \infty$.

We are now prepared to state the main theorem of this section.

Theorem 2.22. *Let (M,g(t)) be a 3-dimensional Ricci flow, defined on a finite maximal time interval $[0, T)$. Then there exists a complete Ricci flow $(\hat{M}, \hat{g}(t), \hat{p})$ such that, after passing to a subsequence, the sequence of parabolically re-scaled Ricci flows $(M_i, g_i(t), p_i)$ converges to $(\hat{M}, \hat{g}(t), \hat{p})$, which is called the blow-up limit.*

Futhermore, $(\hat{M}, \hat{g}(t))$ is a κ-solution, where κ only depends on $g(0)$ and T.

Remark 2.23. Careful readers may notice that the domains of $(M_i, g_i(t), p_i)$ and $(\hat{M}, \hat{g}(t), \hat{p})$ are different, which is not identical to the setting of Theorem 2.21. However, one can apply Theorem 2.21 for $-\infty < a < 0$ first and then let a tend to minus infinity (notice $t_i|\operatorname{Rm}(p_i, t_i)| \to \infty$ as $i \to \infty$).

Proof. (1) Existence of $(\hat{M}, \hat{g}(t), \hat{p})$ and $t \in (-\infty, 0]$. Our strategy is to apply Theorem 2.21 as indicated in Remark 2.23. The bounded curvature condition is naturally satisfied. From Chapter 7 theorem 4.2 of [5], one sees that the non-collapsing theorem gives a positive lower bound for injectivity radii. (2) κ-non-collapsed on all scales. We only need two observations to prove this. First, if g is κ-non-collapsed on scale ρ, then λg is κ-non-collapsed on scale $\rho\sqrt{\lambda}$. Second, if $(M_i, g_i(t), p_i)$ is κ-non-collapsed on scale ρ, then so is $(\hat{M}, \hat{g}(t), \hat{p})$. (3) Completeness and connectedness. They are guaranteed by Theorem 2.21. (4) Curvature norms are bounded and positive. The Boundedness is trivial. The positivity follows from Theorem A.4. (5) Non-flatness. Observe that $|\operatorname{Rm}(p_i, 0)| = 1$. □

3. κ-SOLUTIONS

In this section, we give a complete qualitative classification of κ-solutions. In Subsection 3.1, we introduce new quantities which will be utilized throughout this section. In Subsection 3.2, we prove the existence of asymptotic gradient shrinking solitons in each κ-solution and classify these solitons. In Subsection 3.3, we demonstrate that the space of κ-solutions is compact and establish some uniform geometric control over them. In Subsection 3.4, we classify κ-solutions by combining Theorem 3.17, Theorem 3.26, and Theorem 3.24 with the compactness-contradiction argument as outlined in the Introduction 1.

3.1. Perelman's reduced volume.

We introduce another monotone quantity introduced by Perelman in [23]. For convenience, we will work with backward time, denoted by $\tau := -t$.

Definition 3.1. Let $(M, g(t))$ be an n-dimensional Ricci flow, and let $\gamma : [\tau_1, \tau_2] \to M$ $(0 \leq \tau_1 < \tau_2)$ be a smooth path. We define the L-lengh of γ by

$$L(\gamma) := \int_{[\tau_1,\tau_2]} [R_g(\tau)(\gamma(\tau)) + |\gamma'(\tau)|^2_{g(\tau)}]\sqrt{\tau}d\tau, \tag{3.2}$$

where $\tau := -t$.

Fix a point $(p, 0)$, for a point (q, τ), we define the L-length by $L(q, \tau) = \inf(L(\gamma))$, where the infimum is taken over all smooth paths $\gamma : [0, \tau] \to M$ such that $\gamma(0) = p$ and $\gamma(\tau) = q$. We define the reduced length by $l(q, \tau) = L(q, \tau)/\sqrt{\tau}$, and the reduced volume by

$$\hat{V}(\tau) := \int_M \tau^{-\frac{n}{2}} \exp(-l(q, \tau))dV_{g(\tau)} \tag{3.3}$$

The reduced volume is also a monotone quantity, providing strong geometric control over the Ricci flow.

Theorem 3.4. *(Monotonicity of Reduced Volume) Let $(M, g(t))$ be an n-dimensional Ricci flow with bounded curvature, such that each time slice is a complete Riemannian manifold. Then the integral in the definition of $\hat{V}(\tau)$ is absolutely convergent. The function $\hat{V}(\tau)$ is non-increasing with respect to τ and satisfies $\hat{V}(\tau) \to (4\pi)^{\frac{n}{2}}$ as $\tau \to 0$. If $\hat{V}(\tau)$ is not strictly decreasing, then $(M, g(t))$ is a gradient shrinking soliton.*

Proof. See Theorem 3.2.8 of [6] and Theorem 7.26 of [21]. □

3.2. **Asymptotic gradient shrinking solitons.** We fix a positive number κ and consider an n-dimensional κ-solution $(M, g(t))$ defined on $t \in (-\infty, 0]$. Fix a point $(p, 0)$ as before, and let $q(\tau)$ be a point such that $l(q, \tau)$ attains its minimum at $q(\tau)$ when regarded as a function of q. By corollary 3.2.6 of [6], We also have the bound $l(q(\tau), \tau) \leq \frac{n}{2}$. Define $g_\tau(t) = \frac{1}{\tau} g(\tau t)$ for $t \in (-\infty, 0]$. We now state the main theorem of this subsection.

Theorem 3.5. *(Asymptotic gradient shrinking solitons) Choose a sequence τ_k such that $\tau_k \to \infty$ as k tends to ∞. We denote $g_k := g_{\tau_k}$ and $q_k := q(\tau_k)$. After passing to a subsequence, there exists a non-flat pointed Ricci flow $(M_\infty, g_\infty(t), (q_\infty, -1))$ defined on $t \in (-\infty, 0)$ such that $(M, g_k, (q_k, -1)) \to (M_\infty, g_\infty(t), (q_\infty, -1))$ in the sense of Definition 2.17. Furthermore, $(M_\infty, g_\infty(t)$ is κ-non-collapsed with non-negative curvature operators and is a gradient shrinking soliton. Namely there exists a smooth function $f : M \times (-\infty, 0) \to \mathbb{R}$ such that, for all $t \in (-\infty, 0)$,*

$$\mathrm{Ric}(g_\infty(t)) + \mathrm{Hess}_{g_\infty(t)}(f) = \frac{g_\infty(t)}{2\tau}. \tag{3.6}$$

There are two main nontrivial points to prove. The first one is the existence of $(M_\infty, g_\infty(t)$. The second one is that $(M_\infty, g_\infty(t)$ is an asymptotic gradient shrinking soliton. We will construct f in (3.6) as a limit of reduced length functions.

We first obtain some estimates of the reduced length from Theorem A.5. These estimates will give control over curvature so that we can apply Theorem 2.21 to construct $(M_\infty, g_\infty(t)$.

Lemma 3.7. *Let $(M, g(t))$, $-\tau_0 \leq t \leq 0$, be an n-dimensional Ricci flow such that each time slice is a complete Riemannian manifold with non-negative, bounded* *curvature operator. Then for any $0 < c < 1$ and any $\tau \le (1-c)\tau_0$, we have*

$$|\nabla l(q,\tau)|^2 + R(q,\tau) \le \frac{(1+2c^{-1})l(q,\tau)}{\tau},$$

$$R(q,\tau) - \frac{(1+c^{-1})l(q,\tau)}{\tau} \le \frac{\partial l}{\partial \tau},$$

where ∇ and $|\cdot|$ are defined on M with respect to $g(-\tau)$

Proof. Take a minimal L-geodesic γ from $(p,0)$ to (q,τ). Define

$$H(X) := -\frac{\partial R}{\partial \tau} - \frac{R}{\tau} - 2\langle \nabla R, X\rangle + 2\operatorname{Ric}(X,X)$$

and

$$K^\tau(\gamma) := \int_{[0,\tau]} s^{\frac{3}{2}} H(X) ds,$$

where $X = d\gamma_*(-\frac{d}{dt})$ is a vector field along γ. The quantities R, Ric, ∇, and the pairing of ∇R and X are defined with respect to the metric $g(t) = g(-\tau)$. From Chapter 6 of [21], the following formulas hold:

$$|\nabla l(q,\tau)|^2 = \frac{l(q,\tau)}{\tau} - \frac{K^\tau(\gamma)}{\tau^{3/2}} - R(q,\tau), \tag{3.8}$$

$$\frac{\partial l(q,\tau)}{\partial \tau} = R(q,\tau) - \frac{l(q,\tau)}{\tau} + \frac{K^\tau(\gamma)}{2\tau^{3/2}}. \tag{3.9}$$

We only have to obtain an estimate for $H(X)$. Applying Theorem A.5, we have

$$-\frac{\partial R}{\partial \tau} - \frac{R}{\tau_0 - \tau} - 2\langle \nabla R, X\rangle + 2\operatorname{Ric}(X,X) \ge 0.$$

Combining this with $\tau \le (1-c)\tau_0$, we obtain $H(X) \ge -\frac{c^{-1}}{\tau}R$. This gives

$$K^\tau(\gamma) = \int_{[0,\tau]} s^{\frac{3}{2}} H(X) ds \ge \int_{[0,\tau]} c^{-1} R s^{\frac{1}{2}} ds \ge -2c^{-1}\tau^{\frac{1}{2}} l(q,\tau).$$

Now we only need to plug the inequality for $K^\tau(\gamma)$ into (3.8) and (3.9). ☐

Corollary 3.10. *In the same setting of the above lemma, we assume $\tau_0 = \infty$. Then we have*

$$|\nabla l(q,\tau)|^2 + R(q,\tau) \le \frac{3l(q,\tau)}{\tau}, \tag{3.11}$$

$$-\frac{2l(q,\tau)}{\tau} \le \frac{\partial l(q,\tau)}{\partial \tau} \le \frac{l(q,\tau)}{\tau}. \tag{3.12}$$

Proof. let c tend to 1. We obtain the first two inequalities. For the right-side inequality in the second line, we plug the first inequality into the following formula

$$\frac{\partial l(q,\tau)}{\partial \tau} = -\frac{l(q,\tau)}{2\tau} + \frac{R(q,\tau)}{2}.$$

We now derive the formula above. Let γ be a minimal L-geodesic from the base point of the l function to (q,τ), followed by the constant path $\mu(\tau') = (q,\tau')$ for $\tau_1 \ge \tau' > \tau$. Denote their concatenation by $\gamma * \mu(\hat{\tau})$, where $\hat{\tau} \in [0,\tau_1]$. Then we have

$$l(\gamma * \mu) = \frac{1}{2\sqrt{\tau_1}}\left(L(\gamma) + \int_{[\tau,\tau_1]} \sqrt{s} R(q,s) ds\right)$$

Differentiating at τ gives

$$\frac{\partial l(q,\tau)}{\partial \tau} = -\frac{1}{4\tau^{3/2}}L(\gamma) + \frac{1}{2\sqrt{\tau}}\sqrt{\tau}R(q,\tau) = -\frac{l(q,\tau)}{2\tau} + \frac{R(q,\tau)}{2}.$$

□

We now return to the proof of Theorem 3.5.

Proof. (Existence) We denote the reduced length of $(M, g_k(t))$ by $l_k(q,\tau)$. Since the reduced length is scale invariant, $l_k(q_k,\tau) \leq \frac{n}{2}$ holds. We apply the comparison theorem of ODE to $-\frac{2l(q,\tau)}{\tau} \leq \frac{\partial l(q,\tau)}{\partial \tau}$ when $0 < \tau < 1$, and to $\frac{\partial l(q,\tau)}{\partial \tau} \leq \frac{l(q,\tau)}{\tau}$ when $\tau > 1$. Then we have $l_k(q_k,\tau) \leq \frac{n}{2\tau^2}$ and $\frac{n\tau}{2}$, $(\tau > 0)$. Integrating along a minimal geodesic with respect to the metric $g_k(-\tau)$ from q_k to q and applying Corollary 3.10, we obtain

$$l_k(q,\tau)^{\frac{1}{2}} \leq (3/\tau)^{\frac{1}{2}} d(q_k,q) + C(\tau),$$

$$|\nabla l_k(q,\tau)| \leq (3/\tau) d(q_k,q) + (3/\tau)^{\frac{1}{2}} C(\tau),$$

where d is the distance between q_k and q on the Riemannain manifold $(M, g_k(-\tau))$ and $C(\tau) = \frac{n}{2\tau^2} + \frac{n\tau}{2}$. Plugging these two inequalities into (3.11), we obtain a uniform bound (with respect to k) for scalar curvature R on the geodesic ball $B(q_k, -\tau_0, A)$ for any $A, \tau_0 \in (0,\infty)$. Since the curvature operator of a κ-solution is non-negative, $|\operatorname{Rm}|$ is bounded by the scalar curvature. Thus, the curvature $|\operatorname{Rm}|$ is also uniformly (with respect to k) bounded on the geodesic ball $B(q_k, -\tau_0, A)$. Since $\partial R/\partial t \geq 0$ (applying Theorem A.5 by taking $X = 0$ and $T_0 \to -\infty$), the scalar curvature R and the curvature operator $|\operatorname{Rm}|$ are also uniformly bounded over $B(q_k, -\tau_0, A) \times (-\infty, -\tau_0]$. Applying a slightly modified version of Theorem 2.21, we obtain a Ricci flow $(M_\infty, g_\infty(t)), -\infty < t \leq -\tau_0$. After passing to a subsequence by the diagonalization argument, we obtain $(M_\infty, g_\infty(t)), -\infty < t < 0$.

□

We now prove that $(M_\infty, g_\infty(t))$ is an asymptotic gradient shrinking soliton.

Proof. (Asymptotic gradient shrinking soliton) Since the integral of $\hat{V}$ is positive and $\hat{V}$ is non-increasing, $\lim_{\tau\to\infty} \hat{V}(\tau)$ exists. Here $\hat{V}$ is the reduced volume of $(M, g(t)$. We denote $V_\infty = \lim_{\tau\to\infty} \hat{V}(\tau)$ and $\hat{V}_k(\tau)$ as the reduced volume of $(M, g_k(t))$. By dimensional analysis, we see that $\hat{V}_k(\tau) = \hat{V}(\tau_k\tau)$ and $\lim_{k\to\infty} \hat{V}_k(\tau) = V_\infty < (4\pi)^{\frac{n}{2}}$. Unfortunately, $\lim_{k\to\infty} \hat{V}_k(\tau)$ may not be the reduced volume of (M_∞, g_∞). Otherwise, the reduced volume of (M_∞, g_∞) would be constant, and the theorem would follow from Theorem 3.4. Therefore, we must construct the smooth function f explicitly.

Note that l_k can be defined on compact subsets of $M_\infty \times (-\infty, 0)$. These subsets form an exhausting sequence of $M_\infty \times (-\infty, 0)$. Since l_k are uniformly locally bounded and uniformly locally Lipschitz, we may arrange that l_k converges strongly in $C^{0,\beta}_{loc}$ to a function l_∞. Note that $C^{0,1}_{loc}$ can be compactly embedded into $C^{0,\beta}_{loc}$. We see that l_∞ is also locally Lipschitz, $l_\infty \in W^{1,2}_{loc}$ and l_k converges weakly to l_∞ in $W^{1,2}_{loc}$.

Warning 3.13. We are not claiming that l_∞ is the reduced length of the Ricci flow $(M_\infty, g_\infty(t))$.

Proposition 3.14. *The function* l_∞ *is a smooth function on* $M_\infty \times (-\infty, 0)$ *and satisfies the following equations*

$$\frac{\partial l_\infty}{\partial \tau} + |\nabla l_\infty|^2 - R + \frac{n}{2\tau} - \Delta l_\infty = 0, \tag{3.15}$$

$$2\Delta l_\infty - |\nabla l_\infty|^2 + R + \frac{l_\infty - n}{\tau} = 0. \tag{3.16}$$

Proof. We only need to prove that l_∞ is a weak solution of (3.15) and (3.16). The smoothness follows from standard regularity theory applied to (3.15). By the property of reduced length, we have

$$\frac{\partial l_k}{\partial \tau} + |\nabla l_k|^2 - R + \frac{n}{2\tau} - \Delta l_k \geq 0.$$

Taking limit we see that $\tilde{D} := \frac{\partial l_\infty}{\partial \tau} + |\nabla l_\infty|^2 - R + \frac{n}{2\tau} - \Delta l_\infty \geq 0$, in the sense of distribution. From the estimate of l_∞ in [21], we can extend $\tilde{D}$ to smooth functions with bounded gradient norm but not necessarily compactly supported. In more concrete setting, we have inequality (we denote left side by $H(\phi)$)

$$\int_{[\tau_0,\tau_1]} \int_{M_\infty \times (-\tau)} \left(\frac{\partial l_\infty}{\partial \tau} + |\nabla l_\infty|^2 - R + \frac{n}{2\tau} - \Delta l_\infty \right) \phi \tau^{-\frac{n}{w}} \exp(-l_\infty) dV_{g_\infty} d\tau \geq 0,$$

where $0 < \tau_0 < \tau_1 < \infty$, ϕ is a non-negative, smooth bounded function with bounded gradient norm.

Since $\lim_{k\to\infty} \hat{V}_k(\tau)$ is a constant function of τ, we have for any $0 < \tau_0 < \tau_1 < \infty$

$$\begin{aligned} &\lim_{k\to\infty} \hat{V}_k(\tau_1) - \lim_{k\to\infty} \hat{V}_k(\tau_0) \\ = &\int_{[\tau_0,\tau_1]} \int_{M_\infty \times (-\tau)} \left(\frac{\partial l_\infty}{\partial \tau} - R + \frac{n}{2\tau} \right) \tau^{-\frac{n}{w}} e^{-l_\infty} dV_{g_\infty} d\tau = 0. \end{aligned}$$

By the divergence theorem, we also have

$$\begin{aligned} &\int_{[\tau_0,\tau_1]} \int_{M_\infty \times (-\tau)} \Delta(e^{-l_\infty}) dV_{g_\infty} d\tau \\ = &\int_{[\tau_0,\tau_1]} \int_{M_\infty \times (-\tau)} (|\nabla l_\infty|^2 - \Delta l_\infty) e^{-l_\infty} dV_{g_\infty} d\tau = 0. \end{aligned}$$

Combining the above two equations, we have $H(1) = 0$. For any test function φ, we see that $0 = H(1) = H(1 - \varphi + \varphi) \geq H(\varphi) \geq 0$. Thus l_∞ is a weak solution of (3.15) and is smooth by regularity theory of parabolic equations. Equation (3.16) follows from corollary 6.5 of [21].

∎

We now only need to show that l_∞ satisfies (3.6).

Take l_∞ as f in Theorem 3.5. Using formulas (2.5) (2.6) and (3.16), we have $\mathrm{Ric}(g_\infty(t)) + \mathrm{Hess}_{g_\infty(t)}(l_\infty(\cdot, -t)) = g_\infty(t)/2\tau$, where $l_\infty(\cdot, -t)$ is regarded as a smooth function over M. This completes the proof of Theorem 3.5. ∎

To better understand the structure of κ-solutions, we need to investigate these asymptotic gradient shrinking solitons more closely. Fortunately, we have a complete classification of them.

Theorem 3.17. *Let $(M_0, g_0(t))$, $t \in (-\infty, 0)$, be a κ-solution of dimension three or two. Then any asymptotic gradient shrinking soliton $(M, g(t))$, with bounded curvature operator for each time slice, for $(M_0, g_0(t))$ is one of the following three types.*

(i) The flow $(M, g(t))$, $t \in (-\infty, 0)$, is a shrinking family of compact, constant positive curvature manifolds. We also say $(M, g(t))$ is a shrinking round three-dimensional Ricci flow

(ii) The flow $(M, g(t))$, $t \in (-\infty, 0)$, is a product of a shrinking family of round 2-spheres with the real line.

(iii) The flow $(M, g(t))$, $t \in (-\infty, 0)$, is a quotient family of metrics of the product of a shrinking family of round 2-spheres with the real line under the action of an isometric involution.

The term "shrinking family" is defined in Example A.2.

Remark 3.18. By Theorem 3.26, we know that the asymptotic gradient shrinking soliton constructed at the beginning of Subsection 3.2 is indeed a κ-solution, whose curvature operator is bounded by Definition 2.18. Therefore, we can apply Theorem 3.17 in that case. The proofs of Theorem 3.26 and Theorem 3.17 have similar patterns but are logically independent.

Remark 3.19. For the third case in Theorem 3.17, there are two possibilities: $\mathbb{RP}^2 \times \mathbb{R}$ and the unique nontrivial $\mathbb{R}$-bundle over $\mathbb{RP}^2$, where the flow is constant over the $\mathbb{R}$ part and a shrinking family on the $\mathbb{RP}^2$ part.

We will prove this theorem by dividing it into three cases and addressing each case by a separate proposition. Before the proof, we first recall the construction of a gradient shrinking soliton.

By the uniqueness result in [28][8], we know that $(M, g(t))$ is uniquely determined by the Ricci flow equation and $(M, g(-1))$. We shall construct a gradient shrinking soliton from

$$\mathrm{Ric}(g(-1)) + \mathrm{Hess}_{g(-1)}(f) = \frac{g(-1)}{2},$$

where $f : M \to \mathbb{R}$ is a smooth function. See Example A.3 for the construction. Thus, by the uniqueness results, $(M, g(t))$ is actually a gradient shrinking soliton. In light of Example A.3 again, different time slices are isometric up to re-scaling by a positive constant.

Proposition 3.20. *Let $(M, g(t))$ be as in Theorem 3.17. Assume that M is compact and the curvature is strictly positive. Then $(M, g(t))$ is a shrinking family of compact round manifolds.*

Proof. Since the curvature is strictly positive and M is compact, from Theorem A.8, we know that the curvature becomes singular at time T. If M is three-dimensional, then by Theorem A.4, the ratio of the largest sectional curvature to the smallest one approaches 1 as t tends to T. See Chapter 10 of [30] for more details. Since each time slice is isometric up to re-scaling, for the Riemannian manifold $(M, g(-1))$, the ratio of the largest sectional curvature to the smallest one can be arbitrarily close to 1. Therefore $(M, g(-1))$ is a three-manifold with constant positive sectional curvature and $(M, g(t))$ is a shrinking round manifold.

If m is two-dimensional, we apply Theorem A.6.

□

Proposition 3.21. *Let $(M, g(t))$ be as in Theorem 3.17. Assume that the curvature of M is not strictly positive. Then $(M, g(t))$ is a product of a shrinking family of round 2-spheres with the real line, or a quotient family of metrics of the product of a shrinking family of round 2-spheres with the real line under the action of an isometric involution.*

Proof. Applying Theorem A.9, the manifold $(M, g(t))$ admits a one or two sheeted covering that splits into a product of a shrinking family of round two-manifolds with a trivial flow on a one-dimensional manifold. Let $\hat{f}$ denote the lift of f to this covering, and let X be a unit vector field along the direction of the one-dimensional factor. Thus, we have $\mathrm{Hess}(f)(X, X) = \frac{1}{2}$, which is impossible if this 1-manifold is a circle. Hence, the 1-dimensional factor must be a standard real line. After applying Theorem A.6 to the 2-dimensional factor, the proof is complete. □

The final case presents the greatest challenge.

Proposition 3.22. *Let $(M, g(t))$ be as in Theorem 3.17. Assume that M is non-compact and the curvature is strictly positive. Then such $(M, g(t))$ does not exist.*

Proof. We begin with some computations with respect to some local chart (x_i, ∂_i). We work with the metric $g(-1)$. Taking the trace of our gradient shrinking equation $\mathrm{Ric}_{ij} + \nabla_i \nabla_j f = \frac{g_{ij}}{2}$ gives $R + \Delta f = \frac{n}{2}$. Additionally, taking the covariant derivative of the gradient shrinking equation, we obtain

$$\nabla_k \nabla_i \nabla_j f = -\nabla_k \mathrm{Ric}_{ij},$$

$$\nabla_i \nabla_j \nabla_k f - \nabla_j \nabla_i \nabla_k f = \nabla_j \mathrm{Ric}_{ik} - \nabla_i \mathrm{Ric}_{jk},$$

where ∇_i denotes ∇_{∂_i}. Using the Ricci identity $\nabla_i \nabla_j \nabla_k f - \nabla_j \nabla_i \nabla_k f = R_{ijkl} \nabla_l f$ and the contracted Bianchi identity $\nabla_j \mathrm{Ric}_{ij} = \frac{1}{2} \nabla_i R$, we have

$$\nabla_i R = 2\,\mathrm{Ric}_{ij} \nabla_j f. \tag{3.23}$$

Fix a point $p \in M$. Choose a minimal geodesic $\gamma : [0, \bar{s}] \to M$ with respect to the metric $g(-1)$ such that $|\gamma'(s)| = 1$ and $\gamma(0) = p$. We denote $X(s) = \gamma'(s)$. Using the second variation formula of geodesics, we have

$$\int_{[0,\bar{s}]} \mathrm{Ric}(X(s), X(s)) ds \leq \text{const.},$$

where const. denotes some constant number independent of $\bar{s}$ and γ. For any unit vector field $Y(s)$ along γ that is orthogonal to $X(s)$, we have

$$\int_{[0,\bar{s}]} |\mathrm{Ric}(X, Y)|^2 ds \leq \int_{[0,\bar{s}]} \mathrm{Ric}(X, X)\,\mathrm{Ric}(Y, Y) ds,$$

by applying the Cauchy-Schwarz inequality to $|\mathrm{Ric}(X, Y)|^2$.

Since $|\mathrm{Ric}(Y, Y)|$ and $\int_{[0,\bar{s}]} \mathrm{Ric}(X, X) ds$ are bounded, $\int_{[0,\bar{s}]} |\mathrm{Ric}(X, Y)|^2 ds$ is also bounded. Thus, we have

$$\int_{[0,\bar{s}]} |\mathrm{Ric}(X, Y)| ds \leq \text{const.}(\bar{s}^{\frac{1}{2}} + 1).$$

From the gradient shrinking equation, we have

$$\mathrm{Hess}(f)(X, X) = \nabla_X \nabla_X f = \frac{1}{2} - \mathrm{Ric}(X, X),$$

$$\text{Hess}(f)(X,Y) = \nabla_X \nabla_Y f = 0 - \text{Ric}(X,Y).$$

Integrating these along γ yields

$$X(f(\gamma(\overline{s})) - X(f(\gamma(0))) = \frac{1}{2}\overline{s} - \int_{[0,\overline{s}]} |\operatorname{Ric}(X,X)| ds,$$

$$Y(f(\gamma(\overline{s})) - Y(f(\gamma(0))) = - \int_{[0,\overline{s}]} |\operatorname{Ric}(X,Y)| ds.$$

Thus, we have

$$\langle X(\gamma(\overline{s})), \nabla f(\gamma(\overline{s}))\rangle \geq \frac{\overline{s}}{2} - \text{const. and } \langle Y(\gamma(\overline{s})), \nabla f(\gamma(\overline{s}))\rangle| \leq \text{const.}(\overline{s}^{\frac{1}{2}} + 1).$$

These two estimates tell us that f has no critical point at a large distance from the reference point p and the angle between ∇f and γ' tends to zero as distance grows.

From (3.23), we see that R is strictly increasing along the gradient curve of f. In particular, we have $\overline{R} = \text{limsup} R > 0$, as $d_{g(-1)}(p,x)$ tends to infinity. We choose a sequence of points $(x_\alpha, -1) \in M \times (-\infty, 0)$ such that $R(x_\alpha, -1) \to \overline{R}$ as $\alpha \to \infty$. Since $d_{g(-1)}(p, x_\alpha) \to \infty$ and $d_{g(-1)}(p, x_\alpha) R(x_\alpha, -1) \to \infty$, applying Theorem A.7, we obtain a κ-solution defined on $t \in (-\infty, 0)$ that is the shrinking family $M_\infty = N \times \mathbb{R}$ with scalar curvature $\overline{R}$ at time -1. Here N is a two-dimensional manifold with positive scalar curvature (sphere or $\mathbb{RP}^2$). Since $(M, g(-1))$ is non-compact and has positive curvature, it is diffeomorphic to $\mathbb{R}^3$ by Soul Theorem in [7] and hence does not contain an embedded copy of $\mathbb{RP}^2$. Thus (M_∞, g_∞) is a shrinking family of infinite cylinder $S^2 \times \mathbb{R}$. Since this solution exists on $(-\infty, 0)$, we have $\overline{R} \leq 1$, $R(x,-1) < 1$ for $d_{g(-1)}(p,x)$ large enough, and $R(x,-1) \to 1$ as $d_{g(-1)}(p,x) \to \infty$.

Now we consider a level surface $f^{-1}(a)$ of f that is sufficiently far from the reference point p. Let Z be the unit normal vector of $f^{-1}(a)$. Then we have:

$$\nabla_Z \nabla_Z f = \frac{1}{2} - \operatorname{Ric}(Z,Z) \geq \frac{1}{2} - \overline{R} > 0,$$

and

$$\frac{d}{da}\text{Area}(f^{-1}(a)) = \int_{f^{-1}(a)} \text{div}(\nabla f / |\nabla f|) > \int_{f^{-1}(a)} \frac{1}{|\nabla f|}(1 - \overline{R}) \geq 0.$$

Thus the area of $f^{-1}(a)$ increase as a grows, and is bounded by 8π: the area of the round sphere of scalar curvature one. For more details, see Chapter 6 of [6]. On the other hand, the Gaussian curvature K of level surfaces is less than $\frac{1}{2}$ because we have

$$K = \frac{1}{2}(R - 2\operatorname{Ric}(X,X)) + \frac{\det(\text{Hess}(f))}{|\nabla f|^2},$$

$$\leq \frac{1}{2}(R - 2\operatorname{Ric}(X,X)) + \frac{(1 - R + \operatorname{Ric}(X,X))^2}{2|\nabla f|^2} < \frac{1}{2}.$$

By the Gauss-Bonnet theorem, we draw a contradiction:

$$4\pi = \int_{f^{-1}(a)} K < \frac{1}{2}\text{Area}(f^{-1}(a)) < 4\pi,$$

where a is sufficiently large. □

We have finished the proof of Theorem 3.17.

An application of this classification of asymptotic gradient shrinking solitons is the existence of a universal $\kappa_0 > 0$ for all κ-solutions that are not the shrinking round Ricci flows.

Theorem 3.24. *(Universal κ) There exists a $\kappa_0 > 0$ such that any three-dimensional κ-solution is either κ_0-non-collapsed or a shrinking round three-dimensional Ricci flow.*

Proof. Let $(M, g(t))$ be a $\hat{\kappa}$-solution. Assume that $(M, g(t))$ is not κ-non-collapsed, where $\kappa > \hat{\kappa}$. Our goal is to provide a universal lower bound for κ, assuming $(M, g(t))$ is not a shrinking round three-dimensional Ricci flow.

After re-scaling, we may assume there is a point $(p, 0) \in M \times (-\infty, 0]$, such that $|\operatorname{Rm}(q,t)| \leq 1$ for all (q,t) satisfying $d_{g(0)}(p,q) < 1$ and $t \in [-1, 0]$, with $\operatorname{Vol}(B_{g(0)}(p,1)) < \kappa$, where $B_{g(0)}(p,1)$ denotes the geodesic ball centered at p of radius 1 with respect to the metric $g(0)$. Let $\hat{V}(\tau)$ be the reduced volume defined by $(p, 0)$ (here $\tau = -t$). We have an estimate from [6]:$\hat{V}(\kappa) \leq 3\kappa^{\frac{3}{2}}$. Take a sequence τ_k such that $\tau_k \to \infty$ as $k \to \infty$. Choose q_k satisfying $l(q_k, \tau_k) \leq \frac{3}{2}$ (recall the construction of asymptotic gradient shrinking solitons). Scaling $(M, g(t))$ by q_k and τ_k, the flow converges to either the round $S^2 \times \mathbb{R}$ or one of its $\mathbb{Z}_2$ quotients. Note we are using the following observation:

Observation 3.25. If an asymptotic gradient shrinking soliton of $(M, g(t))$ is a compact shrinking round soliton, then $(M, g(t))$ is a shrinking round three-dimensional Ricci flow.

Proof. According to the Definition 2.20, M is diffeomorphic to its asymptotic gradient shrinking soliton, which is the limit of $(M, g_{\tau_n}(t) := \frac{1}{\tau_n} g(\tau_n t), q(\tau_n))$, where $t \in (-\infty, 0]$ and τ_n tends to infinity. Thus, the metric $g_{\tau_n}(-1)$ is arbitrarily close to a metric of constant positive curvature for n sufficiently large. Thus $(M, g(t))$ is a shrinking round three-dimensional Ricci flow by Theorem A.4. For more details, see [21]. □

For sufficiently large k, we construct a path $\gamma : [0, 2\tau_k] \to M$, starting from p to a given point $q \in M$, such that: $\gamma|_{[0,\tau_k]}$ connects p and q_k satisfying

$$l(q_k, \tau_k) = \frac{1}{2\sqrt{\tau}} \int_{[0,\tau_k]} \sqrt{\tau}(R + |\gamma'(\tau)|^2) d\tau \leq 2,$$

and the second half of $\gamma|_{[\tau_k,\tau_k]}$ is a shortest geodesic connecting q_k and q with respect to the metric $g(-\tau_k)$. By convergence, the re-scaled metric $\frac{1}{\tau_k} g(-\tau)$ over the domain $B_{g(-\tau_k)}(q_k, \tau_k^{1/2}) \times [-2\tau_k, -\tau_k]$ is sufficiently close to the round $S^2 \times \mathbb{R}$ or its quotients. Thus, $(2\sqrt{2\tau_k})^{-1} \int_{[\tau_k, 2\tau_k]} \sqrt{\tau}(R + |\gamma'(\tau)|^2) d\tau$ is universally bounded. We have a universal upper bound $\hat{C}$ for $l(q, 2\tau_k)$ for any $q \in B_{g(-\tau_k)}(q_k, \tau_k^{1/2})$,

$$l(q, 2\tau_k) \leq \frac{1}{2\sqrt{2\tau_k}} \int_{[0,2\tau_k]} \sqrt{\tau}(R + |\gamma'(\tau)|^2) d\tau$$

$$\leq \sqrt{2} + \frac{1}{2\sqrt{2\tau_k}} \int_{[\tau_k, 2\tau_k]} \sqrt{\tau}(R + |\gamma'(\tau)|^2) d\tau \leq \hat{C}.$$

Thus, we have

$$\hat{V}(2\tau_k) \geq \exp(-\hat{C}) \int_{B_{g(-\tau_k)}(q_k, \tau_k^{1/2})} (8\pi\tau_k)^{-\frac{3}{2}} dV_{g(-2\tau_k)} \geq C,$$

where $C > 0$ is some universal constant. We obtain $C \leq \hat{V}(2\tau_k) \leq \hat{V}(\kappa) \leq 3\kappa^{\frac{3}{2}}$. Hence, we finally get a lower bound for such κ. □

3.3. **Compactness of κ-solutions.** In this section, we prove that the space of κ-solutions is compact. We will also yield some direct corollaries.

Theorem 3.26. *(Compactness theorem) The set of κ-solutions is compact modulo scaling. Specifically, for any sequence of such solutions with marked points $(q_k, 0)$ where $R(q_k, 0) = 1$, we can extract a C^∞_{loc} converging subsequence whose limit is also a κ-solution.*

We first state a lemma that will be used in the proof of Theorem 3.26. This lemma essentially allows us to obtain control over curvature and to apply Theorem 2.21.

Lemma 3.27. *There exists a universal, positive, strictly increasing function $\omega : [0, +\infty) \to (0, +\infty)$ with the following properties. Suppose $(M, g(t))$, where $t \in (-\infty, 0]$, is a three-dimensional κ-solution. Then we have*

$$R(x, t) \leq R(y, t)\omega(R(y, t)d_{g(t)}(x, y)),$$

where $x, y \in M$ and $t \in (-\infty, 0]$.

Proof. This lemma is based on Li-Yau-Hamilton inequality. See corollary 11.6 of [23]. □

Now we prove Theorem 3.26.

Sketch of proof. Consider any sequence of κ-solutions with marked points $(q_k, 0)$ such that $R(q_k, 0) = 1$. By Lemma 3.27 and Theorem 2.21, we can extract a subsequence that converges to a limit $(M_\infty, g_\infty(t))$, with $t \in (-\infty, 0]$, which satisfies all conditions of a κ-solution except for the boundedness of curvature. Since $\partial R/\partial t \geq 0$ and the non-negativity of Rm, we only need to prove that the curvature operator of $(M_\infty, g_\infty(0))$ is bounded.

Assume the contrary: the curvature is not bounded. Note we may also assume M_∞ is non-compact and orientable. We can choose a sequence of points $x_k \in M_\infty$, tending to infinity such that $R_{g_\infty(0)}(x_k, 0) \geq k$ and $R_{g_\infty(0)}(x, 0) \leq 4R_{g_\infty(0)}(x_k, 0)$, for $x \in B_{g_\infty(0)}(x_k, k/R_{g_\infty(0)}(x_k, 0)^{\frac{1}{2}})$. See [21] for details. Since $\partial R_{g_\infty(0)}/\partial t \geq 0$, we have $R_{g_\infty(0)}(x, t) \leq 4R_{g_\infty(0)}(x_k, 0)$ for $x \in B_{g_\infty(0)}(x_k, k/R_{g_\infty(0)}(x_k, 0)^{\frac{1}{2}})$ and $t \in (-\infty, 0]$.

Thus, we have the sequence

$$(M_\infty, R_{g_\infty(0)}(x_k, 0)g_\infty(t/R_{g_\infty(0)}(x_k, 0)), (x_k, 0)),$$

which converges. By Theorem A.7 and the argument of Theorem 3.17, the limit flow must be the shrinking round soliton $S^2 \times \mathbb{R}$. This contradicts the fact that there exist $x_k \to \infty$ and $R_{g_\infty(0)}(x_k, 0) \to \infty$. See proposition 2.19 of [21]. □

Corollary 3.28. *Asymptotic gradient shrinking solitons constructed in Theorem 3.5 are κ-solutions.*

Corollary 3.29. *Assuming $\kappa > 0$, there exist $0 < C < \infty$ such that for any 3-dimensional κ-solution $(M, g(t))$, $t \in (-\infty, 0]$, we have*

$$|\nabla R(x,t)| < CR(x,t)^{\frac{3}{2}}, \tag{3.30}$$

$$|dR(x,t)/dt| < CR(x,t)^2. \tag{3.31}$$

Moreover, C is independent of κ.

Recall that the scalar curvature R is always positive for a κ-solution.

Proof. First, for all κ-solutions, by dimensional analysis, we see that

$$\frac{|\nabla R(x,t)|}{R(x,t)^{\frac{3}{2}}} \text{ and } \frac{|dR(x,t)/dt|}{R(x,t)^2}$$

are scale invariant. Thus the inequalities follow from Theorem 3.26 by *reductio ad absurdum*. Note that Corollary 3.29 holds trivially for shrinking family of round metrics. Applying Theorem 3.24, we see that C is independent of $\kappa > 0$. □

3.4. **Canonical neighborhoods.** We introduce the concept of canonical neighborhoods, which pave the way for a qualitative classification of κ-solutions.

First, We present intuitive descriptions for each concept below. All seven concepts below are invariant under re-scaling. An ϵ-neck is approximately a long and thin tube $\mathbb{S}^2 \times \mathbb{R}$. An ϵ-round component is a compact manifold (M, g_M) diffeomorphic to a spherical space-form (S, g_S) so that the metric g_M is very close to g_S. A C-component is diffeomorphic to $\mathbb{S}^3$ or $\mathbb{RP}^3$ with geometric quantities bounded by C. A (C,ϵ)-cap is a union of an ϵ-neck and its core, glued along the $\mathbb{S}^2$ boundaries. The core is a punctured $\mathbb{S}^3$ or $\mathbb{RP}^3$ with geometric quantities bounded by C.

Definition 3.32. (ϵ-neck) Let (N, g) be a Riemannian manifold and $x \in N$ a point. An ϵ-neck structure on (N, g) centered at x consists of a diffeomorphism

$$\varphi : S^2 \times (-\epsilon^{-1}, \epsilon^{-1}) \to N,$$

with $x \in \varphi(S^2 \times \{0\})$, such that the metric $R(x)\varphi^* g$ is ϵ close to the product of the usual Euclidean metric on the open interval with the metric of constant Gaussian curvature $\frac{1}{2}$ on S^2 in the sense of $C^{[\epsilon^{-1}]}$-topology.

Definition 3.33. ((C,ϵ)-cap) Fix constant $\epsilon \in (0, \frac{1}{2})$ and $C < \infty$. Let (M, g) be a Riemannian three-manifold. A (C,ϵ)-cap in (M, g) is an open sub-manifold $(L, g|_L)$ with an open sub-manifold N in L with the following properties.

(1) L is diffeomorphic either to an open 3-ball or to a punctured $\mathbb{RP}^2$.

(2) N is an ϵ-neck with compact complement in L.

(3) $\overline{Y}$ is the complement of N in L. Its interior, Y, is called the core of L. The frontier of Y, which is $\partial\overline{Y}$, is central 2-sphere of an ϵ-neck contained in L.

(4) $R(y) > 0$ for every $y \in L$ and $\operatorname{diam}(L, g|_L) < C(\sup_{y \in L} R(y))^{\frac{1}{2}}$.

(5) $\sup_{x,y \in L}(R(x)/R(y)) < C$.

(6) $\operatorname{Vol}(L) < C(\sup_{y \in L} R(y))^{-\frac{3}{2}}$.

(7) For any $y \in Y$ let r_y be defined so that $\sup_{y' \in B(y,r_y)} R(y') = r_y^{-2}$. Then for each $y \in Y$, the ball $B(y, r_y)$ lies in L and indeed has compact closure in L. Furthermore,

$$C^{-1} < \inf_{y \in Y} \operatorname{Vol} B(y, r_y)/r_y^3.$$

(8)
$$\sup_{y\in L}\frac{|\nabla R(y)|}{R^{\frac{3}{2}}(y)}<C, \text{ and } \sup_{y\in L}\frac{|\Delta R(y)+2|\operatorname{Ric}|^2|}{R^2(y)}<C.$$

Definition 3.34. (C-component) Fix a positive constant $C>0$. A compact connected Riemannian manifold (M,g) is called a C-component if

(1) M is diffeomorphic to either S^3 or $\mathbb{RP}^3$.
(2) (M,g) has positive sectional curvature.
(3) For all 2-planes P in TM, we have
$$C^{-1}\sup_{y\in M}R(y)<K(P),$$
where $K(P)$ denotes the sectional curvature at P.
(4)
$$C^{-1}\sup_{y\in M}R^{-\frac{1}{2}}(y)<\operatorname{diam}(M)<C\inf_{y\in M}R^{-\frac{1}{2}}(y).$$

Definition 3.35. (ϵ-round) Fix $\epsilon>0$. Let (M,g) be a compact, connected three-manifold. Then (M,g) is within ϵ of round in the $C^{[\epsilon^{-1}]}$-topology if there exist a constant $R>0$, a compact manifold (Z,g_0) of constant curvature 1, and a diffeomorphism $\varphi: Z\to M$ with the property that $\varphi^*(Rg)$ is ϵ close to g_0 in $C^{[\epsilon^{-1}]}$-topology.

Definition 3.36. ((C,ϵ)-Canonical neighborhood) Fix $C<\infty$ and $\epsilon>0$. For any Riemannian manifold (M,g), an open neighborhood U of a point $x\in M$ is a (C,ϵ)-canonical neighborhood if one of the following holds:

(1) U is an ϵ-neck in (M,g) centered at x.
(2) U is a (C,ϵ)-cap in $(M.g)$, whose core contains x.
(3) U is a C-component of (M,g).
(4) U is an connected component of (M,g) within ϵ of round in the $C^{[\epsilon^{-1}]}$-topology.

U is a connected component of (M,g) in case (3) and (4) above. In the context of Ricci flow, we need a stronger notion than ϵ-neck.

Definition 3.37. Fix a constant $C<\infty$ and $\epsilon>0$. Let $(M,g(t))$ be a Ricci flow. An evolving ϵ-neck defined for an interval of normalized time of length $t'>0$ centered at a point $x\in(M,g(t_0))$ is an embedding $\phi: S^2\times(-\epsilon^{-1},\epsilon^{-1})\to N$, where N is an open sub-manifold of M and $x\in\phi(S^2\times\{0\})$, such that the pull-back metrics $\phi^*(R_{g(t_0)}(x)g(t))$, $t\in(t_0-R(x)^{-1}t',t_0]$, are within ϵ in the $C^{[1/\epsilon]}$-topology of the standard family $h(t)\times ds^2$, where $t\in(-t',0]$, $h(t)$ is the round metric of scalar curvature $\frac{1}{1-t}$ on S^2 and ds^2 is the standard metric on the interval.

A strong ϵ-neck is an evolving ϵ-neck which is defined for an interval of normalized time of length 1.

Definition 3.38. (strong (C,ϵ)-canonical neighborhood) Let $(M,g(t))$ be a Ricci flow. Let $x\in(M,g(t_0))$ be a point. We say that an open neighborhood U of x in $(M,g(t_0))$ is a strong (C,ϵ)-canonical neighborhood if one of the following holds

(1) U is a strong ϵ-neck centered at x.
(2) U is a (C,ϵ)-cap in $(M,g(t_0))$ whose core contains x.
(3) U is a C-component of $(M,g(T_0))$.
(4) U is an ϵ-round component of $(M,g(t_0))$.

Remark 3.39. Notice that the notion of (strong) (C,ϵ)-Canonical neighborhood is scale-invariant.

Now we begin our classification of κ-solutions.We start with a preliminary classification.

Proposition 3.40. *Let $(M,g(t))$ be a three-dimensional κ-solution. Then one of the following holds:*

(1) For every $t\in(-\infty,0]$ the manifold $(M,g(t))$ has positive curvature.

(2) $(M,g(t))$ is the product of an evolving family of round S^2 times the standard $\mathbb{R}$.

(3) M is diffeomorphic to a line bundle over $\mathbb{RP}^2$, and there is a finite covering of $(M,g(t))$ that is a flow in (2).

Proof. Suppose that $(M,g(t))$ does not have positive curvature for some time t. Then, by the maximum principle (see corollary 4.20 of [21]) as before, there is a covering $\overline{M}$ of M, which is either one or two sheeeted, such that $(\overline{M},g(t))$ is the product of an evolving family of round surfaces with a flat one-manifold (circle or real line). Note that $(\overline{M},g(t))$ is also a κ-solution. However, when the one-dimensional factor is the trivial flow (S^1,ds^2), the flow $(S^2\times S^1,h(t)\times ds^2)$ is κ-collapsing. Thus, $(M,g(t))$ has either a trivial cover or a double cover isometric to the product of a shrinking family of round surfaces with $\mathbb{R}$. If the round surface is S^2, then we have proved the result. If the round surface is $\mathbb{RP}^2$, a further double covering is a product of round S^2 with $\mathbb{R}$. This proves the proposition. □

We now investigate non-compact κ-solutions with strictly positive curvature.

Definition 3.41. (ϵ-tube and C-capped-ϵ-tube) An ϵ-tube T in a Riemannian three-manifold M is a submanifold diffeomorphic to $S^2\times I$, where I is an interval, such that

(1) Each boundary component S of T is the central 2-sphere of an ϵ-neck in M.

(2) T is a union of ϵ-neck and the closed half ϵ-necks whose boundary sphere is a component of ∂T. Furthermore, the central 2-sphere of each of the ϵ-necks is isotopic in T to the S^2 factors of the product structure.

A C-capped-ϵ-tube in M is a connected sub-manifold that is the union of a (C,ϵ)-cap and an ϵ-tube with boundary removed such that their intersection is diffeomorphic to $S^2\times(0,1)$.

A strong ϵ-tube is an ϵ tube such that each point of the tube is the center of a strong ϵ-neck.

Theorem 3.42. *There is a ϵ_0 such that for any $0<\epsilon\le\epsilon_0$, the following holds. There exists $C_0=C_0(\epsilon)$ such that for any $\kappa>0$ and any non-compact three-dimensional κ-solution $(M,g(t))$ not containing a $\mathbb{RP}^2$ with trivial normal bundle, one of the following holds*

(1) $(M,g(0))$ is a strong ϵ-tube.

(2) $(M,g(0))$ is a C_0-capped strong ϵ-tube.

Sketch of proof. If the curvature is not strictly positive, then by Proposition 3.40, we have concrete descriptions for $(M,g(t))$ and the theorem follows.

If the curvature is always positive, we apply the following lemma and Corollary 3.29. □

Before we state the lemma, we remark that the following lemma is not strong enough to prove Theorem 3.42 because not every item in Definition 3.33 is checked. However, this lemma contains the essential spirit of the proof, and a complete proof of Theorem 3.42 is only technically more complicated. For the strong-enough lemma, see lemma 9.85 of [21].

Lemma 3.43. *Let $(M, g(t))$ be a non-compact three-dimensional κ-solution of positive curvature and let $p \in M$. Then there exists $D' < \infty$, possibly depending on $(M, g(0))$ and p, such that every point $q \in M$ not contained in $B(p, 0, D'R(p, 0)^{-\frac{1}{2}})$ is the center of an evolving ϵ-neck in $(M, g(t))$ defined for an interval of normalized time of length 2.*

Furthermore, there exists $D'_1 < \infty$ such that for any point $x \in B(p, 0, D'R(p, 0)^{-\frac{1}{2}})$ and any plane P_x in TM_x we have $(D'_1)^{-1} < K(P_x)/R(p, 0) < D'_1$ where $K(P_x)$ denotes the sectional curvature of P_x.

Proof. Given $(M, g(t))$ and p, suppose that there is no such $D' < \infty$. We can arrange that $R(p, 0) = 1$. Then we can find a sequence of points $p_k \in M$ with $d_{g(0)}(p, p_k) \to \infty$ as $k \to \infty$ such that no p_k is in the center of an evolving ϵ-neck in $(M, g(0))$ defined for an interval of normalized time of length 2. We assume that $\lim_{k\to\infty} d_{g(0)}(p, p_k)$ exists.

Case 1.$\lim_{k\to\infty} d_{g(0)}(p, p_k) = \infty$. According to Theorem 3.26, we may assume that the sequence

$$(M, R(p_k, 0)g(R(p_k, 0)^{-1}t), (p_K, 0))$$

converges. By Theorem A.7 and Theorem A.6, the limit is the standard shrinking round cylinder. Hence, for sufficiently large k, $(p_k, 0)$ lies in an evolving ϵ-neck in $(M, g(0))$ defined for an interval of normalized time of length 2. This is a contradiction.

Case 2.$\lim_{k\to\infty} d_{g(0)}(p, p_k) = l < \infty$. Thus, we have $R(p_k, 0) \to 0$ as $k \to \infty$. We may assume that $d_{g(0)}(p, p_k)R(p_k, 0) < l + 1$ for all k. Consider the sequence of κ-solutions

$$(M, R(p_k, 0)g(R(p_k, 0)^{-1}t)) = (M_k, g_k(t)).$$

For each k we have $p \in B_{g_k(0)}(p_k, l + 1)$ and $R_{g_k}(p, 0) \to \infty$ as $k \to \infty$. This contradicts Lemma 3.27. □

Theorem 3.44. *There exists $\epsilon_1 > 0$ such that for every $0 < \epsilon \leq \epsilon_1$, there exists $C_1 = C_1(\epsilon) < \infty$ such that one of the following holds for any $\kappa > 0$ and any compact three-dimensional κ-solution $(M, g(t))$.*

(1) The manifold M is compact and of constant positive sectional curvature.

(2) The diameter of $(M, g(0))$ is less than $C_1(\max_{x\in M} R(x, 0)^{-\frac{1}{2}})$, and M is diffeomorphic to either S^3 or $\mathbb{RP}^3$. In fact, $(M, g(0))$ is a C-component.

(3) $(M, g(0))$ is a double C_1-capped strong ϵ-neck.

Proof. First, notice that $(M, g(t))$ has strictly positive curvature. If not, the universal covering space of $(M, g(0))$ is a Riemannian product $S^2 \times \mathbb{R}$ and hence $(M, g(0)$ is either non-compact or is a finite quotient of $S^2 \times S^1$, which is κ-collapsing. Both cases cannot happen. Therefore, we know that the fundamental group of M is finite.

We assume that $(M, g(0))$ is not round. Thus, by Theorem 3.24, there exists $\kappa_0 > 0$ such that $(M, g(t))$ is a κ_0-solution. Let $C_0(\epsilon)$ be the constant in Theorem 3.42. We now discuss two cases.

Observation 3.45. There is a constant C_1 for each $\epsilon > 0$ such that if the diameter of $(M, g(0))$ is larger than $C_1(\max_{x\in M} R(x,0)^{-\frac{1}{2}})$, then every point of $(M, g(0))$ is either contained in the core of a $(C_0(\epsilon), \epsilon)$-cap or is the center of a strong ϵ-neck in $(M, g(t))$.

Proof of observation. Suppose that for some $\epsilon > 0$ such C_1 does not exist. We take a sequence of $C_k \to \infty$ as $k \to \infty$ and a sequence $(M_k, g_k(t), (p_k, 0))$ of based κ_0-solutions such that $\mathrm{diam}(M_k, g_k(0)) > C_k(\max_{x\in M_k} R(x,0)^{-\frac{1}{2}})$, and p_k is neither contained in the core of a cap nor the center of some neck. Scale $(M_k, g_k(t))$ by $R(p_k, 0)$. We are able to assume that $R(p_k, 0) = 1$. By compactness of κ_0-solutions, we assume that the sequence converges to $(M_\infty, g_\infty(t), (p_\infty, 0))$. Thus, the diameter of M_∞ is infinity and M_∞ is non-compact. By Theorem 3.42, p_k, for k large enough, must be contained in the core of a $(C_0(\epsilon), \epsilon)$-cap or be the center of a strong ϵ-neck. This is a contradiction. □

Case 1.the diameter of $(M, g(0))$ is larger than $C_1(\max_{x\in M} R(x,0)^{-\frac{1}{2}})$.

From the geometry of canonical neighborhoods (see the appendix of [21]), we know from the observation that M is diffeomorphic to S^3, $\mathbb{RP}^3$, $\mathbb{RP}^3\#\mathbb{RP}^3$, or an S^2 fibration over S^1. However, we know that the fundamental group is finite. So $(M, g(0))$ can only be a double C_1-capped strong ϵ-neck, which is diffeomorphic to S^3 or $\mathbb{RP}^3$.

Case 2.the diameter of $(M, g(0))$ is less than or equal to $C_1(\max_{x\in M} R(x,0)^{-\frac{1}{2}})$.

In this case, we know that its asymptotic gradient shrinking soliton is not compact. Otherwise it would be a round manifold. So for t sufficiently negative, the diameter of $(M, g(t))$ is larger than $C_1(\max_{x\in M} R(x,0)^{-\frac{1}{2}})$. Thus, we revert to Case 1. □

Thus Theorem 3.44 and Theorem 3.42 provide a complete qualitative description of three-dimensional κ-solutions.

Corollary 3.46. *For every $\epsilon > 0$ small enough, there exists a $C = C(\epsilon)$ such that every point in a κ-solution has a strong (C, ϵ)-canonical neighborhood unless the κ-solution is $\mathbb{RP}^2 \times \mathbb{R}$.*

The results in this subsection will be used to prove properties of the general three-dimensional Ricci flow. For example, it can be shown, using the compactness-contradiction argument, every point with large curvature near the finite time singularity of a general Ricci flow has a strong (C, ϵ)-canonical neighborhood. For the proof, see section 7.1 of [6] and Chapter 11 of [21].

4. Ricci flow with surgery

We finally come to the definition of Ricci flow with surgery, the existence of which leads to a complete proof of the geometrization conjecture [24][22]. In Subsection 4.1, we define the surgery procedure at each finite-time singularity. In Subsection 4.2, we define the Ricci flow with surgery. The definition is not given by combining each single finite time surgery naively. On the contrary, we require that additional properties are preserved when passing a surgery time. Such additional properties ensure the flow is well-behaved, and extra arguments are needed to show that it is possible to choose surgery parameters carefully so that these additional properties are always satisfied. In Subsection 4.3, we summarize results that describe the behavior of the flow under some initial assumptions on the fundamental group. This leads to a proof of Thurston's elliptization conjecture, which implies the Poincaré conjecture.

We remark that most of the statements in this section are proved by comparing the general Ricci flows with κ-solutions, which are well understood now. The essential idea is compactness-contradiction argument.

4.1. **Surgeries at singularities.** We first make the discussion at the end of the last section more precise.

Definition 4.1. (Normalized initial data) We say a three-dimensional Ricci flow $(M, g(t))$ has normalized initial data if the following conditions are satisfied

(1) $|\operatorname{Rm}(x, 0)| \le 1$ for all $x \in M$.

(2) For every $x \in M$ we have $\operatorname{Vol} B(x, 0, 1) \ge \omega/2$ where ω is the volume of the unit ball in $\mathbb{R}^3$.

Any $(M, g(0))$ can be normalized by re-scaling. Any Ricci flow $(M, g(t))$ with normalized initial data always has no singularity in $[0, 1+\epsilon]$, where ϵ is some positive number.

Theorem 4.2. *(Singularity structure theorem) Given $\epsilon > 0$ and $1 < T \le T_0$, one can find $r_0(\epsilon, T_0) > 0$ with the following property. Let $(M, g(t))$, $t \in [0, T)$, be a Ricci flow with a finite-time singularity at T. $(M, g(0))$ is normalized. Then for any point (x_0, t_0) with $t_0 \ge 1$ and $R(x_0, t_0) \ge r_0^{-2}$, the solution in*

$$\{(x,t) | d_{g(t_0)}(x, x_0) < \epsilon^{-2} R(x_0, t_0)^{-1}, t_0 - \epsilon^{-2} R(x_0, t_0)^{-1} \le t \le t_0\}$$

is, after scaling by the factor $R(x_0, t_0)$, ϵ-close to the corresponding subset of some κ-solution.

Proof. See [17] [6] [21] for detailed discussion. □

The following results form the foundation for performing surgery.

Theorem 4.3. *(section 7.1 of [6] and Chapter 11 of [21]) Let $(M, g(t))$, where $0 \le t < T < \infty$, be a Ricci flow such that T is the first singular time. Assume that $(M, g(0))$ has normalized initial data. Define the subset Ω of M by*

$$\Omega = \{x \in M | \liminf_{t \to T} R(x, t) < \infty\}$$

Let $\epsilon > 0$ be sufficiently small. Then there exist constants $C > 0$ and $r_0 > 0$ such that the following holds: Ω is an open subset of M and there exists a metric $g(T)$ defined on Ω such that

(1) $g(t)|_\Omega$ converges to $g(T)$ as $t \to T$ in C^∞-topology.

(2) $R_{g(T)} : \Omega \to \mathbb{R}$ is a proper function and is bounded below.

(3) The Ricci flow equation is satisfied on $\hat{M} := M \times [0, T) \cup \Omega \times \{T\}$.

(4) Every end of a connected component of Ω is contained in a strong 2ϵ-tube.

(5) Any point $x \in \Omega \times \{T\}$ with $R(x) > r_0^{-2}$ has a strong $(2C, 2\epsilon)$-canonical neighborhood in $\hat{M}$.

The constant ϵ and C will be fixed through out this section, while r_0 will only be fixed in this subsection.

We now discuss the structure of $(\Omega, g(T))$.

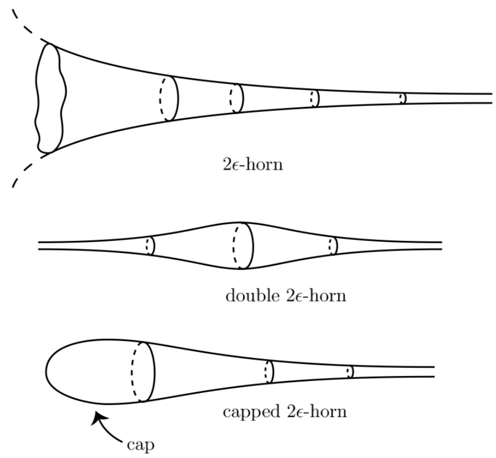

FIGURE 1. Horns

Definition 4.4. (2ϵ-horn) Let ϵ be a positive number, which can be seen as the same ϵ in Theorem 4.3. A 2ϵ-horn in $(\Omega, g(T))$ is a sub-manifold of Ω diffeomorphic to $S^2 \times [0, 1)$ with the following properties

(1) The embedding of $S^2 \times [0, 1)$ into Ω is proper.

(2) Every point of the image of this embedding is the center of a strong 2ϵ-neck in $\hat{M}$.

(3) the image of the boundary $S^2 \times \{0\}$ is the central sphere of a strong 2ϵ-neck.

A double 2ϵ-horn in $(\Omega, g(T))$ is a component of Ω diffeomorphic to $S^2 \times (0, 1)$ with the property that every point in this component is the center of a strong 2ϵ-neck in $\hat{M}$.

For any $C < \infty$, a C-capped-2ϵ-horn in $(\Omega, g(T))$ is a component of Ω that is a union of the core of a $(C, 2\epsilon)$-cap and a 2ϵ-horn.

We denote $\Omega_\rho = \{x \in \Omega | R(x, T) \leq \rho^{-2}\}$, where $\rho < r_0$. Let $\Omega^0(\rho)$ be the union of all components of Ω that intersect Ω_ρ. If $\rho_1 \geq \rho_2$. Then we have $\Omega_{\rho_1} \subseteq \Omega_{\rho_2}$.

The following two lemmas are proved in Chapter 11 [21].

Lemma 4.5. *$\Omega^0(\rho)$ has finitely many connected components and is a union of a compact set and finitely many 2ϵ-horns, each of which is disjoint from Ω_ρ and has its boundary contained in $\Omega_{\frac{\rho}{2C}}$.*

Lemma 4.6. *Fix $\rho > 0$. Then for any $\delta > 0$, there exists $0 < h = h(\delta, \rho) \leq \min(\rho\delta, \frac{\rho}{2C})$, implicitly depending on r_0, C and ϵ, such that for any 2ϵ-horn H of $(\Omega^0(\rho), g(T)|_{\Omega^0(\rho)})$ with boundary contained in $\Omega_{\frac{\rho}{2C}}$, every point $x \in H$ with $R(x, T) \geq h^{-2}$ is the center of a strong δ-neck in $\hat{M}$ contained in H. Furthermore, there is a point $y \in H$ with $R(y, T) = h^{-2}$ such that the central 2-sphere of the δ-neck centered at y cuts off the end of the H disjoint from Ω_ρ.*

We now define what is **the surgery operation at time T using the surgery parameters** δ **and** r_0. Set $\rho = \delta r_0 < r_0$, and set $h = h(\delta, \rho)$ as in Lemma 4.6. Let H_j denote the finitely many 2ϵ-horns of $\Omega^0(\rho)$. Choose $y_j \in H_j$ such that $R(y_j, T) = h^{-2}$, and S_j be the central sphere of the δ-neck N_j centered at y_j as in

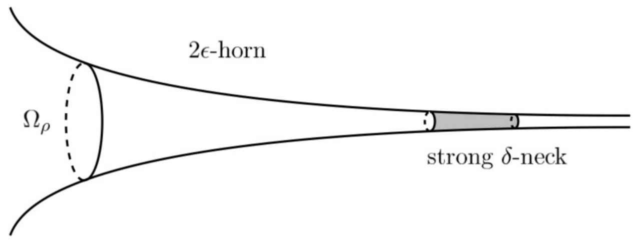

FIGURE 2. Lemma 4.6

Lemma 4.6. Let H_j^* be the unbounded component of $H_j \setminus S_j$. In other words, H_j^* is not contained in any compact subset of $\Omega^0(\rho)$. Define $C_T := \Omega^0(\rho) \setminus \cup H_j^*$.

We may define a standard well-behaved procedure of removing half of a δ-neck (N, S), where S denote the central sphere of the neck, and gluing back a standard solution. See Chapter 13 of [21] and [24]. Now we apply this standard procedure to each S_j. This completes our **surgery operation at time T using the surgery parameters** $0 < \delta < 1/10$ **and** $0 < r_0$. Note that if $\Omega^0(\rho)$ is empty then we will do nothing and stop. In this case, we say the flow becomes extinct. The following proposition is a direct consequence of our definition.

Proposition 4.7. *(Topological relation) Let M_k be a Ricci flow with a finite-time singularity at time T. Suppose M_l is obtained from M_k by performing* ***surgery operation at time T using the surgery parameters*** $0 < \delta < 1/10$ ***and*** $0 < r_0$.. *Then M_k can be topologically reconstructed from M_l by the following procedures:*

(1) Take disjoint union of M_l, finitely many 2-sphere bundle over a circle, and finitely many closed round three-manifolds.

(2) Perform finitely many connected sum operations between them.

Remark 4.8. The only possible topologically nontrivial consequence of a sequence of surgeries is the connected-sum decomposition and throwing away components with finite fundamental groups.

4.2. **Definition of Ricci flow with surgery.** The idea is to perform surgeries, as defined in the previous subsection, at finite-time singularity and continue the Ricci flow on the new manifold. This process yields a sequence $(M_k, g_k(t))$ $(t \in [t_k^-, t_k^+))$, where $t_k^- = t_{k-1}^+$ and $(M_k, g_k(t_k^-))$ is obtained by doing surgery operation at time t_{k-1}^+ for $(M_{k-1}, g_{k-1}(t))$ using specific surgery parameters. According to the definition of surgery operation, both $(\Omega_{k-1}, g_{k-1}(t_{k-1}^+))$ and $(M_k, g_k(t_k^-))$ contain isometric compact three-dimensional sub-manifolds with boundary. We identify these sub-manifolds and denote them as $\hat{\Omega}_k$. Consider the total space $\hat{M} := \cup_{\hat{\Omega}_k} M_k \times [t_k^-, t_k^+)$.

However, without choosing surgery parameters carefully, undesirable outcomes may occur. For example, the sequence t_k^+ might accumulate. Therefore, it is crucial to maintain control over the surgery parameters.

Definition 4.9. (Surgery parameters) We call the following data a set of surgery parameters **S**.

(1) A decreasing sequence $r_k > 0$

(2) A decreasing sequence $\kappa_k > 0$

(3) A decreasing sequence $\delta_k > 0$ and a positive decreasing function $\delta(t)$ such that $0 < \delta(t) < \delta_k$ for $t \in [2^{k-1}\epsilon, 2^k\epsilon)$.

We now define **Ricci flow with S-surgery.** Starting with a normalized metric $(M_1, g_1(0))$ and assuming we already have $(M_k, g_k(t))$ $(t \in [t_k^-, t_k^+))$ for $k < n$. We perform the surgery operation at time t_{n-1}^+ using the surgery parameters $\delta(t_{n-1}^+)$ and r_j if $t_{n-1}^+ \in [2^{j-1}\epsilon, 2^j\epsilon)$ to obtain a new manifold M_n. Restarting the Ricci flow on M_n, we obtain a flow $(M_n, g_n(t))$, where $t \in [t_n^-, t_n^+)$ and the flow develops a singularity at time t_n^+. Continuing this procedure inductively gives a sequence $(M_k, g_k(t))$ $(t \in [t_k^-, t_k^+))$ and a total space $\hat{M}$. We call $\{t_k^+\}$ surgery times. We state properties that $(M_k, g_k(t))$ $(t \in [t_k^-, t_k^+))$ and $\hat{M}$ are supposed to possess.

Assumption 4.10. (Pinching toward positive) There exist a function ϕ, decreasing to zero at infinity, such that $\mathrm{Rm}(x,t) + \phi(R(x,t))R(x,t) \geq 0$, in the sense of operators, for any $(x,t) \in \hat{M}$.

Assumption 4.11. (Strong (C,ϵ)-canonical neighborhood) Any $(x,t) \in \hat{M}$ with $R(x,t) \geq r_j^{-2}$, $t \in [2^{j-1}\epsilon, 2^j\epsilon]$ has a strong (C,ϵ)-canonical neighborhood in $\hat{M}$.

Assumption 4.12. (κ-non-collapsing) The Ricci flow with **S**-surgery is κ_j-non-collapsing for each time slice over $[2^{j-1}\epsilon, 2^j\epsilon]$.

Theorem 4.13. *(Long time existence of Ricci flow with surgery) There exists a set of surgery parameters **S** such that for any normalized initial data, the sequence $(M_k, g_k(t))$ $(t \in [t_k^-, t_k^+))$ is defined for any $t \in [0,\infty)$ and satisfies all three assumptions above. For each $T < \infty$, there are only finitely many surgery times in $[0,T]$.*

Proof. See section 4 and 5 of [24]. □

The proof is analogous to the proof of Theorem 4.2. We will henceforth refer to the sequence of flows $(M_k, g_k(t))$ $(t \in [t_k^-, t_k^+))$ constructed in this theorem as a **Ricci flow with S-surgery.**

4.3. Finite time extinction.

In order to prove the geometrization conjecture, one must further analyze the behavior of the Ricci flow with **S**-surgery near infinity. However, to prove Thurston's elliptization conjecture, it suffices to show that the flow becomes extinct in finite time. This is achieved by analyzing the behavior of homotopy groups under Ricci flow.

Proposition 4.14. *(Extinction of π_2) Starting with any normalized initial data (M,g), we construct a Ricci flow with **S**-surgery $(M_k, g_k(t))$. There exists $\infty > T_2 > 0$ such that the second homotopy group $\pi_2(M_k^i)$ is trivial for $t > T_2$ where M_k^i denotes any connected component of $(M_k, g_k(t))$ and $t \in [t_k^-, t_k^+)$.*

Proof. See [25], Chapter 18 of [21]. □

This means that, after T_2, each component M_k^i is irreducible and is either aspherical or has a finite fundamental group. Being aspherical means $\pi_l(M_k^i)$ is trivial for $l > 1$. If a connected manifold M has finite fundamental group, then its universal covering space is a simply-connected compact three manifold. Thus its covering space is a homotopy sphere and M has nontrivial third homotopy group.

Proposition 4.15. *(Extinction of π_3) Starting with any normalized initial data (M, g) as in Proposition 4.14, there exist $\infty > T_3 > T_2 > 0$ such that the third homotopy group $\pi_3(M_k^i)$ is trivial for $t > T_3$ where M_k^i denotes a connected component of $(M_k, g_k(t))$ and $t \in [t_k^-, t_k^+)$.*

Proof. See [25], Chapter 18 of [21]. □

Thus, the components with finite fundamental groups become extinct after finite time, and only aspherical components survive to infinity. We can see that the effect of Ricci flow with **S**-surgery is

(1) Performing prime decomposition

(2) Discarding connected components that are 2-sphere bundles over circle or closed round three-manifolds

(3) Evolving aspherical factors into infinity.

Now we finally have the following theorem.

Theorem 4.16. *(Thurston's elliptization conjecture) Let (M, g) be a three-dimensional connected orientable Riemannian manifold. Assume (M, g) is normalized. Then the following four conditions are equivalent*

*(1) Ricci flow with **S**-surgery for (M, g) becomes extinct in finite time.*

(2) M is diffeomorphic to a connected sum of three-dimensional spherical space-forms and S^2 bundles over S^1.

(3) The fundamental group of M is a free product of finite groups and infinite cyclic groups.

(4) No prime factor of M is aspherical.

Proof. See corollary 0.5 of [21] for complete proof. (2) is a direct consequence of (1). We show that (3) implies (1). If $\pi_1(M)$ is a nontrivial free product, then M must be a nontrivial connected sum of prime manifolds. After T_2, M is decomposed into prime manifolds whose fundamental groups are finite or isomorphic to $\mathbb{Z}$. But such prime manifolds have nontrivial π_2 or π_3. Thus, after time T_3, all of them become extinct. □

Corollary 4.17. *(Poincaré conjecture) Let M be closed smooth three-manifold with $\pi_1(M) = 0$. Then M is diffeomorphic to S^3.*

Proof. $\pi_1(M)$ satisfies Condition (3) of Theorem 4.16. Thus, M is diffeomorphic to a connected sum of three-dimensional spherical space-forms and S^2 bundles over S^1. However, $\pi_1(M)$ is trivial, which implies that M is diffeomorphic to S^3. □

Appendix A. Ricci flows and comparison geometry

Theorem A.1. *(Short time existence) Given a smooth metric g_0 on a closed manifold M, there exist $\delta > 0$ and a smooth family of metrics $g(t)$ for $t \in [0, \delta]$ such that*

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g)$$

where $t \in [0, \delta], g(0) = g_0$. Moreover, suppose that we have Ricci flows with initial conditions (M, g_0) at time 0 defined respectively on time intervals I and I'. Then these flows agree on $I \cap I'$.

Proof. See [12] [28]. □

We now present two spceial solutions of Ricci flow.

Example A.2. (Einstein manifolds) If we take a metric g_0 such that $\operatorname{Ric}(g_0) = \lambda g_0$ for some constant $\lambda \in \mathbb{R}$, then a solution $g(t)$ of Ricci flow with $g(0) = g_0$ is given by $g(t) = (1-2\lambda t)g_0$. Note that the Ricci curvature $\operatorname{Ric}_{ij} = \partial_l \Gamma^l_{ij} - \partial_j \Gamma^l_{il} + \Gamma^l_{lt}\Gamma^t_{ij} - \Gamma^l_{ip}\Gamma^p_{lj}$ is invariant under re-scaling of the metric. If g_0 is of constant positive curvature, we say $g(t)$ is a **shrinking family** or **shrinking round Ricci flow**.

Example A.3. (Ricci solitons) Any time dependent family of smooth vector fields $X(t)$ on a closed manifold M can generate a family of diffeomorphisms ψ_t. In other words, for a smooth function ϕ, we have $X(\psi_t(y), t)\phi = \partial_t \phi(\psi_t(y))$. If we have a metric g_0, a vector field Y and $\lambda \in \mathbb{R}$ (all independent of time) such that $-2\operatorname{Ric}(g_0) = L_Y g_0 - 2\lambda g_0$, then after setting $g(t) = g_0$ and $\sigma(t) = 1 - 2\lambda t$, if we also integrate the family of vector fields $X(t) := Y/\sigma(t)$ to give a family of ψ_t with ψ_0 the identity, then $\hat{g}(t) := \sigma(t)\psi_t^*(g(t))$ is a Ricci flow with $\hat{g}(0) = g_0$. This follows from:

$$\frac{\partial \hat{g}(t)}{\partial t} = \frac{d\sigma(t)}{dt}\psi_t^*(g_0) + \sigma\psi_t^*(L_X g_0) = \psi_t^*(-2\lambda g_0 + L_Y g_0) = -2\operatorname{Ric}(\psi_t^* g_0) = -2\operatorname{Ric}(\hat{g}).$$

Such a flow is called a **shrinking Ricci soliton** if $\lambda > 0$. If there is a smooth function such that $\nabla f = Y$, where ∇ denotes the gradient operator of g_0, and $\lambda > 0$, then such a flow is called a **gradient shrinking soliton**. In this case, we have $L_{\nabla f} g_0 = 2\operatorname{Hess}_{g_0}(f)$, and $\operatorname{Hess}_{g_0}(f) + \operatorname{Ric}(g_0) = \lambda g_0$.

Theorem A.4. *(Roundness) Let $(M, g(t))$ be a Ricci flow, where M is a closed three-dimensional manifold. Suppose there are positive constants $A < B$ so that at time $t = 0$ we have $A < \operatorname{Ric} < B$ pointwise in the sense of operators. Then for any positive number γ there is a constant C such that*

$$|\operatorname{Ric} - Rg/3| \leq \gamma R + C,$$

where C depends on A, B and γ.

Proof. See [30]. □

Theorem A.5. *(Hamilton's Harnack inequality) Let $(M, g(t))$ be a Ricci flow such that, for each time slice $t_0 \in (T_0, T_1)$, $(M, g(t_0))$ is a complete Riemannian manifold with bounded non-negative curvature operator. Then for any vector field X,*

$$\frac{\partial R}{\partial t} + \frac{R}{t - T_0} + 2\langle \nabla R, X\rangle + 2\operatorname{Ric}(X, X) \geq 0,$$

where R, Ric, ∇, and the pairing of ∇R and X are defined with respect to $g(t)$.

Proof. See [14]. □

Theorem A.6. *(2-dimensional κ-solutions) The only two-dimensional κ-solutions are the shrinking round spheres and shrinking round real projective spaces.*

Proof. See [15]. □

Theorem A.7. *(Splitting off a line) Suppose (M, g) is a complete n-dimensional Riemannian manifold with non-negative sectional curvature. Let $p \in M$ be fixed and $p_k \in M$ a sequence of points and $\mu_k > 0$ a sequence of real numbers such that $d_g(p, p_k) \to \infty$ and $\mu_k d_g(p, p_k) \to \infty$ as $k \to \infty$. Suppose also that $(M, \mu_k^2 g, p_k)$ converge to a Riemannian manifold $\hat{M}$. Then the limit $\hat{M}$ is isometric to a product of form $N \times \mathbb{R}$, where N is a Riemannian manifold with non-negative sectional curvature.*

Proof. See [9]. □

Theorem A.8. *Suppose $(M, g(t))$ is a Ricci flow on a n-dimensional closed manifold M, for $t \in [0, T]$. If $R \geq A$ at time $t = 0$, then for all times $t \in [0, T]$,*

$$R \geq \frac{A}{1 - \frac{2At}{n}}$$

Proof. See [30]. □

Theorem A.9. *Suppose that $(M, g(t))$, $t \in [0, T]$, is a Ricci flow of complete, connected Riemannian 3-manifolds with $\mathrm{Rm} \geq 0$. Suppose that $(M, g(0))$ is not flat and that for some $x \in M$ the endomorphism $\mathrm{Rm}(x, T)$ has a zero eigenvalue. Then M has a cover $\hat{M}$ such that, denoting the induced family of metrics on this cover by $\hat{g}(t)$, we have $(\hat{M}, \hat{g}(t))$ splits as a product*

$$(N, h(t)) \times (\mathbb{R}, ds^2),$$

where $(N, h(t))$ is a surface of positive curvature for all time t. The second factor $(\mathbb{R}, ds^2)$ denotes a trivial flow on $\mathbb{R}$.

Proof. See Corollary 4.20 of [21]. □

Acknowledgments

I would like to express my heartfelt gratitude to my mentor, Ben, who helped me discover this fascinating topic and provided feedback on the paper. I extend my deepest appreciation to Peter May for organizing this exceptional REU program and for taking the time to review my paper.

References

[1] Agol, Ian, Peter Storm, and William Thurston. "Lower bounds on volumes of hyperbolic Haken 3-manifolds." Journal of the American Mathematical Society 20, no. 4 (2007): 1053-1077.

[2] Bamler, Richard H., and Bruce Kleiner. "Uniqueness and stability of Ricci flow through singularities." Acta Mathematica 228, no. 1 (2022): 1-215.

[3] Bamler, Richard, and Bruce Kleiner. "Ricci flow and diffeomorphism groups of 3-manifolds." Journal of the American Mathematical Society 36, no. 2 (2023): 563-589.

[4] Bamler, Richard. "Long-time behavior of 3–dimensional Ricci flow: introduction." Geometry and Topology 22, no. 2 (2018): 757-774.

[5] Danny Calegari. 3-manifolds. in preparation. https://math.uchicago.edu/ dannyc/books/3manifolds/3manifolds.html

[6] Cao, Huai-Dong, and Xi-Ping Zhu. "Hamilton-Perelman's Proof of the Poincaré Conjecture and the Geometrization Conjecture." arXiv preprint math/0612069 (2006).

[7] Cheeger, Jeff, David G. Ebin, and David Gregory Ebin. Comparison theorems in Riemannian geometry. Vol. 9. Amsterdam: North-Holland publishing company, 1975.

[8] Chen, Bing-Long, and Xi-Ping Zhu. "Complete Riemannian manifolds with pointwise pinched curvature." Inventiones mathematicae 140 (2000): 423-452.

[9] Chen, Bing-Long, and Xi-Ping Zhu. "Ricci flow with surgery on four-manifolds with positive isotropic curvature." Journal of Differential Geometry 74, no. 2 (2006): 177-264.

[10] Chow, Bennett, and Dan Knopf. The Ricci Flow: An Introduction: An Introduction. Vol. 1. American Mathematical Soc., 2004.

[11] Donaldson, Simon K. "An application of gauge theory to four-dimensional topology." Journal of Differential Geometry 18, no. 2 (1983): 279-315.

[12] Hamilton, Richard S. "Three-manifolds with positive Ricci curvature." Journal of Differential geometry 17, no. 2 (1982): 255-306.

[13] Hamilton, Richard S. "A compactness property for solutions of the Ricci flow." American journal of mathematics 117, no. 3 (1995): 545-572.
[14] Hamilton, Richard S. "The Harnack estimate for the Ricci flow." Journal of Differential Geometry 37, no. 1 (1993): 225-243.
[15] Hamilton, Richard. "The formations of singularities in the Ricci Flow." Surveys in differential geometry 2, no. 1 (1993): 7-136.
[16] Kleiner, Bruce, and John Lott. "Singular Ricci flows I." Acta Math 219 (2017): 65-134.
[17] Kleiner, Bruce, and John Lott. "Notes on Perelman's papers." Geometry and Topology 12, no. 5 (2008): 2587-2855.
[18] Lowe, Ben, and Andre Neves. "Minimal surface entropy and average area ratio." arXiv preprint arXiv:2110.09451 (2021).
[19] Lee, John M. Introduction to Riemannian manifolds. Vol. 2. Cham: Springer, 2018.
[20] Marques, Fernando Codá. "Deforming three-manifolds with positive scalar curvature." Annals of mathematics (2012): 815-863.
[21] Morgan, John W., and Gang Tian. "Ricci flow and the Poincaré conjecture." arXiv preprint math/0607607 (2006).
[22] Morgan, John, and Gang Tian. The geometrization conjecture. Vol. 5. American Mathematical Soc., 2014.
[23] Perelman, Grisha. "The entropy formula for the Ricci flow and its geometric applications." arXiv preprint math/0211159 (2002).
[24] Perelman, Grisha. "Ricci flow with surgery on three-manifolds." arXiv preprint math/0303109 (2003).
[25] Perelman, Grisha. "Finite extinction time for the solutions to the Ricci flow on certain three-manifolds." arXiv preprint math/0307245 (2003).
[26] Poincaré, Henri. "Sur l'uniformisation des fonctions analytiques." (1908): 1-63.
[27] Rothaus, O. S. "Diffusion on compact Riemannian manifolds and logarithmic Sobolev inequalities." Journal of functional analysis 42, no. 1 (1981): 102-109.
[28] Shi, Wan-Xiong. "Deforming the metric on complete Riemannian manifolds." Journal of Differential Geometry 30, no. 1 (1989): 223-301.
[29] Tao, Terence. "Perelman's proof of the Poincaré conjecture: a nonlinear PDE perspective." arXiv preprint math/0610903 (2006).
[30] Topping, Peter. Lectures on the Ricci flow. Vol. 325. Cambridge University Press, 2006.