THE HAUSDORFF DIMENSION: CONSTRUCTION AND METHODS OF CALCULATION

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ABSTRACT. This paper provides an in-depth overview of the Hausdorff dimension, a fundamental concept in fractal geometry. We begin with an overview of the necessary preliminaries from measure theory, including definitions of σ-algebras and measures. We then introduce the concept of fractal dimensions, focusing specifically on the Hausdorff Dimension where we present three methods for calculating the Hausdorff Dimension: Upper and Lower Bound Estimation, the Projection Theorem, and the Similarity Dimension. The applications of these methods are demonstrated through several examples of fractals such as the Cantor set, Sierpinski's triangle, and the Koch Curve. The paper concludes with a discussion of the broader implications of these methods and their relevance in other fields of mathematics.

CONTENTS

1. INTRODUCTION

Fractals, characterized by their self-similarity and intricate detail at every scale, have emerged as a fascinating and complex subject of study in modern mathematics. These sets, often non-integer dimensional, defy the traditional geometric framework used to describe simpler shapes such as lines, circles, and planes. Instead, fractals exhibit a recursive structure, where each part mirrors the whole, creating infinite complexity within a bounded space. The challenge in analyzing fractals lies not only in understanding their visual intricacies but also in rigorously defining and calculating their dimensional properties.

This paper focuses on the Hausdorff Dimension, exploring its definition, properties, and methods for calculation. Section 2 offers a review of foundational concepts

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in measure theory, which form the basis for understanding the Hausdorff measure and dimension. Section 3 delves into the construction of the Hausdorff Measure and Dimension. Section 4 presents three different methods for calculating the Hausdorff Dimension, including examples of well-known fractals to demonstrate these methods in practice.

2. Overview of Measure Theory

Our following overview of essential concepts in measure theory mainly follows contents from [2]. We shall prove the Caratheodory Theorem, which is central in measure theory and fundamental to the construction of the Hausdorff measure in the next chapter.

Definition 2.1. A *σ-algebra* $\mathcal{F}$ on a set $\mathcal{X}$ is a collection of subsets of $\mathcal{X}$ such that:

(1) $\mathcal{X} \in \mathcal{F}$.
(2) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.
(3) If $\{A_i\} \subset \mathcal{F}$ (countable collection), then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Then, $(\mathcal{X}, \mathcal{F})$ is called a *measurable space*.

Definition 2.2. Let $\mathcal{X}$ be a set. An *outer measure* μ^* is a function that assigns a non-negative number to each subset of $\mathcal{X}$ such that:

(1) **Non-negativity**: $\mu(A) \geq 0$ for all $A \in \mathcal{F}$.
(2) **Null Empty Set**: $\mu(\emptyset) = 0$.
(3) **Monotonicity:** If $A, B \in \mathcal{X}$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$.
(4) **Sub-additivity:** If $\{A_i\} \in \mathcal{F}$ and $A = \bigcup_{i=1}^{\infty} A_i$, then $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$.

Definition 2.3. Let $(\mathcal{X}, \mathcal{F})$ be a measurable space. A *measure* μ is a function that assigns a non-negative number to each set in $\mathcal{F}$ such that:

(1) **Non-negativity**: $\mu(A) \geq 0$ for all $A \in \mathcal{F}$.
(2) **Null Empty Set**: $\mu(\emptyset) = 0$.
(3) **Monotonicity:** If $A, B \in \mathcal{X}$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$.
(4) **Countable Additivity**: For any countable collection of *disjoint* sets $\{A_i\} \subset \mathcal{F}$,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

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Then, $(X, \mathcal{F}, \mu)$ is called a *measure space*.

Remark 2.4. The primary distinction between a measure and an outer measure lies in their respective additive properties. An outer measure satisfies only countable sub-additivity whereas a measure satisfies the stronger condition of countable additivity.

Definition 2.5 (Carathéodory Criterion)**.** Let μ^* be an outer measure on $\mathcal{X}$. A set F is μ^*-measurable if for every set $S \in \mathcal{X}$,

$$\mu^*(S) = \mu^*(S \cap F) + \mu^*(S \cap F^c). \tag{2.6}$$

We now introduce one of the most important theorems in measure theory: the Carathéodory Theorem.

Theorem 2.7 (Carathéodory Theorem)**.** *If μ^* is an outer measure on $\mathcal{X}$, the collection $\mathcal{F}$ of μ^*-measurable sets is a σ-algebra.*

Proof. The following proof comes from [2].

We want to show that $\mathcal{F}$ satisfies all three conditions in Definition 2.1. It is easy to check that $\mathcal{X}$ is in $\mathcal{F}$ and that $\mathcal{F}$ is closed under complements. Thus, we only have to consider closure of $\mathcal{F}$ under countable unions. We do this in two steps.

Step 1: Closure under finite union of sets.

For all $S \subseteq \mathcal{X}$, if $F_1, F_2 \in \mathcal{F}$, the following inequality holds:

$$\begin{aligned}\mu^*(S) &= \mu^*(S \cap F_1) + \mu^*(S \cap F_1^c) \\ &= [\mu^*(S \cap F_1 \cap F_2) + \mu^*(S \cap F_1 \cap F_2^c)] \\ &+ [\mu^*(S \cap F_1^c \cap F_2) + \mu^*(S \cap F_1^c \cap F_2^c)] \\ &\geq \mu^*(S \cap (F_1 \cup F_2)) + \mu^*(S \cap (F_1 \cup F_2)^c) \\ &\geq \mu^*(S)\end{aligned}$$

Thus,

$$\mu^*(S) = \mu^*(S \cap (F_1 \cup F_2)) + \mu^*(S \cap (F_1 \cup F_2)^c). \tag{2.8}$$

By (2.6), this shows that $F_1 \cup F_2 \in \mathcal{F}$.

Step 2: Closure under countable union of pairwise disjoint sets.

Let $\{F_i\}$ be a set of pairwise disjoint sets in $\mathcal{F}$ where $J_n = \bigcup_{i=1}^n F_i$ and $J = \bigcup_{i=1}^\infty F_i$. If $S \subseteq X$,

$$\begin{aligned}\mu^*(S \cap J_n) &= \mu^*(S \cap J_n \cap F_n) + \mu^*(S \cap J_n \cap F_n^c) \\ &= \mu^*(S \cap F_n) + \mu^*(S \cap J_{n-1}) \\ &= \mu^*(S \cap F_n) + \mu^*(S \cap F_{n-1}) + \mu^*(S \cap J_{n-2}) \\ &\vdots \\ &= \mu^*(S \cap F_n) + \mu^*(S \cap F_{n-1}) + \cdots + \mu^*(S \cap F_1) \\ &= \sum_{i=1}^n \mu^*(S \cap F_i).\end{aligned}$$

From Step 1,

$$\begin{aligned}\mu^*(S) &= \mu^*(S \cap J_n) + \mu^*(S \cap J_n^c) \\ &= \sum_{i=1}^n \mu^*(S \cap F_i) + \mu^*(S \cap J_n^c) \\ &\geq \sum_{i=1}^n \mu^*(S \cap F_i) + \mu^*(S \cap J^c).\end{aligned}$$

Then, taking the limit $n \to \infty$ and using the fact that μ^* is an outer measure, we have the following inequalities:

$$\begin{aligned}\mu^*(S) &\geq \sum_{i=1}^{\infty} \mu^*(S \cap F_i) + \mu^*(S \cap J^c) \\ &\geq \mu^*\left(\bigcup_{i=1}^{\infty}(S \cap F_i)\right) + \mu^*(S \cap J^c) \\ &= \mu^*(S \cap J) + \mu^*(S \cap J^c) \\ &\geq \mu^*(S).\end{aligned}$$

Thus, we can conclude that

$$\mu^*(S) = \mu^*(S \cap J) + \mu^*(S \cap J^c) = \mu^*(S \cap \bigcup_{i=1}^{\infty} F_i) + \mu^*(S \cap \bigcup_{i=1}^{\infty} F_i^c). \tag{2.9}$$

Once again, by (2.6), this shows that $\bigcup_{i=1}^{\infty} F_i \in \mathcal{F}$.

Step 3: Closure under countable union of possibly overlapping sets.

Suppose $C_1, C_2, ...$ are sets in $\mathcal{F}$. Let $F_i = C_i - \left(\bigcup_{j=1}^{i-1} F_j\right)$, which is a series of pairwise disjoint sets. Then, applying Step 2, we have

$$\bigcup_{i=1}^{\infty} C_i = \bigcup_{i=1}^{\infty} F_i \in \mathcal{F}.$$

This proves that $\mathcal{F}$ is a σ-algebra.

□

Thus, we have proved the Caratheodory Theorem. We are left with one last corollary that helps us construct a measure out of an outer measure.

Corollary 2.10. *If μ is the restriction of μ^* to $\mathcal{F}$, then μ is a measure. Moreover, $\mathcal{F}$ contains all the null sets.*

Proof. Continuing with the definition of J and $\{F_i\}$ from above, we have already proven that

$$\mu^*(S) = \sum_{i=1}^{\infty} \mu^*(S \cap F_i) + \mu^*(S \cap J^c). \tag{2.11}$$

If we replace S with J,

$$\mu^*(J) = \sum_{i=1}^{\infty} \mu^*(J \cap F_i) + \mu^*(J \cap J^c) = \sum_{i=1}^{\infty} \mu^*(J \cap F_i) = \sum_{i=1}^{\infty} \mu^*(F_i).$$

Thus, μ is countably additive and μ is a measure.

If $\mu^*(C) = 0$,

$$\mu^*(S) = \mu^*(S \cap C) + \mu^*(S \cap C^c) = \mu^*(S \cap C^c) \leq \mu^*(S).$$

Thus, $\mathcal{F}$ contains all the null sets.

□

3. Fractal Dimensions

In this paper, we will focus specifically on the Hausdorff Dimension using [3] and [4].

Let X be a metric space.

Definition 3.1. Let $U \subseteq X$. The *diameter* of U is defined as:

$$|U| = sup\{|x - y| : x, y \in U\}$$

Definition 3.2. Let $E \subseteq X$. A δ-*cover* of E is a set $\{A_i\}$ such that:

(1) $A \subseteq \bigcup_{i=1}^{\infty} A_i$ and $A_i \subseteq X$ for all i.

(2) $|A_i| \leq \delta$ for all i.

Definition 3.3. Let $E \subseteq X$ and let $\{U_i\}$ be all countable δ-covers of E.
$\forall s \geq 0$ and $\delta > 0$, define:

$$\mathcal{H}^s_\delta(E) = \inf \sum_{i=1}^{\infty} |U_i|^s \tag{3.1}$$

As $\delta \to 0$, we get the *Hausdorff s-dimensional outer measure* of E:

$$\mathcal{H}^s(E) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(E) = \sup_{\delta > 0} \mathcal{H}^s_\delta(E). \tag{3.2}$$

We can now apply concepts from Section 2 to the Hausdorff s-dimensional outer measure.

Proposition 3.4. $\mathcal{H}^s(E)$ *is indeed an outer measure.*

Proof. We want to show that $\mathcal{H}^s(E)$ satisfies all four conditions from Definition 2.2.

Again, non-negativity and the null empty set property are straightforward from the definition of $\mathcal{H}^s_\delta(E)$, and monotonicity is obvious since any countable cover $\{V_i\}_{i=1}^{\infty}$ of B is also a countable cover of A. Thus, we focus on countable sub-additivity.

Let $\{A_i\}_{i=1}^{\infty}$ be a countable collection of subsets of X. We want to show that:

$$\mathcal{H}^s\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathcal{H}^s(A_i).$$

For any $\epsilon > 0$ and any i, there exists a countable cover $\{U_{i,j}\}_{j=1}^{\infty}$ of A_i such that $|U_{i,j}| < \delta$ and:

$$\sum_{j=1}^{\infty} |U_{i,j}|^s < \mathcal{H}^s(A_i) + \frac{\epsilon}{2^i}.$$

The union of all these covers $\{U_{i,j}\}_{i,j=1}^{\infty}$ forms a cover for $\bigcup_{i=1}^{\infty} A_i$, and since $|U_{i,j}| < \delta$ for all i, j, this cover satisfies the requirements for $\mathcal{H}^s$ with parameter δ:

$$\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} U_{i,j}.$$

Now consider the total sum over all these coverings:

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |U_{i,j}|^s.$$

Substituting the inequalities for each A_i:

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |U_{i,j}|^s < \sum_{i=1}^{\infty} \left(\mathcal{H}^s(A_i) + \frac{\epsilon}{2^i} \right).$$

Using the linearity of sums:

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |U_{i,j}|^s < \sum_{i=1}^{\infty} \mathcal{H}^s(A_i) + \sum_{i=1}^{\infty} \frac{\epsilon}{2^i}.$$

Since $\sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon$, we have:

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |U_{i,j}|^s < \sum_{i=1}^{\infty} \mathcal{H}^s(A_i) + \epsilon.$$

Taking the infimum over all such covers $\{U_{i,j}\}_{i,j=1}^{\infty}$ and then the limit as $\delta \to 0$, we obtain:

$$\mathcal{H}^s \left(\bigcup_{i=1}^{\infty} A_i \right) \leq \sum_{i=1}^{\infty} \mathcal{H}^s(A_i) + \epsilon.$$

Since this inequality holds for any $\epsilon > 0$, we conclude:

$$\mathcal{H}^s \left(\bigcup_{i=1}^{\infty} A_i \right) \leq \sum_{i=1}^{\infty} \mathcal{H}^s(A_i).$$

Thus, $\mathcal{H}^s(E)$ is an outer measure.

□

Remark 3.5. If we restrict, $\mathcal{H}^s$ to the collection of $\mathcal{H}^s$-measurable sets, by Lemma 2.10, we get the *Hausdorff s-dimensional measure.*

Definition 3.6. For $E \subseteq \mathbb{R}^n$, we define the *Hausdorff dimension* of E $\dim_H E$ as follows:

(1) $\mathcal{H}^s(E) = \infty$ if $0 \leq s < \dim_H E$
(2) $\mathcal{H}^s(E) = 0$ if $\dim_H E < s < \infty$

In other words,

$$\dim_H E = \inf\{s \geq 0 : \mathcal{H}^s(E) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(E) = \infty\}. \tag{3.7}$$

Proposition 3.8. *The Hausdorff dimension exists and is unique.*

Proof. Check [10]. □

4. Methods for Calculating Hausdorff Dimension

We will explore three different methods of calculating the Hausdorff Dimention.

4.1. Upper and Lower Bound Estimation.

The following contents are from [4].

Let A be a set in a metric space.

Step 1: Upper Bound

This is usually a simple and straightforward calculation. We need to find a specific covering such that $\sum |U_i|^s < \infty$. Then, $\dim_H A \leq s$.

Step 2: Lower Bound

Definition 4.1. A *mass distribution* on X is a measure μ that satisfies the following properties:
(1) $\mu(X) > 0$
(2) μ is non-negative
(3) μ is often locally finite

Theorem 4.2. ***Mass Distribution Principle***

Let μ be a mass distribution on A. For some s, there exist $c > 0$ and $\delta > 0$ such that $\mu(U) \leq c|U|^s$ holds for all sets U satisfying $|U| \leq \delta$. It follows that

$$\mathcal{H}^s(A) \geq \frac{\mu(A)}{c} \quad \text{and} \quad s \leq \dim_H A.$$

Proof. Let μ be a measure on A that satisfies the conditions above. We want to show that

$$\mathcal{H}^s(A) \geq \frac{\mu(A)}{c}.$$

Consider any cover $\{U_i\}$ of A with $|U_i| \leq \delta$. Then, by the assumption $\mu(U_i) \leq c|U_i|^s$, if we take the sum over all U_i, we have

$$\sum_i \mu(U_i) \leq c \sum_i |U_i|^s.$$

By Definition 2.3, $\mu(A) \leq \sum_i \mu(U_i)$. It follows that

$$\mu(A) \leq c \sum_i |U_i|^s.$$

Since this is true for any covering $\{U_i\}$ of A with $|U_i| \leq \delta$, we can take the infimum over all such covers:

$$\mu(A) \leq c \inf \left\{ \sum_i |U_i|^s : A \subseteq \bigcup_i U_i,\ |U_i| \leq \delta \right\}.$$

This infimum expression is precisely the definition of $\mathcal{H}^s_\delta(A)$. Thus,

$$\mu(A) \leq c\mathcal{H}^s_\delta(A).$$

As $\delta \to 0$,

$$\mu(A) \leq c\mathcal{H}^s(A),$$

so

$$\mathcal{H}^s(A) \geq \frac{\mu(A)}{c}.$$

Since $\mathcal{H}^s(A) \geq \frac{\mu(A)}{c} > 0$, the Hausdorff s-measure of A is positive. Thus, s is a lower bound on the Hausdorff dimension $\dim_H A$ and $s \leq \dim_H A$.

□

Example 4.3. Square

Let us start with an easy example. Let the square be F.

Upper Bound:

To find an upper bound for the Hausdorff dimension, consider any $s > 2$.

Cover the square with smaller squares of side length ϵ. The area (measure) of each small square is ϵ^2. The total number of smaller squares needed to cover the original square is $\left(\frac{L}{\epsilon}\right)^2$.

For dimension s, the s-dimensional Hausdorff measure is estimated by summing ϵ^s for each small square:

$$\mathcal{H}^s(F) \approx \left(\frac{L}{\epsilon}\right)^2 \cdot \epsilon^s = L^2 \cdot \epsilon^{s-2}.$$

As $\epsilon \to 0$, $\epsilon^{s-2} \to 0$, so $\mathcal{H}^s(F) \to 0$.

This shows that for $s > 2$, the Hausdorff s-dimensional measure is zero, suggesting that the Hausdorff dimension cannot be greater than 2.

Lower Bound:

To find a lower bound for the Hausdorff dimension, we use the Mass Distribution Principle.

Again, cover the square with smaller squares of side length ϵ. The number of smaller squares is $\left(\frac{L}{\epsilon}\right)^2$, so

$$\mathcal{H}^s(F) \approx \left(\frac{L}{\epsilon}\right)^2 \cdot \epsilon^s = \frac{L^2}{\epsilon^{2-s}}.$$

For $s < 2$, as $\epsilon \to 0$, $\mathcal{H}^s(F) \to \infty$. For $s = 2$, $\mathcal{H}^s(F) \approx \frac{L^2}{\epsilon^0} = L^2$, which is finite.

Thus, the Hausdorff measure is finite for $s = 2$, suggesting that the Hausdorff dimension is at least 2.

Combining the upper and lower bounds, we conclude that the Hausdorff dimension of the square is $\dim_H(F) = 2$.

Example 4.4. The Cantor Set

Let the Cantor Set be C.

Construction of the Cantor Set:

Step 1: Start with the interval $[0, 1]$.
Step 2: Remove the open middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$, leaving two intervals: $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$.
Step 3: Remove the middle third of each remaining interval, leaving:

$$\left[0, \frac{1}{9}\right], \left[\frac{2}{9}, \frac{1}{3}\right], \left[\frac{2}{3}, \frac{7}{9}\right], \left[\frac{8}{9}, 1\right].$$

Continue this process indefinitely.

The following figure is from [8].

FIGURE 1. The first seven stages of the Cantor Set

Upper Bound:

To find an upper bound for the Hausdorff dimension, we cover the Cantor Set with intervals whose lengths match the construction process.

At the n-th stage of the construction, there are 2^n intervals, each of length $\left(\frac{1}{3}\right)^n$. The s-dimensional Hausdorff measure is estimated by summing the s-th power of the lengths of these intervals:

$$\mathcal{H}^s(C) \approx \sum_{i=1}^{2^n} \left(\frac{1}{3^n}\right)^s = 2^n \left(\frac{1}{3^n}\right)^s = 2^n \cdot 3^{-ns} \approx (2 \cdot 3^{-s})^n. \tag{4.5}$$

To ensure that $\mathcal{H}^s(C)$ does not tend to infinity as $n \to \infty$, we need:

$$2 \cdot 3^{-s} \leq 1.$$

We now solve for s:

$$3^s \geq 2 \implies s \geq \frac{\log 2}{\log 3}.$$

Thus, if $s > \frac{\log 2}{\log 3}$, the Hausdorff s-measure $\mathcal{H}^s(C) = 0$. This implies that the Hausdorff dimension is at most $\frac{\log 2}{\log 3}$.

Lower Bound:

For the Cantor Set, we do not need to use the Mass Distribution Principle.

To find a lower bound for the Hausdorff dimension, we use the same approach but instead show that the Hausdorff measure becomes infinite for $s < \frac{\log 2}{\log 3}$.

To ensure that $\mathcal{H}^s(C)$ does not tend to zero as $n \to \infty$, we now need:

$$2 \cdot 3^{-s} \geq 1.$$

Once again, we solve for s:

$$3^s \leq 2 \implies s \leq \frac{\log 2}{\log 3}.$$

Thus, if $s < \frac{\log 2}{\log 3}$, the Hausdorff s-measure $\mathcal{H}^s(C) = \infty$. This implies that the Hausdorff dimension is at least $\frac{\log 2}{\log 3}$.

Combining the upper and lower bounds, we conclude that the Hausdorff dimension of the Cantor Set is $\dim_H(C) = \frac{\log 2}{\log 3}$.

4.2. **The Projection Theorem.** The following contents are from [9] and [4].
Let B be a set in a metric space.

Definition 4.6. The Grassmannian $\mathcal{G}_{n,k}$ (or $\mathrm{Gr}(k, n)$) is defined as the set of all k-dimensional linear subspaces of an n-dimensional vector space.

Theorem 4.7. *Marstrand's Projection Theorem.*
For almost all $\Pi \in \mathcal{G}_{n,k}$*,*

$$\dim_H(\mathrm{proj}_\Pi B) = \min\left(\dim_H B, 1\right). \tag{4.8}$$

Proof. A detailed proof is provided in [9]. □

Although this theorem cannot provide a definitive result, it has significant implications for estimating the Hausdorff Dimension of sets. For example, we can project sets in $\mathbb{R}^3$ into $\mathbb{R}^2$ (the 2-dimensional plane) to infer properties of the original set.

Example 4.9. The Sierpinski Triangle

Let the Sierpinski Triangle be S.

Construction of the Sierpinski Triangle

Step 1: Begin with an equilateral triangle.
Step 2: Divide the equilateral triangle into four smaller congruent equilateral triangles.
Step 3: Remove the middle triangle (the one formed in the center) from the shape. For each of the three remaining smaller triangles, repeat the subdivision and removal process.

Continue this process indefinitely.

The following figure is from [8].

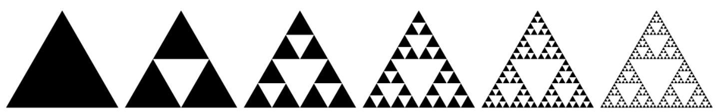

FIGURE 2. The first six stages of the Sierpinski Triangle

For almost all projections of the Sierpinski triangle onto a 1-dimensional line (either vertically or horizontally, or in other directions), the Hausdorff dimension of the projection will be 1. By (4.7), we can estimate that $\dim_H S \geq 1$.

We will later see a straightforward and direct method of calculating the Hausdorff Dimension of the Sierpinski Triangle using its self-similar properties.

4.3. **The Similarity Dimension.** The following contents are from [5], [6], [7], and [8].

Definition 4.10. Let X be a metric space. A *similarity* is a map $T : X \to X$ of the form $T(x) = r\mathcal{O}(x) + b$ such that:
(1) **Scaling Ratio**: $r > 0$,
(2) **Rotation/Reflection**: $\mathcal{O}$ is an orthonormal transformation, and
(3) **Translation Vector**: $b \in \mathbb{R}^n$.

Definition 4.11. Given n similarities $T_1, T_2, ..., T_n$, a set S is *self-similar* if it is unique, compact, non-empty, and satisfies:

$$S = \bigcup_{i=1}^{N} T_i(S).$$

Definition 4.12. Let $\{T_1, \ldots, T_n\}$ be a collection of similarities with corresponding scaling ratios $\{r_1, \ldots, r_n\}$ acting on a set S. We define the *Similarity dimension* of S $\dim_{\text{sim}}(S)$ such that:

$$\sum_{i=1}^{n} r_i^{\dim_{\text{sim}}(S)} = 1. \tag{4.13}$$

If $r_1 = r_2 = \ldots = r_n$, then

$$\dim_{\text{sim}}(S) = \frac{\log n}{\log \frac{1}{r}}. \tag{4.14}$$

Proposition 4.15. *The Similarity dimension exists and is unique.*

Proof. Check [6]. □

Theorem 4.16.

$$\dim_H(F) = \dim_{sim}(S).$$

Proof. A detailed proof is provided in [8]. □

Example 4.17. The Cantor Set

We can now apply Definition 4.12 and Theorem 4.16 for an alternative and simpler method of computing the Hausdorff Dimension of the Cantor Set.

The Cantor Set has two similarities $S_1, S_2 : [0,1] \to [0,1]$ with $S_1 : x \mapsto \frac{1}{3}x$ and $S_2 : x \mapsto \frac{1}{3}x + \frac{2}{3}$. The open set condition holds, which follows by considering the open interval $(0,1)$. $r_1 = r_2 = \frac{1}{3}$, so by (4.12) and Theorem 4.13,

$\dim_H(C) = \frac{\log 2}{\log 3}$.

Example 4.18. Sierpinski's Triangle

Once again, we apply Definition 4.12 and Theorem 4.16 for a more direct method of computing the Hausdorff Dimension of Sierpinski's Triangle.

Sierpinski's Triangle has three similarities with ratio $\frac{1}{2}$. By considering the interior of the first-stage triangle S_0, we verify that the open set condition is satisfied. Again, all ratios are equal, so we conclude that

$\dim_H(S) = \frac{\log 3}{\log 2}$.

Example 4.19. The Koch Curve

Let the Koch Curve be K.

Construction of the Koch Curve:

Step 1: Begin with a straight line segment, which we can denote as ℓ.
Step 2: Divide ℓ into three equal parts.
Step 3: Remove the middle segment and construct an equilateral triangle on this middle segment. This results in a new shape, which consists of four line segments (each of length $\frac{1}{3}$ of the original segment).

Continue this process indefinitely.

The following figure is from [8].

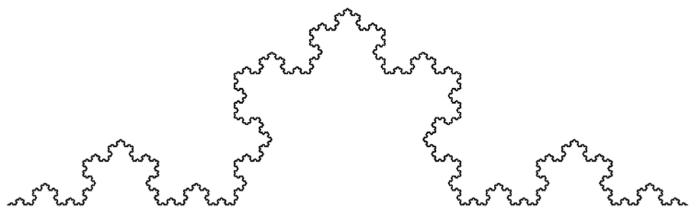

FIGURE 3. The Koch Curve

In the most concise terms possible, the Koch Curve has four similarities all with ratio $\frac{1}{3}$. Thus,

$$\dim_H(K) = \frac{\log(4)}{\log(3)}.$$

5. Conclusion

The study of the Hausdorff dimension offers a powerful framework for analyzing and interpreting the intricate complexities of fractal sets, enhancing our grasp of fractal geometry and prompting further research across various scientific fields. This paper began with an overview of measure theory, followed by an introduction to fractal dimensions with a focus on the Hausdorff dimension, and explored three methods for its calculation: Upper and Lower Bound Estimation, the Projection Theorem, and the Similarity Dimension, illustrated with examples like the Cantor set and Sierpinski's triangle. Applications of these methods go beyond mathematics, providing innovative ways to address complex systems in various different domains such as physics and computer science.

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